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On the Well-Posedness of Stochastic Partial Differential Equations with Locally Lipschitz Coefficients

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Abstract

We consider the stochastic partial differential equation (SPDE)

$$\partial_t u = \tfrac{1}{2}\partial_x^2 u + b(u) + \sigma(u)\dot{W},$$

where $u = u(t,x)$ is defined for $(t,x) \in (0,\infty) \times \mathbb{R}$ and $\dot{W}$ denotes space-time white noise. We prove that this SPDE is well posed solely under the assumptions that the initial condition $u(0)$ is bounded and measurable, and b and σ are locally Lipschitz continuous functions having at most linear growth with regularly behaved local Lipschitz constants. Our method is based on a truncation argument together with moment bounds and tail estimates of the truncated solution. The novelty of our method is in the pointwise nature of the truncation argument.

Keywords SPDEs · Space-time white noise · Existence and uniqueness

Mathematics Subject Classification 60H15 · 60H10 · 60H20

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1 Introduction

We revisit the well-posedness of the solution $u = \{u(t, x)\}_{t \geq 0, x \in \mathbb{R}}$ of the following stochastic PDE (SPDE):

$$\partial_t u(t\,, x) = \tfrac{1}{2}\partial_x^2 u(t\,, x) + b(t\,, u(t\,, x)) + \sigma(t\,, u(t\,, x))\dot{W}(t\,, x), \tag{1.1}$$

where $(t, x) \in (0, \infty) \times \mathbb{R}$, subject to $u(0, x) = u_0(x)$, and the forcing $\dot{W}$ is space-time white noise; that is, $\dot{W}$ is a generalized Gaussian random field with mean zero and

$$\mathrm{Cov}[\dot{W}(t\,, x)\,, \dot{W}(s\,, y)] = \delta_0(t - s)\delta_0(x - y) \quad \text{for all } t, s \geq 0 \text{ and } x, y \in \mathbb{R}.$$

It is well known that (1.1) is well posed when b and σ are Lipschitz in their spatial variable uniformly in their time variable; see for example Dalang [2] and Walsh [11]. We aim to extend this result to the setting where "Lipschitz" is replaced by "local Lipschitz with linear growth."

The present undertaking yields an infinite-dimensional version of one of the foundational results of the theory of stochastic differential equations (SDEs) which asserts that if $Y = \{Y_t\}_{t \geq 0}$ denotes a standard Brownian motion on $\mathbb{R}$ and $x_0 \in \mathbb{R}$ is non-random, then the one-dimensional Itô-type SDE

$$\mathrm{d}X_t = b(t\,, X_t)\,\mathrm{d}t + \sigma(t\,, X_t)\,\mathrm{d}Y_t \qquad (t > 0)$$

subject to $X_0 = x_0$ has a unique solution provided only that $b(t, x)$ and $\sigma(t, x)$ are locally Lipschitz in x with at-most linear growth, all valid uniformly in t. The preceding follows from the more classical result about SDEs with Lipschitz coefficients and a stopping time argument. See for example Exercise (2.10) of Revuz and Yor [8, p. 383]. It is possible to use essentially the same method in order to extend the preceding to the multidimensional setting where Y denotes a d-dimensional Brownian motion, $b : (0\,, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : (0\,, \infty) \times \mathbb{R}^d \to (\mathbb{R}^d)^2$.

An infinite-dimensional extension appears in a forthcoming monograph by Robert C. Dalang and Marta Sanz-Solé [4]. There, Dalang and Sanz-Solé show the well-posedness of (1.1), where instead $(t, x) \in (0, \infty) \times I$ in the case that $I \subset \mathbb{R}$ is a bounded interval (together with any of the usual boundary conditions). A key step in their analysis is the observation that if b and σ are locally Lipschitz in their space variable, uniformly in the time variable, then (1.3) below has a predictable random field solution up to the stopping time

$$\tau(I) = \inf\{t > 0 : \sup_{x \in I} |u(t\,, x)| = \infty\} \qquad [\inf \varnothing = \infty],$$

and $\mathrm{P}\{\tau(I) > 0\} = 1$ when $I \subset \mathbb{R}$ is bounded. Such stopping time arguments are typically not used to study equations on $\mathbb{R}_+ \times \mathbb{R}$ because one expects the resulting solution, if there is one, to be unbounded with probability one. Therefore, in order to produce solutions to the SPDE (1.1) one needs to introduce a different argument.

The purpose of this article is to provide one such argument: We use truncation, as has been done previously, but replace the stopping time argument by pointwise tail probability estimates for the truncated solution. In this way we are able to show that, for a wide family of locally Lipschitz functions b and σ of linear growth, and with high probability, the truncated solution is not large, uniformly in the truncation level.

Our method is elementary as it uses only standard SPDE estimates and more significantly works without need for any unnecessary technical assumptions. But we are quick to mention that other potential approaches to such problems already exist in the literature: It might be for example possible to adapt methods of finite-dimensional SDEs, such as weak existence (in the probabilistic sense) followed by strong uniqueness via a Yamada–Watanabe (see Kurtz [6]) or Gyöngy–Krylov (see Gyöngy and Krylov [5])-type arguments. In the infinite-dimensional context of SPDEs, weak existence for (1.1) was established in Shiga [10, Theorem 2.6] when additionally $b(0) \geq 0$ and $\sigma(0) = 0$. And Mytnik et al [7, Theorem 1.2] establish the weak existence for (1.1) when $b \equiv 0$, $\sigma =$ Hölder continuous with linear growth, all forced by high-dimensional, spatially correlated noise. There is also a method that involves transforming the SPDE into a random integral equation which can in turn be compared with a deterministic PDE; see Salins [9]. This method does not lend itself easily to the case that σ is non-constant unless there are additional constraints such as $b(0) = 0, \sigma(0) = 0$, and more stringent conditions on the initial function; see Chen et al [1] for the current state of the art of that method.

We impose the following assumptions on the initial profile u_0 and the coefficients b and σ in (1.1):

Assumption 1.1 $u_0 : \mathbb{R} \to \mathbb{R}$ is non-random, bounded, and measurable.

Assumption 1.2 The functions $b : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous in their space variable with at most linear growth, uniformly in their time variable. In other words, $0 < \text{Lip}_n(b), \text{Lip}_n(\sigma) < \infty$ and $0 \leq L_b, L_\sigma < \infty$, for all real numbers $n > 0$ where, for every space-time function ψ,

$$L_\psi = \sup_{t>0} \sup_{x\in\mathbb{R}} \frac{|\psi(t\,,x)|}{1+|x|}, \ \text{Lip}_n(\psi) = \sup_{t>0} \sup_{\substack{x,y\in[-n,n] \\ x\neq y}} \frac{|\psi(t\,,x)-\psi(t\,,y)|}{|y-x|}. \tag{1.2}$$

Let us recall that a random field solution to (1.1) is a predictable random field $u = \{u(t, x)\}_{t\geq 0, x\in\mathbb{R}}$ that satisfies the following integral equation:

$$u(t\,,x) = (p_t * u_0)(x) + \mathcal{I}_b(t\,,x) + \mathcal{I}_\sigma(t\,,x), \tag{1.3}$$

where the symbol "$*$" denotes convolution,

$$p_r(z) = (2\pi r)^{-1/2} \exp\{-z^2/(2r)\} \qquad \text{for all } r > 0 \text{ and } z \in \mathbb{R},$$

and $\mathcal{I}_b$ and $\mathcal{I}_\sigma$ are the following random fields,

$$\begin{aligned}\mathcal{I}_b(t\,,x) &= \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)b(s\,,u(s\,,y))\,\mathrm{d}s\,\mathrm{d}y,\\ \mathcal{I}_\sigma(t\,,x) &= \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma(s\,,u(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y),\end{aligned} \tag{1.4}$$

where the second (stochastic) integral is understood in the sense of Walsh [11].

As was mentioned in the Introduction, we will use a truncation argument. For every real number $N > 0$, we define $b_N : (0,\infty)\times\mathbb{R}\to\mathbb{R}$ and $\sigma_N : (0,\infty)\times\mathbb{R}\to\mathbb{R}$ as follows: For all $t > 0$ and $\psi : \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ let

$$\psi_N(t\,,x) = \psi(t\,,x)\mathbf{1}_{\{-\mathrm{e}^N\le x\le \mathrm{e}^N\}} + \psi(t\,,\mathrm{e}^N)\mathbf{1}_{\{x>\mathrm{e}^N\}} + +\psi(t\,,-\mathrm{e}^N)\mathbf{1}_{\{x<-\mathrm{e}^N\}}.$$

We will need the following assumption on the Lipschitz coefficients of b_N and σ_N. Set

$$L_{N,b} = \mathrm{Lip}_{\exp(N)}(b) \quad\text{and}\quad L_{N,\sigma} = \mathrm{Lip}_{\exp(N)}(\sigma), \tag{1.5}$$

Assumption 1.3 If $L_\sigma > 0$ then we assume that

$$L_{N,\sigma} = o(N^{3/8}) \quad\text{and}\quad L_{N,b}/L_{N,\sigma}^4 = \mathcal{O}(1) \qquad\text{as } N\to\infty. \tag{1.6}$$

If σ is bounded, then we assume that

$$L_{N,\sigma} = o(\mathrm{e}^{N/2}) \quad\text{and}\quad L_{N,b}/L_{N,\sigma}^4 = \mathcal{O}(1) \qquad\text{as } N\to\infty. \tag{1.7}$$

It has been widely believed for a long time that Theorem 1.4 ought to hold solely under the assumptions of locally Lipschitz coefficients b and σ with linear growth. The following is the main result of this paper and makes some partial success toward the resolution of this long-standing problem.

Theorem 1.4 *If (b,σ) satisfy Assumption 1.3, then (1.1) has a unique random field solution that satisfies the following:*

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}} \mathrm{E}\left(|u(t\,,x)|^k\right) < \infty \quad \textit{for all } T>0 \textit{ and } k\ge 1.$$

Next we make some remarks in order to highlight the limitations and scope of Theorem 1.4 and its proof.

Remark 1.5 The condition $L_{N,b}/L_{N,\sigma}^4 = \mathcal{O}(1)$ has new content only when $L_{N,\sigma}\to\infty$ as $N\to\infty$. Indeed, if $\sup_N L_{N,\sigma} < \infty$, then the first stated condition ensures that both b and σ are globally Lipschitz continuous.

Remark 1.6 Condition (1.6) is quite restrictive as it necessarily implies that $\mathrm{Lip}_n(\sigma) = o(|\log n|^{3/8})$ and $\mathrm{Lip}_n(b) = o(|\log n|^{3/2})$ as $n\to\infty$; see (1.5).

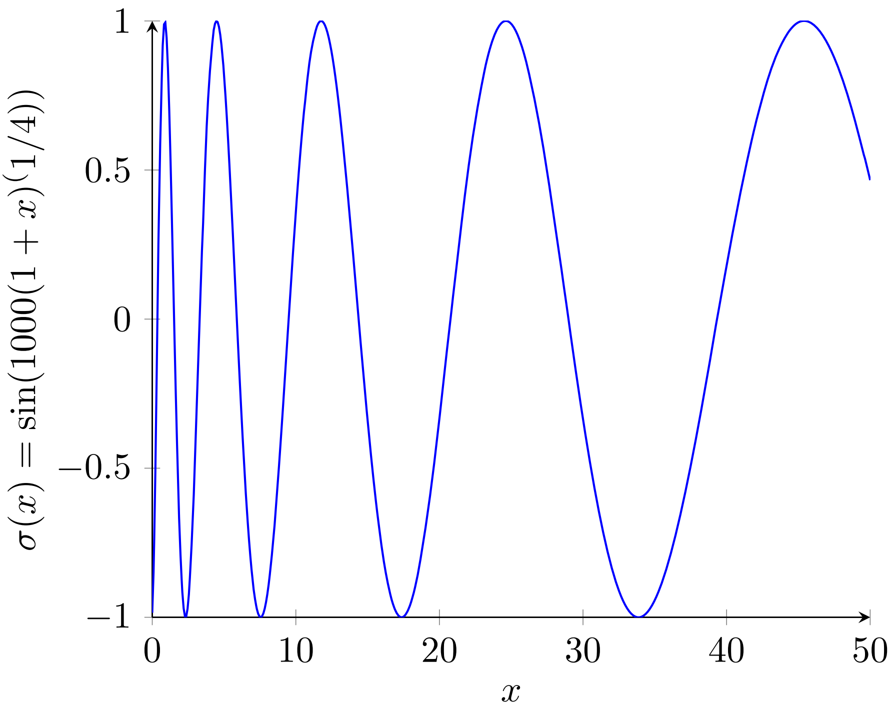

Fig. 1 An example of how Assumption 1.3 is less restrictive for the drift when σ fluctuates wildly

Remark 1.7 (*Oscillatory diffusion coefficients*) Theorem 1.4 has stronger content when σ is bounded for then (1.7) allows for a wide class of drifts b when σ is highly oscillatory. First, note that the first condition in (1.7) is equivalent to the statement that $\mathrm{Lip}_n(\sigma) = o(\sqrt{n})$ as $n \to \infty$. Now consider an oscillatory diffusion coefficient such as $\sigma(x) = \sin(1000(1 + |x|)^{1/4})$ for all $x \in \mathbb{R}$; see Fig. 1. Then, σ satisfies (1.7) together with every drift function that satisfies $\mathrm{Lip}_n(b) = o(n)$.

Remark 1.8 The method of proof of Theorem 1.4 allows for minor improvements of the first parts of (1.6) and (1.7). For example, the first condition in (1.6) can be improved slightly to $L_{N,\sigma} = o(N^{2/3})$ by, instead of choosing the parameter k as in (3.9), choosing it as $k(N) = A_1(N/T)^{1/2} L_{N,\sigma}^{-4/3}$ for a suitably small constant A_1 that is independent of N and T. Then one obtains the main part of the proof—that is (3.1) below—from the fact that $\|u_{N+1}(t,x) - u_N(t,x)\|_k \le \|u_{N+1}(t,x) - u_N(t,x)\|_{k(N)}$ for all large N [Jensen's inequality]. We have omitted the details of such improvements as they appear to be primarily technical in nature.

Let us end the Introduction with a brief outline of the paper. In §2 we introduce the truncated solution and develop some moment and tail estimates. The remaining details of the proof of Theorem 1.4 are gathered in §3 and use the earlier results of the paper.

Throughout this paper, we write $\|X\|_p = \{\mathrm{E}(|X|^p)\}^{1/p}$ for all $p \geq 1$ and $X \in L^p(\Omega)$. For every space-time function $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$, $\mathrm{Lip}(f)$ denotes the optimal Lipschitz constant of f; that is,

$$\mathrm{Lip}(f) = \sup_{t>0} \sup_{a,b\in\mathbb{R}:a\neq b} |f(t\,, b) - f(t\,, a)|/|b - a|.$$

In particular, f is globally Lipschitz continuous in x, uniformly in t, iff $\mathrm{Lip}(f) < \infty$. If f depends only on a spatial variable x, then $\mathrm{Lip}(f)$ still makes sense provided that we extend f to a space-time function as follows $f(t, x) = f(x)$, in the usual way.

2 Truncation and Preliminary Results

In this section, we give more information about the truncation argument referred to verbally in the introduction. We begin by recalling the definition of b_N and σ_N and note that they are globally Lipschitz functions. In fact,

$$\sup_{t>0} \sup_{x\in\mathbb{R}} \frac{|b_N(t\,, x)|}{1 + |x|} \leq L_b < \infty \quad \text{and} \quad \sup_{t>0} \sup_{x\in\mathbb{R}} \frac{|\sigma_N(t\,, x)|}{1 + |x|} \leq L_\sigma < \infty,$$

uniformly in $N > 0$. Thanks to standard theory [2, 11], the following SPDE has a predictable mild solution: For $(t, x) \in (0, \infty) \times \mathbb{R}$,

$$\partial_t u_N(t\,, x) = \tfrac{1}{2}\partial_x^2 u_N(t\,, x) + b_N(t\,, u_N(t\,, x)) + \sigma_N(t\,, u_N(t\,, x))\dot{W}(t\,, x), \quad (2.1)$$

subject to $u_N(0, x) = u_0(x)$. Moreover this solution is unique subject to

$$\sup_{t\in(0,T)} \sup_{x\in\mathbb{R}} \mathrm{E}\left(|u_N(t\,, x)|^k\right) < \infty \quad \text{for all } N, T > 0 \text{ and } k \geq 1$$

and jointly continuous in $L^k(\Omega)$. See Dalang and Sanz-Solé [4, Theorems 4.2.1 and 4.2.8]. As usual, (2.1) is short-hand for the random integral equation,

$$u_N(t\,, x) = (p_t * u_0)(x) + \mathcal{I}_{b_N}^N(t\,, x) + \mathcal{I}_{\sigma_N}^N(t\,, x), \quad (2.2)$$

where $\mathcal{I}_{b_N}^N$ and $\mathcal{I}_{\sigma_N}^N$ are the truncated analogous of the integrals in (1.4) and are given by

$$\begin{aligned} \mathcal{I}_{b_N}^N(t\,, x) &= \int_{(0,t)\times\mathbb{R}} p_{t-s}(y - x) b_N(s\,, u_N(s\,, y))\,\mathrm{d}s\,\mathrm{d}y, \\ \mathcal{I}_{\sigma_N}^N(t\,, x) &= \int_{(0,t)\times\mathbb{R}} p_{t-s}(y - x) \sigma_N(s\,, u_N(s\,, y))\, W(\mathrm{d}s\,\mathrm{d}y). \end{aligned} \quad (2.3)$$

The next result yields a moment estimate of the truncated solution which is similar to that in Theorem 6.3.2 of [4] for more general SPDEs. We describe the details in order to provide the explicit constants, and parameter dependencies, of the bound.

Proposition 2.1 *If $L_\sigma > 0$, then*

$$\sup_{N>0}\sup_{x\in\mathbb{R}} \mathrm{E}\left(|u_N(t\,,x)|^k\right) \le 4^k(\|u_0\|_{L^\infty(\mathbb{R})}+1)^k \mathrm{e}^{128L_\sigma^4 k^3 t}.$$

uniformly for all $t>0$ and $k \ge \max(2\,, L_b^{1/2}L_\sigma^{-2})$.

Proof Choose and fix $N, t>0$ and $x\in\mathbb{R}$. Owing to (2.2), we may write

$$\|u_N(t\,,x)\|_k \le \|u_0\|_{L^\infty(\mathbb{R})} + I_1 + I_2, \tag{2.4}$$

where $I_1 = \|\mathcal{I}_{b_N}^N(t,x)\|_k$ and $I_2 = \|\mathcal{I}_{\sigma_N}^N(t,x)\|_k$. We begin by estimating I_1 as follows. Thanks to (1.2) and the Minkowski inequality and the fact that

$$\|b_N(s\,,u_N(s\,,y))\|_k \le L_b(1+\|u_N(s\,,y)\|_k),$$

we can see that

$$\begin{aligned} I_1 &\le \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ p_{t-s}(y-x)\,\|b_N(s\,,u_N(s\,,y))\|_k \\ &\le L_b\left[t + \int_0^t \sup_{y\in\mathbb{R}} \|u_N(s\,,y)\|_k\,\mathrm{d}s\right]. \end{aligned}$$

For every space-time random field $Z=\{Z(t,x);\ t\ge 0, x\in\mathbb{R}\}$ and for all real numbers $k\ge 1$ and $\beta>0$ define

$$\mathcal{N}_{k,\beta}(Z) = \sup_{t\ge 0}\sup_{x\in\mathbb{R}} \mathrm{e}^{-\beta t}\|Z(t\,,x)\|_k. \tag{2.5}$$

It follows that

$$I_1 \le L_b\left[t + \mathcal{N}_{k,\beta}(u_N)\int_0^t \mathrm{e}^{\beta s}\,\mathrm{d}s\right] \le L_b\left[t+\beta^{-1}\mathrm{e}^{\beta t}\mathcal{N}_{k,\beta}(u_N)\right].$$

Since $t\exp(-\beta t) \le (\mathrm{e}\beta)^{-1} < \beta^{-1}$, this leads to the following bound for I_1:

$$I_1 \le L_b\beta^{-1}\mathrm{e}^{\beta t}\left[1+\mathcal{N}_{k,\beta}(u_N)\right]. \tag{2.6}$$

We bound I_2 using the asymptotically optimal form of the Burkholder–Davis–Gundy inequality (see [3]) as follows:

$$I_2^2 \le 4k\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \left[p_{t-s}(y-x)\right]^2 \|\sigma_N(s\,,u_N(s\,,y))\|_k^2. \tag{2.7}$$

Thanks to (1.2) and the fact that $\|\sigma_N(s, u_N(s, y))\|_k^2 \le 2L_\sigma^2(1 + \|u_N(s, y)\|_k^2)$,

$$I_2^2 \le 8kL_\sigma^2 \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ \left[p_{t-s}(y-x)\right]^2 \left(1 + \|u_N(s\,, y)\|_k^2\right).$$

Basic properties of the heat kernel imply that

$$\|p_r\|_{L^2(\mathbb{R})}^2 = (p_r * p_r)(0) = p_{2r}(0) = \tfrac{1}{2}(\pi r)^{-1/2} \qquad \text{for every } r > 0. \tag{2.8}$$

Therefore,

$$\begin{aligned} I_2^2 &\le \frac{4kL_\sigma^2}{\sqrt{\pi}} \int_0^t \frac{\mathrm{d}s}{\sqrt{t-s}} + \frac{4kL_\sigma^2}{\sqrt{\pi}} \int_0^t \sup_{y\in\mathbb{R}} \|u_N(s\,, y)\|_k^2 \frac{\mathrm{d}s}{\sqrt{t-s}} \\ &\le \frac{4kL_\sigma^2}{\sqrt{\pi}} \int_0^t \frac{\mathrm{d}s}{\sqrt{s}} + \frac{4kL_\sigma^2 \left[\mathcal{N}_{k,\beta}(u_N)\right]^2 \mathrm{e}^{2\beta t}}{\sqrt{\pi}} \int_0^t \frac{\mathrm{e}^{-2\beta(t-s)}}{\sqrt{t-s}} \, \mathrm{d}s. \end{aligned}$$

Since $\int_0^t s^{-1/2}\,\mathrm{d}s \le \exp(2\beta t)\int_0^\infty s^{-1/2}\exp(-2\beta s)\,\mathrm{d}s = \sqrt{\pi/(2\beta)}\exp(2\beta t)$, we are led to the following:

$$I_2^2 \le 4k\mathrm{e}^{2\beta t} L_\sigma^2 \left(1 + \mathcal{N}_{k,\beta}(u_N)^2\right)/\sqrt{2\beta}.$$

We prefer to simplify the preceding slightly more, using the inequality $\sqrt{l^2+n^2} \le |l| + |n|$—valid for all $l, n \in \mathbb{R}$—as follows:

$$I_2 \le 2\sqrt{k}(2\beta)^{-1/4}\mathrm{e}^{\beta t} L_\sigma \left(1 + \mathcal{N}_{k,\beta}(u_N)\right). \tag{2.9}$$

We can combine (2.6) and (2.9) to find that

$$\begin{aligned} &\|u_N(t\,, x)\|_k \\ &\quad\le \|u_0\|_{L^\infty(\mathbb{R})} + L_b\beta^{-1}\mathrm{e}^{\beta t}\left[1 + \mathcal{N}_{k,\beta}(u_N)\right] + 2\sqrt{k}(2\beta)^{-1/4}\mathrm{e}^{\beta t}L_\sigma\left(1 + \mathcal{N}_{k,\beta}(u_N)\right) \\ &\quad\le \|u_0\|_{L^\infty(\mathbb{R})} + \mathrm{e}^{\beta t}\left(L_b\beta^{-1} + 2\sqrt{k}L_\sigma(2\beta)^{-1/4}\right)\left(1 + \mathcal{N}_{k,\beta}(u_N)\right). \end{aligned}$$

This implies that

$$\mathcal{N}_{k,\beta}(u_N) \le \|u_0\|_{L^\infty(\mathbb{R})} + \left(L_b\beta^{-1} + 2\sqrt{k}L_\sigma(2\beta)^{-1/4}\right)\left(1 + \mathcal{N}_{k,\beta}(u_N)\right).$$

Set $\beta = 128k^2L_\sigma^4$ to see that $\beta^{-1}L_b + (2\beta)^{-1/4}2\sqrt{k}L_\sigma \le \frac{3}{4}$ for this particular choice of β. Solve for $\mathcal{N}_{k,128k^2L_\sigma^4}(u_N)$ in order to find that $\mathcal{N}_{k,128k^2L_\sigma^4}(u_N) \le 4(\|u_0\|_{L^\infty(\mathbb{R})} + 1)$. This is another way to state the proposition. □

Remark 2.2 The requirement that $k \ge \max(2, L_b^{1/2}L_\sigma^{-2})$ is a technical consequence of the proof. We can always choose, without incurring loss of generality, L_σ large enough to ensure that $L_b^{1/2} L_\sigma^{-2} \le 2$, in order to ensure that the result stated in the proposition holds for all $k \ge 2$.

The conclusion of Proposition 2.1 can be improved when σ is constant. The following takes care of that case and at the same time improves Proposition 2.1 when σ is bounded but $\mathrm{Lip}(\sigma) > 0$.

Proposition 2.3 *If $\sigma : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is bounded, then*

$$\sup_{N>0} \sup_{x\in\mathbb{R}} \mathrm{E}\left(|u_N(t\,,x)|^k\right) \le 4^k \mathrm{e}^{2L_b kt} \left(\|u_0\|_{L^\infty(\mathbb{R})} + \|\sigma\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} t^{1/4} + 1\right)^k k^{k/2},$$

uniformly for all $t > 0$ and $k \ge 2$.

Proof We modify the proof of Proposition 2.1 by first observing that (2.4) remains valid and so does (2.6). We start with (2.7) and estimate I_2, using (2.8), simply as follows:

$$\begin{aligned} I_2^2 &\le 4k\|\sigma\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \int_0^t \|p_r\|^2_{L^2(\mathbb{R})}\,\mathrm{d}r = 2k\pi^{-1/2}\|\sigma\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \int_0^t \mathrm{d}r/\sqrt{r} \\ &= 4k\|\sigma\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\sqrt{t/\pi} \le 4k\|\sigma\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\sqrt{t}. \end{aligned}$$

This yields

$$\|u_N(t\,,x)\|_k \le \|u_0\|_{L^\infty(\mathbb{R})} + L_b\beta^{-1}\exp(\beta t)\left[1 + \mathcal{N}_{k,\beta}(u_N)\right] + 2\sqrt{k}\|\sigma\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} t^{1/4}.$$

Since the right-hand side does not depend on (t, x), we divide by $\exp(\beta t)$ and optimize over (t, x) to find that

$$\mathcal{N}_{k,\beta}(u_N) \le \|u_0\|_{L^\infty(\mathbb{R})} + 2\sqrt{k}\|\sigma\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} t^{1/4} + L_b\beta^{-1}\left[1 + \mathcal{N}_{k,\beta}(u_N)\right],$$

uniformly for all real numbers $k \ge 2, N, \beta > 0$. Set $\beta = 2L_b$ and solve for $\mathcal{N}_{k,2L_b}(u_N)$ to find that

$$\begin{aligned} \mathcal{N}_{k,2L_b}(u_N) &\le 2\|u_0\|_{L^\infty(\mathbb{R})} + 4\sqrt{k}\|\sigma\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} t^{1/4} + 1 \\ &\le 4\left(\|u_0\|_{L^\infty(\mathbb{R})} + \|\sigma\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} t^{1/4} + 1\right)\sqrt{k}, \end{aligned}$$

which is another way to state the result. □

The next result shows some tail estimates of the truncated solution.

Proposition 2.4 *If $L_\sigma > 0$, then*

$$\mathrm{P}\left\{|u_{N+1}(t\,,x)| \ge \mathrm{e}^N\right\} \le \exp\left(-\frac{N^{3/2}}{64L_\sigma^2\sqrt{t}}\right), \tag{2.10}$$

uniformly for all $t > 0$, $x \in \mathbb{R}$, *and*

$$N \geq 4\log(4(\|u_0\|_{L^\infty(\mathbb{R})} + 1)) \vee 256t \max\left(4L_\sigma^4, L_b\right).$$

If $\sigma \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$, *then*

$$\mathrm{P}\left\{|u_{N+1}(t\,,x)| \geq \mathrm{e}^N\right\} \leq \exp\left(-\frac{\mathrm{e}^{2N-4L_bt}}{32\mathrm{e}\left(\|u_0\|_{L^\infty(\mathbb{R})} + \|\sigma\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}t^{1/4} + 1\right)^2}\right), \tag{2.11}$$

uniformly for all $t > 0$, $x \in \mathbb{R}$, *and*

$$N \geq \tfrac{1}{2}\log 32 + 2L_bt + \tfrac{1}{2} + \log\left(\|u_0\|_{L^\infty(\mathbb{R})} + \|\sigma\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}t^{1/4} + 1\right).$$

Proof Let us first consider the case that $L_\sigma > 0$. Proposition 2.1 and Chebyshev's inequality together ensure that

$$\begin{aligned}\mathrm{P}\left\{|u_{N+1}(t\,,x)| \geq \mathrm{e}^N\right\} &\leq \mathrm{e}^{-kN}\mathrm{E}\left(|u_{N+1}(t\,,x)|^k\right)\\ &\leq \left(4(\|u_0\|_{L^\infty(\mathbb{R})} + 1)\right)^k \mathrm{e}^{-kN+128L_\sigma^4k^3t},\end{aligned}$$

uniformly for all real numbers $N, t > 0, x \in \mathbb{R}$, and all real $k \geq \max(2, L_b^{1/2}L_\sigma^{-2})$. Set $k = \sqrt{AN}$—where the value of $A > 0$ is to be determined—and $C = 4(\|u_0\|_{L^\infty(\mathbb{R})} + 1)$ in order to see that

$$\mathrm{P}\left\{|u_{N+1}(t\,,x)| \geq \mathrm{e}^N\right\} \leq \exp\left(-\left\{\sqrt{A} - 128L_\sigma^4A^{3/2}t - \frac{\sqrt{A}\log C}{N}\right\}N^{3/2}\right).$$

We use this with $A = (256L_\sigma^4t)^{-1}$ and observe that $k \geq \max(2, \sqrt{L_b}L_\sigma^{-2})$ iff $N \geq 256t\max(4L_\sigma^4, L_b)$. It follows that

$$\begin{aligned}\mathrm{P}\left\{|u_{N+1}(t\,,x)| \geq \mathrm{e}^N\right\} &\leq \exp\left(-\left\{\frac{128L_\sigma^4t}{[256L_\sigma^4t]^{3/2}} - \frac{\log C}{N\sqrt{256L_\sigma^4t}}\right\}N^{3/2}\right)\\ &= \exp\left(-\left\{\frac{1}{2} - \frac{\log C}{N}\right\}\frac{N^{3/2}}{(256t)^{1/2}L_\sigma^2}\right),\end{aligned}$$

which has the desired outcome, when $L_\sigma > 0$, provided additionally that $N \geq 4\log C$. The case $\sigma \in L^\infty(\mathbb{R})$ is proved similarly but rests on Proposition 2.3 instead of 2.1, viz.,

$$\mathrm{P}\left\{|u_{N+1}(t\,,x)| \geq \mathrm{e}^N\right\} \leq \mathrm{e}^{-kN}\mathrm{E}\left(|u_{N+1}(t\,,x)|^k\right) \leq C^k\mathrm{e}^{-kN}k^{k/2},$$

valid uniformly for all $N, t > 0$, $x \in \mathbb{R}$, $k \geq 2$, where

$$C = C(t\,,u_0\,,L_b) = 4\mathrm{e}^{2L_bt}\left(\|u_0\|_{L^\infty(\mathbb{R})} + \|\sigma\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}t^{1/4} + 1\right).$$

We apply the preceding with the particular choice $k = C^{-2}\exp\{2N-1\}$ and compute a bit in order to finish. □

Finally, the following real-variable lemma will be helpful to us.

Lemma 2.5 *Consider a function* $f : (0,\infty) \to (0,\infty)$ *and an increasing function* $g : (0,\infty) \to (0,\infty)$. *If there exists* $\beta, T_0 > 0$ *such that*

$$\sup_{t\in(0,T]}\left[\mathrm{e}^{-\beta t} f(t)\right] \le \mathrm{e}^{-\beta T} g(T) \qquad \forall T \in (0\,,T_0),$$

then $\sup_{(0,T]} f \le g(T)$ *for every* $T \in (0, T_0)$.

Proof Since $\mathrm{e}^{-\beta T} f(T) \le \mathrm{e}^{-\beta T} g(T)$, we cancel the exponentials to deduce the result from the monotonicity of g. □

3 Proof of Theorem 1.4

Proof of existence We first prove the theorem in the case that $L_\sigma > 0$. Until we mention to the contrary, we therefore assume tacitly that $L_\sigma > 0$. Thus, we also assume that (1.6) holds.

Let us choose and fix an arbitrary number $T > 0$. Our primary goal is to prove that

$$\sum_{N=1}^{\infty} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \|u_{N+1}(t\,,x) - u_N(t\,,x)\|_k < \infty \qquad \forall k \ge 1. \tag{3.1}$$

Because $T > 0$ can be as large as needed, (3.1) implies that the random variable $u(t,x) = \lim_{N\to\infty} u_N(t,x)$ exists in $L^2(\Omega)$, say, pointwise in (t,x). From there it is not difficult to prove that the thus-defined random field u is a mild solution to (1.1).

With the preceding paragraph in mind, we now adjust (2.5) slightly and define the following local-in-time norms:

$$\mathcal{N}_{k,\beta,T}(Z) = \sup_{t\in(0,T]} \sup_{x\in\mathbb{R}} \mathrm{e}^{-\beta t}\|Z(t\,,x)\|_k, \tag{3.2}$$

defined for every $\beta, T > 0$, $k \ge 1$, and all space-time random fields Z.

Recall (1.5) and note that $N \mapsto \mathrm{Lip}_N(b)$ and $N \mapsto \mathrm{Lip}_N(\sigma)$ are nondecreasing and b and σ are globally Lipschitz, respectively, when $\lim_{N\to\infty}\mathrm{Lip}_N(b) < \infty$ and $\lim_{N\to\infty}\mathrm{Lip}_N(\sigma) < \infty$. Therefore, Condition (1.6) ensures that we can assume without any loss in generality that

$$\lim_{N\to\infty} L_{N,\sigma} = \infty, \tag{3.3}$$

for b and σ will be globally Lipschitz otherwise, in which case there is nothing to prove. From now on, condition (3.3) is assumed to hold.

Thanks to (2.2) we can write, for every $k \geq 1$ [not necessarily an integer], $t > 0$, and $x \in \mathbb{R}$,

$$\|u_{N+1}(t\,,x) - u_N(t\,,x)\|_k \leq I_1 + I_2, \tag{3.4}$$

where

$$I_1 = \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\, \|b_{N+1}(s\,,u_{N+1}(s\,,y)) - b_N(s\,,u_N(s\,,y))\|_k \,\mathrm{d}s\,\mathrm{d}y,$$

$$I_2 = \left\| \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x) \left[\sigma_{N+1}(s\,,u_{N+1}(s\,,y)) - \sigma_N(s\,,u_N(s\,,y))\right] W(\mathrm{d}s\,\mathrm{d}y) \right\|_k .$$

Recall (2.5) and for every $N, s > 0$ and $y \in \mathbb{R}$ consider the event

$$G_{N+1}(s\,,y) = \left\{\omega \in \Omega :\ |u_{N+1}(s\,,y)|(\omega) \leq \mathrm{e}^N\right\}.$$

On the one hand,

$$\begin{aligned}
&\left\| \left[b_{N+1}(s\,,u_{N+1}(s\,,y)) - b_N(s\,,u_N(s\,,y))\right] \mathbf{1}_{G_{N+1}(s\,,y)} \right\|_k \\
&\quad \leq \|b_N(s\,,u_{N+1}(s\,,y)) - b_N(s\,,u_N(s\,,y))\|_k \\
&\quad \leq L_{N,b}\, \|u_{N+1}(s\,,y) - u_N(s\,,y)\|_k \leq L_{N,b}\mathrm{e}^{\beta s} \mathcal{N}_{k,\beta,T}(u_{N+1} - u_N),
\end{aligned}$$

for all $k \geq 1$, β, $N > 0$, $s \in (0, T]$, and $y \in \mathbb{R}$. On the other hand,

$$\begin{aligned}
&\left\| \left[b_{N+1}(s\,,u_{N+1}(s\,,y)) - b_N(s\,,u_N(s\,,y))\right] \mathbf{1}_{\Omega\setminus G_{N+1}(s\,,y)} \right\|_k \\
&\quad \leq \left\| b_{N+1}(s\,,u_{N+1}(s\,,y)) \mathbf{1}_{\Omega\setminus G_{N+1}(s\,,y)} \right\|_k + \left\| b_N(s\,,u_N(s\,,y)) \mathbf{1}_{\Omega\setminus G_{N+1}(s\,,y)} \right\|_k \\
&\quad \leq \left[\|b_{N+1}(s\,,u_{N+1}(s\,,y))\|_{2k} + \|b_N(s\,,u_N(s\,,y))\|_{2k}\right] \left[1 - \mathrm{P}\left(G_{N+1}(s\,,y)\right)\right]^{1/(2k)} .
\end{aligned}$$

We have used the following variation of the Cauchy–Schwarz inequality in the last line: $\|X\mathbf{1}_F\|_k \leq \|X\|_{2k}[\mathrm{P}(F)]^{1/(2k)}$ for all $X \in L^k(\Omega)$ and all events $F \subset \Omega$. Set $C = 4(\|u_0\|_{L^\infty(\mathbb{R})} + 1)$. Thanks to Proposition 2.1,

$$\begin{aligned}
&\|b_{N+1}(s\,,u_{N+1}(s\,,y))\|_{2k} + \|b_N(s\,,u_N(s\,,y))\|_{2k} \\
&\quad \leq L_b \left[\|u_{N+1}(s\,,y)\|_{2k} + \|u_N(s\,,y)\|_{2k}\right] \leq 2CL_b \mathrm{e}^{512 L_\sigma^4 k^2 s},
\end{aligned}$$

uniformly for all $N, s > 0$, $y \in \mathbb{R}$, and $k \geq \max(1, \frac{1}{2} L_b^{1/2} L_\sigma^{-2})$. This yields the following inequality:

$$\begin{aligned}
&\left\| \left[b_{N+1}(s\,,u_{N+1}(s\,,y)) - b_N(s\,,u_N(s\,,y))\right] \mathbf{1}_{\Omega\setminus G_{N+1}(s\,,y)} \right\|_k \\
&\quad \leq 2CL_b \mathrm{e}^{512 L_\sigma^4 k^2 s} \left[\mathrm{P}\left\{|u_{N+1}(s\,,y)| \geq \mathrm{e}^N\right\}\right]^{1/(2k)},
\end{aligned}$$

valid uniformly for all $N, s > 0$, $y \in \mathbb{R}$, and $k \geq 1$. Therefore, Proposition 2.4 yields

$$\left\| \left[b_{N+1}(s\,,u_{N+1}(s\,,y)) - b_N(s\,,u_N(s\,,y)) \right] \mathbf{1}_{\Omega \setminus G_{N+1}(s\,,y)} \right\|_k \\ \le 2L_b C \mathrm{e}^{512 L_\sigma^4 k^2 s} \exp\left(-\frac{N^{3/2}}{128 k L_\sigma^2 \sqrt{s}} \right),$$

valid uniformly for all $s \in (0, T]$, $y \in \mathbb{R}$, $N \ge 4 \log C \vee 256T \max(4L_\sigma^4, L_b)$, and $k \ge 1$. Thus, we find using (3.2) that

$$\begin{aligned} I_1 &\le L_{N,b} \mathrm{e}^{\beta t} \mathcal{N}_{k,\beta,T}(u_{N+1} - u_N) \int_{(0,t)\times\mathbb{R}} \mathrm{e}^{-\beta(t-s)} p_{t-s}(y-x)\,\mathrm{d}s\,\mathrm{d}y \\ &\quad + 2L_b C \int_{(0,t)\times\mathbb{R}} \mathrm{e}^{512 L_\sigma^4 k^2 s} \exp\left(-\frac{N^{3/2}}{128 k L_\sigma^2 \sqrt{s}} \right) p_{t-s}(y-x)\,\mathrm{d}s\,\mathrm{d}y \qquad (3.5) \\ &\le \frac{L_{N,b} \mathrm{e}^{\beta t}}{\beta} \mathcal{N}_{k,\beta,T}(u_{N+1} - u_N) + \frac{L_b C}{128 L_\sigma^4 k^2} \exp\left(512 L_\sigma^4 k^2 t - \frac{N^{3/2}}{128 k L_\sigma^2 \sqrt{t}} \right), \end{aligned}$$

uniformly for all $t \in (0\,,T)$ and $x \in \mathbb{R}$, provided that $N, k \ge c$, for a complicated looking but otherwise unimportant number

$$c = c(\|u_0\|_{L^\infty(\mathbb{R})}\,, L_\sigma\,, T\,, L_b) > 1. \qquad (3.6)$$

For later purposes, we pause to mention that the number c, while fixed, can be chosen to be as large as we wish. Owing to Condition (1.6), we therefore select $c = c(\|u_0\|_{L^\infty(\mathbb{R})}, L_\sigma, T, L_b)$ large enough to additionally ensure that

$$c > \sup_{N \ge N_0} \frac{\sqrt{L_{N,b}}}{L_{N,\sigma}^2}, \quad \text{where} \quad N_0 = \inf\left\{ N > 1 :\ L_{N,\sigma} \ge 1 \right\}. \qquad (3.7)$$

The number N_0 is well defined and finite thanks to (3.3).

Next we study the quantity I_2. Let us appeal to the Burkholder–Davis–Gundy inequality (see [3]) to find that

$$I_2^2 \le 4k \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, [p_{t-s}(y-x)]^2 \left\| \sigma_{N+1}(s\,,u_{N+1}(s\,,y)) - \sigma_N(s\,,u_N(s\,,y)) \right\|_k^2.$$

As in the above truncation, we may write

$$\left\| \left[\sigma_{N+1}(s\,,u_{N+1}(s\,,y)) - \sigma_N(s\,,u_N(s\,,y)) \right] \mathbf{1}_{G_{N+1}(s\,,y)} \right\|_k \\ \le L_{N,\sigma} \mathrm{e}^{\beta s} \mathcal{N}_{k,\beta,T}(u_{N+1} - u_N) \qquad \forall \beta, N > 0,\ s \in (0\,,T],\ y \in \mathbb{R}.$$

Moreover,

$$\left\| \left[\sigma_{N+1}(s\,,u_{N+1}(s\,,y)) - \sigma_N(s\,,u_N(s\,,y)) \right] \mathbf{1}_{\Omega \setminus G_{N+1}(s\,,y)} \right\|_k \\ \le \left[\|\sigma_{N+1}(s\,,u_{N+1}(s\,,y))\|_{2k} + \|\sigma_N(s\,,u_N(s\,,y))\|_{2k} \right] \left[1 - \mathrm{P}\left(G_{N+1}(s\,,y) \right) \right]^{1/(2k)}.$$

Thanks to Proposition 2.1,

$$
\begin{aligned}
&\|\sigma_{N+1}(s\,,u_{N+1}(s\,,y))\|_{2k} + \|\sigma_N(s\,,u_N(s\,,y))\|_{2k}\\
&\quad\le L_\sigma\left[\|u_{N+1}(s\,,y)\|_{2k} + \|u_N(s\,,y)\|_{2k}\right] \le 2L_\sigma C\mathrm{e}^{512L_\sigma^4k^2s},
\end{aligned}
$$

uniformly for all $N, s > 0$, $y\in\mathbb{R}$, and $k\ge 1$. Therefore, Proposition 2.4 yields

$$
\begin{aligned}
&\left\|\left[\sigma_{N+1}(s\,,u_{N+1}(s\,,y)) - \sigma_N(s\,,u_N(s\,,y))\right]\mathbf{1}_{\Omega\setminus G_{N+1}(s\,,y)}\right\|_k\\
&\quad\le 2L_\sigma C\mathrm{e}^{512L_\sigma^4k^2s}\exp\left(-\frac{N^{3/2}}{128kL_\sigma^2\sqrt{s}}\right),
\end{aligned}
$$

uniformly for all $s\in(0,T]$, $y\in\mathbb{R}$, $k\ge 1$, and

$$
N \ge c_T := 4\log C \vee 256T\max\left(4L_\sigma^4, L_b\right). \tag{3.8}
$$

Thus, we find that

$$
\begin{aligned}
I_2^2 &\le 8kL_{N,\sigma}^2\mathrm{e}^{2\beta t}\left[\mathcal{N}_{k,\beta,T}(u_{N+1}-u_N)\right]^2\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\ \mathrm{e}^{-2\beta(t-s)}[p_{t-s}(y-x)]^2\\
&\quad + 32kL_b^2C^2\int_{(0,t)\times\mathbb{R}}\mathrm{e}^{1024L_\sigma^4k^2s}\exp\left(-\frac{N^{3/2}}{128kL_\sigma^2\sqrt{s}}\right)[p_{t-s}(y-x)]^2\,\mathrm{d}s\,\mathrm{d}y\\
&\le \frac{4kL_{N,\sigma}^2\mathrm{e}^{2\beta t}}{\sqrt{\beta}}\left[\mathcal{N}_{k,\beta,T}(u_{N+1}-u_N)\right]^2\\
&\quad + 128kL_\sigma^2C^2\mathrm{e}^{1024L_\sigma^4k^2t}\exp\left(-\frac{N^{3/2}}{128kL_\sigma^2\sqrt{t}}\right)\sqrt{t},
\end{aligned}
$$

for every choice of $\beta>0$, $t\in(0,T)$, and $x\in\mathbb{R}$, provided that $N\ge\max(c\,,c_T)$ and $k\ge c$—in case it helps we recall that $c>1$ and $c_T>0$ were defined, respectively, in (3.6) and (3.8). Combine this last inequality with (3.5) and (3.4) in order to see that

$$
\begin{aligned}
&\|u_{N+1}(t\,,x) - u_N(t\,,x)\|_k\\
&\quad\le \mathrm{e}^{\beta t}\left[L_{N,b}\beta^{-1} + 2\sqrt{k}L_{N,\sigma}\beta^{-1/4}\right]\mathcal{N}_{k,\beta,T}(u_{N+1}-u_N)\\
&\qquad + C\left[\frac{L_b}{128L_\sigma^4k^2} + \sqrt{128kt}^{1/4}L_\sigma\right]\exp\left(512L_\sigma^4k^2t - \frac{N^{3/2}}{128kL_\sigma^2\sqrt{t}}\right),
\end{aligned}
$$

as long as $N\ge\max(c\,,c_T)$ and $k\ge c$.

Next, we make particular choices of $k\ge c$ and $\beta>0$ as follows:

$$
k = c \quad\text{and}\quad \beta = 16A_0^4k^2L_{N,\sigma}^4, \tag{3.9}
$$

where

$$
A_0 = \max\left(\sqrt{8}L_\sigma^4\,,4\right). \tag{3.10}
$$

We pause to emphasize that β depends on (N, T). In this way we find that

$$\sup_{x\in\mathbb{R}} \|u_{N+1}(t\,,x) - u_N(t\,,x)\|_c \le \mathrm{e}^{\beta t}\left[\frac{L_{N,b}}{16A_0^4c^2L_{N,\sigma}^4} + \frac{1}{A_0}\right]\mathcal{N}_{c,\beta,T}(u_{N+1}-u_N)$$
$$+ C\left[\frac{L_b}{128L_\sigma^4c^2} + \sqrt{128c}t^{1/4}L_\sigma\right]\exp\left(512L_\sigma^4c^2t - \frac{N^{3/2}}{128cL_\sigma^2\sqrt{t}}\right),$$

uniformly for all $T > 0$, $t \in (0, T]$, and $N \ge \max(N_0, c, c_T)$; see (3.7). Because $A_0 \ge 4$—see (3.10)—an application of (3.7) yields the following inequality,

$$\begin{aligned}\sup_{x\in\mathbb{R}} \|u_{N+1}(t\,,x) - u_N(t\,,x)\|_c &\le \mathrm{e}^{\beta t}\left[\frac{1}{16A_0^4} + \frac{1}{A_0}\right]\mathcal{N}_{c,\beta,T}(u_{N+1}-u_N)\\ &\quad+ C\left[\frac{L_b}{128L_\sigma^4c^2} + \sqrt{128c}t^{1/4}L_\sigma\right]\exp\left(512L_\sigma^4c^2t - \frac{N^{3/2}}{128cL_\sigma^2\sqrt{t}}\right)\\ &\le \mathrm{e}^{\beta t}\left[\frac{1}{4096} + \frac{1}{4}\right]\mathcal{N}_{c,\beta,T}(u_{N+1}-u_N)\\ &\quad+ C\left[\frac{L_b}{128L_\sigma^4c^2} + \sqrt{128c}t^{1/4}L_\sigma\right]\exp\left(512L_\sigma^4c^2t - \frac{N^{3/2}}{128cL_\sigma^2\sqrt{t}}\right),\end{aligned}$$

valid uniformly for all $T > 0$, $t \in (0, T]$, and $N \ge \max(N_0, c, c_T)$. Since $\frac{1}{4096} + \frac{1}{4} < \frac{1}{2}$, we may divide both sides of the preceding by $\exp(\beta t)$ and optimize over $t \in (0, T]$ in order to find that

$$\begin{aligned}&\mathcal{N}_{c,\beta,T}(u_{N+1}-u_N)\\ &\quad\le 2C\left[\frac{L_b}{128L_\sigma^4c^2} + \sqrt{128c}T^{1/4}L_\sigma\right]\sup_{t\in(0,T]}\exp\left(-\left(\beta - 512L_\sigma^4c^2\right)t - \frac{N^{3/2}}{128cL_\sigma^2\sqrt{t}}\right),\end{aligned} \tag{3.11}$$

uniformly for all $T > 0$, and $N \ge \max(N_0, c, c_T)$. Thanks to (3.7), (3.9), and (3.10),

$$\beta - 512L_\sigma^4c^2 = c^2\left(16A_0^4L_{N,\sigma}^4 - 512L_\sigma^4\right) \ge c^2\left(16A_0^4 - 512L_\sigma^4\right) > 0,$$

uniformly for all $N \ge N_0$. Now, let us consider the following function that appears in the exponent on the right-hand side of (3.11):

$$\psi(t) = \left(\beta - 512L_\sigma^4c^2\right)t + \frac{N^{3/2}}{128kL_\sigma^2\sqrt{t}} \qquad (t > 0).$$

Because

$$\psi'(t) \le \beta - \frac{(N/T)^{3/2}}{256cL_\sigma^2} = 16A_0^4L_{N,\sigma}^4c^2 - \frac{(N/T)^{3/2}}{256cL_\sigma^2} \quad \forall 0 < t \le T,$$

it follows from this that $\psi' < 0$ everywhere on $(0, T]$ provided that

$$\frac{N}{L_{N,\sigma}^{8/3}} > 4096^{2/3} A_0^{8/3} c^2 L_\sigma^{4/3} T =: N_T; \tag{3.12}$$

the left-hand side being well defined for example when $N_T > N_0$. Condition (1.6) ensures that the left-hand side tends to infinity as $N \to \infty$. Therefore, (3.12) holds for every $N > \max(N_0, N_T)$. It follows that

$$\inf_{t\in(0,T]} \psi(t) = \psi(T) \quad \forall N > \max(N_0\,, N_T),\ \ T > 0,$$

whence it follows from (3.11) that, for every $T_0 > 0$ and $N \geq \max(N_0\,, N_{T_0}\,, c_0\,, c_{T_0})$,

$$\begin{aligned}
&\mathcal{N}_{c,\beta,T}(u_{N+1} - u_N)\\
&\quad \leq 2C\left[\frac{L_b}{128L_\sigma^4 c^2} + \sqrt{128c}T^{1/4}L_\sigma\right]\exp\left(-\left(\beta - 512L_\sigma^4 c^2\right)T - \frac{N^{3/2}}{128cL_\sigma^2\sqrt{T}}\right),
\end{aligned}$$

uniformly for every $T \in (0, T_0)$. In order to ensure the uniformity statement of T, we have also used the fact that $T \mapsto c_T$ is increasing; see (3.8). We now apply Lemma 2.5 in order to deduce from the above and (3.2) that, for every $T > 0$ fixed,

$$\limsup_{N\to\infty} N^{-3/2} \log \sup_{t\in(0,T]}\sup_{x\in\mathbb{R}} \|u_{N+1}(t\,, x) - u_N(t)\|_k < 0 \qquad \forall k \geq 1. \tag{3.13}$$

When $k \in [1, c]$ this follows from the preceding. For general k it follows from a relabeling $[k \leftrightarrow c]$, and an appeal to the fact that c can be as large as we wish—see the comments that follow (3.6). This proves our original goal (3.1). The remainder of the argument is technically simpler.

The preceding proves that

$$u(t\,, x) = \lim_{N\to\infty} u_N(t\,, x) \quad \text{exists in } L^k(\Omega) \qquad \forall k \geq 1, \tag{3.14}$$

and the rate of convergence does not depend on $t \in (0, T)$ nor on $x \in \mathbb{R}$. As a direct consequence u is $L^k(\Omega)$-continuous. This ensures that u has a predictable version.

Therefore it remains to prove that, for every $t \in (0\,, T)$ and $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathcal{I}_{b_N}^N(t\,, x) = \mathcal{I}_b(t\,, x) \quad \text{and} \quad \lim_{N\to\infty} \mathcal{I}_{\sigma_N}^N(t\,, x) = \mathcal{I}_\sigma(t\,, x), \tag{3.15}$$

where both of the limits hold in $L^2(\Omega)$, and the random fields $\mathcal{I}_b$, $\mathcal{I}_\sigma$, $\mathcal{I}_b^N$, and $\mathcal{I}_\sigma^N$ were defined in (1.4) and (2.3). Thanks to (3.14) and (2.2), this proves that u is a mild solution to (1.1).

Observe that

$$\begin{aligned}
|b_N(t\,, z) - b(t\,, z)| &\leq \mathbf{1}_{\mathbb{R}\setminus[-\exp(N),\exp(N)]}(z)(|b(t\,, z)| + |b_N(t\,, z)|)\\
&\leq 2L_b(1 + |z|)\mathbf{1}_{\mathbb{R}\setminus[-\exp(N),\exp(N)]}(z),
\end{aligned}$$

for all $z \in \mathbb{R}$ and $t, N > 0$. This yields

$$\begin{aligned}
&\left\| \mathcal{I}_{b_N}^N(t\,,x) - \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x) b(s\,,u_N(s\,,y))\,\mathrm{d}s\,\mathrm{d}y \right\|_2 \\
&\le \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\, \| b_N(s\,,u_N(s\,,y)) - b(s\,,u_N(s\,,y))\|_2\,\mathrm{d}s\,\mathrm{d}y \\
&\le 2L_b \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x) \left(\sqrt{\mathrm{E}\left(1+|u_N(s\,,y)|^2;\, |u_N(s\,,y)| > \mathrm{e}^N\right)} \right) \mathrm{d}s\,\mathrm{d}y.
\end{aligned}$$

If $X \ge 0$ is a random variable and $A > 0$ is a constant, then we apply the Cauchy–Schwarz inequality and Chebyshev's inequality back to back in order to find that

$$\mathrm{E}(X^2;\, X > A) \le \sqrt{\mathrm{E}(X^4)\mathrm{P}\{|X| > A\}} \le A^{-2}\mathrm{E}(X^4).$$

Therefore, Propositions 2.1 and 2.3 assure us that

$$\lim_{N\to\infty} \sup_{s\in(0,T)} \sup_{y\in\mathbb{R}} \mathrm{E}\left(1+|u_N(s\,,y)|^2;\, |u_N(s\,,y)| > \mathrm{e}^N\right) = 0,$$

and hence $\mathcal{I}_{b_N}^N(t\,,x) - \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x) b(s\,,u_N(s\,,y))\,\mathrm{d}s\,\mathrm{d}y \to 0$ as $N \to \infty$, where the convergence takes place in $L^2(\Omega)$. Therefore, the first assertion of (3.15) would follow once we can show that

$$\lim_{N\to\infty} \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x) b(s\,,u_N(s\,,y))\,\mathrm{d}s\,\mathrm{d}y = \mathcal{I}_b(t\,,x), \tag{3.16}$$

where the convergence takes place in $L^2(\Omega)$. Since b has at-most linear growth and $\|b(s\,,u_N(s\,,y)) - b(s\,,u(s\,,y))\|_2 \le \|b(s\,,u_N(s\,,y))\|_2 + \|b(s\,,u(s\,,y))\|_2$, Propositions 2.1 and 2.3 ensure that $\|b(s, u_N(s, y)) - b(s, u(s, y))\|_2$ is bounded uniformly in $s \in (0, T)$, $N > 0$, and $y \in \mathbb{R}$. Thanks to (3.14), uniform integrability, and the continuity of b,

$$\lim_{N\to\infty} b(s\,,u_N(s\,,y)) = b(s\,,u(s\,,y)) \quad \text{in } L^2(\Omega), \text{ for every } s > 0 \text{ and } y \in \mathbb{R}^d.$$

Therefore, the dominated convergence theorem yields

$$\lim_{N\to\infty} \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\, \|b(s\,,u_N(s\,,y)) - b(s\,,u(s\,,y))\|_2\,\mathrm{d}s\,\mathrm{d}y = 0,$$

for all $t \in (0\,,T)$ and $x \in \mathbb{R}$. The triangle inequality now yields (3.16) and hence the first assertion of (3.15). Similarly,

$$|\sigma_N(t\,,z) - \sigma(t\,,z)| \le 2L_\sigma(1+|z|)\mathbf{1}_{\mathbb{R}\setminus[-\exp(N),\exp(N)]}(z)$$

for all $z \in \mathbb{R}$ and $t, N > 0$. Then, by the $L^2(\Omega)$-isometry of stochastic integrals,

$$\begin{aligned}&\left\| \mathcal{I}^N_{\sigma_N}(t\,,x) - \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma(s\,,u_N(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y)\right\|_2^2\\ &\le \int_{(0,t)\times\mathbb{R}} [p_{t-s}(y-x)]^2\,\|\sigma_N(s\,,u_N(s\,,y)) - \sigma(s\,,u_N(s\,,y))\|_2^2\,\mathrm{d}s\,\mathrm{d}y\\ &\le 4L_\sigma^2\int_{(0,t)\times\mathbb{R}} [p_{t-s}(y-x)]^2\mathrm{E}\left(1+|u_N(s\,,y)|^2;\,|u_N(s\,,y)|>\mathrm{e}^N\right)\mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

As before, Propositions 2.1 and 2.3 assure us that

$$\lim_{N\to\infty}\sup_{s\in(0,T)}\sup_{y\in\mathbb{R}}\mathrm{E}\left(1+|u_N(s\,,y)|^2;\,|u_N(s\,,y)|>\mathrm{e}^N\right)=0,$$

and hence $\mathcal{I}^N_{\sigma_N}(t\,,x) - \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma(s\,,u_N(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y) \to 0$ in $L^2(\Omega)$ as $N\to\infty$. Therefore, we are left to show that

$$\lim_{N\to\infty}\int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma(s\,,u_N(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y) = \mathcal{I}_\sigma(t\,,x),$$

where the convergence takes place in $L^2(\Omega)$. Since σ has at-most linear growth and

$$\|\sigma(s\,,u_N(s\,,y)) - \sigma(s\,,u(s\,,y))\|_2 \le \|\sigma(s\,,u_N(s\,,y))\|_2 + \|\sigma(s\,,u(s\,,y))\|_2,$$

Propositions 2.1 and 2.3 ensure that $\|\sigma(s, u_N(s, y)) - \sigma(s, u(s, y))\|_2$ is bounded uniformly in $s \in (0, T)$, $N > 0$, and $y \in \mathbb{R}$. Thanks to (3.14), uniform integrability, and the continuity of σ, $\lim_{N\to\infty}\sigma(s, u_N(s, y)) = \sigma(s, u(s, y))$ in $L^2(\Omega)$, for every $s > 0$ and $y \in \mathbb{R}^d$. Therefore,

$$\begin{aligned}&\left\|\int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma(s\,,u_N(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y) - \mathcal{I}_\sigma(t\,,x)\right\|_2^2\\ &\le \int_{(0,t)\times\mathbb{R}} [p_{t-s}(y-x)]^2\,\|\sigma(s\,,u_N(s\,,y)) - \sigma(s\,,u(s\,,y))\|_2^2\,\mathrm{d}s\,\mathrm{d}y,\end{aligned}$$

which converges to 0 as $N\to\infty$ by dominated convergence, for all $t \in (0\,,T]$ and $x \in \mathbb{R}$. The triangle inequality now yields (3.16). This verifies the second assertion of (3.15) and completes the proof of Theorem 1.4 in the case that $L_\sigma > 0$. It remains to study the same problem when σ is bounded.

The proof in the case that σ is bounded is structurally similar to the derivation of the first part, except we replace the tail bound (2.10) with (2.11) in order to obtain the following variant of (3.13):

$$\limsup_{N\to\infty}\mathrm{e}^{-2N}\log\sup_{t\in(0,T]}\sup_{x\in\mathbb{R}}\|u_{N+1}(t\,,x) - u_N(t)\|_k < 0 \qquad \forall k\ge 1.$$

The remainder of our derivation of the σ-bounded case is the same as in the first portion of the proof $[L_\sigma > 0]$ and therefore omitted. This completes the proof. □

Proof of uniqueness We only give an outline of the proof in the case that $L_\sigma > 0$, since the proof of uniqueness is essentially a simplified modification of the proof of existence, because the other case where σ is bounded is similar.

Let u, v be two solutions to (1.1) with the same initial condition u_0 satisfying Assumption 1.1. Recall that u and v satisfy the mild formulation (1.3) with σ and b satisfying Assumption 1.2. Then, using Burkholder–Davis–Gundy inequality (see [3]), we find that for all $t > 0$, $x \in \mathbb{R}$, and $k \geq 1$,

$$\|u(t\,,x) - v(t\,,x)\|_k \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x) \left\|(b(s\,,u(s\,,y)) - b(s\,,v(s\,,y)))\mathbf{1}_{A_N(s,y)}\right\|_k \mathrm{d}s\,\mathrm{d}y,\\ I_2^2 &= 4k\int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, [p_{t-s}(y-x)]^2 \left\|(\sigma(s\,,u(s\,,y)) - \sigma(s\,,v(s\,,y)))\mathbf{1}_{A_N(s,y)}\right\|_k^2,\\ I_3 &= \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x) \left\|(b(s\,,u(s\,,y)) - b(s\,,v(s\,,y)))\mathbf{1}_{\Omega\setminus A_N(s,y)}\right\|_k \mathrm{d}s\,\mathrm{d}y,\\ I_4^2 &= 8k\int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, [p_{t-s}(y-x)]^2 \left\|(\sigma(s\,,u(s\,,y)) - \sigma(s\,,v(s\,,y)))\mathbf{1}_{\Omega\setminus A_N(s,y)}\right\|_k^2, \end{aligned}$$

where for $N, s > 0$ and $y \in \mathbb{R}$,

$$A_N(s\,,y) = \left\{\omega \in \Omega : \ |u(s\,,y)|(\omega) \leq \mathrm{e}^N,\ \ |v(s\,,y)|(\omega) \leq \mathrm{e}^N\right\}.$$

Observe that the moment bounds obtained in Propositions 2.1 and 2.3 only use the linear growth constants L_σ and L_b. Therefore, they also hold when u_N is replaced by u or v. The same is true for the tail estimates obtained Proposition 2.4, which also hold replacing u_{N+1} by u or v. Therefore, we can proceed as in the proof of existence but appeal to the local Lipschitz condition on b and σ; that is, we recall (1.5) and write

$$\left\|[b(s\,,u(s\,,y)) - b(s\,,v(s\,,y))]\,\mathbf{1}_{A_N(s,y)}\right\|_k \leq L_{N,b}\,\|u(s\,,y) - v(s\,,y)\|_k\,,$$

and proceed similarly for σ. Because $\mathrm{P}(\Omega\setminus A_N(s, y))$ is at-most $\mathrm{P}\{|u(t, y)| \geq \mathrm{e}^N\} + \mathrm{P}\{|v(t, y)| \geq \mathrm{e}^N\}$, similar computations to those in the proof of existence imply that for all $T > 0$,

$$\begin{aligned} \|u(t\,,x) - v(t\,,x)\|_k &\leq \mathrm{e}^{\beta t}\left[L_{N,b}\beta^{-1} + 2\sqrt{k}L_{N,\sigma}\beta^{-1/4}\right]\mathcal{N}_{k,\beta,T}(u-v)\\ &\quad + C\left[\frac{L_b}{128L_\sigma^4k^2} + \sqrt{128k}T^{1/4}L_\sigma\right]\exp\left(512L_\sigma^4k^2t - \frac{N^{3/2}}{128kL_\sigma^2\sqrt{t}}\right), \end{aligned}$$

uniformly for all $\beta > 0, t \in (0, T), x \in \mathbb{R}$, and $N, k \geq c$, where $C = 4(\|u_0\|_{L^\infty(\mathbb{R})}+1)$ and $c = c(\|u_0\|_{L^\infty(\mathbb{R})}, L_\sigma, T, L_b) > 0$. Now we fix the parameters β and k as in (3.9) and analyze the preceding exactly as was done in the proof of existence in order to conclude that

$$\sum_{N=1}^{\infty} \sup_{t\in(0,T]} \sup_{x\in\mathbb{R}} \|u(t\,,x) - v(t\,,x)\|_2 < \infty \qquad \forall T > 0.$$

Since the summand does not depend on N, it must be zero. This concludes the proof. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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