Stabilisation of Hybrid Systems by Event-triggered Control based on Discrete-time State Observations

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Abstract

Given an unstable hybrid stochastic differential equation (SDE) $dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t)$, we can design a feedback control $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$ so that the controlled SDE $dx(t) = \big(f(x(t), r(t), t) + u(x(\lfloor t/\tau \rfloor \tau), r(t), t)\big)dt + g(x(t), r(t), t)dB(t)$ becomes stable, were $\tau > 0$ and $\lfloor t/\tau \rfloor$ is the integer part of t/τ. Noting that the control $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$ is based on the state observations at discrete times $0, \tau, 2\tau, \cdots$, we see that this stabilisation problem is significantly different from the classical one where the feedback control $u(x(t), r(t), t)$ is based on continuous-time state observations. This stabilisation problem was initiated by Mao in 2013 and has been intensively studied by many authors for the past ten years. In this paper we will further develop an event-triggered control based on a trajectory of the state or output observations at discrete times. This is significantly different from most of the existing papers where an event-triggered mechanism for SDEs is designed based on a criterion in terms of the mathematical expectation of the system state. The novel approach in this paper is to relate the stability of the event-triggered controlled SDE to the stability of the corresponding uncertain SDE.

Key words: Brownian motion, Markov chain, Event-triggered control, Exponential stability, Lypunov function.

1 Introduction

An important class of hybrid systems is the class of hybrid stochastic differential equations (SDEs) (also known as SDEs with Markovian switching), which have frequently been used in

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many branches of science and industry (see, e.g., [17, 24, 25, 32]). The hybrid SDEs can be described by

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t). \tag{1.1}$$

Here the state $x(t)$ takes values in $\mathbb{R}^n$ and the mode $r(t)$ is a Markov chain taking values in a finite space $\mathbb{S} = \{1, 2, \cdots, N\}$, $B(t)$ is a Brownian motion and f and g are referred to as the drift and diffusion coefficient, respectively. One of the important issues in the study of hybrid SDEs is the analysis of stability (see, e.g., [13, 17, 18, 23, 27, 28, 29, 40]).

The following stabilisation problem by a feedback control based on discrete-time state observations was initiated in [19]: Assume that the given hybrid SDE (1.1) is not stable and the observations of the state $x(t)$ are only available at discrete times $0, \tau, 2\tau, \cdots$, while the mode $r(t)$ is available at any time $t \geq 0$, can we design a feedback control $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$ so that the controlled system

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\lfloor t/\tau \rfloor \tau), r(t), t)\big)dt + g(x(t), r(t), t)dB(t) \tag{1.2}$$

becomes stable? Here $\tau > 0$ is a constant which stands for the duration between two consecutive state observations, and $\lfloor t/\tau \rfloor$ is the integer part of t/τ. This is significantly different from the stabilisation by a continuous-time (regular) feedback control $u(x(t), r(t), t)$, because the regular feedback control requires the continuous observations of the state $x(t)$ for all $t \geq 0$, while the feedback control $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$ needs only the discrete observations of the state $x(t)$ at times $0, \tau, 2\tau, \cdots$. The latter is clearly more realistic and costs less in practice. This stabilisation problem has been studied by many authors for the past ten years (see, e.g., [7, 20, 21, 26, 34, 35, 39]). It should be pointed out that we here assume that the mode $r(t)$ of the system is obvious (i.e., fully observable at no cost), for example, in a financial system where the mode represents the interest rate (see, e.g., [37, 41]). There is hence no cost for the communication of the mode $r(t)$ too. When design the feedback control, we should of course make full use of the available information on it, namely allow the feedback control to depend on the mode $r(t)$.

To explain how this problem will be developed in this paper, we set $t_k = k\tau$ for $k \in \mathbb{N} := \{0, 1, 2, \cdots\}$, which are the discrete times when the state observations are made. The feedback control $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$ in (1.2) can be written as

$$u(x(t_k), r(t), t) \quad \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}. \tag{1.3}$$

That is, the state input $x(t_k)$ in the control is held constant during $[t_k, t_{k+1})$ but is updated to $x(t_{k+1})$ at time t_{k+1} and remains constant during $[t_{k+1}, t_{k+2})$ while the control will be recomputed according to $u(x(t_{k+1}), r(t), t)$. In other words, the state input is updated at every time t_k and the control has to be recomputed accordingly. We wonder whether this is absolutely necessary? Imaging that $x(t_{k+1})$ might be close to $x(t_k)$, we would expect it is unnecessary to update the state input from $x(t_k)$ to $x(t_{k+1})$ at time t_{k+1}. In other words, in this situation, we might still use the control $u(x(t_k), r(t), t)$ for the next duration $[t_{k+1}, t_{k+2})$ to avoid communicating the new state observation $x(t_{k+1})$ and recomputing the new control $u(x(t_{k+1}), r(t), t)$ and hence to reduce the overall control cost.

To develop this idea into a new theory, we first need to describe precisely how $x(t_{k+1})$ is said to be close to $x(t_k)$. We will use the following simple definition in this paper.

Definition 1.1. *Let $\Delta \in (0, 1)$. Let $y = (y_1, \cdots, y_n)^T$ and $z = (z_1, \cdots, z_n)^T$ be two vectors in $\mathbb{R}^n$. If*

$$|y_i - z_i| \leq \Delta |z_i| \quad \textit{for all } i = 1, \cdots, n, \tag{1.4}$$

then z is said to be close to y in the order of Δ. We will not mention "in the order of Δ" any more from now on. We will call Δ the event-triggered parameter.

We next need to form our new controlled system. Let $X_0 = x(0)$, the initial value. The control is $u(X_0, r(t), t)$ for $t \in [0, t_1)$. At time t_1, we observe the state $x(t_1)$. If $x(t_1)$ is close to X_0, we will not update the state input in the control; otherwise update it. Hence the control is $u(X_1, r(t), t)$ for $t \in [t_1, t_2)$, where

$$X_1 = \begin{cases} X_0 & \text{if } x(t_1) \text{ is close to } X_0, \\ x(t_1) & \text{otherwise.} \end{cases} \tag{1.5}$$

In general, we define X_k for $k = 1, 2, \cdots$ by

$$X_k = \begin{cases} X_{k-1} & \text{if } x(t_k) \text{ is close to } X_{k-1}, \\ x(t_k) & \text{otherwise.} \end{cases} \tag{1.6}$$

Then the control is $u(X_k, r(t), t)$ for $t \in [t_k, t_{k+1})$. Therefore our new controlled system has the following form

$$dx(t) = \big(f(x(t), r(t), t) + u(X_{\lfloor t/\tau \rfloor}, r(t), t)\big)dt + g(x(t), r(t), t)dB(t) \tag{1.7}$$

on $t \geq 0$.

The idea above is motivated by the popular event-triggered controls. In the field of ordinary differential equations (ODEs), the event-triggered controls have been studied intensively and we only mention [4, 5, 8, 30, 38] among many others. In the field of SDEs, they have received more and more attention recently (see, e.g., [2, 9, 12, 33, 36]). In most of the existing papers, an event-triggered mechanism for SDEs is designed based on a criterion in terms of the mathematical expectation of the system state. Such a mechanism can only be implemented if a sample of large size of the system state, at any observation time, is available so that the mathematical expectation involved in the event-triggered criterion can be calculated approximately. There are some systems in this category, for example, a stochastic particle system [11]. However, in much more real world SDE systems, e.g., an SDE model for a stock price, the state observation is just a single-size sample at a time, namely the state observations at discrete times form a trajectory of the solution to the underlying SDE. In such a situation, the event-triggered criterion should be designed based on a trajectory of the state observations at discrete times. To our best knowledge, there are a few papers in this direction so far (e.g., [12]). There is therefore an urgent need to develop the event-triggered control theory based on a trajectory of the state observations at discrete times.

It is necessary to explain how the results in [12] differ from ours in this paper. The event-triggered mechanism in [12] differs from ours. In [12], z is said to be close to y if $|z-y| \leq \Delta|z|$. The difference between theirs and our Definition 1.1 looks minor. However, the significant effect is that in their case, there is no form of (2.6) between y and z (see Section 2 below please). We give a counter example: Let $z = (10, 0)^T$ and $y = (10.1, 0.1)^T$ and $\Delta = 0.1$. Then $|z - y| = 0.1\sqrt{2} \leq 0.1|z| = 1$. But we cannot write y in terms of z in the form of (2.6). Let us now explain why the form of (2.6) is so crucial in this paper. In fact, it is due to the form of (2.6) that we will be able to establish our novel approach which is completely different from that in [12]. Our approach is to relate the event-triggered controlled SDE (1.7) to the corresponding uncertain SDE

$$dz(t) = \big(f(z(t), r(t), t) + u((I_n + \phi(t))z(t), r(t), t)\big)dt + g(z(t), r(t), t)dB(t) \tag{1.8}$$

on $t \geq 0$, where I_n is the $n \times n$ identity matrix and $\phi(t) = \text{diag}(\phi_1(t), \cdots, \phi_n(t))$ is an $\mathcal{F}_t$-adapted process such that its every element $\phi_i(t) \in [-\Delta, \Delta]$ a.s. Please note that the real values of every $\phi_i(t)$ are unknown though they are within $[-\Delta, \Delta]$. In words, $\phi_i(t)$ represents the system uncertainty. We will show that under some regular conditions on the coefficients f, g, u, if the uncertain SDE (1.8) is exponentially stable in mean square, then there is a positive number τ^* such that the event-triggered controlled SDE (1.7) is also exponentially stable in mean square as long as $\tau < \tau^*$. That is, we establish a useful link between the stability of the event-triggered controlled SDE (1.7) and that of the uncertain SDE (1.8). There is no such a link before. It is the discovery of this new link that makes the significant impact.

Let us now explain the impact. First of all, the stability of the uncertain SDEs has been well established (see, e.g., [10, 16, 24]). The techniques and methods in the study of the uncertain SDEs can now be developed to study the event-triggered controlled SDEs. In other words, the discovery of the link provides the researchers in the area of uncertain SDEs with a new opportunity to develop their research in the area of event-triggered controlled SDEs (as we do in this paper). This is even more significant given that there are only a few papers on the event-triggered controls based on a trajectory of the state (though much more papers based on conditions in terms of the mathematical expectation of the system state). Further more, on the one hand, we observe that the stabilisation problem (1.2) is a stochastic differential delay equation (SDDE) and that is why many existing papers have applied the method of Lyapunov functionals to study it (see, e.g., [7, 39]). Our underlying event-triggered controlled system (1.7) is a generalised version of (1.2) and is hence an SDDE too. It will be natural to develop the method of Lyapunov functionals to study (1.7). On the other hand, our novel approve links systems (1.7) and (1.8) together. Noting that system (1.8) is an SDE (but not an SDDE), we realise that the link provides us with a new opportunity to develop the method of Lyapunov functions to study system (1.7). As the design of Lyapunov functions is simpler than that of Lyapunov functionals, our new results can be applied more easily in practice.

What we have discussed so far is the case where state $x(t_k)$ is observable at the discrete times $\{t_k\}_{k\in\mathbb{N}}$ and the control is known as the state feedback (see, e.g., [22]). However, there is another important case in practice, where the output $y(t_k) := G_{r(t_k)}x(t_k)$ is observable but state $x(t_k)$ itself is not, in which $G_i \in \mathbb{R}^{l\times n}$ is known for every $i \in \mathbb{S}$ and, in general, $l < n$. In this case, the event-triggered control should be designed based on $y(t_k)$ but not $x(t_k)$. In this paper we will investigate this case too but [12] did not. This is one more of our contributions.

Let us summarise our main contributions before we develop our main results:

- It is first time that the stability of the event-triggered controlled SDE is linked to the stability of the corresponding uncertain SDE.
- The techniques and methods in the study of the uncertain SDEs can now be developed to study the event-triggered controlled SDEs as we demonstrate in this paper.
- In particular, the link provides us with a new opportunity to develop the method of Lyapunov functions, instead of Lyapunov functionals, to study the event-triggered controlled SDEs.
- We do not only investigate the event-triggered state feedback control for the case where the state is observable but also the event-triggered output injection for the case where the output is observable (and the latter was not studied in [12]). We explain how both cases differ from each other significantly and hence different mathematical techniques have been developed to study them.

2 Stabilisation by Event-triggered State Feedback

2.1 Notation

Throughout this paper, unless otherwise specified, we use the following notation. If A is a vector or matrix, its transpose is denoted by A^T. For $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while it operator norm is denoted by $\|A\| = \max\{|Ax| : |x| = 1\}$. Let I_n be the $n \times n$ identity matrix. Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{N} = \{0, 1, 2, \cdots\}$. If $A \in \mathbb{R}^{n\times n}$ is symmetric, denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively, while by $A \leq 0$ and $A < 0$, we mean A is non-positive and negative definite, respectively. Moreover, $A \geq 0$ and $A > 0$ mean $-A \leq 0$ and $-A < 0$, respectively. For symmetric matrices A_1 and A_2, by $A_1 \leq A_2$ and $A_1 < A_2$, we mean that $A_1 - A_2 \leq 0$ and $A_1 - A_2 < 0$, respectively. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. If A is a subset of Ω, denote by I_A its indicator function; that is, $I_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. By $X := Y$ or $X =: Y$, we mean X is defined or denoted by Y, respectively.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t + h) = j | r(t) = i\} = \begin{cases} \gamma_{ij}h + o(h) & \text{if } i \neq j, \\ 1 + \gamma_{ii}h + o(h) & \text{if } i = j, \end{cases}$$

where $h > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $-\gamma_{ii} = \sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost all sample paths of $r(t)$ are piecewise constant except for a finite number of simple jumps in any finite subinterval of $\mathbb{R}_+$. We stress that almost all sample paths of $r(t)$ are right continuous.

2.2 Controlled system

Consider the given unstable n-dimensional hybrid SDE (1.1), where $f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ are Borel measurable functions. Assume that the state $x(t)$ is observable at the discrete times $\{t_k\}_{k\in\mathbb{N}}$ in all modes or some modes and the Markov chain $r(t)$ is observable at any time (e.g., the Markov chain models the interest rates [6]). Of course, we can consider the situation where the Markov china $r(t)$ is only observable at the discrete times but we will report the corresponding results else where due to the page limit here. We need to design the control function $u : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ so that the controlled SDE (1.7) becomes stable. To make our new theory more understandable, we will only consider the linear control function of the form

$$u(x, i, t) = A_i x \quad \text{with} \quad A_i = F_i G_i, \tag{2.1}$$

where $F_i \in \mathbb{R}^{n\times l}$ and $G_i \in \mathbb{R}^{l\times n}$ for some positive integer l which is less than n in general. If in mode i, state $x(t)$ is not observable, we set $A_i = 0$ (see Example 4.2). This is known as the structure control, which has two cases: (i) state feedback control where F_i's are known while G_i's need to be designed; (ii) output injection where G_i's are known while F_i's need to

be designed (see, e.g., [22]). In this section we consider case (i) but leave case (ii) to the next section. We will explain how to design G_i's in Section 2.5 but in the meanwhile we assume they are known and so are A_i's. Consequently, our controlled SDE (1.7) becomes

$$dx(t) = \big(f(x(t), r(t), t) + A_{r(t)}X_{\lfloor t/\tau \rfloor}\big)dt + g(x(t), r(t), t)dB(t) \tag{2.2}$$

on $t \geq 0$, with initial data $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in \mathbb{S}$ at time zero. To make this SDE well defined, we impose the following standing hypotheses.

Assumption 2.1. *Assume that the coefficients f and g are locally Lipschitz continuous (see, e.g., [16, 24]). Moreover, they satisfy the following linear growth condition*

$$|f(x, i, t)| \leq K_1|x| \quad \text{and} \quad |g(x, i, t)| \leq K_2|x| \tag{2.3}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where both K_1 and K_2 are positive numbers.

Please note that condition (2.3) forces that $f(0, i, t) = 0$ and $g(0, i, t) = 0$ so 0 is an equilibrium state. We claim that under Assumption 2.1, the controlled SDE (2.2) with any initial data $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in \mathbb{S}$ has a unique global solution $x(t)$ such that $\mathbb{E}|x(t)|^2 < \infty$ for all $t \geq 0$. In fact, restricting $t \in [0, t_1]$, we see that the controlled SDE (2.2) becomes

$$dx(t) = \big(f(x(t), r(t), t) + A_{r(t)}X_0\big)dt + g(x(t), r(t), t)dB(t), \tag{2.4}$$

where $X_0 = x_0$. It is well known (see, e.g., [24]) that this SDE has a unique solution $x(t)$ on $t \in [0, t_1]$ such that $\sup_{t\in[0,t_1]} \mathbb{E}|x(t)|^2 < \infty$. At time t_1, $x(t_1)$ is observed and X_1 can be defined by (1.5). For $t \in [t_1, t_2]$, the controlled SDE (2.2) becomes

$$dx(t) = \big(f(x(t), r(t), t) + A_{r(t)}X_1\big)dt + g(x(t), r(t), t)dB(t) \tag{2.5}$$

with initial data $x(t_1)$ and $r(t_1)$ at time t_1. It is well known that this SDE has a unique solution $x(t)$ on $t \in [t_1, t_2]$ such that $\sup_{t\in[t_1,t_2]} \mathbb{E}|x(t)|^2 < \infty$. At time t_2, $x(t_2)$ is observed and X_2 can be defined by (1.6). Repeating the above arguments for $t \in [t_2, t_3]$ etc. we see what we claimed.

To perform the theoretical analysis, we need to re-write the controlled SDE (2.2) in a modified form. Let $z = (z_1, \cdots, z_n)^T$ be close to $y = (y_1, \cdots, y_n)^T$. By Definition 1.1, for each $i = 1, \cdots, n$, there is a unique number $\sigma_i \in [-\Delta, \Delta]$ such that $y_i = (1 + \sigma_i)z_i$. In fact, $\sigma_i = (y_i - z_i)/z_i$ when $z_i \neq 0$ and we set $\sigma_i = 0$ when $z_i = 0$. Hence

$$y = (I_n + \sigma)z, \tag{2.6}$$

where $\sigma = \text{diag}(\sigma_1, \cdots, \sigma_n)$ (and I_n is the $n \times n$ identity matrix as defined before). It is also easy to see

$$|\sigma z| \leq \Delta|z| \quad \text{and} \quad (1 - \Delta)|z| \leq |y| \leq (1 + \Delta)|z|. \tag{2.7}$$

Applying (2.6) to (1.6), we see that for $k = 1, 2, \cdots$,

$$X_k = \begin{cases} (I_n + \bar{\Delta}(t_k))x(t_k) & \text{if } x(t_k) \text{ is close to } X_{k-1}, \\ x(t_k) & \text{otherwise,} \end{cases} \tag{2.8}$$

where $\bar{\Delta}(t_k) = \text{diag}(\bar{\Delta}_1(t_k), \cdots, \bar{\Delta}_n(t_k)$ with $\bar{\Delta}_i(t_k) \in [-\Delta, \Delta]$ for all $i = 1, \cdots, n$, while $\bar{\Delta}(t_k)$ is $\mathcal{F}_{t_k}$-measurable as it is determined by $x(t_k)$ and X_{k-1} only. We can then further write $X_k = (I_n + \hat{\Delta}(t_k))x(t_k)$, where

$$\hat{\Delta}(t_k) = \begin{cases} \bar{\Delta}(t_k) & \text{if } x(t_k) \text{ is close to } X_{k-1}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

which is $\mathcal{F}_{t_k}$-measurable of course. By (2.7), we also see that $(\bar{\Delta}(t_k))^2 - \Delta^2 I_n \leq 0$,

$$|\hat{\Delta}(t_k)x(t_k)| \leq \Delta|x(t_k)| \quad \text{and} \quad (1-\Delta)|x(t_k)| \leq |(I_n + \hat{\Delta}(t_k))x(t_k)| \leq (1+\Delta)|x(t_k)| \tag{2.10}$$

amost surely. We set $\hat{\Delta}(t_0) = 0$ so that $\hat{\Delta}(t_k)$ is now defined for all $k \in \mathbb{N}$. Furthermore, we will define $\delta_t = \lfloor t/\tau \rfloor \tau$ for $t \geq 0$. With these notations, the controlled SDE (2.2) can be written in the following modified form

$$dx(t) = \big(f(x(t), r(t), t) + A_{r(t)}(I_n + \hat{\Delta}(\delta_t))x(\delta_t)\big)dt + g(x(t), r(t), t)dB(t). \tag{2.11}$$

It should be pointed out that this form is convenient when we perform mathematical analysis from next section on by making use of (2.10). However, in practice, we should use (2.2) because once X_k is determined by (1.6), the controlled SDE (2.2) can already be implemented without knowing $\hat{\Delta}(t_k)$.

We will see that the stability of this controlled SDE is related to the robust stability of the following auxiliary uncertain SDE

$$dz(t) = \big(f(z(t), r(t), t) + A_{r(t)}(I_n + \phi(t))z(t)\big)dt + g(z(t), r(t), t)dB(t) \tag{2.12}$$

on $t \geq 0$, where $\phi(t) = \text{diag}(\phi_1(t), \cdots, \phi_n(t))$ is an $\mathcal{F}_t$-adapted process such that its every element $\phi_i(t) \in [-\Delta, \Delta]$ a.s. Please note that the real values of every $\phi_i(t)$ are unknown though they are within $[-\Delta, \Delta]$. In words, $\phi_i(t)$ represents the system uncertainty.

2.3 Lemma

Let us begin this section by stating a key assumption.

Assumption 2.2. *There exist positive numbers β, β_i ($i \in \mathbb{S}$) and symmetric positive definite matrices $Q_i \in \mathbb{R}^{n\times n}$ such that*

$$\begin{aligned} &2x^T Q_i f(x,i,t) + \text{trace}[g(x,i,t)^T Q_i g(x,i,t)] \\ &+ x^T \Big(Q_i A_i + A_i^T Q_i + \beta_i Q_i A_i A_i^T Q_i + (\Delta^2/\beta_i) I_n + \sum_{j=1}^{N} \gamma_{ij} Q_j\Big) x \leq -\beta|x|^2 \end{aligned} \tag{2.13}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

It should be pointed out that if (2.13) holds, then Assumption 2.2 holds for λQ_i with any $\lambda > 0$. That is, the choices of Q_i's are scale-independent. We will explain in detail in Section 2.5 how to choose matrices Q_i and free parameters β_i and then design the corresponding matrices G_i's and hence A_i's (recalling F_i's are known) and find the positive number β for (2.13) to hold. In this section, we just assume that Assumption 2.2 holds. By [24, Theorem 5.8 on page 166] (with $V(x,i,t) = x^T Q_i x$ there), we see that Assumption 2.2 guarantees the mean-square exponential stability of the auxiliary uncertain SDE (2.12).

From now on we will fix initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$ arbitrarily and will not mention them any more unless necessary. Moreover, we will set

$$K_A = \max_{i\in\mathbb{S}} \|A_i\|,\ K_{QA} = \max_{i\in\mathbb{S}} \|Q_i A_i\|,\ \lambda_m = \min_{i\in\mathbb{S}} \lambda_{\min}(Q_i),\ \lambda_M = \max_{i\in\mathbb{S}} \lambda_{\max}(Q_i). \tag{2.14}$$

Before we state our first main theorem in this paper, let us present a useful lemma.

Lemma 2.3. *Let Assumption 2.1 hold. If $\tau < 1/(K_A(1+\Delta))$, then the solution of the controlled SDE (2.2) has the property that*

$$\mathbb{E}(|x(t)||x(t) - x(\delta_t)|) \le H_1(\tau)\mathbb{E}|x(t)|^2 + H_2(\tau)\int_{\delta_t}^t \mathbb{E}|x(s)|^2 ds \tag{2.15}$$

for all $t \ge 0$, where

$$H_1(\tau) = \frac{\tau[0.5K_1 + K_A(1+\Delta)] + 0.5K_2\sqrt{\tau}}{1 - \tau K_A(1+\Delta)} \tag{2.16}$$

and

$$H_2(\tau) = \frac{0.5(K_1 + K_2/\sqrt{\tau})}{1 - \tau K_A(1+\Delta)}. \tag{2.17}$$

Proof. For $t \in [t_k, t_{k+1})$ for some $k \in \mathbb{N}_+$, by Assumption 2.1 and property (2.10), it follows from (2.11) that

$$\begin{aligned}
&\mathbb{E}(|x(t)||x(t) - x(\delta_t)|) = \mathbb{E}(|x(t)||x(t) - x(t_k)|) \\
\le&\mathbb{E}\Big(|x(t)|\int_{t_k}^t (K_1|x(s)| + K_A(1+\Delta)|x(t_k)|)ds + |x(t)||M_k(t)|\Big),
\end{aligned} \tag{2.18}$$

where $M_k(t) = \int_{t_k}^t g(x(s), r(s), s)dB(s)$. But

$$\begin{aligned}
&\mathbb{E}\Big(|x(t)|\int_{t_k}^t (K_1|x(s)| + K_A(1+\Delta)|x(t_k)|)ds\Big) \\
&\le \mathbb{E}\int_{t_k}^t \Big(K_1|x(t)||x(s)| + K_A(1+\Delta)|x(t)|[|x(t)| + |x(t) - x(t_k)|]ds \\
&\le \tau[0.5K_1 + K_A(1+\Delta)]\mathbb{E}|x(t)|^2 + 0.5K_1\mathbb{E}\int_{t_k}^t |x(s)|^2 ds \\
&+ \tau K_A(1+\Delta)\mathbb{E}(|x(t)||x(t) - x(t_k)|),
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
\mathbb{E}|x(t)||M_k(t)| &\le 0.5K_2\sqrt{\tau}\mathbb{E}|x(t)|^2 + (0.5/K_2\sqrt{\tau})\mathbb{E}|M_k(t)|^2 \\
&\le 0.5K_2\sqrt{\tau}\mathbb{E}|x(t)|^2 + (0.5K_2/\sqrt{\tau})\mathbb{E}\int_{t_k}^t |x(s)|^2 ds.
\end{aligned} \tag{2.20}$$

Substituting these into (3.14) yields

$$\begin{aligned}
&\mathbb{E}(|x(t)||x(t) - x(\delta_t)|) \\
&\le (\tau[0.5K_1 + K_A(1+\Delta)] + 0.5K_2\sqrt{\tau})\mathbb{E}|x(t)|^2 + 0.5(K_1 + K_2/\sqrt{\tau})\mathbb{E}\int_{t_k}^t |x(s)|^2 ds \\
&+ \tau K_A(1+\Delta)\mathbb{E}(|x(t)||x(t) - x(t_k)|).
\end{aligned} \tag{2.21}$$

This implies the desired assertion (2.15) as $\tau K_A(1+\Delta) < 1$. □

2.4 Theorems

Theorem 2.4. *Let Assumptions 2.1 and 2.2 hold. Let $\tau^* \in (0, (K_A(1+\Delta))^{-1})$ be the unique root to the following equation of τ*

$$\beta = \frac{2K_{QA}(1+\Delta)}{1-\tau K_A(1+\Delta)}\big(\tau[K_1 + K_A(1+\Delta)] + K_2\sqrt{\tau}\big) \tag{2.22}$$

and it has the explicit form (2.26) below. If $\tau < \tau^*$, then the solution of the controlled SDE (2.2) has the properties that

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^2) \le -\varepsilon, \tag{2.23}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -\frac{\varepsilon}{2} \quad a.s., \tag{2.24}$$

where ε is a unique positive number such that

$$\beta = \varepsilon\lambda_M + 2K_{QA}(1+\Delta)\big(H_1(\tau) + \tau e^{\varepsilon\tau}H_2(\tau)\big), \tag{2.25}$$

in which $H_1(\tau)$ and $H_2(\tau)$ have been defined in Lemma 2.3. That is, the controlled SDE (2.2) is not only exponentially stable in mean square but also almost surely.

Proof. We observe that as τ increases from 0 to $(K_A(1+\Delta))^{-1}$, the right-hand-side of (2.22) increases strictly and continuously from 0 to positive infinity. Equation (2.22) must therefore have a unique root $\tau^* \in (0, (K_A(1+\Delta))^{-1})$. It is straightforward to obtain

$$\tau^* = \Big(\frac{-b + \sqrt{b^2 + 4a\beta}}{2a}\Big)^2, \tag{2.26}$$

where $a = 2K_{QA}(1+\Delta)[K_1 + K_A(1+\Delta)] + \beta K_A(1+\Delta)$ and $b = 2K_2K_{QA}(1+\Delta)$.

Fix $\tau < \tau^*$ arbitrarily. We then have

$$\beta > \frac{2K_{QA}(1+\Delta)}{1-\tau K_A(1+\Delta)}\big(\tau[K_1 + K_A(1+\Delta)] + K_2\sqrt{\tau}\big). \tag{2.27}$$

Recalling the definitions of $H_1(\tau)$ and $H_2(\tau)$ in Lemma 2.3, we observe that

$$2K_{QA}(1+\Delta)\big(H_1(\tau) + \tau H_2(\tau)\big) = \frac{2K_{QA}(1+\Delta)}{1-\tau K_A(1+\Delta)}\big(\tau[K_1 + K_A(1+\Delta)] + K_2\sqrt{\tau}\big).$$

Making use of (2.27), we see that there is a unique positive number ε for (2.25) to hold.

Applying the generalized Itô formula (see, e.g., [18, 24]) to the Lyapunov function

$$V(x(t), r(t)) := x^T(t)Q_{r(t)}x(t),$$

we obtain

$$dV(x(t), r(t)) = J_1(t)dt + dM(t) \tag{2.28}$$

for $t \ge 0$, where $M(t)$ is a continuous martingale with $M(0) = 0$ (the explicit form of $M(t)$ is of no use in this paper so we do not state it here) and

$$\begin{aligned} J_1(t) &:= 2x^T(t)Q_{r(t}[f(x(t), r(t), t) + A_{r(t)}(I_n + \hat{\Delta}(\delta_t))x(\delta_t)] \\ &+ \text{trace}[g^T(x(t), r(t), t)Q_{r(t)}g(x(t), r(t), t)] + \sum_{j=1}^{N}\gamma_{r(t),j}x^T(t)Q_jx(t). \end{aligned} \tag{2.29}$$

Making use of (2.10), we can derive that

$$\begin{aligned}
&2x^T(t)Q_{r(t)}A_{r(t)}(I_n+\hat{\Delta}(\delta_t))x(\delta_t)\\
&=2x^T(t)Q_{r(t)}A_{r(t)}(I_n+\hat{\Delta}(\delta_t))\big(x(t)-[x(t)-x(\delta_t)]\big)\\
&\le x^T(t)\Big(Q_{r(t)}A_{r(t)}+A^T_{r(t)}Q_{r(t)}+\beta_{r(t)}Q_{r(t)}A_{r(t)}A^T_{r(t)}Q_{r(t)}+(\Delta^2/\beta_{r(t)})I_n\Big)x(t)\\
&+2K_{QA}(1+\Delta)|x(t)||x(t)-x(\delta_t)|. \qquad (2.30)
\end{aligned}$$

Substituting this into (2.28) and then applying condition (2.13), we get

$$J_1(t)\le -\beta|x(t)|^2+2K_{QA}(1+\Delta)|x(t)||x(t)-x(\delta_t)|. \qquad (2.31)$$

Applying the generalized Itô formula to the Lyapunov function $e^{\varepsilon t}V(x(t),r(t))$, we obtain that

$$e^{\varepsilon t}\mathbb{E}V(x(t),r(t))=V(x_0,r_0)+\mathbb{E}\int_0^t e^{\varepsilon s}\Big(\varepsilon V(x(s),r(s))+J_1(s)\Big)ds \qquad (2.32)$$

for $t\ge 0$. By (2.31), Lemma 2.3, etc., we then derive

$$\begin{aligned}
&\lambda_m e^{\varepsilon t}\mathbb{E}|x(t)|^2\le\lambda_M|x_0|^2\\
&+\int_0^t e^{\varepsilon s}\Big(-(\beta-\varepsilon\lambda_M)\mathbb{E}|x(s)|^2+2K_{QA}(1+\Delta)\mathbb{E}(|x(s)||x(s)-x(\delta_s)|)\Big)ds\\
&\le\lambda_M|x_0|^2-\big(\beta-\varepsilon\lambda_M-2K_{QA}(1+\Delta)H_1(\tau)\big)\int_0^t e^{\varepsilon s}\mathbb{E}|x(s)|^2ds\\
&+2K_{QA}(1+\Delta)H_2(\tau)\int_0^t e^{\varepsilon s}\Big(\int_{\delta_s}^s\mathbb{E}|x(v)|^2dv\Big)ds. \qquad (2.33)
\end{aligned}$$

But

$$\begin{aligned}
&\int_0^t e^{\varepsilon s}\Big(\int_{\delta_s}^s\mathbb{E}|x(v)|^2dv\Big)ds\le\int_0^t e^{\varepsilon s}\Big(\int_{(s-\tau)\vee 0}^s\mathbb{E}|x(v)|^2dv\Big)ds\\
&\le\int_0^t\mathbb{E}|x(v)|^2\Big(\int_v^{v+\tau}e^{\varepsilon s}ds\Big)dv\le\int_0^t\mathbb{E}|x(v)|^2\tau e^{\varepsilon(v+\tau)}dv\\
&=\tau e^{\varepsilon\tau}\int_0^t e^{\varepsilon v}\mathbb{E}|x(v)|^2dv. \qquad (2.34)
\end{aligned}$$

Substituting this into (2.33) yields

$$\begin{aligned}
&\lambda_m e^{\varepsilon t}\mathbb{E}|x(t)|^2\le\lambda_M|x_0|^2\\
&-\big(\beta-\varepsilon\lambda_M-2K_{QA}(1+\Delta)[H_1(\tau)+\tau e^{\varepsilon\tau}H_2(\tau)\big)\int_0^t e^{\varepsilon s}\mathbb{E}|x(s)|^2ds. \qquad (2.35)
\end{aligned}$$

Recalling (2.25) we obtain

$$\lambda_m e^{\varepsilon t}\mathbb{E}|x(t)|^2\le\lambda_M|x_0|^2, \qquad (2.36)$$

which implies the required assertion (2.23) immediately.

To prove the second assertion (2.24), we can show easily from (2.11) using the Burkholder-Davis-Gundy inequality (see, e.g., [24, pp.70–76]) etc. that for any $k \in \mathbb{N}_+$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)\\
\le&4\mathbb{E}|x(t_k)|^2+4\mathbb{E}\int_{t_k}^{t_{k+1}}\Big(\tau K_1^2|x(s)|^2+\tau K_A^2(1+\Delta)^2|x(t_k)|^2+4K_2^2|x(s)|^2\Big)ds\\
=&4(1+\tau K_A^2(1+\Delta)^2)\mathbb{E}|x(t_k)|^2+4(\tau K_1^2+4K_2^2)\int_{t_k}^{t_{k+1}}\mathbb{E}|x(s)|^2ds.
\end{aligned}$$

By (2.36), we obtain

$$\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)\le K_3e^{-\varepsilon t_k},$$

where K_3 is a positive constant independent of k. Let $\bar\varepsilon\in(0,\varepsilon)$ be arbitrary. Then

$$\sum_{k=0}^{\infty}\mathbb{P}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|>e^{-0.5\bar\varepsilon t_k}\Big)\le\sum_{k=0}^{\infty}K_3e^{-(\varepsilon-\bar\varepsilon)t_k}<\infty.$$

The well-known Borel-Cantelli lemma (see, e.g., [24, p.10]) shows that for almost all $\omega\in\Omega$, there is positive integer $k_0=k_0(\omega)$ such that

$$\sup_{t_k\le t\le t_{k+1}}|x(t)|\le e^{-0.5\bar\varepsilon t_k},\quad k\ge k_0.$$

Hence, for almost all $\omega\in\Omega$,

$$\frac{1}{t}\log(|x(t)|)\le-\frac{0.5\bar\varepsilon\tau k}{\tau(k+1)},\quad t\in[t_k,t_{k+1}],\ k\ge k_0.$$

This implies

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|)\le-0.5\bar\varepsilon<0\quad a.s.$$

This implies the second assertion (2.24) as $\bar\varepsilon\in(0,\varepsilon)$ is arbitrary. The proof is complete. □

In Section 2.5 we will see that the use of Assumption 2.2 requires us to identity the matrices Q_i and parameters β_i wisely so that the corresponding matrices A_i (via G_i and F_i) and the positive number β can be found for (2.13) to hold. On the one side, this gives us flexibility as we have very wide choices of Q_i and β_i. On the other side, more search is required to identity these Q_i and β_i. Let us now present an alternative assumption.

Assumption 2.5. *There exist real numbers b_i ($i\in\mathbb{S}$) such that*

$$\mathcal{A}:=-\mathrm{diag}(b_1+2\Delta\|A_1\|,\cdots,b_N+2\Delta\|A_N\|)-\Gamma \tag{2.37}$$

is a non-singular M-matrix and

$$2x^Tf(x,i,t)+|g(x,i,t)|^2+x^T(A_i+A_i^T)x\le b_i|x|^2 \tag{2.38}$$

for all $(x,i,t)\in\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}_+$.

In comparison with Assumption 2.2, there are two advantages of this new one: (i) we only need to design the matrices A_i (or G_i) but not Q_i in application; (ii) it is described in terms of M-matrices which can be verified easily. Of course, we will see that this new assumption is less general than Assumption 2.2. In fact, we now explain that under this assumption, the matrices Q_i and parameter β can be identified automatically so that Assumption 2.2 is satisfied. Set

$$(q_1, \cdots, q_N)^T := \mathcal{A}^{-1}(1, \cdots, 1)^T. \tag{2.39}$$

By the theory of M-matrices (see, e.g., [24]), all q_i are positive. Moreover,

$$(b_i + 2\Delta\|A_i\|)q_i + \sum_{j=1}^{N} \gamma_{ij} q_j = -1, \ \forall i \in \mathbb{S}. \tag{2.40}$$

For each $i \in \mathbb{S}$, set $Q_i = q_i I_n$. Note that two terms on the left-hand-side of (2.13), namely

$$x^T(\beta_i Q_i A_i A_i^T Q_i + (\Delta^2/\beta_i) I_n)x \le 2\Delta\|Q_i A_i\||x|^2 = 2q_i\Delta\|A_i\||x|^2$$

by letting $\beta_i = \Delta/\|Q_i A_i\|$ when $A_i \neq 0$, while they can be treated as 0 by letting $\beta_i \to \infty$ when $A_i = 0$. Alternatively, we observe from (2.30) that these two terms are the upper bound for $2x^T Q_i A_i \hat{\Delta}(\cdot)x$. But we clearly have $2x^T Q_i A_i \hat{\Delta}(\cdot)x \le 2q_i\Delta\|A_i\||x|^2$ almost surely no matter whether $A_i = 0$ or not. Consequently,

$$\begin{aligned}
&\text{the left-hand-side of (2.13)} \\
&\le 2q_i x^T f(x,i,t) + q_i|g(x,i,t)|^2 + q_i x^T(A_i + A_i^T)x \\
&+ 2q_i\Delta\|A_i\||x|^2 + \sum_{j=1}^{N} \gamma_{ij} q_j |x|^2 \\
&\le (b_i + 2\Delta\|A_i\|)q_i|x|^2 + \sum_{j=1}^{N} \gamma_{ij} q_j |x|^2 \\
&= -|x|^2,
\end{aligned}$$

where (2.38) and (2.40) have been used. In other words, we have just shown that Assumption 2.5 implies Assumption 2.2 with $Q_i = q_i I_n$ for $i \in \mathbb{S}$ and $\beta = 1$ while β_i's are specified above but have no further use. We therefore have the following theorem.

Theorem 2.6. *Let Assumptions 2.1 and 2.5 hold. Let q_i's be defined by (2.39) and re-set $\beta = 1$, $K_{QA} = \max_{i\in\mathbb{S}} q_i\|A_i\|$, $\lambda_m = \min_{i\in\mathbb{S}} q_i$, $\lambda_M = \max_{i\in\mathbb{S}} q_i$ but leave K_A the same as before. Let $\tau^* \in (0, (K_A(1+\Delta))^{-1})$ be the unique root to equation (2.22). Assume that $\tau < \tau^*$ and let $\varepsilon > 0$ be the unique root to equation (2.25). Then assertions (2.23) and (2.24) of Theorem 2.4 hold.*

This theorem is particularly useful when we deal with the situation where the feedback control could only be used for some modes but not every mode. We will illustrate this in detail in Example 4.2.

2.5 Design of matrices G_i's

The implementation of the controlled SDE (2.2) depends on the design of N matrices G_i's so that we can determine $A_i = F_i G_i$. In this section we will use the technique of linear matrix

inequalities (LMIs, see, e.g., [40]) to design G_i's. Under Assumption 2.1, we will design G_i's in two steps.

Step 2.5.1. Find N pairs of symmetric matrices Q_i and $\hat{Q}_i$ ($i \in \mathbb{S}$) with $Q_i > 0$ such that

$$2x^T Q_i f(x,i,t) + \mathrm{trace}[g(x,i)^T Q_i g(x,i)] \le x^T \hat{Q}_i x \tag{2.41}$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

There are plenty of choices for these matrices. In fact, for each symmetric positive-definite matrix Q_i, we can let $\hat{Q}_i = \|Q_i\|(2K_1 + K_2^2)I_n$ to make (2.41) hold. However, it is wise to find alternative matrices in order to make use of the given structures of f and g so that Step 2 below can be done more easily. In the case when the given SDE (1.1) is linear, $\hat{Q}_i$'s can be determined explicitly once Q_i's are chosen. In fact, assume that the coefficients f and g in the given SDE (1.1) have the linear forms

$$f(x,i,t) = U_i x \quad \text{and} \quad g(x,i) = (V_{1i}x, \cdots, V_{mi}x) \tag{2.42}$$

for $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $U_i, V_{1i}, \cdots, V_{mi} \in \mathbb{R}^{n\times n}$. Then

$$\begin{aligned} &2x^T Q_i f(x,i,t) + \mathrm{trace}[g(x,i)^T Q_i g(x,i)] \\ &= x^T \Big(Q_i U_i + U_i^T Q_i + \sum_{k=1}^m V_{ki}^T Q_i V_{ki} \Big) x. \end{aligned} \tag{2.43}$$

We hence have $\hat{Q}_i = Q_i U_i + U_i^T Q_i + \sum_{k=1}^m V_{ki}^T Q_i V_{ki}$.

To explain the second step, we observe that condition (2.13) is now equivalent to

$$\hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + \beta_i Q_i F_i G_i G_i^T F_i^T Q_i + (\Delta^2/\beta_i) I_n + \sum_{j=1}^N \gamma_{ij} Q_j < 0, \quad i \in \mathbb{S}. \tag{2.44}$$

Letting $y_i = 1/\beta_i > 0$, we can re-write the matrix inequalities above as

$$\begin{aligned} &\hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + Q_i F_i G_i (y_i I_n)^{-1} (Q_i F_i G_i)^T \\ &\quad + y_i \Delta^2 I_n + \sum_{j=1}^N \gamma_{ij} Q_j < 0, \quad i \in \mathbb{S}. \end{aligned} \tag{2.45}$$

But, by the well-known Schur complements (see, e.g., [24, Theorem 2.8 on page 64]), these are equivalent to the following LMIs of G_i and y_i:

$$\begin{pmatrix} Z_i & Q_i F_i G_i \\ G_i^T F_i^T Q_i & -y_i I_n \end{pmatrix} < 0, \quad i \in \mathbb{S}, \tag{2.46}$$

where $Z_i := \hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + y_i \Delta^2 I_n + \sum_{j=1}^N \gamma_{ij} Q_j$. We hence arrive at our second step.

Step 2.5.2. Find a solution $(G_i, y_i)_{i\in\mathbb{S}}$ of the LMIs (2.46) with $G_i \in \mathbb{R}^{l\times n}$ and $y_i > 0$.

These two steps ensure that Assumption 2.2 is satisfied with $A_i = F_i G_i$ and $\beta_i = 1/y_i$ ($i \in \mathbb{S}$) as well as

$$\begin{aligned} \beta = -\max_{i\in\mathbb{S}} \lambda_{\max} \Big(&\hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i \\ &+ Q_i F_i G_i (y_i I_n)^{-1} (Q_i F_i G_i)^T + y_i \Delta^2 I_n + \sum_{j=1}^N \gamma_{ij} Q_j \Big). \end{aligned} \tag{2.47}$$

We hence have the following corollary.

Corollary 2.7. *Under Assumption 2.1, find matrices G_i and positive numbers y_i ($i \in \mathbb{S}$) in two steps as described above. Then Theorem 2.4 holds with β being defined by (2.47) and $A_i = F_iG_i$.*

3 Stabilisation by Event-triggered Output Injection

3.1 Stabilisation

Let us now return to the given SDE (1.1). Instead assuming the state is observable, we now consider the situation where the output $y(t_k) := G_{r(t_k)}x(t_k)$ is observable at discrete times $\{t_k\}_{k\in\mathbb{N}}$, in which $G_i \in \mathbb{R}^{l\times n}$ is known for every $i \in \mathbb{S}$ (in general, $l < n$). In this situation, the event-triggered control should be designed based on $y(t_k)$ but not $x(t_k)$. More precisely, given we only consider linear control functions in this paper, we need to design N matrices $F_i \in \mathbb{R}^{n\times l}$ so that the event-triggered controlled SDE

$$dx(t) = \big(f(x(t), r(t), t) + F_{r(t)}Y_{\lfloor t/\tau\rfloor}\big)dt + g(x(t), r(t), t)dB(t) \tag{3.1}$$

becomes stable, where $Y_0 = y(0)$ and for $k = 1, 2, \cdots$,

$$Y_k = \begin{cases} Y_{k-1} & \text{if } y(t_k) \text{ is close to } Y_{k-1}, \\ y(t_k) & \text{otherwise.} \end{cases} \tag{3.2}$$

The control $F_{r(t)}Y_{\lfloor t/\tau\rfloor}$ is called the event-triggered output injection. We can show in the same way as we did in Section 2 that under Assumption 2.1, the controlled SDE (3.1) with any initial data $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in \mathbb{S}$ has a unique global solution $x(t)$ such that $\mathbb{E}|x(t)|^2 < \infty$ for all $t \geq 0$.

By Definition 1.1, we can show in the same way as we did in Section 2 that $Y_k = (I_n + \tilde{\Delta}(t_k))G_{r(t_k)}x(t_k)$, where $\tilde{\Delta}(t_k) = \mathrm{diag}(\tilde{\Delta}_1(t_k), \cdots, \tilde{\Delta}_n(t_k))$ is $\mathcal{F}_{t_k}$-measurable and $\tilde{\Delta}_i(t_k) \in [-\Delta, \Delta]$ a.s. for every $i \in \mathbb{S}$. The controlled SDE (3.1) can therefore be written in the modified form

$$dx(t) = \big(f(x(t), r(t), t) + F_{r(t)}(I_n + \tilde{\Delta}(\delta_t))G_{r(\delta_t)}x(\delta_t)\big)dt + g(x(t), r(t), t)dB(t). \tag{3.3}$$

The stability of this controlled SDE is very much related to the robust stability of the auxiliary uncertain SDE

$$dz(t) = \big(f(z(t), r(t), t) + F_{r(t)}(I_n + \varphi(t))G_{r(t)}z(t)\big)dt + g(z(t), r(t), t)dB(t), \tag{3.4}$$

where $\varphi(t) = \mathrm{diag}(\varphi_1(t), \cdots, \varphi_n(t))$ is an $\mathcal{F}_t$-adapted process such that its every element $\varphi_i(t) \in [-\Delta, \Delta]$ a.s. We impose the following assumption for the mean square exponential stability of this uncertain SDE (see, e.g., [24]).

Assumption 3.1. *There exist positive numbers β, β_i ($i \in \mathbb{S}$) and symmetric positive definite matrices $Q_i \in \mathbb{R}^{n\times n}$ such that*

$$\begin{aligned} &2x^TQ_if(x, i, t) + \mathrm{trace}[g(x, i, t)^TQ_ig(x, i, t)] \\ &+ x^T\Big(Q_iF_iG_i + G_i^TF_i^TQ_i + \beta_iQ_iF_iF_i^TQ_i + (\Delta^2/\beta_i)G_i^TG_i + \sum_{j=1}^N \gamma_{ij}Q_j\Big)x \leq -\beta|x|^2 \end{aligned} \tag{3.5}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

In what follows, we let λ_m and λ_M be the same as defined by (2.14) and set

$$K_F = \max_{i\in\mathbb{S}} \|F_i\|,\ K_G = \max_{i\in\mathbb{S}} \|G_i\|, K_{QF} = \max_{i\in\mathbb{S}} \|Q_iF_i\|,\ D_G = \max_{i,j\in\mathbb{S}} \|G_i - G_j\|^2. \tag{3.6}$$

We will see that the stabilisation by the event-triggered output injection is more complicated than the stabilisation by the event-triggered state feedback. Let us present three lemmas before we form our main result in this section.

Lemma 3.2. *Set $\bar{\gamma} = \max_{i\in\mathbb{S}}(-\gamma_{ii})$. Then*

$$\mathbb{E}(\|G_{r(t)} - G_{r(\delta_t)}\|^2) \le D_G(1 - e^{-\bar{\gamma}\tau}),\ \forall t \ge 0. \tag{3.7}$$

Proof. Let $t \ge 0$ be arbitrary and $k = \lfloor t/\tau \rfloor$ so $t_k = \delta_t \le t < t_{k+1}$. Given $r(t_k) = i$, define the stopping time

$$\eta_i = \inf\{s \ge t_k : r(s) \ne i\},$$

where and throughout this paper we set $\inf \emptyset = \infty$ (in which $\emptyset$ denotes the empty set as usual). It is well known (see, e.g., [1]) that $\eta_i - t_k$ has the exponential distribution with parameter $-\gamma_{ii}$. Hence

$$\begin{aligned}\mathbb{P}(r(s) \ne i \text{ for some } s \in [t_k, t_{k+1}]|r(t_k) = i) &= \mathbb{P}(\eta_i - t_k \le \tau | r(t_k) = i)\\ &= \int_0^\tau (-\gamma_{ii})e^{\gamma_{ii}s}ds = 1 - e^{\gamma_{ii}\tau} \le 1 - e^{-\bar{\gamma}\tau}.\end{aligned}$$

Consequently, $\mathbb{P}(r(t) \ne i|r(t_k) = i) \le 1 - e^{-\bar{\gamma}\tau}$. Therefore

$$\begin{aligned}\mathbb{E}(\|G_{r(t)} - G_{r(\delta_t)}\|^2) &= \sum_{i\in\mathbb{S}} \mathbb{P}(r(t_k) = i)\mathbb{E}(\|G_{r(t)} - G_i\|^2|r(t_k) = i)\\ &\le \sum_{i\in\mathbb{S}} \mathbb{P}(r(t_k) = i)D_G\mathbb{P}(r(t) \ne i|r(t_k) = i)\\ &\le D_G(1 - e^{-\bar{\gamma}\tau})\end{aligned} \tag{3.8}$$

as required. □

Lemma 3.3. *Let Assumption 2.1 hold. If $\tau < 1/[\sqrt{6}K_FK_G(1 + \Delta)]$, then the solution of the controlled SDE (3.1) has the property that*

$$\mathbb{E}(|x(t) - x(\delta_t)|^2) \le H_3(\tau)\mathbb{E}|x(t)|^2 + H_4(\tau)\int_{\delta_t}^t \mathbb{E}|x(s)|^2 ds \tag{3.9}$$

for all $t \ge 0$, where

$$H_3(\tau) = \frac{6[\tau K_FK_G(1 + \Delta)]^2}{1 - 6[\tau K_FK_G(1 + \Delta)]^2} \quad \text{and} \quad H_4(\tau) = \frac{3(\tau K_1^2 + K_2^2)}{1 - 6[\tau K_FK_G(1 + \Delta)]^2}. \tag{3.10}$$

Proof. Let $t \in [t_k, t_{k+1})$ for some $k \in \mathbb{N}_+$. By Assumption 2.1, it follows easily from (3.3) that

$$\begin{aligned}\mathbb{E}(|x(t) - x(\delta_t)|^2) &\le 3(\tau K_1^2 + K_2^2)\int_{t_k}^t \mathbb{E}|x(s)|^2 ds + 3[\tau K_FK_G(1 + \Delta)]^2\mathbb{E}|x(t_k)|^2\\ &\le 3(\tau K_1^2 + K_2^2)\int_{t_k}^t \mathbb{E}|x(s)|^2 ds + 6[\tau K_FK_G(1 + \Delta)]^2(\mathbb{E}|x(t)|^2 + \mathbb{E}|x(t) - x(\delta_t)|^2).\end{aligned}$$

This implies the required assertion 3.9 as $6[\tau K_FK_G(1 + \Delta)]^2 < 1$. □

Lemma 3.4. *Let Assumption 2.1 hold. If $\tau < 1/(K_F K_G(1+\Delta))$, then the solution of the controlled SDE (3.1) has the property that*

$$\mathbb{E}(|x(t)||x(t)-x(\delta_t)|) \le H_5(\tau)\mathbb{E}|x(t)|^2 + H_6(\tau)\int_{\delta_t}^{t}\mathbb{E}|x(s)|^2 ds \tag{3.11}$$

for all $t \ge 0$, where

$$H_5(\tau) = \frac{\tau[0.5K_1 + K_F K_G(1+\Delta)] + 0.5K_2\sqrt{\tau}}{1-\tau K_F K_G(1+\Delta)} \tag{3.12}$$

and

$$H_6(\tau) = \frac{0.5(K_1 + K_2/\sqrt{\tau})}{1-\tau K_F K_G(1+\Delta)}. \tag{3.13}$$

Proof. For $t \in [t_k, t_{k+1})$ for some $k \in \mathbb{N}_+$, by Assumption 2.1 and property (2.10), it follows from (3.3) that

$$\begin{aligned}&\mathbb{E}(|x(t)||x(t)-x(\delta_t)|) = \mathbb{E}(|x(t)||x(t)-x(t_k)|)\\ \le&\mathbb{E}\Big(|x(t)|\int_{t_k}^{t}(K_1|x(s)| + K_F K_G(1+\Delta)|x(t_k)|)ds + |x(t)||M_k(t)|\Big),\end{aligned} \tag{3.14}$$

where $M_k(t) = \int_{t_k}^{t} g(x(s), r(s), s)dB(s)$. We can then prove the lemma in the same way as Lemma 2.3 was proved. □.

Theorem 3.5. *Let Assumptions 2.1 and 3.1 hold. Choose a free parameter $\theta \in (0, \beta/[K_{QF}(1+\Delta)])$ and fix it. Set $\sigma_1 = 2K_{QF}K_G(1+\Delta)$, $\sigma_2 = \sigma_2(\theta) = K_{QF}(1+\Delta)\theta$, $\sigma_3 = \sigma_3(\theta) = K_{QF}(1+\Delta)/\theta$ (so $\beta > \sigma_2$). Let $H_3(\tau), \cdots, H_6(\tau)$ be the same as defined in Lemmas 3.3 and 3.4. Set $\bar{\gamma} = \max_{i\in\mathbb{S}}(-\gamma_{ii})$ and define*

$$H_7(\tau) = \beta - \sigma_2 - \sigma_1 H_5(\tau) - 2\sigma_3 D_G(1-e^{-\bar{\gamma}\tau})[1+H_3(\tau)], \tag{3.15}$$

$$H_8(\tau) = \sigma_1 H_6(\tau) + 2\sigma_3 D_G(1-e^{-\bar{\gamma}\tau})H_4(\tau). \tag{3.16}$$

Let $\tau^* \in (0, [\sqrt{6}K_F K_G(1+\Delta)]^{-1})$ be the unique root to the following equation of τ:

$$H_7(\tau) - \tau H_8(\tau) = 0. \tag{3.17}$$

If $\tau < \tau^*$, then the solution of the controlled SDE (3.1) has the properties that

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^2) \le -\bar{\varepsilon}, \tag{3.18}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|) \le -\frac{\bar{\varepsilon}}{2} \quad a.s., \tag{3.19}$$

where $\bar{\varepsilon} > 0$ is the unique root to the equation

$$H_7(\tau) - \bar{\varepsilon}\lambda_M - \tau e^{\bar{\varepsilon}\tau}H_8(\tau) = 0. \tag{3.20}$$

Proof. Noting that $H_7(\tau) - \tau H_8(\tau)$ will decrease strictly and continuously from $\beta - \sigma_2 > 0$ to negative infinity as τ changes from 0 to $1/[\sqrt{6}K_F K_G(1+\Delta)]$ and hence τ^* is well defined. Given that $\tau < \tau^*$, we have $H_7(\tau) - \tau H_8(\tau) > 0$ and hence there must have a unique root $\bar{\varepsilon} > 0$ to equation (3.20).

Applying the generalized Itô formula (see, e.g., [18, 24]) to $V(x(t), r(t)) := x^T(t)Q_{r(t)}x(t)$, we obtain

$$dV(x(t), r(t)) = J_2(t)dt + dM(t) \tag{3.21}$$

for $t \geq 0$, where $M(t)$ is a continuous martingale with $M(0) = 0$ and

$$\begin{aligned} J_2(t) :=& 2x^T(t)Q_{r(t}[f(x(t), r(t), t) + F_{r(t)}(I_n + \hat{\Delta}(\delta_t))G_{r(\delta_t)}x(\delta_t)] \\ &+ \text{trace}[g^T(x(t), r(t), t)Q_{r(t)}g(x(t), r(t), t)] + \sum_{j=1}^{N} \gamma_{r(t),j}x^T(t)Q_j x(t). \end{aligned} \tag{3.22}$$

But

$$\begin{aligned} &2x^T(t)Q_{r(t)}F_{r(t)}(I_n + \hat{\Delta}(\delta_t))G_{r(\delta_t)}x(\delta_t) \\ &= 2x^T(t)Q_{r(t)}F_{r(t)}(I_n + \hat{\Delta}(\delta_t))G_{r(t)}x(t) + J_3(t) \\ &\leq x^T(t)\Big(Q_{r(t)}F_{r(t)}G_{r(t)} + G_{r(t)}^T F_{r(t)}^T Q_{r(t)} + \beta_{r(t)}Q_{r(t)}F_{r(t)}F_{r(t)}^T Q_{r(t)} \\ &\quad + (\Delta^2/\beta_{r(t)})G_{r(t)}^T G_{r(t)}\Big)x(t) + J_3(t), \end{aligned} \tag{3.23}$$

where

$$J_3(t) = -2x^T(t)Q_{r(t)}F_{r(t)}(I_n + \hat{\Delta}(\delta_t))[G_{r(t)}x(t) - G_{r(\delta_t)}x(\delta_t)].$$

Substituting this into (3.22) and then applying condition (3.5), we get

$$J_2(t) \leq -\beta|x(t)|^2 + J_3(t). \tag{3.24}$$

But

$$\begin{aligned} &|J_3(t)| \leq 2K_{QF}(1+\Delta)|x(t)||G_{r(t)}(x(t) - x(\delta_t)) + (G_{r(t)} - G_{r(\delta_t)})x(\delta_t)| \\ &\leq 2K_{QF}K_G(1+\Delta)|x(t)||x(t) - x(\delta_t)| + 2K_{QF}(1+\Delta)|x(t)||x(\delta_t)|\|G_{r(t)} - G_{r(\delta_t)}\| \\ &\leq 2K_{QF}K_G(1+\Delta)|x(t)||x(t) - x(\delta_t)| \\ &+ K_{QF}(1+\Delta)\Big(\theta\|x(t)|^2 + \frac{1}{\theta}|x(\delta_t)|^2\|G_{r(t)} - G_{r(\delta_t)}\|^2\Big) \\ &= \sigma_1|x(t)||x(t) - x(\delta_t)| + \sigma_2|x(t)|^2 + \sigma_3|x(\delta_t)|^2\|G_{r(t)} - G_{r(\delta_t)}\|^2, \end{aligned}$$

where $\sigma_1, \sigma_2, \sigma_3$ have been defined in the statement of the theorem. We hence have

$$\begin{aligned} \mathbb{E}J_2(t) \leq& -(\beta - \sigma_2)\mathbb{E}|x(t)|^2 + \sigma_1\mathbb{E}(|x(t)||x(t) - x(\delta_t)|) \\ &+ \sigma_3\mathbb{E}(|x(\delta_t)|^2\|G_{r(t)} - G_{r(\delta_t)}\|^2). \end{aligned} \tag{3.25}$$

Noting that $x(\delta_t)$ is independent of $G_{r(t)} - G_{r(\delta_t)}$ while applying Lemmas 3.2 and 3.3, we derive that

$$\begin{aligned} &\mathbb{E}(|x(\delta_t)|^2\|G_{r(t)} - G_{r(\delta_t)}\|^2) = \mathbb{E}|x(\delta_t)|^2\mathbb{E}\|G_{r(t)} - G_{r(\delta_t)}\|^2 \\ \leq& 2(\mathbb{E}|x(t)|^2 + \mathbb{E}|x(t) - x(\delta_t)|^2)\mathbb{E}\|G_{r(t)} - G_{r(\delta_t)}\|^2 \\ \leq& 2D_G(1 - e^{-\bar{\gamma}\tau})\Big([1 + H_3(\tau)]\mathbb{E}|x(t)|^2 + H_4(\tau)\int_{\delta_t}^{t}\mathbb{E}|x(s)|^2 ds\Big). \end{aligned} \tag{3.26}$$

Substituting this into (3.25) and applying Lemma 3.4, we get

$$\mathbb{E}J_2(t) \le -H_7(\tau)\mathbb{E}|x(t)|^2 + H_8(\tau)\int_{\delta_t}^t \mathbb{E}|x(s)|^2 ds \tag{3.27}$$

for all $t \ge 0$. An application of the generalized Itô formula to $e^{\bar{\varepsilon}t}V(x(t), r(t))$ gives

$$e^{\bar{\varepsilon}t}\mathbb{E}V(x(t), r(t)) = V(x_0, r_0) + \int_0^t e^{\bar{\varepsilon}s}\Big(\bar{\varepsilon}\mathbb{E}V(x(s), r(s)) + \mathbb{E}J_2(s)\Big)ds. \tag{3.28}$$

This, together with (3.27), implies

$$\begin{aligned}\lambda_m e^{\bar{\varepsilon}t}\mathbb{E}|x(t)|^2 \le \lambda_M|x_0|^2 &- \int_0^t e^{\bar{\varepsilon}s}[H_7(\tau) - \bar{\varepsilon}\lambda_M]\mathbb{E}|x(s)|^2 ds \\ &+ H_8(\tau)\int_0^t e^{\bar{\varepsilon}s}\Big(\int_{\delta_s}^s \mathbb{E}|x(v)|^2 dv\Big)ds.\end{aligned} \tag{3.29}$$

But, recalling (2.34), we have

$$\int_0^t e^{\bar{\varepsilon}s}\Big(\int_{\delta_s}^s \mathbb{E}|x(v)|^2 dv\Big)ds \le \tau e^{\bar{\varepsilon}\tau}\int_0^t e^{\varepsilon v}\mathbb{E}|x(v)|^2 dv. \tag{3.30}$$

Therefore

$$\lambda_m e^{\bar{\varepsilon}t}\mathbb{E}|x(t)|^2 \le \lambda_M|x_0|^2 - \int_0^t e^{\bar{\varepsilon}s}[H_7(\tau) - \bar{\varepsilon}\lambda_M - \tau e^{\bar{\varepsilon}\tau}H_8(\tau)]\mathbb{E}|x(s)|^2 ds = \lambda_M|x_0|^2, \tag{3.31}$$

where (3.20) has been used. This implies the first assertion (3.18). The second assertion (3.19) can be proved in the same way as Theorem 2.4 was proved. The proof is complete. □

Remark 3.6. *In Theorem 3.5, we introduce a free parameter $\theta \in (0, \beta/[K_{QF}(1+\Delta)])$. We observe that τ^* determined there depends on θ. In application, it is better to have a larger τ^*. At this moment, we do not know how to choose a better θ so that the corresponding τ^* is larger. We will tackle this open problem in the future.*

3.2 Design of matrices F_i's

Let us now explain how to design F_i's in two steps.

Step 3.2.1. The same as Step 2.5.1.

Once this first step is done, we observe that condition (3.5) is now equivalent to

$$\hat{Q}_i + Q_iF_iG_i + G_i^TF_i^TQ_i + \beta_iQ_iF_iF_i^TQ_i + (\Delta^2/\beta_i)G_i^TG_i + \sum_{j=1}^N \gamma_{ij}Q_j < 0, \quad i \in \mathbb{S}. \tag{3.32}$$

By setting $y_i = 1/\beta_i > 0$, the matrix inequalities above becomes

$$\begin{aligned}\hat{Q}_i + Q_iF_iG_i + G_i^TF_i^TQ_i + Q_iF_i(y_iI_n)^{-1}(Q_iF_i)^T \\ + y_i\Delta^2G_i^TG_i + \sum_{j=1}^N \gamma_{ij}Q_j < 0, \quad i \in \mathbb{S}.\end{aligned} \tag{3.33}$$

By the well-known Schur complements, these are equivalent to the following LMIs of F_i and y_i:

$$\begin{pmatrix} W_i & Q_iF_i \\ F_i^TQ_i & -y_iI_n \end{pmatrix} < 0, \quad i \in \mathbb{S}, \tag{3.34}$$

where $W_i := \hat{Q}_i + Q_iF_iG_i + G_i^TF_i^TQ_i + y_i\Delta^2G_i^TG_i + \sum_{j=1}^N \gamma_{ij}Q_j$. We hence arrive at our second step.

Step 3.1.2. Find a solution $(F_i, y_i)_{i\in\mathbb{S}}$ of the LMIs (3.34) with $F_i \in \mathbb{R}^{n\times l}$ and $y_i > 0$.

These two steps ensure that Assumption 3.1 is satisfied with $\beta_i = 1/y_i$ $(i \in \mathbb{S})$ as well as

$$\begin{aligned}\beta = -\max_{i\in\mathbb{S}} \lambda_{\max}\Big(&\hat{Q}_i + Q_iF_iG_i + G_i^TF_i^TQ_i \\ &+ Q_iF_i(y_iI_n)^{-1}(Q_iF_i)^T + y_i\Delta^2G_i^TG_i + \sum_{j=1}^N \gamma_{ij}Q_j\Big).\end{aligned} \tag{3.35}$$

We hence have the following corollary.

Corollary 3.7. *Under Assumption 2.1, find matrices G_i and positive numbers y_i $(i \in \mathbb{S})$ in two steps as described above. Then Theorem 3.5 holds with β being defined by (3.35).*

4 Examples

To illustrate our theoretical results, we will discuss a couple of examples.

Example 4.1. Let us consider the linear hybrid SDE

$$dx(t) = U_{r(t)}x(t)dt + V_{r(t)}x(t)dB(t) \tag{4.1}$$

on $t \geq 0$. Here $x(t) = (x_1(t), x_2(t))^T$, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix},$$

and the system matrices are

$$U_1 = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \quad U_2 = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The computer simulation (Figure 4.1) shows this hybrid SDE is not almost surely exponentially stable and hence it is not exponentially stable in mean square either.

Let us now design an event-triggered feedback control to stabilize the system. Let the event-triggered parameter $\Delta = 0.1$. Assume that the controlled SDE has the form

$$dx(t) = \big(U_{r(t)}x(t) + F_{r(t)}G_{r(t)}X_{\lfloor t/\tau\rfloor}\big)dt + V_{r(t)}x(t)dB(t) \tag{4.2}$$

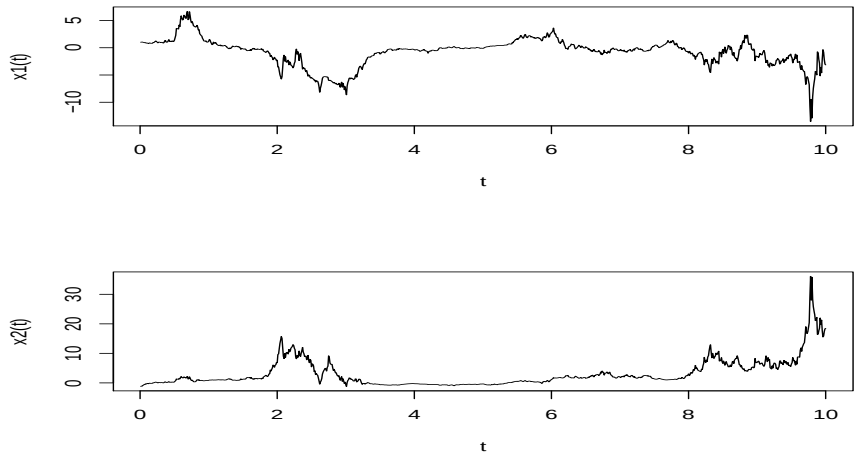

Figure 4.1: The computer simulation of the sample paths of the solution of the SDE (4.1) using the Euler–Maruyama method with step size 10^{-3} and initial data $x(0) = (1, -1)^T$, $r(0) = 1$.

on $t \geq 0$, which is equivalent to

$$dx(t) = \big(U_{r(t)}x(t) + F_{r(t)}G_{r(t)}(I_2 + \hat{\Delta}(\delta_t))x(\delta_t)\big)dt + V_{r(t)}x(t)dB(t). \tag{4.3}$$

Assume

$$F_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

but we need to determine G_1 and G_2 in $R^{1\times 2}$ and then make sure τ is sufficiently small for the controlled SDE (4.2) to be exponentially stable in mean square and almost surely as well. It is easy to see that Assumption 2.1 holds with $K_1 = 5.236$ and $K_2 = \sqrt{2}$. To apply Corollary 2.7, we apply two steps in Section 4 to determine G_1 and G_2. In Step 1, we choose $Q_1 = Q_2 = I_2$. Then $\hat{Q}_i = U_i + U_i^T + V_i^T V_i$, namely

$$\hat{Q}_1 = \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix} -8 & 0 \\ 0 & 4 \end{bmatrix}.$$

It is easy to verify that

$$G_1 = (-6, 0), \quad G_2 = (0, -6), \quad y_1 = y_2 = 60$$

form a solution of the LMIs (2.46). By (2.14) and (2.47), we compute $K_A = K_{QA} = 6$ and $\beta = 6.8$. Consequently, equation (2.22) becomes

$$6.8 = 13.2(11.836\tau + \sqrt{2\tau})/(1 - 6.6\tau),$$

which has its unique solution $\tau^* = 0.02051$ in $(0, 0.1515152)$. By Corollary 2.7, if we set G_i as above and make sure that $\tau < 0.02051$, then the controlled SDE (4.2) is exponentially stable in mean square and almost surely as well. We perform a computer simulation (Figure 4.2) with $\tau = 0.01$. It supports the theoretical results clearly.

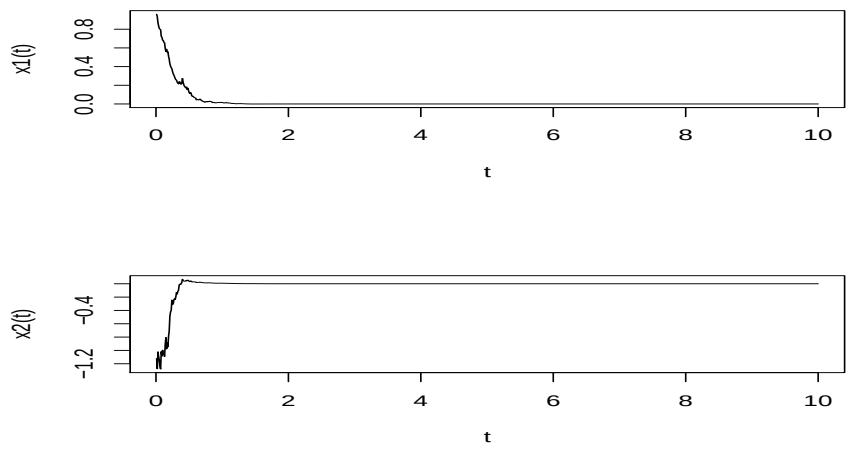

Figure 4.2: The computer simulation of the sample paths of the solution of the controlled SDE (4.2) with $\tau = 0.01$ using the Euler–Maruyama method with step size 10^{-3} and initial data $x(0) = (1, -1)^T$, $r(0) = 1$.

Example 4.2. Let us now apply our theory to the stochastic Hopfield neural networks (see, e.g., [3, 14, 15, 31, 34, 35]). For illustration, it is sufficient to consider the following Hopfield neural network with two neurons

$$dx(t) = [-D_{r(t)}x(t) + U_{r(t)}\varphi(x(t))]dt + V_{r(t)}x(t)dB(t) \tag{4.4}$$

on $t \geq 0$. Here $x(t) = (x_1(t), x_2(t))^T$, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix},$$

$\varphi(x) = (\tanh(x_1), \tanh(x_2))^T$ and the system matrices are $D_1 = \mathrm{diag}(0.1, 0.2)$, $D_2 = \mathrm{diag}(0.2, 0.1)$,

$$U_1 = \begin{bmatrix} 0.5 & -0.1 \\ 0.1 & 0.5 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.5 & 0.1 \\ -0.1 & 0.5 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & -0.1 \end{bmatrix}.$$

Matrices U_1 and U_2 consist of the transfer coefficients between two neurons and themselves in mode 1 and 2, respectively, and they are illustrated in Figure 4.3. The computer simulation (Figure 4.4) shows this hybrid SDE is not almost surely exponentially stable and hence it is not exponentially stable in mean square either.

We now design an event-triggered feedback control to stabilize the system. Let the event-triggered parameter $\Delta = 0.1$. Consider the case where the system is observable only in mode 1 but not in mode 2 so we could only use a feedback control in mode 1 (namely we have to set the control function to be 0 in mode 2). Assume the controlled SDE has the form

$$dx(t) = \big(- D_{r(t)}x(t) + U_{r(t)}\varphi(x(t)) + A_{r(t)}X_{\lfloor t/\tau \rfloor}\big)dt + V_{r(t)}x(t)dB(t) \tag{4.5}$$

Figure 4.3: Hopefield neural network with two neurons. Left: mode 1. Right: mode 2.

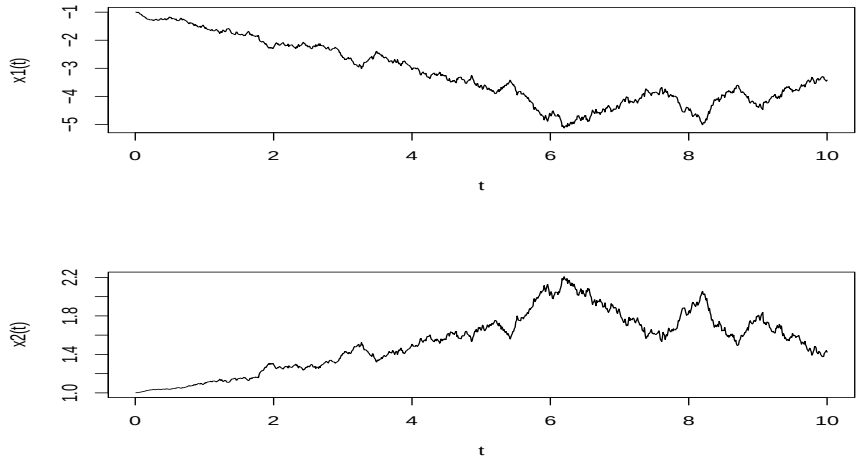

Figure 4.4: The computer simulation of the sample paths of the solution of the SDE (4.4) using the Euler–Maruyama method with step size 10^{-3} and initial data $x(0) = (-1, 1)^T$, $r(0) = 1$.

on $t \geq 0$, where $A_2 = 0$ and $A_1 = \text{diag}(-2.1, -2)$. It is easy to see that Assumption 2.1 holds with $K_1 = 0.7099$ and $K_2 = 0.1414$. To verify Assumption 2.5, we observe that

$$\begin{aligned} &\text{the left hand side of (2.38) when } i = 1 \\ &\leq -0.2x_1^2 - 0.4x_2^2 + 1.4398|x|^2 - 4.2x_1^2 - 4x_2^2 \leq -2.9602|x|^2, \end{aligned}$$

while

$$\begin{aligned} &\text{the left hand side of (2.38) when } i = 2 \\ &\leq -0.4x_1^2 - 0.2x_2^2 + 1.4398|x|^2 \leq 1.0398|x|^2. \end{aligned}$$

We hence have that $b_1 = -2.9602$, $b_1 + 2\Delta\|A_1\| = -2.5402$ while $b_2 = 1.0398$, $b_2 + 2\Delta\|A_2\| = 1.0398$. Consequently, the matrix $\mathcal{A}$ defined by (2.37) becomes

$$\mathcal{A} = \begin{pmatrix} 3.5402 & -1 \\ -3 & 1.0602 \end{pmatrix}$$

which is a non-singular M-matrix. That is, Assumption 2.5 is satisfied. To apply Theorem 2.6, we compute $q_1 = 2.734827$ and $q_2 = 8.681834$ by (2.40). Then $\beta = 1$, $K_A = 2.1$ and $K_{QA} = 5.743137$. Consequently, equation (2.22) becomes

$$1 = 12.6349(3.0199\tau + 0.1414\sqrt{\tau})/(1 - 4.62\tau),$$

which has its unique solution $\tau^* = 0.0178$ in $(0, 0.2164502)$. By Theorem 2.6, if we set $A_1 = \text{diag}(-2.1, -2)$ and $A_2 = 0$ and make sure that $\tau < 0.0178$, then the controlled SDE (4.5) is exponentially stable in mean square and almost surely as well. We perform a computer simulation (Figure 4.5) with $\tau = 0.01$. It supports the theoretical once again.

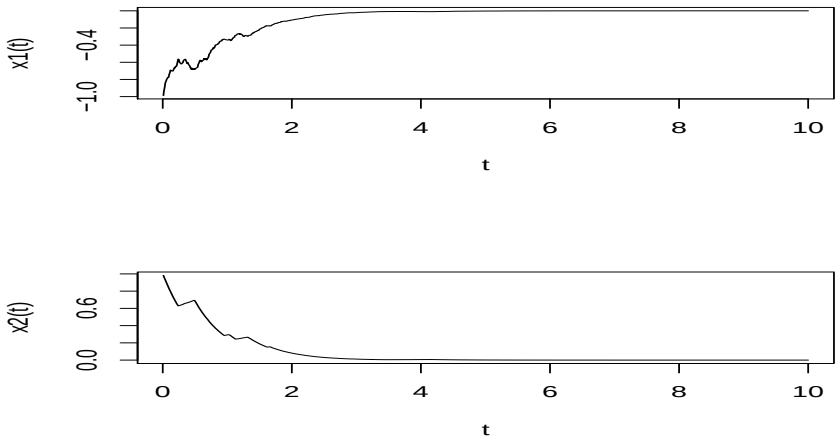

Figure 4.5: The computer simulation of the sample paths of the solution of the controlled SDE (4.5) using the Euler–Maruyama method with step size 10^{-3} and initial data $x(0) = (-1, 1)^T$, $r(0) = 1$.

5 Conclusion

In this paper we have established the new theory on how to design an event-triggered control based on a trajectory of the state or output observations at discrete times to stabilise a given unstable hybrid SDE. We have explained clearly how our event-triggered control based on the trajectory is significantly different from most of the existing papers where an event-triggered mechanism for SDEs is designed based on a criterion in terms of the mathematical expectation of the system state. We have also explained how the case where the state is observable differs significantly from the case where the output is observable. Accordingly, we have developed the event-triggered state feedback control for the former while the event-triggered output injection for the latter. In both cases, we have shown that the stability of the event-triggered controlled SDE is related to the stability of the corresponding uncertain SDE. With such a novel approach, we have successfully established our new theory on the stabilisation of hybrid SDEs by event-triggered control based on a trajectory of the state or output observations at discrete times. Two examples have been discussed to illustrate the new theory.

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