Space-Time Covariance Matrix Factorisation and Estimation for Broadband Multichannel Problems, Part 2: Eigenvalue Decomposition

Ian K. Proudler

Department of Electronic & Electrical Engineering
University of Strathclyde, Glasgow, Scotland

Tutorial at 23rd IEEE Statistical Signal Processing Workshop
Edinburgh, Scotland

8 June 2025

Eigenvalue Decomposition

1. Overview
2. Analytic eigenvalue decomposition
3. Polynomial eigenvalue decomposition
4. Time domain algorithms
5. DFT domain algorithms
6. Summary

2. Analytic Eigenvalue Decomposition

2.1 Ordinary Eigenvalue Decomposition

- For a Hermitian matrix $\mathbf{R} = \mathbf{R}^{\mathrm{H}}$, we know that an eigenvalue decomposition (EVD) $\mathbf{R} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{\mathrm{H}}$ exists [32, 35];
- for eigenvalues $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_M\}$ and eigenvectors $\mathbf{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_M]$:

$$\mathbf{R}\mathbf{q}_m = \lambda_m \mathbf{q}_m$$

- eigenvalues $\lambda \in \mathbb{R}$;
- eigenvectors can be chosen as orthonormal, but may have an arbitary phase shift: $\mathbf{q}'_m = e^{j\varphi}\mathbf{q}_m$ is also an eigenvector;
- in case of an algebraic multiplicity C: $\lambda_m = \lambda_{m+1} = \cdots = \lambda_{m+C-1}$, only a C-dimensional subspace is defined, within which the eigenvectors can form an arbitrary orthonormal basis, with any unitary $\mathbf{V}$:

$$[\mathbf{q}'_m, \ \ldots \ \mathbf{q}'_{m+C-1}] = [\mathbf{q}_m, \ \ldots \ \mathbf{q}_{m+C-1}]\,\mathbf{V}\ . \tag{1}$$

2.2 Existence of an Analytic EVD on a Real Interval

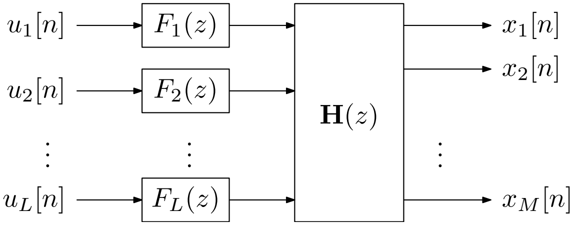
- A standard EVD can diagonalise $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ \mathbf{R}[\tau]$ only for one specific value of z or of τ, respectively;
- Franz Rellich (1939, [67]) for a self-adjoint, analytic $\boldsymbol{R}(t) = \boldsymbol{R}^{\mathrm{H}}(t)$, $t \in \mathbb{R}$:

$$\boldsymbol{R}(t) = \boldsymbol{Q}(t)\boldsymbol{\Lambda}(t)\boldsymbol{Q}^{\mathrm{H}}(t) \;;$$

- $\boldsymbol{Q}(t)$ and $\boldsymbol{\Lambda}(t)$ *can* be chosen analytic;
- similarly for an arbitrary (i.e. not necessarily Hermitian or square) analytic matrix, de Moor & Boyd (1989, [24]) and Bunse-Gerstner (1991, [8]) established an analytic SVD.

Analyticity of $\boldsymbol{R}(z)$

- The analyticity of $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ\, \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\}$ can be tied to a source model [63, 84] with white $u_\ell[n] \in \mathcal{N}(0,1)$:



- the innovation filters $F_\ell(z)$, $\ell = 1,\dots,L$ describe the spectral shape of the L contributing source signals;
- a convolutive mixing system $\boldsymbol{H}(z) : \mathbb{C} \to \mathbb{C}^{M\times N}$ models the transfer paths between the L sources and M sensors;
- if $F_\ell(z)$ and $\boldsymbol{H}(z)$ are stable and causal, then $\boldsymbol{R}(z) = \boldsymbol{H}(z)\boldsymbol{F}(z)\boldsymbol{F}^{\mathrm{P}}(z)\boldsymbol{H}^{\mathrm{P}}(z)$ is analytic.

Analytic EVD on the Unit Circle

- Analyticity of $\boldsymbol{R}(z)$ permits a restricted evaluation on the unit circle $z = \mathrm{e}^{\mathrm{j}\Omega}$;
- due to Rellich [67]:

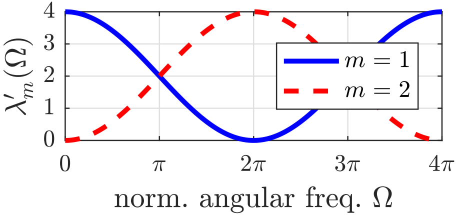
$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{Q}(\Omega)\ \boldsymbol{\Lambda}(\Omega)\ \boldsymbol{Q}^{\mathrm{H}}(\Omega)\ , \qquad (2)$$

- unfortunately, while analytic in $\Omega \in \mathbb{R}$, $\boldsymbol{\Lambda}(\Omega)$ and $\boldsymbol{Q}(\Omega)$ can be $2\pi L$-periodic, with some $L \in \mathbb{Z}$ [85, 6];
- example [20, 72, 85]:

$$\boldsymbol{R}(z) = \frac{1}{2}\begin{bmatrix} 2 & 1+z^{-1} \\ 1+z & 2 \end{bmatrix}\ ,$$

$$\rightarrow \lambda_{1,2}(z) = 2 \pm (z^{\frac{1}{2}} + z^{-\frac{1}{2}})\ ,$$

$$\lambda'_{1,2}(\Omega) = 2 \pm \cos(\Omega/2)\ ;$$

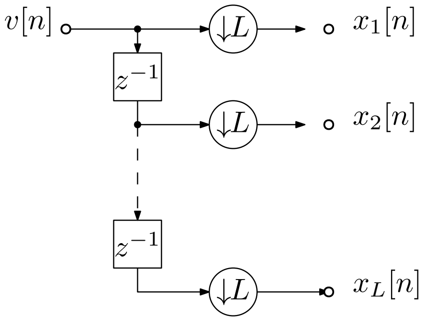


- while $\cos(\Omega/2)$ is analytic in Ω, a fractional delay $z^{-\frac{1}{2}}$ is not analytic: its time domain equivalent decays too slowly [47].

Non-Existence of an Analytic EVD of $\boldsymbol{R}(z)$

- The case $L > 1$ can be tied to multiplexing operation [85];
- assume $\mathbf{x}[n]$ is L-fold demultiplexed,
  $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ\, \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\}$;
- $\boldsymbol{R}(z)$ will be pseudo-circulant [75] with modulated eigenvalues [85, 6];
- $\boldsymbol{Q}(\Omega)$ and $\boldsymbol{\Lambda}(\Omega)$ will be $2\pi L$-periodic;
- as such, we can only find an analytic EVD if $\boldsymbol{R}(z)$ is L-fold oversampled [85]:

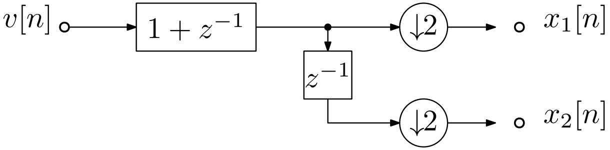
$$\boldsymbol{R}(z^L) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \; . \qquad (3)$$



- a similar effect occurs with the analytic SVD [6, 88], which can be linked to the analytic EVD [79].

Return to Example

- The previous example of $\boldsymbol{R}(z) = [2, \ \ 1 + z^{-1}; 1 + z, \ \ 2]$ arises from the following arrangement with white $v[n] \in \mathcal{N}(0,1)$:



- therefore we require oversampling by $L = 2$:

$$\boldsymbol{R}(z^2) = \begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix} \begin{bmatrix} z + 2 + z^{-1} & \\ & -z + 2 - z^{-1} \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 1 & -z^{-1} \end{bmatrix} ;$$

- multiplexing is linked to a pseudo-circulant property of $\boldsymbol{R}(z)$ [74]; this may be obscured by paraunitary similarity transforms [85, 6].

Analytic EVD of a Parahermitian Matrix

- For an analytic parahermitian matrix $\boldsymbol{R}(z)$, $z \in \mathbb{C}$, that is not connected to multiplexing, we can find [84, 85]

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z)\ \boldsymbol{\Lambda}(z)\ \boldsymbol{Q}^{\mathrm{P}}(z)\ , \tag{4}$$

  with analytic factors;
- $\boldsymbol{Q}(z) = [\boldsymbol{q}_1(z), \ldots, \boldsymbol{q}_M(z)]$ must be paraunitary [74, 76], such that

$$\boldsymbol{Q}(z)\boldsymbol{Q}^{\mathrm{P}}(z) = \boldsymbol{Q}^{\mathrm{P}}(z)\boldsymbol{Q}(z) = \mathbf{I}\ ; \tag{5}$$

- $\boldsymbol{\Lambda}(z) = \mathrm{diag}\{\lambda_1(z), \ldots, \lambda_M\}$ must be diagonal and parahermitian;
- the parahermitian property implies that on the unit circle, $\lambda(\mathrm{e}^{\mathrm{j}\Omega}) = \lambda(z)|_{z=\mathrm{e}^{\mathrm{j}\Omega}} \in \mathbb{R}$.

2.3 Properties: Uniqueness and Ambiguities

- For the analytic EVD [84, 85, 6]

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z) \cdot \boldsymbol{\Lambda}(z) \cdot \boldsymbol{Q}^{\mathrm{P}}(z) \; ; \tag{6}$$

- the eigenvalues in $\boldsymbol{\Lambda}(z)$ are unique up to a permutation;
- if eigenvalues are distinct, then eigenvectors are unique up to an allpass filter $\psi_\ell(z)$;
- with $\boldsymbol{\Psi}(z) = \text{diag}\{\psi_1(z), \ldots, \psi_M(z)\}$,

$$\begin{aligned} \boldsymbol{R}(z) &= \boldsymbol{Q}(z)\boldsymbol{\Psi}(z)\boldsymbol{\Lambda}(z)\boldsymbol{\Psi}^{\mathrm{P}}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \\ &= \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{\Psi}(z)\boldsymbol{\Psi}^{\mathrm{P}}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \\ &= \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \; ; \end{aligned}$$

- an analytic allpass $\psi_m(z)$ does not affect analyticity, but will affects the support of $\mathbf{Q}[n] \circ\!\!-\!\!\bullet\, \boldsymbol{Q}(z)$.

Properties: Support of EVD Factors

- We calculate the analytic EVD of an arbitrary parahermitian $\boldsymbol{R}(z) : \mathbb{C} \rightarrow \mathbb{C}^{2\times 2}$;
- eigenvalues $\gamma_{1,2}(z)$ can be directly computed in the z-domain as the roots of

$$\det\{\gamma(z)\mathbf{I} - \boldsymbol{R}(z)\} = \gamma^2(z) - T(z)\gamma(z) + D(z) = 0$$

- determinant $D(z) = \det\{\boldsymbol{R}(z)\}$ and trace $T(z) = \mathrm{trace}\{\boldsymbol{R}(z)\}$;
- this leads to

$$\gamma_{1,2}(z) = \frac{1}{2}T(z) \pm \frac{1}{2}\sqrt{T(z)T^{\mathrm{P}}(z) - 4D(z)}\;; \tag{7}$$

- awkward: $T(z)T^{\mathrm{P}}(z) - 4D(z) = S(z)S^{\mathrm{P}}(z)$ is parahermitian, but so must be the result of the square root.

Exact Calculation cont'd

- Maclaurin series: for every root of $S(z)$,

$$\sqrt{1-\beta z^{-1}} = \sum_{n=0}^{\infty} \xi_n \beta^n z^{-n} \tag{8}$$

$$\frac{1}{\sqrt{1-\alpha z^{-1}}} = \left(\sum_{n=0}^{\infty} \xi_n \alpha^n z^{-n}\right)^{-1} = \sum_{n=0}^{\infty} \chi_n \alpha^n z^{-n} \tag{9}$$

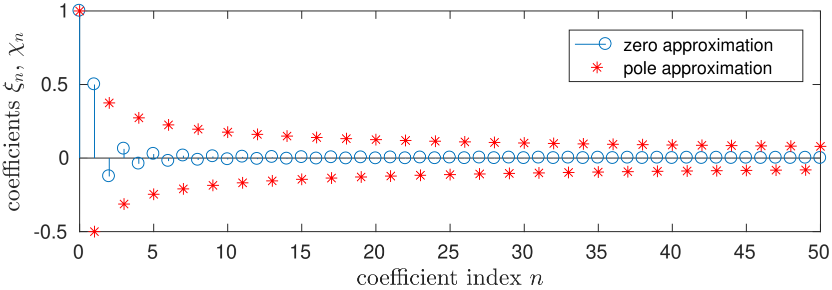
- with coefficients [7]

$$\xi_n = (-1)^n \binom{\frac{1}{2}}{n} = \frac{(-1)^n}{n!} \prod_{i=0}^{n-1} \left(\frac{1}{2} - i\right) , \tag{10}$$

$$\chi_n = (-1)^n \binom{-\frac{1}{2}}{n} = \frac{(-1)^{n-1}}{n!} \prod_{i=0}^{n-1} \left(\frac{1}{2} + i\right) . \tag{11}$$

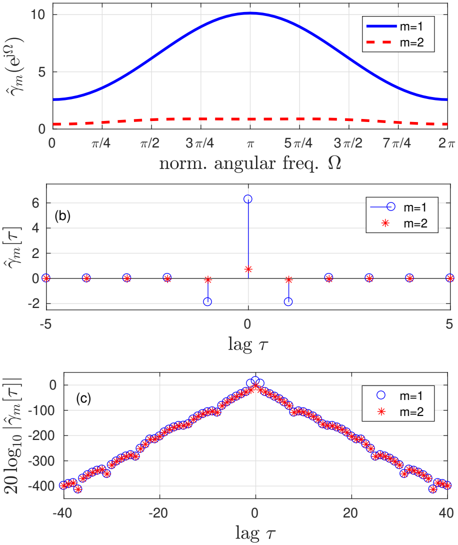
Maclaurin Series

- Coefficients ξ_n and χ_n for $n = 0 \ldots 50$ [84]:



- these coefficients additionally dampen a geometric series in (8) and (9);
- only if $S(z)$ has double zeros (and double poles) is a polynomial (rational) solution possible;
- in general, the result are transcendental eigenvalues.

Numerical Example



- Example from Icart & Comon (2012, [34]):

$$\boldsymbol{R}(z) = \begin{bmatrix} 1 & 1 \\ 1 & -2z + 6 - 2z^{-1} \end{bmatrix}$$

- (top) solution on unit circle;
- (middle) coefficients of analytic eigenvalues;
- (bottom) decay of coefficients;
- solution generally can be transcendental, i.e. neither finite nor rational.

3. Polynomial Eigenvalue Decomposition

3.1 Polynomial EVD and Spectral Majorisation

- Polynomial EVD or McWhirter decomposition [49] of the CSD matrix

$$\boldsymbol{R}(z) \approx \boldsymbol{U}(z)\, \boldsymbol{\Gamma}(z)\, \boldsymbol{U}^{\mathrm{P}}(z) \qquad (12)$$

- with paraunitary, polynomial $\boldsymbol{U}(z)$, s.t. $\boldsymbol{U}(z)\boldsymbol{U}^{\mathrm{P}}(z) = \mathbf{I}$;
- diagonal Laurent polynomial matrix

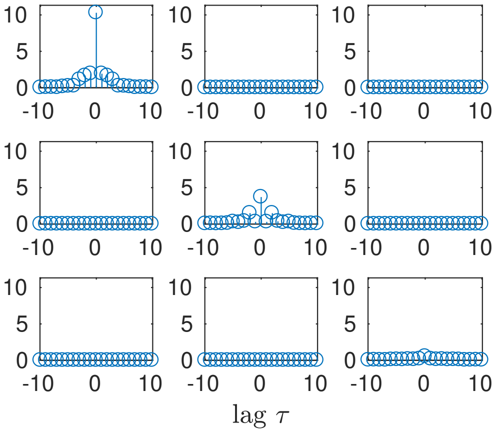
$$\boldsymbol{\Gamma}(z) = \mathrm{diag}\{\gamma_1(z),\ \ldots,\ \gamma_M(z)\}\ , \qquad (13)$$

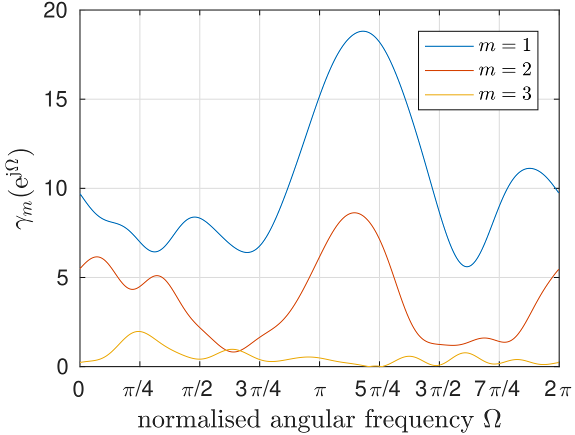
- approximation sign due to restriction to polynomials [34];
- the eigenvalues are spectrally majorised [73], i.e. on the unit circle satisfy

$$\gamma_m(\mathrm{e}^{\mathrm{j}\Omega}) \geq \gamma_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})\ , \qquad \forall\Omega, \quad m = 1, \ldots (M-1)\ . \qquad (14)$$

Polynomial Eigenvalues and Spectral Majorisation

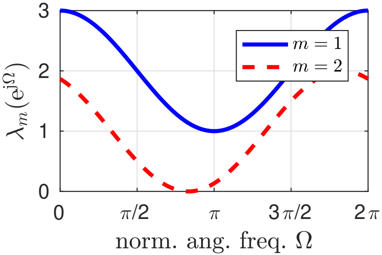
- Example for polynomial eigenvalues $\gamma_m[\tau] \circ\!\!-\!\!\bullet\; \gamma_m(\mathrm{e}^{\mathrm{j}\Omega})$ of a 3×3 matrix:

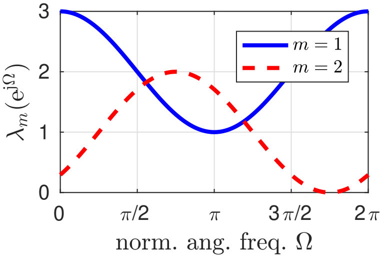




3.2 Relation to Analytic EVD

- If the analytic eigenvalues do not intersect on the unit circle, then analytic EVD and polynomial EVD (with sufficiently high order) are 'identical';
- the polynomial EVD has a strict ordering of eigenvalues;
- specific polynomial/analytic eigenvector solution may differ — recall the allpass ambiguity;





- if analytic eigenvalues intersect, then the solutions of analytic EVD and polynomial EVD differ;
- we explore by way of an example ...

3.3 Numerical Example

- We pick our own eigenvalues (order 2) and eigenvectors (order 1):

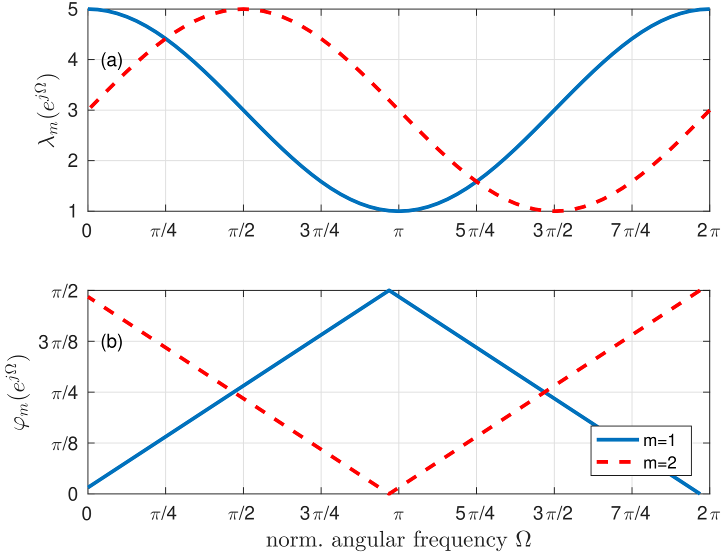
$$\boldsymbol{\Lambda}(z) = \begin{bmatrix} z + 3 + z^{-1} & \\ & -jz + 3 + jz^{-1} \end{bmatrix}$$

$$\boldsymbol{Q}(z) = [\boldsymbol{q}_1(z),\, \boldsymbol{q}_2(z)] \qquad \text{with} \qquad \boldsymbol{q}_{1,2}(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm z^{-1} \end{bmatrix} ;$$

- parahermitian matrix $\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z)$:

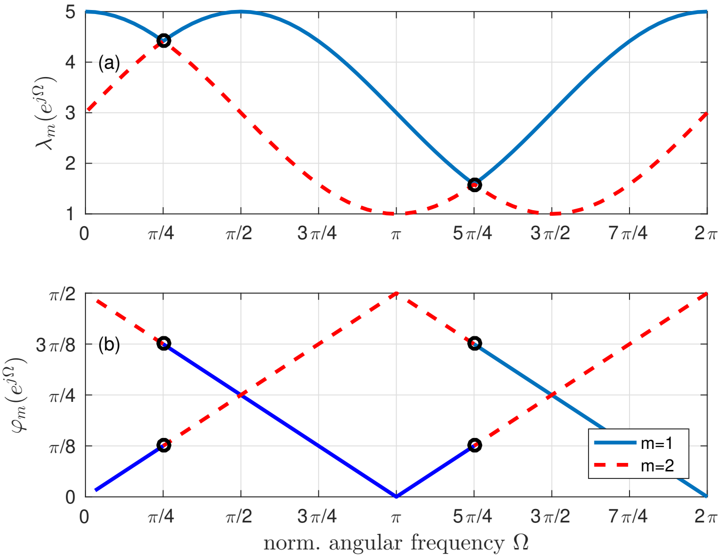
$$\boldsymbol{R}(z) = \begin{bmatrix} \frac{1-j}{2}z + 3 + \frac{1+j}{2}z^{-1} & \frac{1+j}{2}z^{2} + \frac{1-j}{2} \\ \frac{1+j}{2} + \frac{1-j}{2}z^{-2} & \frac{1-j}{2}z + 3 + \frac{1+j}{2}z^{-1} \end{bmatrix} .$$

Numerical Example — Analytic Solution



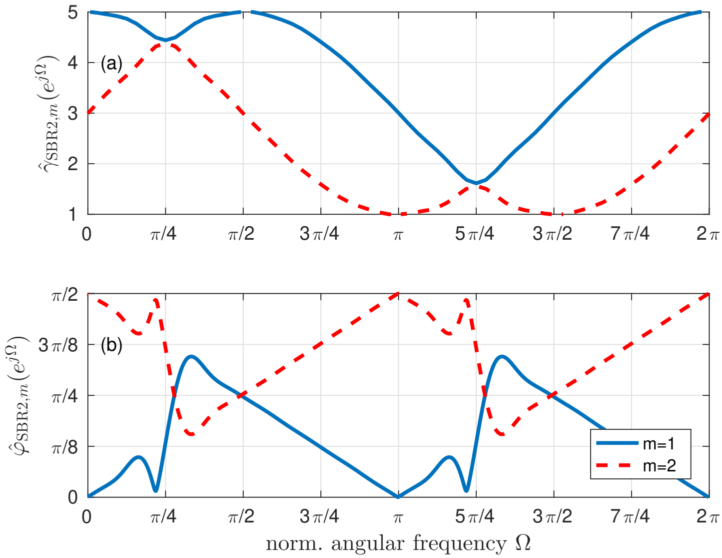
- Analytic (and therefore infinitely differentiable) eigenvalues $\lambda_m(\mathrm{e}^{\mathrm{j}\Omega})$;
- smooth Hermitian angles $\cos\varphi_m = |\boldsymbol{q}_1^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}0}) \cdot \boldsymbol{q}_m(\mathrm{e}^{\mathrm{j}\Omega})|$.

Numerical Example — Ideal Spectral Majorisation



- Analytic eigenvalues are permuted where they intersect;
- resulting spectrally majorised eigenvalues are piecewise analytic but not differentiable;
- corresponding eigenvectors are piecewise analytic but not continuous.

Numerical Example — PEVD Algorithmic Solution



- Using the SBR2 algorithm in [49] to approximate the McWhirter factorisation;
- trimming is applied to PEVD factors [15, 31, 16, 71];
- spectrally majorised eigenvalues $\mathbf{\Gamma}(z)$ of order 24;
- corresponding eigenvectors in $\boldsymbol{U}(z)$ of order 84;
- SBR2's spectral majorisation can be beneficial or detrimental in applications.

4. Time Domain Algorithms

1. Overview
2. Analytic eigenvalue decomposition
3. Polynomial eigenvalue decomposition
4. Time domain algorithms
   - 4.1 iterative PEVD approaches
   - 4.2 second order sequential best rotation (SBR) algorithm
   - 4.3 sequential matrix diagonalisation algorithm
   - 4.4 comparison
5. DFT domain algorithms
6. Summary

4.1 Iterative PEVD Approach

- Second order sequential best rotation (SBR2, McWhirter 2007, [49, 63, 66, 77]);
- iterative approach based on an elementary paraunitary operation:

$$\begin{aligned} \boldsymbol{S}^{(0)}(z) &= \boldsymbol{R}(z) \\ &\vdots \\ \boldsymbol{S}^{(i+1)}(z) &= \left\{\boldsymbol{H}^{(i+1)}(z)\right\}^{\mathrm{P}} \boldsymbol{S}^{(i)}(z) \boldsymbol{H}^{(i+1)}(z) \end{aligned}$$

- $\boldsymbol{H}^{(i)}(z)$ is an elementary paraunitary operation, which at the ith step eliminates the largest off-diagonal element in $\boldsymbol{S}^{(i-1)}(z)$;
- stop after I iterations:

$$\hat{\boldsymbol{\Gamma}}(z) = \boldsymbol{S}^{(I)}(z) \qquad , \qquad \hat{\boldsymbol{U}}(z) = \prod_{i=1}^{I} \boldsymbol{H}^{(i)}(z)$$

- sequential matrix diagonalisation (SMD) [10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 58, 64] follows a similar scheme.

Elementary Paraunitary Operation

- An elementary paraunitary matrix [74] is defined as

$$\boldsymbol{H}^{(i)}(z) = \mathbf{I} - \mathbf{v}^{(i)}\mathbf{v}^{(i),\mathrm{H}} + z^{-1}\mathbf{v}^{(i)}\mathbf{v}^{(i),\mathrm{H}} \quad , \quad \|\mathbf{v}^{(i)}\|_2 = 1$$

- we utilise a different definition:

$$\boldsymbol{H}^{(i)}(z) = \boldsymbol{D}^{(i)}(z)\mathbf{Q}^{(i)}$$

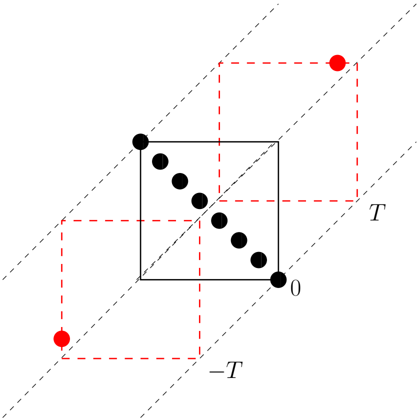
- $\boldsymbol{D}^{(i)}(z)$ is a delay matrix:

$$\boldsymbol{D}^{(i)}(z) = \mathrm{diag}\{1 \ \ldots \ 1 \ z^{-\tau} \ 1 \ \ldots \ 1\}$$

- $\mathbf{Q}^{(i)}(z)$ is a Givens rotation.

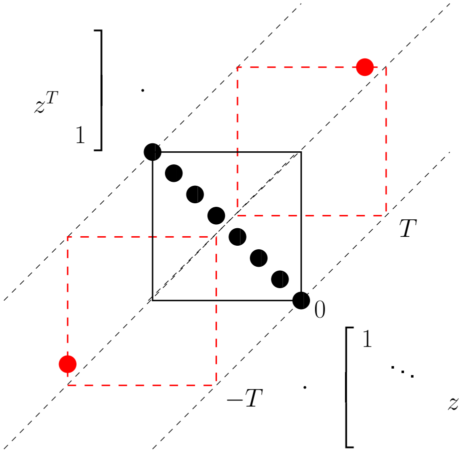
4.2 Sequential Best Rotation Algorithm (McWhirter [49])

- At iteration i, consider $\boldsymbol{S}^{(i-1)}(z) \circ\!\!-\!\!\bullet\ \mathbf{S}^{(i-1)}[\tau]$ (max. off-diag. value: ●)



4.2 Sequential Best Rotation Algorithm (McWhirter [49])

- $\tilde{\boldsymbol{D}}^{(i)}(z)\boldsymbol{S}^{(i-1)}(z)\boldsymbol{D}^{(i)}(z)$

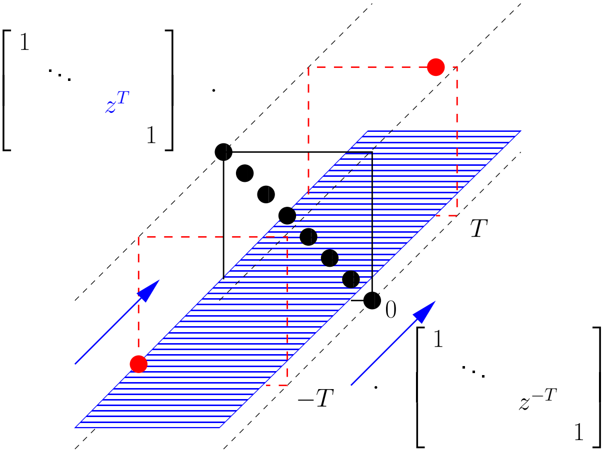


$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & z^{T} & \\ & & & 1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & z^{-T} & \\ & & & 1 \end{bmatrix}$$

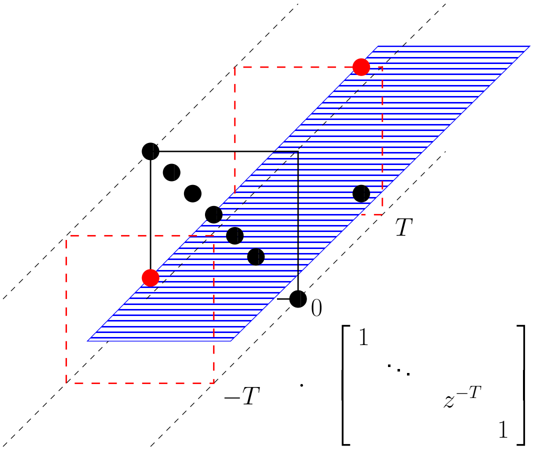
4.2 Sequential Best Rotation Algorithm (McWhirter [49])

- $\tilde{\boldsymbol{D}}^{(i)}(z)$ advances a row-slice of $\boldsymbol{S}^{(i-1)}(z)$ by T



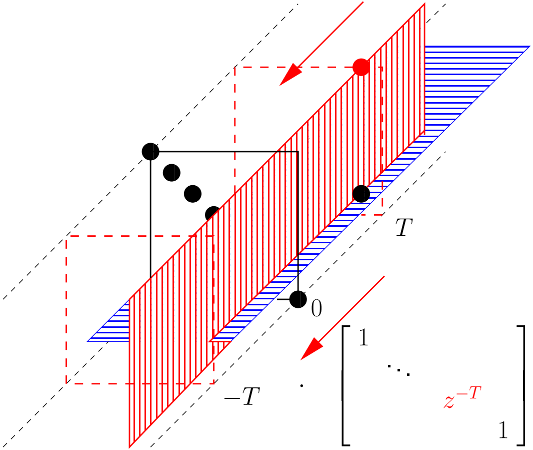
4.2 Sequential Best Rotation Algorithm (McWhirter [49])

- the off-diagonal element at $-T$ has now been translated to lag zero



4.2 Sequential Best Rotation Algorithm (McWhirter [49])

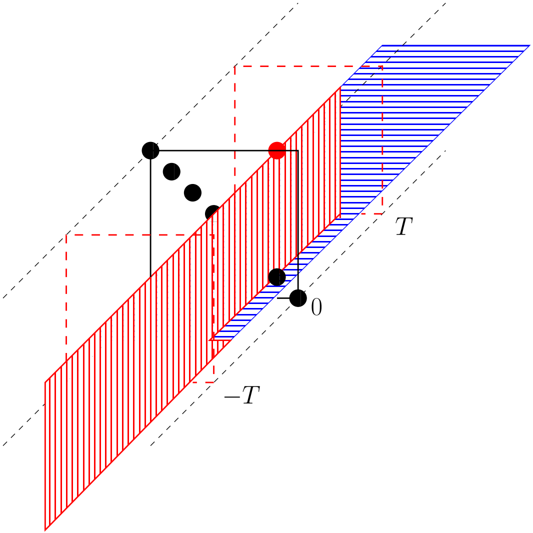
- $\mathbf{D}^{(i)}(z)$ delays a column-slice of $\boldsymbol{S}^{(i-1)}(z)$ by T



$$\cdot \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & z^{-T} & \\ & & & 1 \end{bmatrix}$$

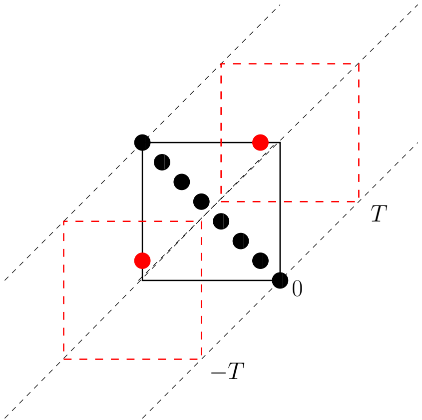
4.2 Sequential Best Rotation Algorithm (McWhirter [49])

- the off-diagonal element at $-T$ has now been translated to lag zero



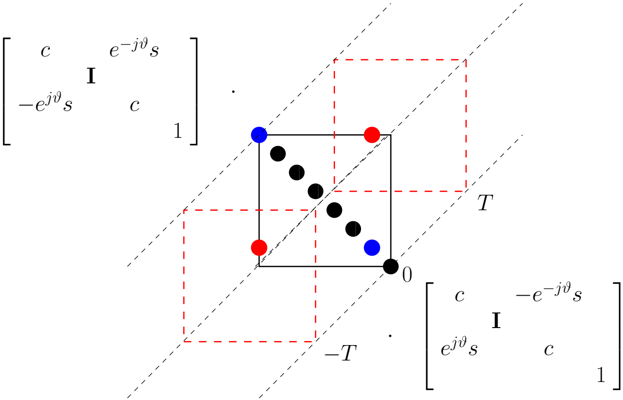
4.2 Sequential Best Rotation Algorithm (McWhirter [49])

- the step $\tilde{\boldsymbol{D}}^{(i)}(z)\boldsymbol{S}^{(i-1)}(z)\boldsymbol{D}_{(i)}(z)$ has brought the largest off-diagonal elements to lag 0.



4.2 Sequential Best Rotation Algorithm (McWhirter [49])

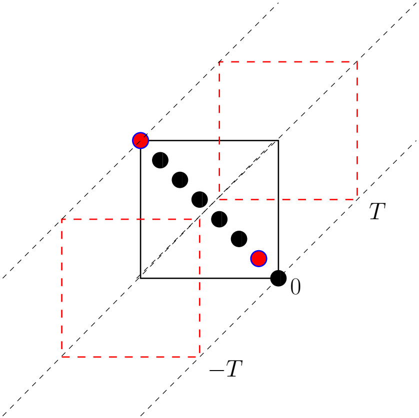
- Jacobi step to eliminate largest off-diagonal elements by $\mathbf{Q}^{(i)}$



4.2 Sequential Best Rotation Algorithm (McWhirter [49])

- iteration i is completed, having performed

$$\boldsymbol{S}^{(i)}(z) = \mathbf{Q}^{(i)} \boldsymbol{D}^{(i)}(z) \boldsymbol{S}^{(i-1)}(z) \tilde{\boldsymbol{D}}^{(i)}(z) \tilde{\mathbf{Q}}^{(i)}(z)$$



SBR2 Outcome

- At the ith iteration, the zeroing of off-diagonal elements achieved during previous steps may be partially undone;
- the algorithm has proven convergence, transfering energy onto the main diagonal at every step [49];
- after I iterations, we reach an approximate diagonalisation

$$\hat{\boldsymbol{\Gamma}}(z) = \boldsymbol{S}^{(L)}(z) = \hat{\boldsymbol{U}}^{\mathrm{P}}(z)\boldsymbol{R}(z)\hat{\boldsymbol{U}}(z)$$

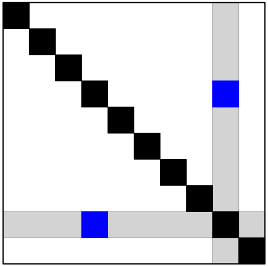
with

$$\hat{\boldsymbol{U}}(z) = \prod_{i=1}^{I} \boldsymbol{D}^{(i)}(z)\mathbf{Q}^{(i)}$$

- the factors may require trimming of trailing zeros or very small coefficients [15, 16, 31, 37, 71].

SBR2 — Givens Rotation

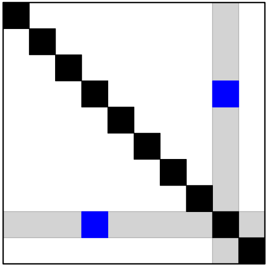
- A Givens rotation eliminates the maximum off-diagonal element once brought onto the lag-zero matrix;
- note I: in the lag-zero matrix, one column and one row are modified by the shift:

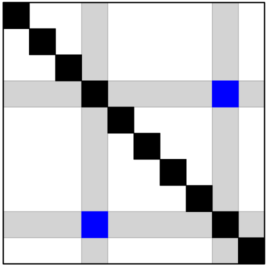


- note II: a Givens rotation only affects two columns and two rows in every matrix;
- Givens rotation is relatively low in computational cost, but the zero lag component is not diagonal.

SBR2 — Givens Rotation

- A Givens rotation eliminates the maximum off-diagonal element once brought onto the lag-zero matrix;
- note I: in the lag-zero matrix, one column and one row are modified by the shift:

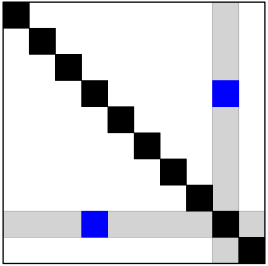




- note II: a Givens rotation only affects two columns and two rows in every matrix;
- Givens rotation is relatively low in computational cost, but the zero lag component is not diagonal.

4.3 Sequential Matrix Diagonalisation (SMD, [64])

- Main idea — the zero-lag matrix is diagonalised in every step;
- initialisation: diagonalise $\mathbf{R}[0]$ by EVD and apply modal matrix to all matrix coefficients $\longrightarrow \boldsymbol{S}^{(0)}$;
- at the ith step as in SBR2, the maximum element (or column with max. norm) is shifted to the lag-zero matrix:



- an EVD is used to re-diagonalise the zero-lag matrix;
- a full modal matrix is applied at all lags — more costly than SBR2.

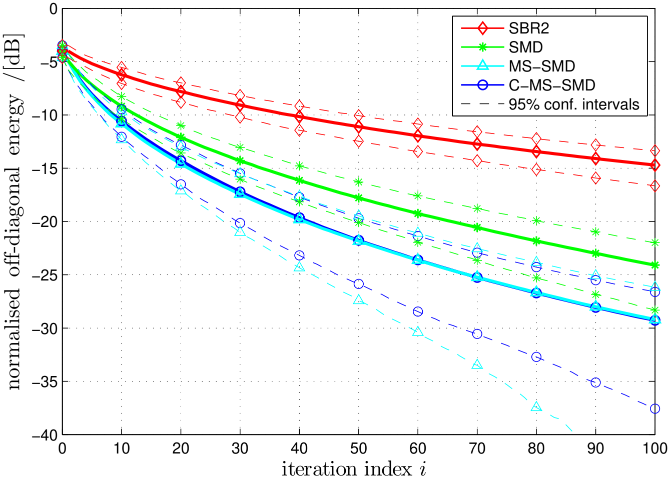
4.3 Sequential Matrix Diagonalisation (SMD, [64])

- Main idea — the zero-lag matrix is diagonalised in every step;
- initialisation: diagonalise $\mathbf{R}[0]$ by EVD and apply modal matrix to all matrix coefficients $\longrightarrow \boldsymbol{S}^{(0)}$;
- at the ith step as in SBR2, the maximum element (or column with max. norm) is shifted to the lag-zero matrix:

- an EVD is used to re-diagonalise the zero-lag matrix;
- a full modal matrix is applied at all lags — more costly than SBR2.

4.4 Comparison: SBR2/SMD Convergence

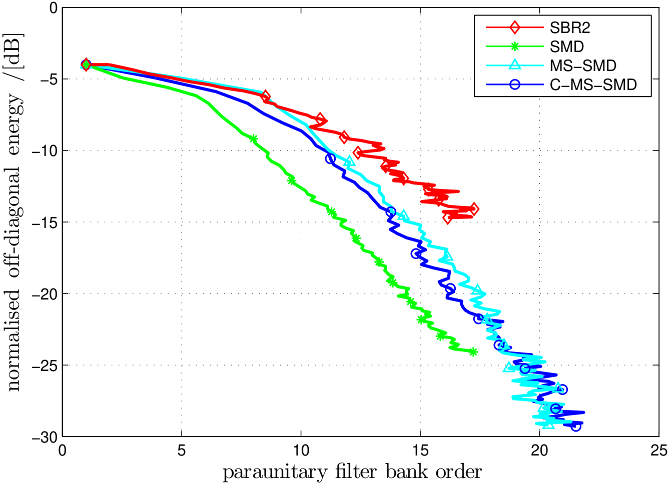
- Measuring the remaining normalised off-diagonal energy over an ensemble of space-time covariance matrices:



- SMD [64] and evolved schemes (multiple shift SMD [12] and constrained multiple shift SMD [13]) transfering additional energy on the zero lag component converge faster per iteration;
- the computation cost per iteration is not comparable: SMD and variants are significantly more complex.

SBR2/SMD Application Cost 1

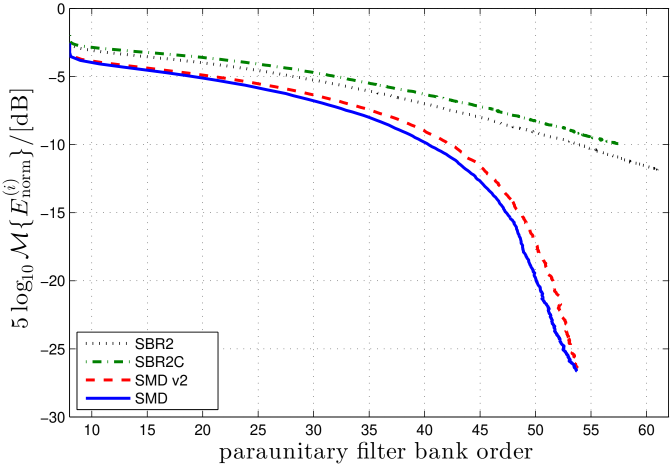
- Ensemble average of remaining off-diagonal energy vs. order of paraunitary filter banks to decompose 4x4 matrices of order 15:



- for modestly-sized matrices, SMD a better diagonalisation with lower orders of $\boldsymbol{Q}(z)$ than other variants;
- this reflects the cost when applying the PEVD factors e.g. for a subspace decomposition.

SBR2/SMD Application Cost 2

- Ensemble average of remaining off-diagonal energy vs. order of paraunitary filter banks to decompose 8x8 matrices of order 63:

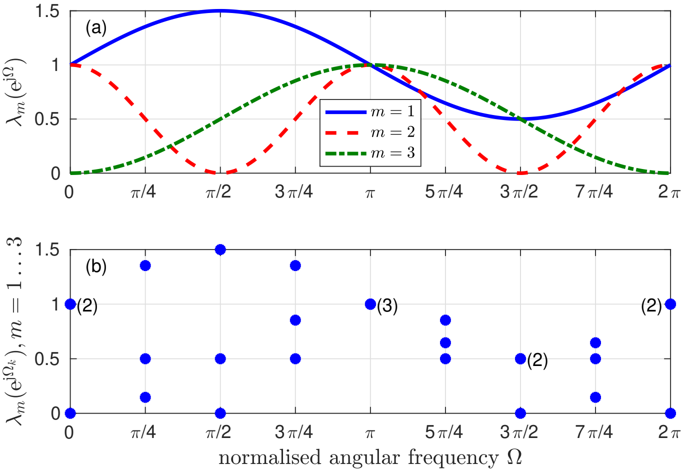


- for larger matrices of higher order, SMD (and variants) provide a better diagonalisation than SBR2 [49] (and variants [63, 66]) for any given order of $\boldsymbol{Q}(z)$.

5. DFT Domain Algorithms

5.1 Analytic Eigenvalue Extraction

- Idea for DFT-based algorithms: calculate an EVD in every DFT bin adn join the dots;



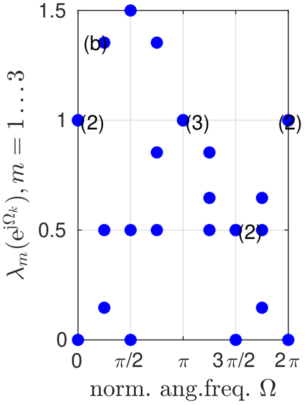
- spectral coherence must be re-established across bins;
- we exploit that the solution must be analytic, i.e. infinitely differentiable;
- we first extract eigenvalues, which are less volatile under perturbation [35];

Analytic Eigenvalue Extraction Algorithm I

- Bin-wise EVD yields:

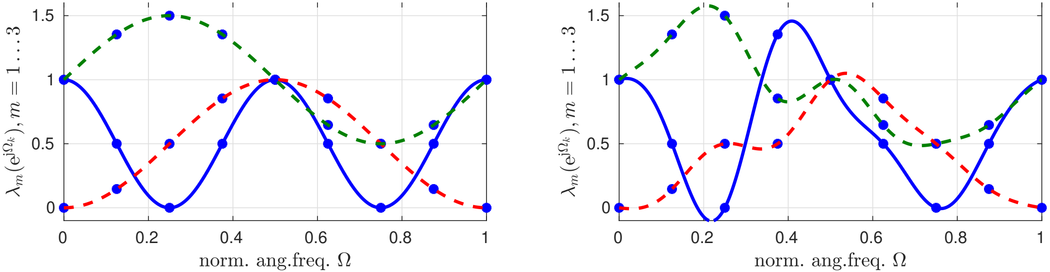
$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{Q}_k \boldsymbol{\Lambda}_k \mathbf{Q}_k^{\mathrm{H}} = \underbrace{\mathbf{Q}_k \boldsymbol{\Psi}_k \mathbf{P}_k}_{\boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}} \boldsymbol{\Lambda}_k \mathbf{P}_k}_{\boldsymbol{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}} \boldsymbol{\Psi}_k^{\mathrm{H}} \mathbf{Q}_k^{\mathrm{H}}}_{\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega_k})}$$

- $\mathbf{P}_k$ is a permutation matrix, since in the analytic EVD, eigenvalues can intersect and are not necessarily majorised;
- for distinct eigenvalues: $\boldsymbol{\Psi}_k$ is a diagonal matrix that accounts for the phase ambiguity of eigenvectors;
- in case of a C-fold algebraic multiplicity: $\boldsymbol{\Psi}_K$ is block diagonal, with a $C \times C$ unitary matrix accounting for eigenvectors forming an arbitrary basis within a C-dimensional subspace;
- a predecessor algorithm [72] can fail on this;



Analytic Eigenvalue Extraction Algorithm II

- To find the smoothest association of M functions across K frequency bins, we compare the power in the p-th derivative of a Dirichlet interpolation [83, 91, 96]:



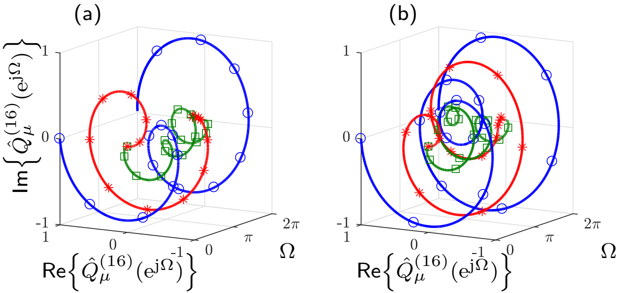
- for an exhaustive search, there would be $M!^{K-1}$ associations to check;
- a Viterbi-type scheme operates iteratively across bins [96], and only retains viable associations [89, 90];
- DFT length K can be increased until a criterion based on time-domain aliasing is met.

5.2 Analytic Eigenvector Extraction

- From the eigenvalue extraction, the correct association across frequency bins defines $\mathbf{P}_k$ [86]:

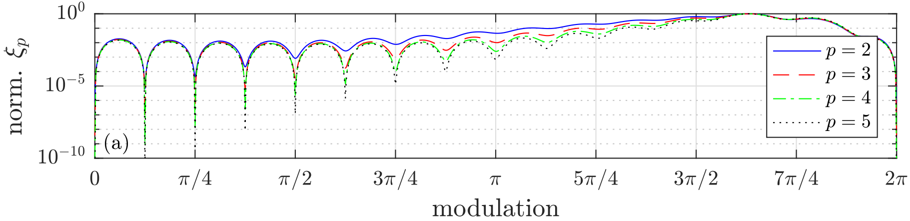
$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{Q}_k\boldsymbol{\Lambda}_k\mathbf{Q}_k^{\mathrm{H}} = \underbrace{\mathbf{Q}_k\boldsymbol{\Psi}_k\mathbf{P}_k}_{\boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{P}_k}_{\boldsymbol{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}}\boldsymbol{\Psi}_k^{\mathrm{H}}\mathbf{Q}_k^{\mathrm{H}}}_{\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega_k})}$$

- for the eigenvalue extraction, it remains to find the correct phase adjustment $\boldsymbol{\Psi}_k$;
- again the smoothness of a Dirichlet interpolation can lead to the analytic solution;
- example: $M = 3$ components of one eigenvector with different $\boldsymbol{\Psi}_k$.



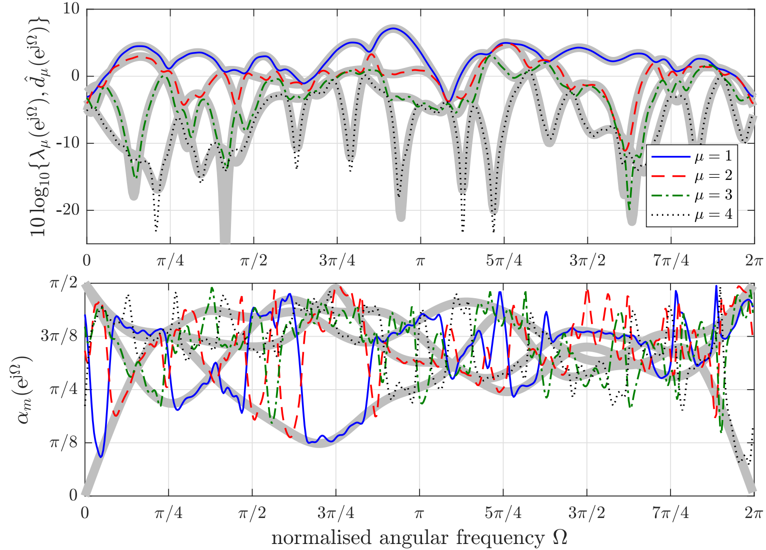
Analytic Eigenvector Extraction Algorithm

- The main task for the extraction of analytic eigenvectors is the adjustment of a smooth phase progression $\boldsymbol{\Psi}_k$ across bins;
- this problem is NP hard [68], but the specific smoothness cost function possesses — for a sufficient DFT size — stationary points that are approximately separated by a modulation [87];
- cut through cost function ξ_p for powers of different derivatives p:



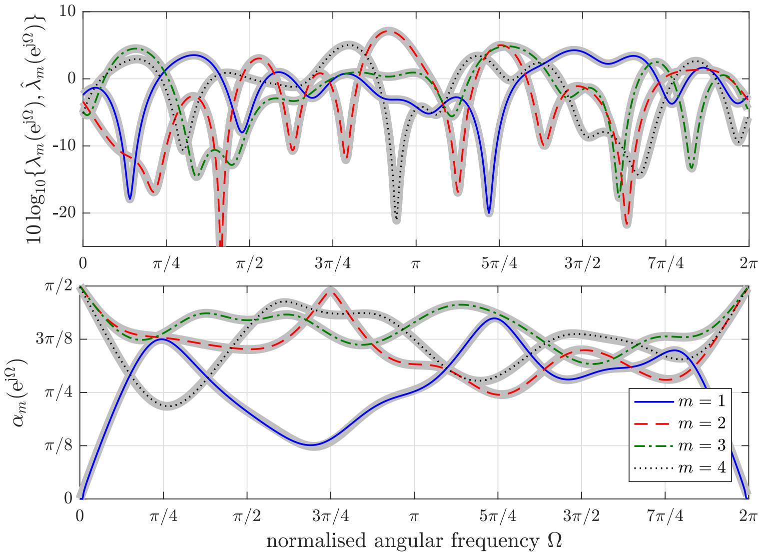
- an iterative algorithm is proven to converge [87], increasing the DFT length until a reconstruction error is minimised;

5.3 Comparison — SMD Algorithm Example



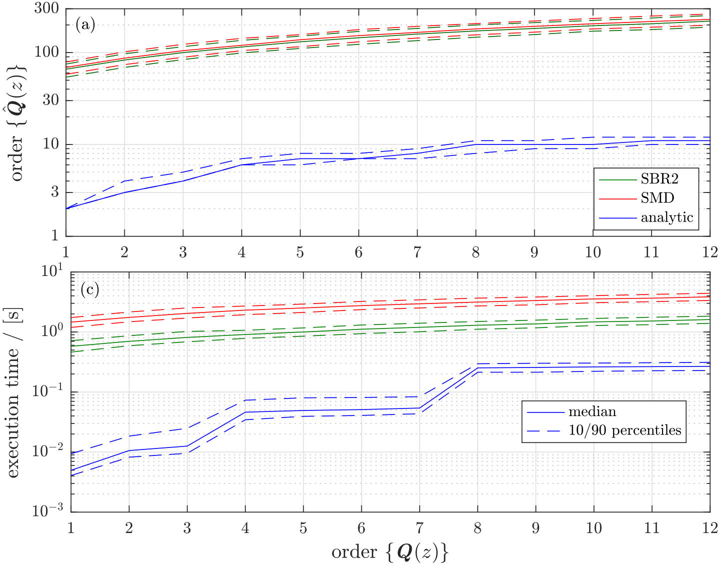
- $\boldsymbol{R}(z) : \mathbb{C} \rightarrow \mathbb{C}^{4\times4}$ of order 47;
- SMD algorithm [64] yields approximate spectral majorisation [50];
- Hermitian angles of eigenvectors to a reference vector indicate approximation of piecewise analytic functions.

Analytic EVD Extraction Example



- same matrix, but utilising analytic eigen-value [23, 90, 89] and -vector extraction [86, 87, 42];
- extracted analytic EVD factors are close to ground truth;
- lower order compared to SMD result.

Comparison — Ensemble Results



- ensemble results over matrices with different ground truth, and for various orders;
- above: application cost — the order of the extracted paraunitary matrices, required e.g. for a subspace projection;
- below: execution time of the algorithms.

6. Summary I

- An analytic EVD exists in almost all cases — when data is not multiplexed;
- a polynomial EVD (PEVD) differs if eigenvalues intersect; it targets spectrally majorised eigenvalues;
- spectral majorisation in the EVD is desirable in coding or communications applications;
- the PEVD is supported by a number of well-established algorithms [22, 49, 63];
- analytic EVD algorithms are useful where low-order factors or low-perturbed subspace methods matter;
- if a space-time covariance matrix is estimated from limited data [25, 26, 27, 28, 39], time- and DFT-domain algorithms target the same factors (within the allpass ambiguity of the eigenvectors) with probability one [46].

Summary II

- To calculate an analytic eigenvalue decomposition, a number of efficient implementations have been suggested [22, 45, 40, 41, 43];
- such algorithms find applications in coding [63, 92], angle of arrival estimation [2, 1, 29, 33, 44, 78], beamforming [3, 4, 54, 65, 80], subspace detection [59, 60, 61, 81], speech enhancement [55, 57, 56, 53], communications [51, 52, 70, 69] and others [62, 82, 93, 95, 94];
- the idea of an analytic decomposition extends beyond the EVD; similarly the existence has been proven for an analytic SVD [6, 5, 88, 79], and is suspected for the QR decomposition [9];
- algorithms have been extended to such further decompositions, including the SVD [30, 36, 48, 49, 72, 88, 79, 93, 95, 94] or the QR decomposition [9, 21, 30, 38].

References I

[1] M. Alrmah, J. Corr, A. Alzin, K. Thompson, and S. Weiss.
Polynomial subspace decomposition for broadband angle of arrival estimation.
In *Sensor Signal Processing for Defence*, pages 1–5, Edinburgh, Scotland, Sept 2014.

[2] M. Alrmah, S. Weiss, and S. Lambotharan.
An extension of the MUSIC algorithm to broadband scenarios using polynomial eigenvalue decomposition.
In *19th European Signal Processing Conference*, pages 629–633, Barcelona, Spain, August 2011.

[3] A. Alzin, F. Coutts, J. Corr, S. Weiss, I. Proudler, and J. Chambers.
Polynomial matrix formulation-based Capon beamformer.
In *IMA International Conference on Signal Processing in Mathematics*, Birmingham, UK, December 2016.

[4] A. Alzin, F. Coutts, J. Corr, S. Weiss, I. K. Proudler, and J. A. Chambers.
Adaptive broadband beamforming with arbitrary array geometry.
In *IET/EURASIP Intelligent Signal Processing*, London, UK, December 2015.

[5] G. Barbarino.
On the periodicity of singular vectors and the holomorphic block-circulant SVD on the unit circumference.
Linear Algebra and its Applications, 721:465–483, Sept. 2025.

[6] G. Barbarino and V. Noferini.
On the Rellich eigendecomposition of para-Hermitian matrices and the sign characteristics of $*$-palindromic matrix polynomials.
Linear Algebra and its Applications, 672:1–27, Sept. 2023.

[7] I. Bronshtein and K. Semendyayew.
Handbook of Mathematics.
Springer, Heidelberg, 2015.

[8] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nicols.
Numerical computation of an analytic singular value decomposition of a matrix valued function.
Numer. Math, 60:1–40, 1991.

References II

[9] D. Cescato and H. Bölcskei.
QR decomposition of laurent polynomial matrices sampled on the unit circle.
IEEE Transactions on Information Theory, 56(9):4754–4761, Sept 2010.

[10] J. Corr, K. Thompson, S. Weiss, J. McWhirter, and I. Proudler.
Maximum energy sequential matrix diagonalisation for parahermitian matrices.
In *48th Asilomar Conference on Signals, Systems and Computers*, pages 470–474, Pacific Grove, CA, USA, November 2014.

[11] J. Corr, K. Thompson, S. Weiss, J. McWhirter, and I. Proudler.
Performance trade-offs in sequential matrix diagonalisation search strategies.
In *IEEE 6th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, pages 25–28, Cancun, Mexico, December 2015.

[12] J. Corr, K. Thompson, S. Weiss, J. McWhirter, S. Redif, and I. Proudler.
Multiple shift maximum element sequential matrix diagonalisation for parahermitian matrices.
In *IEEE Workshop on Statistical Signal Processing*, pages 312–315, Gold Coast, Australia, June 2014.

[13] J. Corr, K. Thompson, S. Weiss, J. G. McWhirter, and I. K. Proudler.
Causality-Constrained multiple shift sequential matrix diagonalisation for parahermitian matrices.
In *22nd European Signal Processing Conference*, pages 1277–1281, Lisbon, Portugal, September 2014.

[14] J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. McWhirter.
Reduced search space multiple shift maximum element sequential matrix diagonalisation algorithm.
In *IET/EURASIP Intelligent Signal Processing*, London, UK, December 2015.

[15] J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. McWhirter.
Row-shift corrected truncation of paraunitary matrices for PEVD algorithms.
In *23rd European Signal Processing Conference*, pages 849–853, Nice, France, August/September 2015.

[16] J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. McWhirter.
Shortening of paraunitary matrices obtained by polynomial eigenvalue decomposition algorithms.
In *Sensor Signal Processing for Defence*, Edinburgh, Scotland, September 2015.

References III

[17] F. Coutts, J. Corr, K. Thompson, I. Proudler, and S. Weiss.
Divide-and-conquer sequential matrix diagonalisation for parahermitian matrices.
In *Sensor Signal Processing for Defence Conference*, pages 1–5, London, UK, December 2017.

[18] F. Coutts, J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. McWhirter.
Complexity and search space reduction in cyclic-by-row PEVD algorithms.
In *50th Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, CA, November 2016.

[19] F. Coutts, K. Thompson, I. Proudler, and S. Weiss.
Restricted update sequential matrix diagonalisation for parahermitian matrices.
In *IEEE 7th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Curacao, December 2017.

[20] F. Coutts, K. Thompson, S. Weiss, and I. Proudler.
A comparison of iterative and DFT-based polynomial matrix eigenvalue decompositions.
In *IEEE 7th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Curacao, December 2017.

[21] F. K. Coutts, J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. G. McWhirter.
Memory and complexity reduction in parahermitian matrix manipulations of PEVD algorithms.
In *24th European Signal Processing Conference*, Budapest, Hungary, Aug. 2016.

[22] F. K. Coutts, I. K. Proudler, and S. Weiss.
Efficient implementation of iterative polynomial matrix evd algorithms exploiting structural redundancy and parallelisation.
IEEE Transactions on Circuits and Systems I: Regular Papers, 66(12):4753–4766, Dec. 2019.

[23] F. K. Coutts, K. Thompson, J. Pestana, I. Proudler, and S. Weiss.
Enforcing eigenvector smoothness for a compact DFT-based polynomial eigenvalue decomposition.
In *10th IEEE Workshop on Sensor Array and Multichannel Signal Processing*, pages 1–5, July 2018.

[24] B. De Moor and S. Boyd.
Analytic properties of singular values and vectors.
Technical report, KU Leuven, 1989.

References IV

[25] C. Delaosa, F. K. Coutts, J. Pestana, and S. Weiss.
Impact of space-time covariance estimation errors on a parahermitian matrix EVD.
In *10th IEEE Workshop on Sensor Array and Multichannel Signal Processing*, pages 1–5, July 2018.

[26] C. Delaosa, J. Pestana, N. J. Goddard, S. Somasundaram, and S. Weiss.
Sample space-time covariance matrix estimation.
In *IEEE International Conference on Acoustics, Speech and Signal Processing*, pages 8033–8037, Brighton, UK, May 2019.

[27] C. Delaosa, J. Pestana, N. J. Goddard, S. D. Somasundaram, and S. Weiss.
Support estimation of a sample space-time covariance matrix.
In *Sensor Signal Processing for Defence*, Brighton, UK, March 2019.

[28] C. Delaosa, J. Pestana, I. K. Proudler, and S. Weiss.
Impact of space-time covariance matrix estimation on bin-wise eigenvalue and eigenspace perturbations.
Signal Processing, 233:109946, Aug. 2025.

[29] C. Delaosa, J. Pestana, S. Weiss, and I. Proudler.
Subspace perturbation bounds with an application to angle of arrival estimation using the MUSIC algorithm.
In *International Conference on Sensor Signal Processing for Defence*, Edinburgh, UK, 2020.

[30] J. Foster, J. McWhirter, M. Davies, and J. Chambers.
An algorithm for calculating the qr and singular value decompositions of polynomial matrices.
IEEE Transactions on Signal Processing, 58(3):1263–1274, March 2010.

[31] J. Foster, J. G. McWhirter, and J. Chambers.
Limiting the order of polynomial matrices within the SBR2 algorithm.
In *IMA International Conference on Mathematics in Signal Processing*, Cirencester, UK, December 2006.

[32] G. H. Golub and C. F. Van Loan.
Matrix Computations.
John Hopkins University Press, Baltimore, Maryland, 3rd edition, 1996.

References V

[33] A. Hogg, V. Neo, S. Weiss, C. Evers, and P. Naylor.
A polynomial eigenvalue decomposition MUSIC approach for broadband sound source localization.
In *IEEE Workshop on Applications of Signal Processing to Audio and Acoustics*, New Paltz, NY, Oct. 2021.

[34] S. Icart and P. Comon.
Some properties of Laurent polynomial matrices.
In *9th IMA Conference on Mathematics in Signal Processing*, Birmingham, UK, December 2012.

[35] T. Kato.
Perturbation Theory for Linear Operators.
Springer, 1980.

[36] F. Khattak, I. Proudler, J. McWhirter, and S. Weiss.
Generalised sequential matrix diagonalisation for the SVD of polynomial matrices.
In *International Conference on Defence for Signal Processing*, Edinburgh, Scotland, Sept. 2023.

[37] F. Khattak, I. K. Proudler, and S. Weiss.
Support estimation of analytic eigenvectors of parahermitian matrices.
In *International Conference on Recent Advances in Electrical Engineering and Computer Sciences*, Islamabad, Pakistan, October 2022.

[38] F. A. Khattak, M. A. Bakhit, I. K. Proudler, and S. Weiss.
Smooth qr decomposition of polynomial matrices.
In *9th IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Los Sueños, Costa Rica, Dec. 2023.

[39] F. A. Khattak, I. K. Proudler, and S. Weiss.
Enhanced space-time covariance estimation based on a system identification approach.
In *Sensor Signal Processing for Defence Conference*, London, UK, Sept. 2022.

[40] F. A. Khattak, I. K. Proudler, and S. Weiss.
Extension of power method to para-Hermitian matrices: polynomial power method.
In *31st European Signal Processing Conference*, pages 1564–1568, Helsinki, Finland, Sept. 2023.

References VI

[41] F. A. Khattak, I. K. Proudler, and S. Weiss.
Low-rank para-hermitian matrix evd via polynomial power method with deflation.
In *9th IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Los Sueños, Costa Rica, Dec. 2023.

[42] F. A. Khattak, I. K. Proudler, and S. Weiss.
Scalable analytic eigenvalue extraction algorithm.
IEEE Access, 12:166652–166659, Dec. 2024.

[43] F. A. Khattak, I. K. Proudler, and S. Weiss.
Scalable approach for analytic polynomial subspace projection matrices for a space-time covariance matrix.
In *IEEE High Performance Extreme Computing Conf.*, Boston, MA, Sept. 2024. IEEE.

[44] F. A. Khattak, S. Weiss, M. A. Bakhit, and I. K. Proudler.
Improved polynomial music algorithm for low-complexity and high-accuracy broadband aoa estimation.
In *23rd IEEE Statistical Signal Processing Workshop*, Edinburgh, Scotland, June 2025.

[45] F. A. Khattak, S. Weiss, and I. K. Proudler.
Fast givens rotation approach to second order sequential best rotation algorithms.
In *International Conference in Sensor Signal Processing for Defence*, pages 40–44, Edinburgh, Scotland, September 2021.

[46] F. A. Khattak, S. Weiss, I. K. Proudler, and J. G. McWhirter.
Space-time covariance matrix estimation: Loss of algebraic multiplicities of eigenvalues.
In *56th Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, Oct. 2022.

[47] T. I. Laakso, V. Välimäki, M. Karjalainen, and U. K. Laine.
Splitting the Unit Delay.
IEEE Signal Processing Magazine, 13(1):30–60, January 1996.

[48] J. G. McWhirter.
An algorithm for polynomial matrix SVD based on generalised Kogbetliantz transformations.
In *18th European Signal Processing Conference*, pages 457–461, Aalborg, Denmark, August 2010.

References VII

[49] J. G. McWhirter, P. D. Baxter, T. Cooper, S. Redif, and J. Foster.
An EVD Algorithm for Para-Hermitian Polynomial Matrices.
IEEE Transactions on Signal Processing, 55(5):2158–2169, May 2007.

[50] J. G. McWhirter and Z. Wang.
A novel insight to the SBR2 algorithm for diagonalising para-hermitian matrices.
In *11th IMA Conference on Mathematics in Signal Processing*, Birmingham, UK, December 2016.

[51] A. Mertins.
MMSE Design of Redundant FIR Precoders for Arbitrary Channel Lengths.
IEEE Transactions on Signal Processing, 51(9):2402–2409, September 2003.

[52] A. A. Nagy and S. Weiss.
Channel equalisation of a MIMO FBMC/OQAM system using a polynomial matrix pseudo-inverse.
In *10th IEEE Workshop on Sensor Array and Multichannel Signal Processing*, April 2018.

[53] V. Neo, C. Evers, S. Weiss, and P. Naylor.
Signal compaction using polynomial EVD for spherical array processing with applications.
IEEE/ACM Transactions on Audio, Speech, and Language Processing, 31:3537–3549, 2023.

[54] V. W. Neo, E. d'Olne, A. H. Moore, and P. A. Naylor.
Fixed beamformer design using polynomial eigenvalue decomposition.
In *2022 International Workshop on Acoustic Signal Enhancement (IWAENC)*, pages 1–5, Sep. 2022.

[55] V. W. Neo, C. Evers, and P. A. Naylor.
Speech enhancement using polynomial eigenvalue decomposition.
In *2019 IEEE Workshop on Applications of Signal Processing to Audio and Acoustics (WASPAA)*, pages 125–129, New Paltz, NY, Oct 2019.

[56] V. W. Neo, C. Evers, and P. A. Naylor.
Enhancement of noisy reverberant speech using polynomial matrix eigenvalue decomposition.
IEEE/ACM Transactions on Audio, Speech, and Language Processing, 29:3255–3266, 2021.

References VIII

[57] V. W. Neo, C. Evers, and P. A. Naylor.
Speech dereverberation performance of a polynomial-evd subspace approach.
In *2020 28th European Signal Processing Conference (EUSIPCO)*, pages 221–225, 2021.

[58] V. W. Neo and P. A. Naylor.
Second order sequential best rotation algorithm with householder reduction for polynomial matrix eigenvalue decomposition.
In *IEEE International Conference on Acoustics, Speech and Signal Processing*, pages 8043–8047, Brighton, UK, May 2019.

[59] V. W. Neo, S. Weiss, S. W. McKnight, A. O. T. Hogg, and P. A. Naylor.
Polynomial eigenvalue decomposition-based target speaker voice activity detection in the presence of competing talkers.
In *17th International Workshop on Acoustic Signal Enhancement*, Bamberg, Germany, Sept. 2022.

[60] V. W. Neo, S. Weiss, and P. A. Naylor.
A polynomial subspace projection approach for the detection of weak voice activity.
In *Sensor Signal Processing for Defence Conference*, pages 1–5, London, UK, Sept. 2022.

[61] C. A. D. Pahalson, L. H. Crockett, and S. Weiss.
Detection of weak transient broadband signals using a polynomial subspace and likelihood ratio test approach.
In *32nd European Signal Processing Conference*, pages 1312–1316, Lyon, France, Aug. 2024.

[62] C. A. D. Pahalson and S. Weiss.
Broadband array time reversal method for jamming with protective constraints.
In *23rd IEEE Statistical Signal Processing Workshop*, Edinburgh, Scotland, June 2025.

[63] S. Redif, J. McWhirter, and S. Weiss.
Design of FIR paraunitary filter banks for subband coding using a polynomial eigenvalue decomposition.
IEEE Transactions on Signal Processings, 59(11):5253–5264, November 2011.

[64] S. Redif, S. Weiss, and J. McWhirter.
Sequential matrix diagonalization algorithms for polynomial EVD of parahermitian matrices.
IEEE Transactions on Signal Processing, 63(1):81–89, January 2015.

References IX

[65] S. Redif, S. Weiss, and J. McWhirter.
Relevance of polynomial matrix decompositions to broadband blind signal separation.
Signal Processing, 134:76–86, May 2017.

[66] S. Redif, S. Weiss, and J. G. McWhirter.
An approximate polynomial matrix eigenvalue decomposition algorithm for para-hermitian matrices.
In *11th IEEE International Symposium on Signal Processing and Information Technology*, pages 421–425, Bilbao, Spain, December 2011.

[67] F. Rellich.
Störungstheorie der Spektralzerlegung. III. Mitteilung. Analytische, nicht notwendig beschränkte Störung.
Mathematische Annalen, 116:555–570, 1939.

[68] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao, and M. Segev.
Phase retrieval with application to optical imaging: A contemporary overview.
IEEE Signal Processing Magazine, 32(3):87–109, May 2015.

[69] C. H. Ta and S. Weiss.
A design of precoding and equalisation for broadband MIMO systems.
In *Forty-First Asilomar Conference on Signals, Systems and Computers*, pages 1616–1620, Pacific Grove, CA, USA, Nov. 2007.

[70] C. H. Ta and S. Weiss.
A Jointly Optimal Precoder and Block Decision Feedback Equaliser Design With Low Redundancy.
In *15th European Signal Processing Conference*, pages 489–492, Poznan, Poland, September 2007.

[71] C. H. Ta and S. Weiss.
Shortening the order of paraunitary matrices in SBR2 algorithm.
In *6th International Conference on Information, Communications & Signal Processing*, pages 1–5, Singapore, Dec. 2007.

[72] M. Tohidian, H. Amindavar, and A. M. Reza.
A DFT-based approximate eigenvalue and singular value decomposition of polynomial matrices.
EURASIP Journal on Advances in Signal Processing, 2013(1):1–16, 2013.

References X

[73] P. Vaidyanathan.
Theory of optimal orthonormal subband coders.
IEEE Transactions on Signal Processing, 46(6):1528–1543, June 1998.

[74] P. P. Vaidyanathan.
Multirate Systems and Filter Banks.
Prentice Hall, Englewood Cliffs, 1993.

[75] P. P. Vaidyanathan and T. H. Chen.
Structure for Anticausal Inverses and Applications in Multirate Filter Banks.
IEEE Transaction on Signal Processing, 46(2):507–514, February 1998.

[76] M. Vetterli.
A theory of multirate filter banks.
IEEE Transactions on Acoustics Speech and Signal Processing, Vol.35(No.3):pp.356–372, March 1987.

[77] Z. Wang, J. G. McWhirter, J. Corr, and S. Weiss.
Multiple shift second order sequential best rotation algorithm for polynomial matrix EVD.
In *23rd European Signal Processing Conference*, pages 844–848, Nice, France, September 2015.

[78] S. Weiss, M. Alrmah, S. Lambotharan, J. McWhirter, and M. Kaveh.
Broadband angle of arrival estimation methods in a polynomial matrix decomposition framework.
In *IEEE 5th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, pages 109–112, Dec 2013.

[79] S. Weiss and G. Barbarino.
Relations between analytic spectral and singularvalue decompositions.
In *33rd European Signal Processing Conference*, Isola delle Femmine, Italy, Sept. 2025.

[80] S. Weiss, S. Bendoukha, A. Alzin, F. Coutts, I. Proudler, and J. Chambers.
MVDR broadband beamforming using polynomial matrix techniques.
In *23rd European Signal Processing Conference*, pages 839–843, Nice, France, September 2015.

References XI

[81] S. Weiss, C. Delaosa, J. Matthews, I. Proudler, and B. Jackson.
Detection of weak transient signals using a broadband subspace approach.
In *International Conference on Sensor Signal Processing for Defence*, pages 65–69, Edinburgh, Scotland, Sept. 2021.

[82] S. Weiss, N. J. Goddard, S. Somasundaram, I. K. Proudler, and P. A. Naylor.
Identification of broadband source-array responses from sensor second order statistics.
In *Sensor Signal Processing for Defence Conference*, pages 1–5, London, UK, December 2017.

[83] S. Weiss and M. D. Macleod.
Maximally smooth Dirichlet interpolation from complete and incomplete sample points on the unit circle.
In *IEEE International Conference on Acoustics, Speech, and Signal Processing*, Brighton, UK, May 2019.

[84] S. Weiss, J. Pestana, and I. K. Proudler.
On the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix.
IEEE Transactions on Signal Processing, 66(10):2659–2672, May 2018.

[85] S. Weiss, J. Pestana, I. K. Proudler, and F. K. Coutts.
Corrections to "on the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix".
IEEE Transactions on Signal Processing, 66(23):6325–6327, Dec 2018.

[86] S. Weiss, I. Proudler, F. Coutts, and J. Deeks.
Extraction of analytic eigenvectors from a parahermitian matrix.
In *International Conference on Sensor Signal Processing or Defence*, Edinburgh, UK, 2020.

[87] S. Weiss, I. Proudler, F. Coutts, and F. Khattak.
Eigenvalue decomposition of a parahermitian matrix: extraction of analytic eigenvectors.
IEEE Transactions on Signal Processing, 71:1642–1656, Apr. 2023.

[88] S. Weiss, I. K. Proudler, G. Barbarino, J. Pestana, and J. G. McWhirter.
On properties and structure of the analytic singular value decomposition.
IEEE Transactions on Signal Processing, 72:2260–2275, 2024.

References XII

[89] S. Weiss, I. K. Proudler, and F. K. Coutts.
Eigenvalue decomposition of a parahermitian matrix: extraction of analytic eigenvalues.
IEEE Transactions on Signal Processing, 69:722–737, 2021.

[90] S. Weiss, I. K. Proudler, F. K. Coutts, and J. Pestana.
Iterative approximation of analytic eigenvalues of a parahermitian matrix EVD.
In *IEEE International Conference on Acoustics, Speech and Signal Processing*, Brighton, UK, May 2019.

[91] S. Weiss, I. K. Proudler, and M. D. Macleod.
Measuring smoothness of real-valued functions defined by sample points on the unit circle.
In *Sensor Signal Processing in Defence Conference*, Brighton, UK, May 2019.

[92] S. Weiss, S. Redif, T. Cooper, C. Liu, P. Baxter, and J. McWhirter.
Paraunitary oversampled filter bank design for channel coding.
EURASIP Journal on Advances in Signal Processing, 2006:1–10, 2006.

[93] S. Weiss, S. Schlecht, O. Das, and E. De Sena.
Polynomial Procrustes problem: Paraunitary approximation of matrices of analytic functions.
In *31st European Signal Processing Conference*, pages 1629–1633, Helsinki, Finland, 2023.

[94] S. Weiss, S. J. Schlecht, and M. Moonen.
Best least squares paraunitary approximation: Analytic procrustes problem.
IEEE Transactions on Signal Processing, 2025.
submitted.

[95] S. Weiss, S. J. Schlecht, and M. Moonen.
Polynomial procrustes method for randomly perturbed near-paraunitary systems.
In *23rd IEEE Statistical Signal Processing Workshop*, Edinburgh, Scotland, June 2025.

[96] S. Weiss, J. Selva, and M. D. Macleod.
Measuring smoothness of trigonometric interpolation through incomplete sample points.
In *28th European Signal Processing Conference*, pages 2319–2323, Amsterdam, Netherlands, Jan. 2021.