Aperiodically Delay Intermittent Control for Lévy Noise driven Highly Nonlinear Stochastic Hybrid Systems based on Discrete-Time Observations

Wenrui Li, Henglei Xu, Chen Fei, Xuerong Mao, and Weiyin Fei

Abstract— To tackle the challenge of effectively stabilizing an inherently unstable highly nonlinear stochastic hybrid system while simultaneously minimizing costs, this paper introduces a novel control strategy termed aperiodically delayed intermittent control (ADIC), grounded in discrete-time observations. Diverging from the conventional stochastic hybrid system driven by Brownian motion, we adopt Lévy noise to characterize the random perturbations, which possess the capacity to capture the discontinuous, heavy tail, and peak pulse features associated with random jumps. Leveraging the Lyapunov functional approach and the strong ergodicity theory of Markov chains, we establish the mean-square exponential stability of a controlled highly nonlinear stochastic hybrid system subjected to Lévy noise. Furthermore, the range of the average time rate for ADIC, and the range of duration separating two consecutive observations are given. One particular note is our introduction of a novel technique for estimating the difference between the current state (or mode) and the discrete-time state (or mode). This innovation leads to less stringent conditions imposed on both the underlying system and the control function. Finally, we apply our theoretical findings to the modified FitzHugh-Nagumo models and the stochastic Gilpin-Ayala model.

Index Terms— Aperiodically delayed intermittent control, Highly nonlinear, Stochastic hybrid systems, Lévy noise, Exponential stability.

I. Introduction

Highly nonlinear stochastic hybrid systems (HNSHSs) represent a sophisticated dynamic system characterized by coefficients demonstrating superlinear growth and the concurrent manifestation of both continuous and discrete behaviors. This system provides a more realistic description of instances that experience abrupt structural and parametric changes in different scientific and industrial fields (see, e.g., [1]–[3]).

Due to the widespread application of control theory in engineering, which significantly contributes to improving efficiency, reducing costs, and optimizing resource utilization, and thereby driving socio-economic development, the stability of HNSHSs has emerged as one of the most captivating research areas, garnering significant attention and investigation from scholars (see, e.g., [4]–[7] and references therein). Among others, in [8], Fei et al. studied the following unstable HNSHSs

$$dZ(t) = f(Z(t), \Lambda(t), t)dt + g(Z(t), \Lambda(t), t)dB(t), \quad (1)$$

where $Z(t) \in \mathbb{R}^n$, $\Lambda(t)$ is a Markov chain taking values in $\mathbb{S} = \{1, \cdots, M\}$, $B(t)$ is a Brownian motion. To stabilize the given unstable system (1), they first devised a feedback control $u(Z([t/\delta]\delta), \Lambda(t), t)$ based on the discrete-time observations of the state $Z(t)$ at times $0, \delta, 2\delta, \cdots$ for the controlled system

$$\begin{aligned} dZ(t) =& \big[f(Z(t), \Lambda(t), t) + u(Z([t/\delta]\delta), \Lambda(t), t)\big]dt \\ &+ g(Z(t), \Lambda(t), t)dB(t) \end{aligned} \quad (2)$$

to achieve stability. Here $[t/\delta]$ denotes the integer part of t/δ. Subsequent research, specifically [9], [10], has further expanded upon the findings of [8]. It is noteworthy that the control functions employed in these systems are grounded solely on the observations of the system state $Z([t/\delta]\delta)$. Nonetheless, from a practical standpoint, it is occasionally imperative to formulate the control function not only based on the state $Z([t/\delta]\delta)$ but also on the system mode $\Lambda([t/\delta]\delta)$ (see, e.g. [11]–[14]). In a recent study, Li et al. [15] formulated a feedback control strategy predicated on the historical discrete-time state pair $(Z([t/\delta]\delta - \tau), \Lambda([t/\delta]\delta - \tau))$. Specifically, they designed $u(Z(\eta(t)), \Lambda(\eta(t)), t)$, where $\eta(t) := [t/\delta]\delta - \tau$ with $\delta > 0$ denoting the duration between two successive observations, and τ representing the time delay between the moment the system state is observed and the point at which the feedback control influences the system. They further advanced a stabilization theory tailored for unstable quasi-linear and highly nonlinear stochastic hybrid systems.

This work was supported in part by the National Natural Science Foundation of China (62273003,72301173), the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship). (Corresponding author: Chen Fei and Weiyin Fei.)

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Considerable advancements have been achieved in the stabilization theory concerning the design of control functions for unstable HNSHSs. To enhance the practicality and applicability of the theory, the following three issues require further research and discussion.

Q1. Random perturbations are characterized by Lévy noise.

Due to its excellent mathematical properties, such as continuity and the tendency of statistical behavior to follow a normal distribution, along with its high degree of randomness, Brownian motion is widely employed to describe random phenomena in the natural world (see, e.g., [16], [17]). However, numerous real systems are susceptible to random, jump-type disturbances, exemplified by disruptions in communication networks [18], the abrupt and irregular evolution of vortices in turbulence [19], exciton and charge transport in conformational motion in a polymer [20]. In such instances, models relying solely on Brownian motion fail to capture the complexities of reality. To construct more realistic systems, Lévy noise (LN) has been integrated into stochastic systems (see, e.g., [21]–[25]). The issue of stabilizing stochastic hybrid systems with LN has attracted widespread attention among researchers, as evidenced by references such as [26]–[29]. To the best of our knowledge, the systems examined in this context are inherently linear, and the control function does not concurrently rely on both the discrete-time state observations and the discrete-time mode observations.

Our objective is to integrate LN into HNSHSs. This endeavor is made to improve the description of real-world dynamics impacted by abrupt events, thereby furnishing a more exhaustive and authentic framework for analysis and simulation.

Q2. The control function is ADIC.

Despite the beneficial effects of employing feedback control based on discrete-time observation to stabilize stochastic hybrid systems, the necessity for the control function to remain active throughout the entire control period poses limitations, demanding real-time high-frequency feedback information. Designing feedback control for the entire duration is inherently intricate and involves higher costs and resource demands. However, selectively activating feedback control during specific system conditions or triggers and deactivating it during other intervals has proven to be a more feasible approach, leading to cost reduction. ADIC meet with this requirement, which considers the impact of delays and offering potential advantages in terms of cost-effectiveness and efficiency improvement.

According to the currently results, intermittent control has extensively been employed to investigate the stabilization of linear stochastic hybrid systems (see, e.g., [30]–[32]). While the primary objective of this paper is to design ADIC based on discrete-time observations to stabilize highly nonlinear stochastic hybrid systems with Lévy noise (HNSHSs-LN) that are inherently unstable.

Q3. A condition of greater leniency for the estimation of discrete-time observation.

Lastly, it is imperative to emphasize that many results on the discrete feedback stabilization problem rely significantly on estimating the difference between the current state $Z(t)$ (or mode $\Lambda(t)$) and the discrete-time state $Z([t/\delta]\delta)$ (or mode $\Lambda([t/\delta]\delta)$) (see, e.g., [8], [9], [11], [15], [33]–[35]). This difference arises directly from the influence of discrete-time observations. Presently, a widely adopted approach is to compute $\mathbb{E}|Z(t)-Z([t/\delta]\delta)|^2$ and $\mathbb{E}|G(\Lambda(s))-G(\Lambda([t/\delta]\delta))|^2$ (the details of $G(\cdot)$ will be given in Assumption 3.2), with the expectation that the term $\mathbb{E}|Z(t)-Z([t/\delta]\delta)|^2$ and $\mathbb{E}|G(\Lambda(s))-G(\Lambda([t/\delta]\delta))|^2$ could be sufficiently minimized if the observation duration δ is suitably small. However, this approach leads to some stringent conditions. For instance, in [11], to estimate the difference between the current mode $\Lambda(t)$ and the discrete-time mode $\Lambda([t/\delta]\delta)$, the authors inevitably require the existence of a constant ε_δ depending on δ such that

$$\mathbb{E}|G(\Lambda(s))-G(\Lambda([t/\delta]\delta))|^2\leq\varepsilon_\delta.$$

In order to control the value of $\mathbb{E}|G(\Lambda(s))-G(\Lambda([t/\delta]\delta))|^2$, they need to impose more conditions on δ (see [11, Theorem III.3]). In [15], to control the deviation from time delay in mean square i.e., the value of $\mathbb{E}|Z(t)-Z(\eta(t))|^2$, the authors inevitably require the following two conditions:

$$z^\top f(z,t,i)+\frac{1}{2}|g(z,t,i)|^2\leq\beta_1(i)|z|^2-\beta_2(i)|z|^{r+1},\quad(3)$$

$$z^\top f(z,t,i)+\frac{r}{2}|g(z,t,i)|^2\leq\bar{\beta}_1(i)|z|^2-\bar{\beta}_2(i)|z|^{r+1},\quad(4)$$

with $r>1$. Is it feasible for us to adjust the estimation so that these conditions are relaxed? In this paper, we will provide an affirmative response to this inquiry (see Assumption 3.1, Lemma 3.1 and Lemma 3.2 below).

This paper is dedicated to addressing the aforementioned three issues. Theoretically, the conditions imposed on the original system and the control function are less stringent. In practice, this approach not only extends coverage to a broader class of stochastic systems but also results in a substantial reduction in control costs. We emphasize several key contributions of this paper in comparison to existing works:

- In this paper, we, for the first time, formulate an ADIC utilizing discrete-time observations of states and modes to stabilize an unstable HNSHSs-LN.
- A novel estimation for the difference between the current state $Z(t)$ (or mode $\Lambda(t)$) and discrete-time state $Z(\eta(t))$ (or mode $\Lambda(\eta(t))$) is introduced, aiming to relax conditions and simplify the analysis.
- The lower bound on δ^* and the range of the average control rate ρ are derived explicitly, ensuring the control function $u(Z(\eta(t)),\Lambda(\eta(t)),t)$ can stabilize unstable HNSHSs-LN under conditions $\delta+\tau<\delta^*$ with $\rho\in(\frac{-\psi_2}{\psi_1-\psi_2},1)$. The impact of the values of δ, τ, and ρ on the Lyapunov exponents is also investigated.
- To illustrate the practical application of the theoretical results, we consider the stabilization of modified FitzHugh-Nagumo models and the stochastic Gilpin-Ayala model.

To close this introduction, we would like to briefly describe the main challenges of this paper. Our first main challenge is to solve the problem of exponential stability of stochastic hybrid systems driven by Lévy noise under highly nonlinear conditions. To solve this problem, utilizing Lyapunov functionals (see (35)), we establish delay-dependent conditions (see (42)) through M-matrix, ensuring that HNSHSs-LN achieve exponential stability (see Theorem 3.1). Our second main challenge is to solve the aperiodic, discrete-time observations and time delay problems. To cope with the mixture of the continuous-time state $Z(t)$ and mode $\Lambda(t)$ and the discrete-time state $Z(\eta(t))$ and mode $\Lambda(\eta(t))$ in the same system, we have developed some new techniques (see Lemma 3.1 and 3.2). In addition, we apply the average control rate to construct an aperiodic delay intermittent control mechanism, which can relax the conservative (see Assumption 2.4).

II. Preliminary and model descriptions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. $A^\top$ is the transpose of a matrix or a vector A. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^\top A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Let $\mathbb{R}_+ = [0, +\infty)$. For a pair of real numbers a and b, we use $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. For any set B, denote by I_B its indicator function, that is, $I_B(\omega) = 1$ if $\omega \in B$ and 0 otherwise.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space satisfying the usual condition. Let $B(t)$ be an n-dimensional standard $\mathcal{F}_t$-adapted Brownian motion. Let $N(dt, dx)$ be an $\mathcal{F}_t$-adapted Poisson random measure defined on $\mathbb{R}_+ \times X$, where $X \subseteq \mathbb{R}^n - \{0\}$. The compensated Poisson random measure is denoted by $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$, where ν is a Lévy measure satisfying $\int_X (1 \wedge |x|)\nu(dx) < \infty$. Let $\Lambda(t)$, $t \geq 0$, be a right-continuous Markov chain with finite state space $\mathbb{S} = \{1, 2, \cdots, M\}$ on the probability space. The generator of $\{\Lambda(t)\}_{t\geq 0}$ is defined by $\Gamma = (\gamma_{il})_{M\times M}$, so that for a sufficiently small $\Delta > 0$,

$$\mathbb{P}\{\Lambda(t+\Delta) = l | \Lambda(t) = i\} = \begin{cases} \gamma_{il}\Delta + o(\Delta) & \text{if } i \neq l, \\ 1 + \gamma_{il}\Delta + o(\Delta) & \text{if } i = l. \end{cases}$$

Here γ_{il} is the transition rate from i to l satisfying $\gamma_{il} > 0$ if $i \neq l$ while $\gamma_{ii} = -\sum_{i\neq l} \gamma_{il}$. Here, we assume Γ is irreducible. For a sequence $\{q_i\}_{1\leq i\leq M}$, we will often write $q(i) = q_i$ and set $q = (q_1, \cdots, q_M)^\top$, define $\hat{q} = \min_{\{1\leq i\leq M\}} q_i$ and $\check{q} = \max_{\{1\leq i\leq M\}} q_i$. Denote by $\mathcal{G}$ the σ-algebra generated by $\{\Lambda(t)\}_{0\leq t<\infty}$. We denote the conditional expectation $\mathbb{E}(\cdot|\mathcal{G})$ by $\mathbb{E}_{\mathcal{G}}(\cdot)$. We also assume that $B(\cdot)$, $\tilde{N}(\cdot, \cdot)$ and $\Lambda(\cdot)$ are mutually independent.

Suppose that the underlying system is described by the HNSHSs-LN

$$\begin{aligned} dZ(t) =& f(Z(t), \Lambda(t), t)dt + g(Z(t), \Lambda(t), t)dB(t) \\ &+ \int_X h(Z(t^-), \Lambda(t), t, x)\tilde{N}(dt, dx) \end{aligned} \tag{5}$$

with the initial data $Z(0) = Z_0 \in \mathbb{R}^n$ and $\Lambda(0) = i_0 \in \mathbb{S}$, where the drift coefficient $f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$, the diffusion coefficient $g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, the jump coefficient $h : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \times X \to \mathbb{R}^n$ are Borel measurable functions. This type of model has been extensively studied, for details see the literature [27], [36], [37]. In this paper, we consider the HNSHSs-LN (5) as characterized by coefficients exhibiting superlinear growth. Consequently, we establish local Lipschitz conditions and polynomial growth conditions for the drift f, diffusion g, and jump coefficients h. These conditions are detailed in the ensuing assumption.

Assumption 2.1: For any positive real number R, there exists a nonnegative number K_R such that

$$\begin{aligned} &|f(z, i, t) - f(\bar{z}, i, t)|^2 \vee |g(z, i, t) - g(\bar{z}, i, t)|^2 \\ &\vee \int_X |h(z, i, t, x) - h(\bar{z}, i, t, x)|^2 \nu(dx) \leq K_R |z - \bar{z}|^2 \end{aligned} \tag{6}$$

for all $(t, i) \in \mathbb{R}_+ \times \mathbb{S}$, z, $\bar{z} \in \mathbb{R}^n$ with $|z| \vee |\bar{z}| \leq m$. Moreover, $f(0, i, t) = 0$, $g(0, i, t) = 0$ and $h(0, i, t, x) = 0$ for any $t \geq 0$. Assume also that there exist six positive constants K_j $(j = 1, \ldots, 6)$ and $r > 1$ such that

$$\begin{aligned} |f(z, i, t)| \leq& K_1|z| + K_2|z|^r, \\ |g(z, i, t)|^2 \leq& K_3|z|^2 + K_4|z|^{r+1}, \\ \int_X |h(z, i, t, x)|^2 \nu(dx) \leq& K_5|z|^2 + K_6|z|^{r+1}. \end{aligned} \tag{7}$$

This assumption, while necessary, is insufficient to guarantee the existence of a unique global solution for the HNSHSs-LN (5). We therefore introduce an additional Khasminskii-type condition for the coefficients.

Assumption 2.2: Assume that there exist positive constants $p \geq r + 1$ and α such that

$$\begin{aligned} &|z|^{p-2}\big(z^\top f(z, i, t) + \frac{p-1}{2}|g(z, i, t)|^2\big) \\ &+ \int_X \big(|z + h(z, i, t, x)|^p - |z|^p \\ &- p|z|^{p-2} z^\top h(z, i, t, x)\big)\nu(dx) \leq \alpha|z|^p. \end{aligned} \tag{8}$$

Under Assumptions 2.1 and 2.2, the HNSHSs-LN (5) possesses a unique global solution denoted as $Z(t)$, satisfying the condition $\sup_{0\leq s\leq t} \mathbb{E}|Z(s)|^p < \infty$ for any initial data $Z(0) = Z_0 \in \mathbb{R}^n$. The proof of this assertion can be established by employing a reasoning analogous to that presented in [16].

Assuming the HNSHSs-LN (5) does not have the desired stability property, it becomes imperative to implement feedback control mechanisms to stabilize the unstable system (5). To enhance adaptability to system changes and capitalize on potential benefits related to cost-effectiveness and efficiency improvement, we design ADIC based on discrete-time observations as follows:

$$u(Z(\eta(t)), \Lambda(\eta(t)), t) = \bar{u}(Z(\eta(t)), \Lambda(\eta(t)), t)R(t), \tag{9}$$

where $R(t) = \sum_{k=0}^{\infty} I_{[t_k, s_k)}(t)$ and $\eta(t) = [t/\delta]\delta - \tau$, $\delta > 0$ denotes the duration between two successive observations, τ represents the time delay between the moment when the system state is observed and the point at which the feedback control influences the system, t_k, s_k are intermittent instants, $t_{k+1} - s_k$ is the rest width and $s_k - t_k$

is the work width. Let $\inf_k(s_k - t_k)$ denote the minimum proportion of the work width such that $\inf_k(s_k - t_k) > \delta$. This means that at least one discrete observation occurs within the control interval. It is evident that the aperiodic delay control function $u(Z(\eta(t)), \Lambda(\eta(t)), t)$ manifests periodicity under the conditions $t_k = kT_0$ and $s_k = (k+\theta)T_0$, where $T_0 > 0$ denotes the control period, and $\theta \in (0, 1)$ signifies the control rate.

For the control function $\bar{u}$, the following assumption must be satisfied.

Assumption 2.3: Assume that there exists a positive constant L such that for any $(z, \bar{z}, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$,

$$|\bar{u}(z, i, t) - \bar{u}(\bar{z}, i, t)| \leq L|z - \bar{z}|, \tag{10}$$

with $\bar{u}(0, i, t) \equiv 0$.

Next, we present the concept of the average control rate, which characterizes the distribution of control intervals in aperiodically intermittent control. This metric provides a comprehensive representation of the temporal patterns associated with control actions, revealing the subtle dynamics of intermittent control strategies. Using wind power generation as an example, it is typically completely aperiodic and intermittent due to the irregularity of wind patterns. Demanding a fixed duration for each wind time is often impractical. However, we can always approximate the average duration of wind within a unit of time. Therefore, considering the aperiodicity intermittent control from an average perspective holds greater significance.

Assumption 2.4: [30] For any $t > s \geq 0$, there are constants $\rho \in (0, 1)$ and $S_0 \geq 0$ such that

$$S(t, s) \geq \rho(t - s) - S_0, \qquad \forall t > s \geq 0, \tag{11}$$

where $S(t, s)$ denotes the total control interval length on $[s, t)$ and S_0 is called the elasticity number. In this regard, aperiodically intermittent control in (9) is said to have an average control rate $\rho \in (0, 1)$.

Remark 2.1: Let $S^c(t, s)$ stand for the total rest interval length on $[s, t)$. Clearly, $S(t, s) + S^c(t, s) = t - s$. Then, it is easy to obtain from Assumption 2.4 that

$$S^c(t, s) = (t - s) - S(t, s) \leq (1 - \rho)(t - s) + S_0. \tag{12}$$

To illustrate the mechanism of aperiodically intermittent control, Fig. 1 is presented. One can see that the lower bound of control intervals $[i.e., \inf_m(s_m - t_m)]$ is a, which can be very small. Additionally, the upper bound of control periods $[i.e., \sup_m(t_{m+1} - t_m)]$ is determined by $\max\{a + S_1, b + S_2\}$, potentially reaching a significant magnitude. The average control rate is given by $(a + S_2)/(a + b + S_1 + S_2)$, providing a comprehensive understanding of the control dynamics. Subsequently, in the numerical simulations, we also provide examples of Assumption 2.4 applies to controlled systems.

To incorporate ADIC $u(Z(\eta(t)), \Lambda(\eta(t)), t)$ based on discrete-time observations into the drift coefficient component of the unstable HNSHSs-LN (5) (It is possible to put the feedback control in the diffusion part but, in this paper, we will only consider the case where the control is put into the drift part), our controlled system takes the following form:

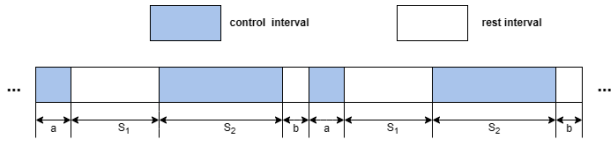

Fig. 1. Simple aperiodically intermittent control.

$$\begin{aligned} dZ(t) =& \big[f(Z(t), \Lambda(t), t) + u(Z(\eta(t)), \Lambda(\eta(t)), t)\big]dt \\ &+ g(Z(t), \Lambda(t), t)dB(t) \\ &+ \int_X h(Z(t^-), \Lambda(t), t, x)\tilde{N}(dt, dx) \end{aligned} \tag{13}$$

with the initial data

$$Z(t) = Z_0 \in \mathbb{R}^n, \ \Lambda(t) = i_0 \in \mathbb{S}, \ -\bar{\tau} \leq t \leq 0, \tag{14}$$

where $u : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ is Borel measurable, $\bar{\tau} = \delta + \tau$.

In what follows, we establish the existence and uniqueness of solution to the controlled system (13). Additionally, the moment finiteness of the solution is given.

Theorem 2.1: Let Assumptions 2.1-2.3 hold. The controlled system (13) with any initial data (14) has a unique global solution $Z(t)$. Moreover, the solution $Z(t)$ obeys

$$\sup_{-\bar{\tau} \leq s \leq t} \mathbb{E}|Z(s)|^p < \infty. \tag{15}$$

Proof: In fact, we notice that the controlled system (13) is a hybrid stochastic delay differential equation driven by Lévy noise, featuring a bounded variable delay. To elaborate, if we define the bounded variable delay $\omega : \mathbb{R}_+ \to [\tau, \delta + \tau]$ by

$$\omega(t) = t - m\delta + \tau, \quad m\delta \leq t \leq (m + 1)\delta, \ m = 0, 1, 2, \cdots,$$

then the controlled system (13) can be reexpressed as

$$\begin{aligned} dZ(t) =& [f(Z(t), \Lambda(t), t) + u(Z(t - \omega(t)), \Lambda(t - \omega(t)), t)]dt \\ &+ g(Z(t), \Lambda(t), t)dB(t) \\ &+ \int_X h(Z(t^-), \Lambda(t), t, x)\tilde{N}(dt, dx) \end{aligned} \tag{16}$$

on $t \geq 0$. Let $Q(z) = |z|^p$, it follows from the Itô formula and (16) that

$$\begin{aligned} dQ(Z(t)) =& LQ(Z(t), Z(t - \omega(t)), \Lambda(t), \Lambda(t - \omega(t)), t)dt \\ &+ p|Z(t)|^{p-2}Z^\top(t)g(Z(t), \Lambda(t), t)dB(t) \\ &+ \int_X \big[|Z(t^-) + h(Z(t^-), \Lambda(t), t, x)|^p \\ &\quad - |Z(t^-)|^p\big]\tilde{N}(dt, dx), \end{aligned}$$

where $LQ : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\begin{aligned} &LQ(z, \bar{z}, i, \bar{i}, t) \\ =& p|z|^{p-2}z^\top\big[f(z, i, t) + u(\bar{z}, \bar{i}, t)\big] + \frac{p}{2}|z|^{p-2}|g(z, i, t)|^2 \\ &+ \frac{p(p-2)}{2}|z|^{p-4}|z^\top g(z, i, t)|^2 + \int_X \big[|z + h(z, i, t, x)|^p \\ &- |z|^p - p|z|^{p-2}z^\top h(z, i, t, x)\big]\nu(dx). \end{aligned}$$

By Assumptions 2.2, 2.3 and the well-known Young inequality, we get

$$\begin{aligned}&LQ(z,\bar{z},i,\bar{i},t)\\ \leq&p|z|^{p-2}\big[z^\top[f(z,i,t)+u(\bar{z},\bar{i},t)]+\frac{p-1}{2}|g(z,i,t)|^2\big]\\ &+\int_X\big[|z+h(z,i,t,x)|^p-p|z|^{p-2}z^\top h(z,i,t,x)\\ &-|z|^p\big]\nu(dx)\leq\big(\alpha+(p-1)L\big)|z|^p+L|\bar{z}|^p.\end{aligned}$$

The remaining proof is similar to that in [8] and is thus omitted. ∎

III. ADIC STABILIZATION PROBLEM

In the above section, we developed the ADIC based on discrete-time observations and deduced the controlled HNSHSs-LN (13). Moreover, we systematically established the existence, uniqueness, and moment finiteness of solutions for (13). In this section, our central focus is on examining the mean-square exponential stability of the controlled HNSHSs-LN (13). Additionally, we discuss the range of the average time rate for ADIC as well as the duration separating two consecutive observations.

To advance our inquiry, it is imperative to introduce additional constraints governing both the coefficients and the control function.

Assumption 3.1: Assume that there exist positive constants r, $\beta_1(i)$ and $\beta_2(i)$ such that

$$\begin{aligned}&z^\top f(z,i,t)+\frac{1}{2}|g(z,i,t)|^2+\int_X|h(z,i,t,x)|^2\nu(dx)\\ \leq&\beta_1(i)|z|^2-\beta_2(i)|z|^{r+1}.\end{aligned}\tag{17}$$

Remark 3.1: For $Q3$ posed in the introduction, we now provide a positive answer. As indicated by Assumption 3.1, in contrast to the requirement for two conditions in existing literature, this paper only necessitates one condition.

Assumption 3.2: Assume that there exists a positive constant $G(i)$ such that

$$z^\top\bar{u}(z,i,t)\leq-G(i)|z|^2.\tag{18}$$

For simply, write $G_i=G(i)$, $\beta_{1i}=\beta_1(i)$, $\beta_{2i}=\beta_2(i)$ and $c_i=c(i)$. We also assume that $\mathcal{B}=-2\text{diag}\{\beta_{11}-G_1,\cdots,\beta_{1M}-G_M\}-\Gamma$ is a nonsingular M-matrix. Set $(c_1,\cdots,c_M)^\top=\mathcal{B}^{-1}(1,\cdots,1)^\top$. As $\mathcal{B}$ is a non-singular M-matrix, all c_i are positive.

Remark 3.2: Under Assumption 3.1, a myriad of control functions can be devised to meet the criteria outlined in Assumption 2.3 and Assumption 3.2. For instance, one may option to design the control function in a linear form, such as $\bar{u}(z,i,t)=-2\beta_1(i)z$. Consequently, Assumption 2.3 is fulfilled with $L=2\max_{i\in\mathbb{S}}\{\beta_1(i)\}$, Assumption 3.2 is valid with $G_i=2\beta_1(i)$, while $\mathcal{B}=2\text{diag}\{\beta_{11},\cdots,\beta_{1M}\}-\Gamma$ is a nonsingular M-matrix.

In the investigation of discrete state feedback stabilization problems, it becomes essential to assess the disparity between the current state, denoted as $Z(t)$, and the state at discrete time instances, represented by $Z(\eta(t))$. This disparity is expressed mathematically as $|Z(t)-Z(\eta(t))|$. We formally articulate this assessment as the following Lemma.

Lemma 3.1: Under Assumptions 2.1 and 2.3, we can conclude that for any $t\geq0$ and $\bar{\tau}<\frac{1}{L}$,

$$\begin{aligned}&\mathbb{E}_{\mathcal{G}}[|Z(t)||Z(\eta(t))-Z(t)|]\\ \leq&\int_{t-\bar{\tau}}^t\big(\phi_1\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\phi_2\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds\\ &+\phi_3\mathbb{E}_{\mathcal{G}}|Z(t)|^2+\phi_4\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1},\end{aligned}\tag{19}$$

where $\phi_1=\frac{1}{1-L\bar{\tau}}(\frac{K_1}{2}+\frac{K_3+K_5}{2\varepsilon})$, $\phi_2=\frac{1}{1-L\bar{\tau}}(\frac{K_2r}{r+1}+\frac{K_4+K_6}{2\varepsilon})$, $\phi_3=\frac{1}{1-L\bar{\tau}}(\varepsilon+\frac{(K_1+2L)\bar{\tau}}{2})$, $\phi_4=\frac{1}{1-L\bar{\tau}}\frac{K_2\bar{\tau}}{r+1}$ and $\varepsilon>0$.

Proof: For any $t\geq0$, there is a unique integer $m\geq0$ such that $t\in[m\delta,(m+1)\delta)$. Moreover, $\eta(s)=m\delta-\tau$ and $Z([s/\delta]\delta-\tau)=Z(m\delta-\tau)$ for any $s\in[m\delta,t]$. To simplify, we denote $H(t)=|Z(t)||Z(t)-Z(m\delta-\tau)|$. From (13), we derive that

$$\begin{aligned}H(t)=&|Z(t)|\Big|\int_{m\delta-\tau}^t\big[f(Z(s),\Lambda(s),s)\\ &+u(Z(\eta(s)),\Lambda(\eta(s)),s)\big]ds\\ &+\int_{m\delta-\tau}^t g(Z(s),\Lambda(s),s)dB(s)\\ &+\int_{m\delta-\tau}^t\int_X h(Z(s^-),\Lambda(s),s,x)\tilde{N}(ds,dx)\Big|\\ \leq&|Z(t)|\Big(|H_1|+\int_{m\delta-\tau}^t\big(|f(Z(s),\Lambda(s),s)|\\ &+|u(Z(\eta(s)),\Lambda(\eta(s)),s)|\big)ds+|H_2|\Big),\end{aligned}\tag{20}$$

where $H_1=\int_{m\delta-\tau}^t g(Z(s),\Lambda(s),s)dB(s)$ and $H_2=\int_{m\delta-\tau}^t\int_X h(Z(s^-),\Lambda(s),s,x)\tilde{N}(ds,dx)$. According to Assumptions 2.1, 2.3, and by introducing the free positive number ε, we have

$$\begin{aligned}H(t)\leq&|Z(t)|\Big(\int_{m\delta-\tau}^t\big(K_1|Z(s)|+K_2|Z(s)|^r\\ &+L|Z(m\delta-\tau)|\big)ds+|H_1|+|H_2|\Big)\\ \leq&\int_{m\delta-\tau}^t\big(|Z(t)|(K_1|Z(s)|+K_2|Z(s)|^r)\\ &+L|Z(t)||Z(m\delta-\tau)|\big)ds\\ &+\varepsilon|Z(t)|^2+\frac{1}{2\varepsilon}|H_1|^2+\frac{1}{2\varepsilon}|H_2|^2\\ \leq&\int_{m\delta-\tau}^t\big(\frac{K_1}{2}|Z(s)|^2+\frac{K_2r}{r+1}|Z(s)|^{r+1}\big)ds\\ &+L\bar{\tau}|Z(t)||Z(t)-Z(m\delta-\tau)|\\ &+\big(\varepsilon+\frac{(K_1+2L)\bar{\tau}}{2}\big)|Z(t)|^2\\ &+\frac{K_2\bar{\tau}}{r+1}|Z(t)|^{r+1}+\frac{1}{2\varepsilon}(|H_1|^2+|H_2|^2).\end{aligned}\tag{21}$$

Taking expectations on both sides and using the Itô isometry formula (see Lemma 4.2.2 in [21] or Theorem 1.17 in [22]), along with $\bar{\tau}<\frac{1}{L}$, we take a further step to

obtain

$$\begin{aligned}
&\mathbb{E}_{\mathcal{G}}H(t)\\
\leq&\frac{1}{1-L\bar{\tau}}\Big(\int_{m\delta-\tau}^{t}\big(\frac{K_1}{2}\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\frac{K_2 r}{r+1}\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds\\
&+\big(\varepsilon+\frac{(K_1+2L)\bar{\tau}}{2}\big)\mathbb{E}_{\mathcal{G}}|Z(t)|^2+\frac{K_2\bar{\tau}}{r+1}\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}\\
&+\frac{1}{2\varepsilon}\int_{m\delta-\tau}^{t}\mathbb{E}_{\mathcal{G}}|g(Z(s),\Lambda(s),s)|^2ds\\
&+\frac{1}{2\varepsilon}\int_{m\delta-\tau}^{t}\int_{X}\mathbb{E}_{\mathcal{G}}|h(Z(s^-),\Lambda(s),s,x)|^2\nu(dx)ds\Big)\\
\leq&\frac{1}{1-L\bar{\tau}}\Big(\int_{m\delta-\tau}^{t}\big(\frac{K_1}{2}\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\frac{K_2 r}{r+1}\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds\\
&+\big(\varepsilon+\frac{(K_1+2L)\bar{\tau}}{2}\big)\mathbb{E}_{\mathcal{G}}|Z(t)|^2+\frac{K_2\bar{\tau}}{r+1}\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}\\
&+\frac{1}{2\varepsilon}\int_{m\delta-\tau}^{t}\big(K_3\mathbb{E}_{\mathcal{G}}|Z(s)|^2+K_4\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds\\
&+\frac{1}{2\varepsilon}\int_{m\delta-\tau}^{t}\big(K_5\mathbb{E}_{\mathcal{G}}|Z(s)|^2+K_6\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds\Big)\\
=&\int_{m\delta-\tau}^{t}\big(\phi_1\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\phi_2\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds\\
&+\phi_3\mathbb{E}_{\mathcal{G}}|Z(t)|^2+\phi_4\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}\\
\leq&\int_{t-\bar{\tau}}^{t}\big(\phi_1\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\phi_2\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds\\
&+\phi_3\mathbb{E}_{\mathcal{G}}|Z(t)|^2+\phi_4\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1},
\end{aligned}\tag{22}$$

where, in the last step, we use $t-\delta\leq m\delta$. The proof is hence complete. ∎

Remark 3.3: Traditionally, the difference between the current state $Z(t)$ and the discrete-time state $Z(\eta(t))$ is estimated by computing $\mathbb{E}_{\mathcal{G}}|Z(t)-Z(\eta(t))|^2$. The expectation is that the term $\mathbb{E}_{\mathcal{G}}|Z(t)-Z(\eta(t))|^2$ would be sufficiently small if the observation duration δ is small. However, this conventional approach occasionally imposes unwarranted constraints (such as (4)) and complicates the selection of Lyapunov functionals later on. The new estimation method we propose in (19) relaxes this limitation.

In order to further control the derivation for $\mathbb{E}_{\mathcal{G}}|Z(t)||Z(\eta(t))-Z(t)|$, i.e. the value of $\int_{t-\bar{\tau}}^{t}\big(\phi_1\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\phi_2\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds+\phi_3\mathbb{E}_{\mathcal{G}}|Z(t)|^2+\phi_4\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}$, we define an auxiliary functional

$$\Phi(\bar{Z}_t,t)=b_3\int_{t-\bar{\tau}}^{t}\int_{s}^{t}\big[b_1|Z(v)|^2+b_2|Z(v)|^{r+1}\big]dvds,\tag{23}$$

where $\bar{Z}_t:=\{Z(t+\theta):-\bar{\tau}\leq\theta\leq 0\}$, $\check{c}=\max_{\{1\leq i\leq M\}}c_i$, $\bar{\tau}<\frac{1}{L}$, $b_3=\frac{\check{c}L}{1-L\bar{\tau}}$, b_1 and b_2 are two free positive numbers. Define $Z(\theta)=Z_0$ for $\theta\in[-\bar{\tau},-\tau)$, so that $\bar{Z}_t$ have a clear definition on $0\leq t<\bar{\tau}$. A direct calculation arrives at

$$d\Phi(\bar{Z}_t,t)=\Phi_1(t)dt-\Phi_2(t)dt,\tag{24}$$

where

$$\Phi_1(t)=b_3\bar{\tau}\big(b_1|Z(t)|^2+b_2|Z(t)|^{r+1}\big),\tag{25}$$

$$\Phi_2(t)=b_3\int_{t-\bar{\tau}}^{t}\big(b_1|Z(s)|^2+b_2|Z(s)|^{r+1}\big)ds.\tag{26}$$

Remark 3.4: From (26), we can see that the integrals of $|Z(s)|^2$ and $|Z(s)|^{r+1}$, are the same as the first two terms on the right-hand-side of (19). This is the reason why we establish the functional in the form of (23). Especially in the later proof, we will choose the appropriate parameters b_1, b_2 and b_3 in (23) to eliminate the influence of the integrals of $|Z(s)|^2$ and $|Z(s)|^{r+1}$ in (19).

On the other hand, according to (13), it is evident that the ADIC relies on the Markov chain $\Lambda(\eta(t))$. It is crucial to emphasize that in the Markov jump process, when a jump occurs at time t, $\Lambda(t)$ and $\Lambda(t^-)=\lim_{s\uparrow t}\Lambda(t)$ are distinguishable. Furthermore, it is noteworthy that diminishing the disparity between $\Lambda(\eta(t))$ and $\Lambda(t)$ by reducing τ to infinitesimal values is not a feasible approach. To address this challenge, we must introduce several new notations and Lemma.

First we redefine the Markov chains. Let $n_0=[\frac{\tau}{\delta}]$, $\delta_0=(n_0+1)\delta-\tau$, $\tilde{\Lambda}(t):=\Lambda(t+(n_0+1)\delta)$ for $t\geq 0$. Obviously $\Lambda(n\delta)$ is a skeleton process of Markov chain $\{\Lambda(t)\}_{t\geq 0}$ for any integer $n\geq 0$, which is a discrete-time homogeneous Markov chain on $\mathbb{S}$. To highlight the initial values, we let $\{\Lambda^i(t)\}_{t\geq 0}$ be the Markov chains starting from state i at $t=0$.

Lemma 3.2: Let $G=(G_1,\cdots,G_M)^\top$. For any constant $q>0$, there is a positive constant $K_7=e^{\bar{\tau}\bar{\gamma}(e^{4q\bar{\tau}\check{G}}-1)}$ such that for any $t\geq 0$

$$\mathbb{E}\big(e^{q\int_0^t|G(\Lambda(u+\tau))-G(\Lambda([\frac{u}{\delta}]\delta))|du}\big)\leq K_7e^{\bar{\gamma}(e^{4q\bar{\tau}\check{G}}-1)t},\tag{27}$$

where $\bar{\gamma}=\max_{i\in\mathbb{S}}\{-\gamma_{ii}\}$ and $\check{G}=\max_{i\in\mathbb{S}}G_i$.

Proof: By the Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}\big(e^{q\int_0^t|G(\Lambda(u+\tau))-G(\Lambda([\frac{u}{\delta}]\delta))|du}\big)\\
=&\mathbb{E}\big(e^{q\int_0^t|G(\Lambda(u+\tau))-G(\Lambda(u))+G(\Lambda(u))-G(\Lambda([\frac{u}{\delta}]\delta))|du}\big)\\
\leq&\Big(\mathbb{E}\big(e^{2q\int_0^t|G(\Lambda(u+\tau))-G(\Lambda(u))|du}\big)\Big)^{\frac{1}{2}}\\
&\times\Big(\mathbb{E}\big(e^{2q\int_0^t|G(\Lambda(u))-G(\Lambda([\frac{u}{\delta}]\delta))|du}\big)\Big)^{\frac{1}{2}}.
\end{aligned}\tag{28}$$

For any nonnegative integer k, using the Jensen inequality, we know that

$$\begin{aligned}
&\mathbb{E}\big(e^{2q\int_0^t|G(\Lambda(u+\tau))-G(\Lambda(u))|du}\big)\\
\leq&\mathbb{E}\big(e^{2q\sum_{k=0}^{[\frac{t}{\tau}]}\int_{k\tau}^{(k+1)\tau}|G(\Lambda(u+\tau))-G(\Lambda(u))|du}\big)\\
=&\mathbb{E}\big(\prod_{k=0}^{[\frac{t}{\tau}]}e^{2q\int_{k\tau}^{(k+1)\tau}|G(\Lambda(u+\tau))-G(\Lambda(u))|du}\big)\\
\leq&\mathbb{E}\big(\prod_{k=0}^{[\frac{t}{\tau}]}\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}e^{2q\tau|G(\Lambda(u+\tau))-G(\Lambda(u))|}du\big).
\end{aligned}\tag{29}$$

By virtue of the homogeneousness of $\Lambda(\cdot)$ and the waiting time for the next jump of the Markov chain $\Lambda(\cdot)$ from current state i obeys the exponential distribution with parameter $-\gamma_{ii}$, we can show that

$$\mathbb{E}\big(\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}e^{2q\tau|G(\Lambda(u+\tau))-G(\Lambda(u))|}du|\Lambda(k\tau)\big)$$

$$
\begin{aligned}
&=\frac{1}{\tau}\sum_{i\in\mathbb{S}} I_{\{\Lambda(k\tau)=i\}}\\
&\quad\times\int_{k\tau}^{(k+1)\tau}\mathbb{E}\big(e^{2q\tau|G(\Lambda(u+\tau))-G(\Lambda(u))|}|\Lambda(k\tau)=i\big)du\\
&=\frac{1}{\tau}\sum_{i\in\mathbb{S}} I_{\{\Lambda(k\tau)=i\}}\\
&\quad\times\int_{k\tau}^{(k+1)\tau}\mathbb{E}\big(e^{2q\tau|G(\Lambda^i(u-(k-1)\tau))-G(\Lambda^i(u-k\tau))|}\big)du
\end{aligned}
$$

while

$$
\begin{aligned}
&\mathbb{E}\big(e^{2q\tau|G(\Lambda(u-(k-1)\tau))-G(\Lambda(u-k\tau))|}\big)\\
=&\mathbb{E}\Big(\mathbb{E}\big(e^{2q\tau|G(\Lambda(u-(k-1)\tau))-G(\Lambda(u-k\tau))}|\mathcal{F}_{u-k\tau}\big)\Big)\\
\le&\mathbb{E}\Big(\mathbb{E}\big(I_{\{\Lambda(u-k\tau)=\Lambda(u-(k-1)\tau)\}}\\
&+e^{4q\tau\check{G}}I_{\{\Lambda(u-k\tau)\neq\Lambda(u-(k-1)\tau)\}}|\mathcal{F}_{u-k\tau}\big)\Big)\\
=&\mathbb{E}\Big(\mathbb{E}\big(\sum_{l\in\mathbb{S}} I_{\{\Lambda(u-k\tau)=l\}}\big(I_{\{\Lambda(u-(k-1)\tau)=l\}}\\
&+e^{4q\tau\check{G}}I_{\{\Lambda(u-(k-1)\tau)\neq l\}}\big)|\mathcal{F}_{u-k\tau}\big)\Big)\\
=&\mathbb{E}\Big(\sum_{l\in\mathbb{S}} I_{\{\Lambda(u-k\tau)=l\}}\big(\mathbb{P}\big(\Lambda(u-(k-1)\tau)=l|\Lambda(u-k\tau)=l\big)\\
&+e^{4q\tau\check{G}}\mathbb{P}\big(\Lambda(u-(k-1)\tau)\neq l|\Lambda(u-k\tau)=l\big)\big)\Big)\\
\le&\mathbb{E}\Big(\sum_{l\in\mathbb{S}} I_{\{\Lambda(u-k\tau)=l\}}\big(e^{\gamma_{ll}\tau}+e^{4q\tau\check{G}}(1-e^{\gamma_{ll}\tau})\big)\Big)\\
=&\mathbb{E}\Big(\sum_{l\in\mathbb{S}} I_{\{\Lambda(u-k\tau)=l\}}\big(1+(e^{4q\tau\check{G}}-1)(1-e^{\gamma_{ll}\tau})\big)\Big)\\
\le&1+\bar{\gamma}\tau(e^{4q\tau\check{G}}-1).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbb{E}\big(e^{2q\int_0^t|G(\Lambda(u+\tau))-G(\Lambda(u))|du}\big)\\
\le&\mathbb{E}\Big(\prod_{k=0}^{[\frac{t}{\tau}]-1}\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}e^{2q\tau|G(\Lambda(u+\tau))-G(\Lambda(u))|}du\\
&\times\mathbb{E}\big(\frac{1}{\tau}\int_{[\frac{t}{\tau}]}^{([\frac{t}{\tau}]+1)\tau}e^{2q\tau|G(\Lambda(u+\tau))-G(\Lambda(u))|}du|\Lambda([\frac{t}{\tau}]\tau)\big)\Big)\\
\le&\big(1+\bar{\gamma}\tau(e^{4q\tau\check{G}}-1)\big)^{[\frac{t}{\tau}]+1}\\
\le&e^{([\frac{t}{\tau}]+1)\bar{\gamma}\tau(e^{4q\tau\check{G}}-1)}\le e^{(t+\tau)\bar{\gamma}(e^{4q\tau\check{G}}-1)}.
\end{aligned}\tag{30}
$$

On the other hand

$$
\begin{aligned}
&\mathbb{E}\big(e^{2q\int_0^t|G(\Lambda(u))-G(\Lambda([\frac{u}{\delta}]\delta))|du}\big)\\
\le&\mathbb{E}\big(e^{2q\sum_{j=0}^{[\frac{t}{\delta}]}\int_{j\delta}^{(j+1)\delta}|G(\Lambda(u))-G(\Lambda(j\delta))|du}\big)\\
=&\mathbb{E}\big(\prod_{j=0}^{[\frac{t}{\delta}]}e^{2q\int_{j\delta}^{(j+1)\delta}|G(\Lambda(u))-G(\Lambda(j\delta))|du}\big).
\end{aligned}\tag{31}
$$

For any nonnegative integer j, by the similar way as (29), we can show that

$$
\begin{aligned}
&\mathbb{E}\big(e^{2q\int_{j\delta}^{(j+1)\delta}|G(\Lambda(u))-G(\Lambda(j\delta))|du}|\Lambda(j\delta)\big)\\
\le&\mathbb{E}\big(\frac{1}{\delta}\int_{j\delta}^{(j+1)\delta}e^{2q\delta|G(\Lambda(u))-G(\Lambda(j\delta))|}du|\Lambda(j\delta)\big)\\
\le&\frac{1}{\delta}\sum_{i\in\mathbb{S}} I_{\{\Lambda(j\delta)=i\}}\int_{j\delta}^{(j+1)\delta}\mathbb{E}\big(I_{\{\Lambda(z)=i\ \forall z\in[j\delta,(j+1)\delta]\}}\\
&+e^{4q\delta\check{G}}I_{\{\exists z\in[j\delta,(j+1)\delta],\ \Lambda(z)\neq i\}}|\Lambda(j\delta)=i\big)du\\
\le&\frac{1}{\delta}\sum_{i\in\mathbb{S}} I_{\{\Lambda(j\delta)=i\}}\int_{j\delta}^{(j+1)\delta}\big(\\
&\mathbb{P}(\Lambda(z)=i\ \forall z\in[j\delta,(j+1)\delta]|\Lambda(j\delta)=i)\\
&+e^{4q\delta\check{G}}\mathbb{P}(\exists z\in[j\delta,(j+1)\delta],\ \Lambda(z)\neq i|\Lambda(j\delta)=i)\big)du\\
\le&\frac{1}{\delta}\sum_{i\in\mathbb{S}} I_{\{\Lambda(j\delta)=i\}}\int_{j\delta}^{(j+1)\delta}\big(e^{\gamma_{ii}\delta}+e^{4q\delta\check{G}}(1-e^{\gamma_{ii}\delta})\big)du\\
\le&1+\bar{\gamma}\delta(e^{4q\delta\check{G}}-1).
\end{aligned}\tag{32}
$$

Inserting this inequality into (31) yields that

$$
\begin{aligned}
&\mathbb{E}\big(e^{2q\int_0^t|G(\Lambda(u))-G(\Lambda([\frac{u}{\delta}]\delta))|du}\big)\\
\le&\mathbb{E}\Big(\mathbb{E}\big(\prod_{j=0}^{[\frac{t}{\delta}]}e^{2q\int_{j\delta}^{(j+1)\delta}|G(\Lambda(u))-G(\Lambda(j\delta))|du}|\Lambda([\frac{t}{\delta}]\delta)\big)\Big)\\
\le&\mathbb{E}\Big(\prod_{j=0}^{[\frac{t}{\delta}]-1}e^{2q\int_{j\delta}^{(j+1)\delta}|G(\Lambda(u))-G(\Lambda(j\delta))|du}\\
&\times\mathbb{E}\big(e^{2q\int_{[\frac{t}{\delta}]\delta}^{([\frac{t}{\delta}]+1)\delta}|G(\Lambda(u))-G(\Lambda([\frac{t}{\delta}]\delta))|du}|\Lambda([\frac{t}{\delta}]\delta)\big)\Big)\\
\le&\big(1+\bar{\gamma}\delta(e^{4q\delta\check{G}}-1)\big)^{[\frac{t}{\delta}]+1}\\
\le&e^{([\frac{t}{\delta}]+1)\bar{\gamma}\delta(e^{4q\delta\check{G}}-1)}\le e^{(t+\delta)\bar{\gamma}(e^{4q\delta\check{G}}-1)}.
\end{aligned}\tag{33}
$$

Thus, inserting (30) and (33) into (28) we obtain

$$
\mathbb{E}\big(e^{q\int_0^t|G(\Lambda(u+\tau))-G(\Lambda([\frac{u}{\delta}]\delta))|du}\big)\le K_7e^{\bar{\gamma}(e^{4q\bar{\tau}\check{G}}-1)t}.\tag{34}
$$

The proof is hence complete. ∎

The key technique used in this paper is the method of Lyapunov functionals. The specific form is outlined as follows:

$$
U(\bar{Z}_t,\Lambda(t),t)=c(\Lambda(t))|Z(t)|^2+\Phi(\bar{Z}_t,t).\tag{35}
$$

It is useful to observe that

$$
\Phi(\bar{Z}_t,t)\le\bar{\tau}\Phi_2(t),\tag{36}
$$

$$
\hat{c}|Z(t)|^2\le U(\bar{Z}_t,\Lambda(t),t)\le\check{c}|Z(t)|^2+\bar{\tau}\Phi_2(t).\tag{37}
$$

where $\hat{c}=\min_{\{1\le i\le M\}}c_i$. Applying the Itô formula to the Lyapunov functional defined by (35) yields

$$
dU(\bar{Z}_t,\Lambda(t),t)=\mathbb{L}U(\bar{Z}_t,\Lambda(t),\Lambda(\eta(t)),t))dt+dM(t),\tag{38}
$$

where

$$
\begin{aligned}
&\mathbb{L}U(\bar{Z}_t,\Lambda(t),\Lambda(\eta(t)),t)\\
=&2c(\Lambda(t))\Big[Z^\top(t)[f(Z(t),\Lambda(t),t)\\
&+\bar{u}(Z(\eta(t)),\Lambda(\eta(t)),t)R(t)]+\frac{1}{2}|g(Z(t),\Lambda(t),t)|^2\Big]\\
&+c(\Lambda(t))\int_X\Big[|Z(t)+h(Z(t),\Lambda(t),t,x)|^2\\
&-|Z(t)|^2-2Z^\top(t)h(Z(t),\Lambda(t),t,x)\Big]\nu(dx)
\end{aligned}
$$

$$+\Phi_1(t)-\Phi_2(t)+\sum_{l=1}^{M}\gamma_{\Lambda(t)l}c(l)|Z(t)|^2$$

and

$$\begin{aligned}M(t)=&\int_0^t 2c(\Lambda(t))Z^\top(s)g(Z(s),\Lambda(s),s)dB(s)\\&+\int_0^t\int_X[U(\bar{Z}_s+h(Z(s^-),\Lambda(s),s,x),\Lambda(s),s)\\&-U(\bar{Z}_s,\Lambda(s),s)]\tilde{N}(ds,dx)\\&+\int_0^t\int_{\mathbb{R}}[U(\bar{Z}_s,\Lambda(0)+y(\Lambda(s),r),s),\Lambda(s),s)\\&-U(\bar{Z}_s,\Lambda(s),s)]\upsilon(ds,dr),\end{aligned}$$

where the function y from $\mathbb{S}\times\mathbb{R}$ to $\mathbb{R}$ is define by

$$y(i,r)=\begin{cases}l-i, & \text{if } r\in\Delta_{il},\\ 0, & \text{otherwise,}\end{cases}$$

and $\upsilon(ds,dr)=\upsilon^*(ds,dr)-ds\times\varphi(dr)$ is a martingale measure. Here $\upsilon^*(ds,dr)$ is a Poisson measure on $\mathbb{R}_+\times\mathbb{R}$ with intensity $ds\times\varphi(dr)$, in which $\varphi(dr)$ is the Lebesgue measure on $\mathbb{R}$ and Δ_{il} with length γ_{il} is consecutive, left closed, right open intervals of the real line. Furthermore, $M(t)$ is a real-valued local martingale with $M(0)=0$ (see, e.g., [21], [36]).

Using (25), (26), Assumption 2.3, 3.1 and 3.2, we have

$$\begin{aligned}&\mathbb{L}U(\bar{Z}_t,\Lambda(t),\Lambda(\eta(t)),t)\\\le&2c(\Lambda(t))\big[\beta_1(\Lambda(t))|Z(t)|^2-\beta_2(\Lambda(t))|Z(t)|^{r+1}\big]\\&+2c(\Lambda(t))Z^\top(t)\bar{u}(Z(\eta(t)),\Lambda(\eta(t)),t)R(t)\\&+b_1b_3\bar{\tau}|Z(t)|^2+b_2b_3\bar{\tau}|Z(t)|^{r+1}\\&-\Phi_2(t)+\sum_{l=1}^{M}\gamma_{\Lambda(t)l}c(l)|Z(t)|^2\\=&2c(\Lambda(t))\big[\beta_1(\Lambda(t))|Z(t)|^2-\beta_2(\Lambda(t))|Z(t)|^{r+1}\big]\\&+2c(\Lambda(t))Z^\top(t)\bar{u}(Z(t),\Lambda(\eta(t)),t)R(t)\\&-2c(\Lambda(t))Z^\top(t)[\bar{u}(Z(t),\Lambda(\eta(t)),t)\\&-\bar{u}(Z(\eta(t)),\Lambda(\eta(t)),t)]R(t)+b_1b_3\bar{\tau}|Z(t)|^2\\&+b_2b_3\bar{\tau}|Z(t)|^{r+1}+\sum_{l=1}^{M}\gamma_{\Lambda(t)l}c(l)|Z(t)|^2\\&-b_3\int_{t-\bar{\tau}}^{t}\big(b_1|Z(s)|^2+b_2|Z(s)|^{r+1}\big)ds\\\le&2c(\Lambda(t))\big[\beta_1(\Lambda(t))|Z(t)|^2-\beta_2(\Lambda(t))|Z(t)|^{r+1}\big]\\&-2c(\Lambda(t))G(\Lambda(\eta(t)))|Z(t)|^2R(t)\\&+2\check{c}L|Z(t)||Z(\eta(t))-Z(t)|R(t)\\&+b_1b_3\bar{\tau}|Z(t)|^2+b_2b_3\bar{\tau}|Z(t)|^{r+1}+\sum_{l=1}^{M}\gamma_{\Lambda(t)l}c(l)|Z(t)|^2\\&-b_3\int_{t-\bar{\tau}}^{t}\big(b_1|Z(s)|^2+b_2|Z(s)|^{r+1}\big)ds\\\le&\Big(2c(\Lambda(t))\big(\beta_1(\Lambda(t))-G(\Lambda(t))\big)\\&+\sum_{l=1}^{M}\gamma_{\Lambda(t)l}c(l)\Big)|Z(t)|^2R(t)+\big(2c(\Lambda(t))\beta_1(\Lambda(t))\\&+\sum_{l=1}^{M}\gamma_{\Lambda(t)l}c(l)\big)|Z(t)|^2(1-R(t))\\&-2c(\Lambda(t))\beta_2(\Lambda(t))|Z(t)|^{r+1}+2\check{c}LH(t)R(t)\\&+2c(\Lambda(t))|G(\Lambda(t))-G(\Lambda(\eta(t)))||Z(t)|^2\\&+b_1b_3\bar{\tau}|Z(t)|^2+b_2b_3\bar{\tau}|Z(t)|^{r+1}\\&-b_3\int_{t-\bar{\tau}}^{t}\big(b_1|Z(s)|^2+b_2|Z(s)|^{r+1}\big)ds\\=&\frac{1}{c(\Lambda(t))}\Big(2c(\Lambda(t))\big(\beta_1(\Lambda(t))-G(\Lambda(t))\big)\\&+\sum_{l=1}^{M}\gamma_{\Lambda(t)l}c(l)+\frac{1}{2}\Big)c(\Lambda(t))|Z(t)|^2R(t)\\&+2c(\Lambda(t))|G(\Lambda(t))-G(\Lambda(\eta(t)))||Z(t)|^2\\&-\frac{1}{2}|Z(t)|^2-2c(\Lambda(t))\beta_2(\Lambda(t))|Z(t)|^{r+1}\\&+\frac{1}{c(\Lambda(t))}\big(2c(\Lambda(t))\beta_1(\Lambda(t))+\sum_{l=1}^{M}\gamma_{\Lambda(t)l}c(l)\\&+\frac{1}{2}\big)c(\Lambda(t))|Z(t)|^2(1-R(t))+2\check{c}LH(t)R(t)\\&+b_1b_3\bar{\tau}|Z(t)|^2+b_2b_3\bar{\tau}|Z(t)|^{r+1}\\&-b_3\int_{t-\bar{\tau}}^{t}\big(b_1|Z(s)|^2+b_2|Z(s)|^{r+1}\big)ds.\end{aligned}$$

By $(c_1,\cdots,c_M)^\top=\mathcal{B}^{-1}(1,\cdots,1)^\top$ with $\mathcal{B}=-2\text{diag}\{\beta_{11}-G_1,\cdots,\beta_{1M}-G_M\}-\Gamma$, we hence have

$$\begin{aligned}&\mathbb{L}U(\bar{Z}_t,\Lambda(t),\Lambda(\eta(t)),t)\\\le&\frac{1}{c(\Lambda(t))}(-1+\frac{1}{2})c(\Lambda(t))|Z(t)|^2R(t)\\&+\frac{1}{c(\Lambda(t))}\big(2c(\Lambda(t))\beta_1(\Lambda(t))\\&+\sum_{l=1}^{M}\gamma_{\Lambda(t)l}c(l)+\frac{1}{2}\big)c(\Lambda(t))|Z(t)|^2(1-R(t))\\&+2|G(\Lambda(t))-G(\Lambda(\eta(t)))|c(\Lambda(t))|Z(t)|^2\\&-\frac{1}{2}|Z(t)|^2-2\hat{c}\hat{\beta}_2|Z(t)|^{r+1}+b_1b_3\bar{\tau}|Z(t)|^2\\&+b_2b_3\bar{\tau}|Z(t)|^{r+1}+2\check{c}LH(t)R(t)\\&-b_3\int_{t-\bar{\tau}}^{t}\big(b_1|Z(s)|^2+b_2|Z(s)|^{r+1}\big)ds\\\le&\big(\psi_1R(t)+\psi_2(1-R(t))\\&+2|G(\Lambda(t))-G(\Lambda(\eta(t)))|\big)c(\Lambda(t))|Z(t)|^2+\psi(t),\end{aligned}\quad(39)$$

where $\psi_1=-\frac{1}{2\check{c}}$, $\psi_2=\max\limits_{i\in\mathbb{S}}\{\frac{2c_i\beta_{1i}+\sum_{l=1}^{M}\gamma_{il}c_l+1/2}{c_i}\}$, and $\psi(t)=-\frac{1}{2}|Z(t)|^2-2\hat{c}\hat{\beta}_2|Z(t)|^{r+1}+b_1b_3\bar{\tau}|Z(t)|^2+b_2b_3\bar{\tau}|Z(t)|^{r+1}-b_3\int_{t-\bar{\tau}}^{t}\big(b_1|Z(s)|^2+b_2|Z(s)|^{r+1}\big)ds+2\check{c}LH(t)R(t)$. Taking the conditional expectation with respect to the σ-algebra $\mathcal{G}$ for $\psi(t)$ and using Lemma 3.1, we get that

$$\begin{aligned}&\mathbb{E}_{\mathcal{G}}\psi(t)\\\le&-\frac{1}{2}\mathbb{E}_{\mathcal{G}}|Z(t)|^2-2\hat{c}\hat{\beta}_2\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}+b_1b_3\bar{\tau}\mathbb{E}_{\mathcal{G}}|Z(t)|^2\\&+b_2b_3\bar{\tau}\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}+2\check{c}L\phi_3R(t)\mathbb{E}_{\mathcal{G}}|Z(t)|^2\end{aligned}$$

$$
\begin{aligned}
&- b_3\int_{t-\bar\tau}^{t}\big(b_1\mathbb{E}_{\mathcal{G}}|Z(s)|^2+b_2\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds\\
&+2\check{c}LR(t)\int_{t-\bar\tau}^{t}\big(\phi_1\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\phi_2\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\big)ds\\
&+2\check{c}L\phi_4R(t)\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}\\
\le&-\big(\frac{1}{2}-b_1b_3\bar\tau-2b_3(\varepsilon+\frac{(K_1+2L)\bar\tau}{2})R(t)\big)\mathbb{E}_{\mathcal{G}}|Z(t)|^2\\
&-\big(2\hat{c}\hat{\beta}_2-b_2b_3\bar\tau-\frac{2b_3K_2\bar\tau}{r+1}R(t)\big)\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}\\
&-\big(b_1b_3-2b_3(\frac{K_1}{2}+\frac{K_3+K_5}{2\varepsilon})R(t)\big)\int_{t-\bar\tau}^{t}\mathbb{E}_{\mathcal{G}}|Z(s)|^2ds\\
&-\big(b_2b_3-2b_3(\frac{K_2r}{r+1}+\frac{K_4+K_6}{2\varepsilon})R(t)\big)\\
&\times\int_{t-\bar\tau}^{t}\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}ds. \qquad (40)
\end{aligned}
$$

Remark 3.5: The Lyapunov functional method is widely recognized as one of the most effective tools for studying stability, as evidenced by various papers (see, e.g., [8], [15], [36]). However, it is crucial to emphasize that this method is heavily dependent on the construction of Lyapunov functionals, and the proficient development of these functionals presents a substantial challenge. In highly nonlinear domains, the most frequently employed expression for the Lyapunov functional is as follows:

$$
\begin{aligned}
\bar{U}(\bar{Z}_t,\bar{\Lambda}_t,t)=&\zeta_1(\Lambda(t))|Z(t)|^2+|Z(t)|^{r+1}\\
&+\zeta_2(\Lambda(t))\int_{-\bar\tau}^{0}\int_{t+s}^{t}\big[\bar\tau|f(Z(y),\Lambda(y),y)\\
&+u(Z(\eta(y)),\Lambda(\eta(y)),y)|^2+|g(Z(y),\Lambda(y),y)|^2\\
&+\int_X|h(Z(y^-),\Lambda(y),y,x)|^2\nu(dx)\big]dyds,
\end{aligned}
$$

where $\zeta_1(\Lambda(t)),\zeta_2(\Lambda(t))>0$. In comparison to this particular functional, the one we developed is both simpler and more efficient.

To ensure brevity throughout the paper, we define the set of four free parameters

$$
\begin{aligned}
\Theta=\{(b_1,b_2,\varepsilon,\rho):\,&b_1>K_1+\frac{K_3+K_5}{\varepsilon},0<\varepsilon<\frac{1}{4\check{c}L},\\
&b_2>2(\frac{K_2r}{r+1}+\frac{K_4+K_6}{2\varepsilon}),\frac{-\psi_2}{\psi_1-\psi_2}<\rho<1\} \qquad (41)
\end{aligned}
$$

and set

$$
\begin{aligned}
\delta^*=\sup_{(b_1,b_2,\varepsilon,\rho)\in\Theta}\Big\{&\frac{2\check{c}(b_1-K_1-\frac{K_3+K_5}{\varepsilon})}{b_1}\\
&\wedge\frac{2\check{c}(b_2-\frac{2K_2r}{r+1}-\frac{K_4+K_6}{\varepsilon})}{b_2}\\
&\wedge\frac{1}{8\bar{G}}\ln(-\frac{\rho(\psi_1-\psi_2)+\psi_2}{2\bar\gamma}+1)\\
&\wedge\frac{\frac{1}{2}-2\check{c}L\varepsilon}{\frac{L}{2}+b_1\check{c}L+\check{c}L(K_1+2L)}\\
&\wedge\frac{2\hat{c}\hat{\beta}_2}{b_2\check{c}L+2\check{c}L\frac{K_2}{r+1}+2\hat{c}\hat{\beta}_2L}\wedge\frac{1}{L}\Big\}. \qquad (42)
\end{aligned}
$$

These technical parameters, and the significance of δ^* will be elucidated subsequently.

Combined with the above assumptions and Lemma, we are now ready to present the main theorem of this paper.

Theorem 3.1: Under Assumptions 2.1, 2.2 and 3.1, we can design a control function u to satisfy Assumptions 2.3 and 3.2 and then have an average control rate ρ satisfy Assumptions 2.4. If $0<\delta+\tau<\delta^*$ and $\frac{-\psi_2}{\psi_1-\psi_2}<\rho<1$, then the solution of the controlled system (13) with initial data (14) has the property that

$$
\limsup_{t\to\infty}\frac{1}{t}\ln(\mathbb{E}|Z(t)|^2)\le\frac{\rho(\psi_1-\psi_2)+\psi_2}{2}, \qquad (43)
$$

where $\psi_1=-\frac{1}{2\check{c}}$ and $\psi_2=\max_{i\in\mathbb{S}}\{\frac{2c_i\beta_{1i}+\sum_{l=1}^{M}\gamma_{il}c_l+1/2}{c_i}\}$, while δ^* is defined by (42).

Proof: As $\delta+\tau<\delta^*$, we can choose four positive parameters $(b_1,b_2,\varepsilon,\rho)\in\Theta$ for

$$
\begin{aligned}
\bar\tau<&\frac{2\check{c}(b_1-K_1-\frac{K_3+K_5}{\varepsilon})}{b_1}\wedge\frac{2\check{c}(b_2-\frac{2K_2r}{r+1}-\frac{K_4+K_6}{\varepsilon})}{b_2}\\
&\wedge\frac{1}{8\bar{G}}\ln(-\frac{\rho(\psi_1-\psi_2)+\psi_2}{2\bar\gamma}+1)\\
&\wedge\frac{\frac{1}{2}-2\check{c}L\varepsilon}{\frac{L}{2}+b_1\check{c}L+\check{c}L(K_1+2L)}\\
&\wedge\frac{2\hat{c}\hat{\beta}_2}{b_2\check{c}L+2\check{c}L\frac{K_2}{r+1}+2\hat{c}\hat{\beta}_2L}\wedge\frac{1}{L}. \qquad (44)
\end{aligned}
$$

It then follows from (40) that

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{G}}\psi(t)\\
\le&-\Big[\bar{b}_1\mathbb{E}_{\mathcal{G}}|Z(t)|^2+\bar{b}_2\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}\\
&+(\bar{b}_3+\frac{b_1}{2}\frac{\bar\tau L}{1-L\bar\tau})\int_{t-\bar\tau}^{t}\mathbb{E}_{\mathcal{G}}|Z(s)|^2ds\\
&+(\bar{b}_4+\frac{b_2}{2}\frac{\bar\tau L}{1-L\bar\tau})\int_{t-\bar\tau}^{t}\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}ds\Big]R(t)\\
&-\Big[\bar{b}_5\mathbb{E}_{\mathcal{G}}|Z(t)|^2+\bar{b}_6\mathbb{E}_{\mathcal{G}}|Z(t)|^{r+1}+(\bar{b}_7\\
&+\frac{b_1}{2}\frac{\bar\tau L}{1-L\bar\tau})\int_{t-\bar\tau}^{t}\mathbb{E}_{\mathcal{G}}|Z(s)|^2ds+(\bar{b}_8+\frac{b_2}{2}\\
&\times\frac{\bar\tau L}{1-L\bar\tau})\int_{t-\bar\tau}^{t}\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}ds\Big](1-R(t)), \qquad (45)
\end{aligned}
$$

where $\bar{b}_1=\frac{1}{2}-b_1b_3\bar\tau-2b_3(\varepsilon+\frac{(K_1+2L)\bar\tau}{2})$, $\bar{b}_2=2\hat{c}\hat{\beta}_2-b_2b_3\bar\tau-\frac{2b_3K_2\bar\tau}{r+1}$, $\bar{b}_3=b_1b_3-2b_3(\frac{K_1}{2}+\frac{K_3+K_5}{2\varepsilon})-\frac{b_1}{2}\frac{\bar\tau L}{1-L\bar\tau}$, $\bar{b}_4=b_2b_3-2b_3(\frac{K_2r}{r+1}+\frac{K_4+K_6}{2\varepsilon})-\frac{b_2}{2}\frac{\bar\tau L}{1-L\bar\tau}$, $\bar{b}_5=\frac{1}{2}-b_1b_3\bar\tau$, $\bar{b}_6=2\hat{c}\hat{\beta}_2-b_2b_3\bar\tau$, $\bar{b}_7=b_1b_3-\frac{b_1}{2}\frac{\bar\tau L}{1-L\bar\tau}$ and $\bar{b}_8=b_2b_3-\frac{b_2}{2}\frac{\bar\tau L}{1-L\bar\tau}$. We see from (44) that they are all positive. Since $\Pi(s):=\psi_1R(s)+\psi_2(1-R(s))+2|G(\Lambda(s))-G(\Lambda(\eta(s)))|$ has a finite number of jumps in any finite interval $[0,t]$, $\Pi(s)$ is differential. Therefore, applying the Itô formula to $e^{-\int_0^t\Pi(s)ds}U(\bar{Z}_t,\Lambda(t),t)$, where the functional $U(\bar{Z}_t,\Lambda(t),t)$ is given by $c(\Lambda(t))|Z(t)|^2+\Phi(\bar{Z}_t,t)$ in (35), we obtain that

$$
\begin{aligned}
&d\Big(e^{-\int_0^t\Pi(s)ds}U(\bar{Z}_t,\Lambda(t),t)\Big)\\
=&e^{-\int_0^t\Pi(s)ds}\Big(-\Pi(t)U(\bar{Z}_t,\Lambda(t),t)dt+dU(\bar{Z}_t,\Lambda(t),t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&=e^{-\int_0^t \Pi(s)ds}\Big(-\Pi(t)(c(\Lambda(t))|Z(t)|^2+\Phi(\bar{Z}_t,t))\Big)dt\\
&+e^{-\int_0^t \Pi(s)ds}\Big(\mathbb{L}U(\bar{Z}_t,\Lambda(t),\Lambda(\eta(t)),t))dt+dM(t)\Big). \quad (46)
\end{aligned}
$$

Integrating (46) on both sides, taking the conditional expectation with respect to the σ-algebra $\mathcal{G}$ and using (45), we get

$$
\begin{aligned}
&e^{-\int_0^t \Pi(s)ds}\mathbb{E}_{\mathcal{G}}U(\bar{Z}_t,\Lambda(t),t)-U(\bar{Z}_0,\Lambda(0),0)\\
&\le\int_0^t e^{-\int_0^s \Pi(v)dv}\Big(-\Pi(s)\mathbb{E}_{\mathcal{G}}\Phi(\bar{Z}_s,s)-\Big[\bar{b}_1\mathbb{E}_{\mathcal{G}}|Z(s)|^2\\
&+\bar{b}_2\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}+(\bar{b}_3+\frac{b_1b_3\bar{\tau}}{2})\int_{s-\bar{\tau}}^s\mathbb{E}_{\mathcal{G}}|Z(v)|^2dv\\
&+(\bar{b}_4+\frac{b_2b_3\bar{\tau}}{2})\int_{s-\bar{\tau}}^s\mathbb{E}_{\mathcal{G}}|Z(v)|^{r+1}dv\Big]R(s)\\
&-\Big[\bar{b}_5\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\bar{b}_6\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\\
&+(\bar{b}_7+\frac{b_1b_3\bar{\tau}}{2})\int_{s-\bar{\tau}}^s\mathbb{E}_{\mathcal{G}}|Z(v)|^2dv+(\bar{b}_8\\
&+\frac{b_2b_3\bar{\tau}}{2})\int_{s-\bar{\tau}}^s\mathbb{E}_{\mathcal{G}}|Z(v)|^{r+1}dv\Big](1-R(s))\Big)ds. \quad (47)
\end{aligned}
$$

One observes from (36) that

$$
-\Pi(s)\Phi(\bar{Z}_s,s)\le\frac{1}{2\check{c}}\Phi(\bar{Z}_s,s)\le\frac{\bar{\tau}}{2\check{c}}\Phi_2(s), \quad (48)
$$

we can then obtain from (47) that

$$
\begin{aligned}
&e^{-\int_0^t \Pi(s)ds}\mathbb{E}_{\mathcal{G}}U(\bar{Z}_t,\Lambda(t),t)-U(\bar{Z}_0,\Lambda(0),0)\\
&\le\int_0^t e^{-\int_0^s \Pi(v)dv}\Big(-\Big[\bar{b}_1\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\bar{b}_2\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}\\
&+\bar{b}_3\int_{s-\bar{\tau}}^s\mathbb{E}_{\mathcal{G}}|Z(v)|^2dv+\bar{b}_4\int_{s-\bar{\tau}}^s\mathbb{E}_{\mathcal{G}}|Z(v)|^{r+1}dv\Big]R(s)\\
&-\Big[\bar{b}_5\mathbb{E}_{\mathcal{G}}|Z(s)|^2+\bar{b}_6\mathbb{E}_{\mathcal{G}}|Z(s)|^{r+1}+\bar{b}_7\int_{s-\bar{\tau}}^s\mathbb{E}_{\mathcal{G}}|Z(v)|^2dv\\
&+\bar{b}_8\int_{s-\bar{\tau}}^s\mathbb{E}_{\mathcal{G}}|Z(v)|^{r+1}dv\Big](1-R(s))\Big)ds. \quad (49)
\end{aligned}
$$

This implies that

$$
\mathbb{E}_{\mathcal{G}}U(\bar{Z}_t,\Lambda(t),t)\le e^{\int_0^t \Pi(s)ds}U(\bar{Z}_0,\Lambda(0),0). \quad (50)
$$

Taking expectation on both sides yields

$$
\mathbb{E}U(\bar{Z}_t,\Lambda(t),t)\le U(\bar{Z}_0,\Lambda(0),0)\mathbb{E}\big(e^{\int_0^t \Pi(s)ds}\big). \quad (51)
$$

According to Assumption 2.4, direct calculation yields

$$
\begin{aligned}
&\int_0^t\big(\psi_1R(s)+\psi_2(1-R(s))\big)ds\\
&=\psi_1S(t,0)+\psi_2S^c(t,0)\\
&\le\rho\psi_1t-\psi_1S_0+(1-\rho)\psi_2t+\psi_2S_0\\
&=\big(\rho(\psi_1-\psi_2)+\psi_2\big)t-(\psi_1-\psi_2)S_0. \quad (52)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathbb{E}U(\bar{Z}_t,\Lambda(t),t)\\
&\le K_8e^{(\rho(\psi_1-\psi_2)+\psi_2)t}\mathbb{E}\big(e^{2\int_0^t|G(\Lambda(s))-G(\Lambda(\eta(s)))|ds}\big) \quad (53)
\end{aligned}
$$

where $K_8=e^{-(\psi_1-\psi_2)S_0}U(\bar{Z}_0,\Lambda(0),0)$. Next we will estimate $\mathbb{E}\big(e^{2\int_0^t|G(\Lambda(s))-G(\Lambda(\eta(s)))|ds}\big)$. Using the homogeneousness of $\Lambda(\cdot)$ and the definition of $\tilde{\Lambda}(\cdot)$, we know that for any $t\ge0$, $s>0$

$$
\begin{aligned}
&\mathbb{E}\big(e^{2\int_0^t|G(\tilde{\Lambda}(u+s))-G(\tilde{\Lambda}([\frac{u+s}{\delta}]\delta-\tau))|du}\big)\\
&=\sum_{l\in\mathbb{S}}\mathbb{E}\Big(I_{\{\tilde{\Lambda}([\frac{s}{\delta}]\delta-\tau)=l\}}\\
&\times\mathbb{E}\big(e^{2\int_0^t|G(\tilde{\Lambda}(u+s))-G(\tilde{\Lambda}([\frac{u+s}{\delta}]\delta-\tau))|du}|\tilde{\Lambda}([\frac{s}{\delta}]\delta-\tau)=l\big)\Big)\\
&=\sum_{l\in\mathbb{S}}\mathbb{E}\Big(I_{\{\Lambda([\frac{s}{\delta}]\delta+\delta_0)=l\}}\\
&\times\mathbb{E}\big(e^{2\int_0^t|G(\Lambda^l(u+s-[\frac{s}{\delta}]\delta+\tau))-G(\Lambda^l([\frac{u+s-[\frac{s}{\delta}]\delta}{\delta}]\delta))|du}\big)\Big). \quad (54)
\end{aligned}
$$

From $0\le s-[\frac{s}{\delta}]\delta<\delta$, one can see that

$$
\begin{aligned}
&\int_0^t|G(\Lambda^l(u+s-[\frac{s}{\delta}]\delta+\tau))-G(\Lambda^l([\frac{u+s-[\frac{s}{\delta}]\delta}{\delta}]\delta))|du\\
&=\int_{s-[\frac{s}{\delta}]\delta}^{t+s-[\frac{s}{\delta}]\delta}|G(\Lambda^l(u+\tau))-G(\Lambda^l([\frac{u}{\delta}]\delta))|du\\
&=\int_0^t|G(\Lambda^l(u+\tau))-G(\Lambda^l([\frac{u}{\delta}]\delta))|du\\
&+\int_t^{t+s-[\frac{s}{\delta}]\delta}|G(\Lambda^l(u+\tau))-G(\Lambda^l([\frac{u}{\delta}]\delta))|du\\
&+\int_0^{s-[\frac{s}{\delta}]\delta}|G(\Lambda^l(u+\tau))-G(\Lambda^l([\frac{u}{\delta}]\delta))|du\\
&\le\int_0^t|G(\Lambda^l(u+\tau))-G(\Lambda^l([\frac{u}{\delta}]\delta))|du+4\check{G}\delta. \quad (55)
\end{aligned}
$$

Inserting (55) into (54), then using the Lemma 3.2, we obtain that for any $s\ge0$,

$$
\mathbb{E}\big(e^{2\int_0^t|G(\tilde{\Lambda}(u+s))-G(\tilde{\Lambda}([\frac{u+s}{\delta}]\delta-\tau))|du}\big)\le K_7e^{4\check{G}\delta}e^{\bar{\gamma}(e^{8\bar{\tau}\check{G}}-1)t}.
$$

This, together with the definition of $\tilde{\Lambda}(t)$, implies that for any $s\in[(n_0+1)\delta,\infty)$

$$
\mathbb{E}\big(e^{2\int_s^{t+s}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du}\big)\le K_7e^{4\check{G}\delta}e^{\bar{\gamma}(e^{8\bar{\tau}\check{G}}-1)t}. \quad (56)
$$

For any $s\in[0,(n_0+1)\delta)$

$$
\begin{aligned}
&\int_s^{t+s}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du\\
&=\int_s^{(n_0+1)\delta}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du\\
&+\int_{(n_0+1)\delta}^{t+(n_0+1)\delta}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du\\
&+\int_{t+(n_0+1)\delta}^{t+s}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du\\
&\le\int_0^{(n_0+1)\delta}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du\\
&+\int_{(n_0+1)\delta}^{t+(n_0+1)\delta}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{t+s}^{t+(n_0+1)\delta}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du\\
\leq&\int_{(n_0+1)\delta}^{t+(n_0+1)\delta}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du\\
&+2(n_0+1)\delta\check{G}. \qquad (57)
\end{aligned}
$$

This together with (56) implies that for any $s\in[0,\infty)$

$$
\begin{aligned}
&\mathbb{E}\big(e^{2\int_s^{t+s}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du}\big)\\
\leq&\mathbb{E}\big(e^{2\int_{(n_0+1)\delta}^{t+(n_0+1)\delta}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du+4(n_0+1)\delta\check{G}}\big)\\
\leq&K_7e^{4\check{G}\delta+4(n_0+1)\delta\check{G}}e^{\bar{\gamma}(e^{8\bar{\tau}\check{G}}-1)t}\\
\leq&K_7e^{4\check{G}(2\delta+\tau)}e^{\bar{\gamma}(e^{8\bar{\tau}\check{G}}-1)t},
\end{aligned}
$$

where we used $n_0=[\frac{\tau}{\delta}]$ in the last step. Thus

$$
\begin{aligned}
&\mathbb{E}\big(e^{2\int_0^{t}|G(\Lambda(u))-G(\Lambda(\eta(u)))|du}\big)\\
\leq&K_7e^{4\check{G}(2\delta+\tau)}e^{\bar{\gamma}(e^{8\bar{\tau}\check{G}}-1)t}. \qquad (58)
\end{aligned}
$$

By condition $\delta+\tau<\delta^*$, we see $\frac{-\psi_2}{\psi_1-\psi_2}<\rho<1$ and $\bar{\gamma}(e^{8\bar{\tau}\check{G}}-1)\leq-\frac{\rho(\psi_1-\psi_2)+\psi_2}{2}$. It then follows from (53) and (58) that

$$
\mathbb{E}U(\bar{Z}_t,\Lambda(t),t)\leq\hat{c}K_9e^{\frac{(\rho(\psi_1-\psi_2)+\psi_2)t}{2}}, \qquad (59)
$$

where $K_9=\frac{K_8K_7e^{4\check{G}(2\delta+\tau)}}{\hat{c}}$. Thus, we conclude that

$$
\mathbb{E}|Z(t)|^2\leq K_9e^{\frac{(\rho(\psi_1-\psi_2)+\psi_2)t}{2}}. \qquad (60)
$$

Clearly, (60) means that

$$
\limsup_{t\to\infty}\frac{1}{t}\ln(\mathbb{E}|Z(t)|^2)\leq\frac{\rho(\psi_1-\psi_2)+\psi_2}{2}. \qquad (61)
$$

The proof is hence complete. ∎

Remark 3.6: In accordance with Theorem (3.1), it becomes evident that, unlike previous investigations into the stability of intermittent control, our focus lies solely on constraining the average control rate ρ for ADIC. Importantly, there is no need to impose specific limitations on individual controls, intervals of rest, or the maximal proportion of rest width within each control period.

In the context of NNSHSs-LN, it is imperative to note that moment exponential stability, in general, does not necessarily imply almost sure exponential stability. Assumption 2.2, as presently formulated, ensures the pth moment finiteness of the global solution but does not offer assurance regarding its uniform boundedness in the pth moment over an infinite time horizon. If we substitute Assumption 2.2 with the condition

$$
\begin{aligned}
&|z|^{p-2}\big(z^Tf(z,i,t)+\frac{p-1}{2}|g(z,i,t)|^2\big)+\int_X\big(-|z|^p\\
&+|z+h(z,i,t,x)|^p-p|z|^{p-2}z^Th(z,i,t,x)\big)\nu(dx)\\
\leq&\alpha_1|z|^p-\alpha_2|z|^{p+r-1}, \qquad (62)
\end{aligned}
$$

where $\alpha_1,\alpha_2>0$, as articulated in [15, Theorem 4.2], we can promptly derive the following result.

Theorem 3.2: Let Assumptions 2.1, 2.3 and (62) hold. The controlled system (16) with any initial data (14) has a unique global solution $Z(t)$. Moreover, the solution $Z(t)$ obeys

$$
\sup_{0\leq t<\infty}\mathbb{E}|Z(t)|^p<\infty. \qquad (63)
$$

In the interest of conciseness, we omit the detailed proof as it closely aligns with existing proofs in the literature. While the system in the existing literature does not incorporate Lévy noise, our system does. However, this divergence does not impact the proof. During the reevaluation of expectations, the introduced disturbance is effectively treated as zero.

Under the condition (62), the moment exponential stability implies almost sure exponential stability. We formulate it as the following theorem.

Theorem 3.3: Let all the conditions in Theorem 3.1 hold, except Assumption 2.2 replaced with (62). Then the solution of the controlled system (13) with initial data (14) has the property that

$$
\limsup_{t\to\infty}\frac{1}{t}\ln(|Z(t)|)<0,\quad a.s. \qquad (64)
$$

We can utilize the same analysis as presented in Theorem 4.3 of [36] to prove this theorem, and thus, it is omitted.

IV. Practical applications and numerical simulations

In this section, we will validate the effectiveness of our theoretical findings through two numerical examples. More precisely, we first design an ADIC for the unstable stochastic FitzHugh-Nagumo model and examine the mean-square stability of the controlled system. Next, we apply our results to the stochastic Gilpin-Ayala model, examining both the mean-square stability and almost sure stability of the stochastic Gilpin-Ayala model with ADIC.

Example 4.1: Consider a modified highly nonlinear stochastic FitzHugh-Nagumo models with Markovian switching and Lévy noise:

$$
\begin{cases}
dF(t)=\big(w(\Lambda(t))(F(t)+J(t)-\frac{F^3(t)}{3})\big)dt\\
\qquad+\eta(\Lambda(t))F^2(t)dB(t)+\theta(\Lambda(t))F(t)d\tilde{N}(t),\\
dJ(t)=\big(\frac{-1}{w(\Lambda(t))}(F(t)+J(t))+\sigma(\Lambda(t))J^3(t)\big)dt\\
\qquad+\eta(\Lambda(t))J^2(t)dB(t)+\theta(\Lambda(t))J(t^-)d\tilde{N}(t),
\end{cases} \qquad (65)
$$

where $F,J\in\mathbb{R}$, $B(t)$ represents the one-dimensional Brownian motion, $\tilde{N}(t)$ denotes a one-dimensional compensated Poisson process with a jump intensity of $\lambda=1$, $\Lambda(t)$ is a Markov chain on the state space $S=\{1,2\}$ with its generator $\Gamma=\begin{pmatrix}-5&5\\3&-3\end{pmatrix}$.

The FitzHugh-Nagumo models, which provide a simplified mathematical representation elucidating neuronal activity, bear significant relevance across various applications, spanning biomedicine, neuroscience, and neural networks (see, e.g., [1], [38]). Incorporating the impact of external environmental noise, the stochastic FitzHugh-Nagumo model more accurately captures the inherent

randomness and uncertainty observed in specific neural systems.

For the convenience of analyzing, we transform (65) to a simpler form. Let $Z = (F, J)^\top$, $g(Z,i,t) = (\eta(i)F^2, \eta(i)J^2)^\top$, $h(Z,i,t) = (\theta(i)F, \theta(i)J)^\top$, and

$$f(Z,i,t) = \begin{pmatrix} w(i)(F + J - \frac{F^3}{3}) \\ -\frac{1}{w(i)}(F+J) + \sigma(i)J^3 \end{pmatrix}.$$

Then we can rewrite modified Fitzthgh-Nagumo models (65) as a general type of HNSHSs-LN

$$\begin{aligned} dZ(t) =& f(Z(t),\Lambda(t),t)dt + g(Z(t),\Lambda(t),t)dB(t) \\ &+ h(Z(t^-),\Lambda(t),t)d\tilde{N}(t). \end{aligned} \tag{66}$$

When choosing the parameters of the system (66) as: $w(1) = 6, w(2) = 9, \eta(1) = 1, \eta(2) = \sqrt{2}, \sigma(1) = -2, \sigma(2) = -3, \theta(1) = 2, \theta(2) = 1$, it is easy to verify that Assumptions 2.1 and 2.2 are satisfied with $K_1 = 15.781$, $K_2 = 5.196$, $K_3 = 0$, $K_4 = 2$, $K_5 = 4$, $K_6 = 0$, $p = 4$, $r = 3$ and $\alpha = 44.36$. Taking $T = 4$ and step size 10^{-6}, we plot the second moment of the solution of (65) produced by the truncated Euler-Maruyama method [39]. Here the expectations are approximated by the Monte Carlo approximation with 100 sample paths. As is exhibited in Fig. 2, we can find that HNSHSs-LN (66) is unstable. In order to ensure the stability of the HNSHSs-

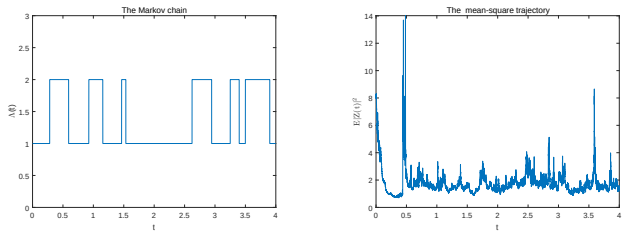

Fig. 2. Computer simulations of the sample path of the Markov chain and the second moment of the solution of HNSHSs-LN (66) with $\boldsymbol{T = 4}$ and $\boldsymbol{\Delta = 10^{-6}}$.(Example 4.1)

LN (66), our objective is to devise an ADIC based on discrete-time observations of the state and mode. The intention is to stabilize the aforementioned HNSHSs-LN (66). Consequently, the controlled HNSHSs-LN can be represented as:

$$\begin{aligned} dZ(t) =& \big[f(Z(t),\Lambda(t),t) + u(Z(\eta(t)),\Lambda(\eta(t)),t)\big]dt \\ &+ g(Z(t),\Lambda(t),t)dB(t) \\ &+ h(Z(t^-),\Lambda(t),t)d\tilde{N}(t), \end{aligned} \tag{67}$$

where

$$u(z,i,t) = \begin{cases} -G_i z, & t \in [t_k, s_k), \\ 0, & t \in [s_k, t_{k+1}). \end{cases} \tag{68}$$

Let $G_1 = 16$ and $G_2 = 16$. It is easily observed that Assumption 2.3 holds with $L = 16$. Select the control intervals as $\bigcup_{k=0}^{\infty}[t_k, s_k) = \bigcup_{k=0}^{\infty}([2k, 2k + 0.95)\bigcup[2k + 1, 2k+1.95))$. Then, it can be obtained that average control rate $\rho = 0.95$. For any initial data $Z_0 \in \mathbb{R}^2$ and $i_0 \in \mathbb{S}$, the controlled system (67) has a unique global solution for $t \in \mathbb{R}^+$ with the property that $\mathbb{E}|Z(t)|^4 < \infty$. However, this alone is not sufficient for stability purposes. Let us now examine Assumptions 3.1 and 3.2. For $(z,i) \in \mathbb{R}^2 \times \mathbb{S}$, we have

$$\begin{aligned} & z^\top f(z,i,t) + \frac{1}{2}|g(z,i,t)|^2 + |h(z,i,t)|^2 \\ \le & \begin{cases} 13.083|z|^2 - 0.5|z|^4, & i = 1, \\ 14.556|z|^2 - 0.5|z|^4, & i = 2, \end{cases} \end{aligned}$$

and

$$z^\top \bar{u}(z,i,t) = \begin{cases} -16|z|^2, & i = 1, \\ -16|z|^2, & i = 2. \end{cases}$$

It is easy to see that

$$\beta_1(1) = 13.083,\ \beta_2(1) = 0.5, \beta_1(2) = 14.556,\ \beta_2(2) = 0.5,$$
$$\mathcal{B}_1 = \begin{pmatrix} 4.58 & -1 \\ -2 & 3.68 \end{pmatrix},\ c_1 = 0.2232,\quad \psi_1 = -1.764,$$
$$c_2 = 0.2835,\quad \psi_2 = 30.238.$$

where $\mathcal{B}_1$ is non-singular M-matrices. Then (41) becomes

$$\begin{aligned} \Theta = \{(b_1, b_2, \varepsilon, \rho) : b_1 > 15.871 + \frac{4}{\varepsilon}, b_2 > 7.794 + \frac{2}{\varepsilon}, \\ 0 < \varepsilon < 0.055, \rho = 0.95\} \end{aligned}$$

and by (42),

$$\begin{aligned} \delta^* = \sup_{(b_1,b_2,\varepsilon,\rho)\in\Theta}\Big\{ & \frac{0.567(b_1 - 15.871 - \frac{4}{\varepsilon})}{b_1} \\ & \wedge \frac{0.567(b_2 - 7.794 - \frac{2}{\varepsilon})}{b_2} \wedge 0.00013 \\ & \wedge \frac{0.5 - 9.072\varepsilon}{8 + 4.536b_1 + 4.536(15.871 + 32)} \\ & \wedge \frac{0.2232}{4.536b_2 + 15.356} \wedge \frac{1}{16}\Big\}. \end{aligned}$$

Let $\varepsilon = 0.05$, $b_1 = 100.871$, $b_2 = 50.794$. From this, we obtain $\delta^* = 0.00013$. In light of Theorem 3.1, the controlled system (67) is therefore exponentially stable in mean-square for any $\delta + \tau < \delta^*$. In other words, the solution of the controlled system (67) with the initial conditions $Z_0 \in \mathbb{R}^2$ and $i_0 \in \mathbb{S}$ possesses the property that

$$\limsup_{t\to\infty} \frac{1}{t}\ln(\mathbb{E}|Z(t)|^2) \le \frac{\rho(\psi_1 - \psi_2) + \psi_2}{2} = -0.082 < 0.$$

Taking the initial data $Z_0 = (2.5, -2.5)^T$, $\Lambda(0) = 1$, $\delta = 0.00001$, $\tau = 0.0001$, $T = 4$ and step size 10^{-6}. Fig. 3 depicts mean-square trajectories $\mathbb{E}|Z(t)|^2$ of the controlled system (67) with average control rates $\rho = 0.95$ for 100 sample paths. It can be shown that the controlled system (67) demonstrates mean-square stability whenever the average control rate satisfies the condition $\frac{-\psi_2}{\psi_1 - \psi_2} < \rho < 1$, which is consistent with the conclusion of Theorem 3.1.

Example 4.2: Consider the classical stochastic Gilpin-Ayala model (see, e.g., [40], [41]):

$$\begin{aligned} dZ(t) =& Z(t)(a(\Lambda(t)) - \sigma(\Lambda(t))Z^2(t))dt \\ &+ \eta(\Lambda(t))Z(t)dB(t) + \theta(\Lambda(t))Z(t^-)d\tilde{N}(t) \end{aligned} \tag{69}$$

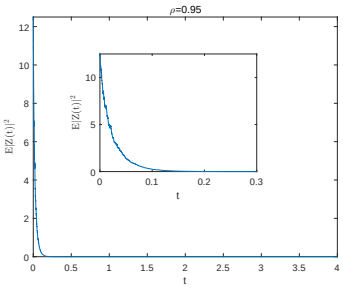

Fig. 3. Mean-square stability of the controlled system (67). (Example 4.1)

on $t \geq 0$, where $B(t)$, $\tilde{N}(t)$ and $\Lambda(t)$ are the same as in Example 4.1.

The stochastic Gilpin-Ayala model serves as a valuable tool for the analysis and prediction of specific biological phenomena within organisms, encompassing aspects like gene expression, cell proliferation, and stochastic events in diverse biological processes. Investigation into the stochastic Gilpin-Ayala model assumes noteworthy significance in propelling scientific knowledge and enhancing medical applications within the realm of biological control. These models furnish a robust methodology for comprehending and navigating the intricacies inherent in biological systems. In this example, we will apply our results to the stochastic Gilpin-Ayala model and show that, under coefficients satisfying stronger conditions, the moment exponential stability of the model implies almost sure exponential stability.

Take

$$a(1) = 2, \sigma(1) = 0.5, \eta(1) = 1, \theta(1) = 2,$$
$$a(2) = 2, \sigma(2) = 1, \eta(2) = 2, \theta(2) = 0.5.$$

Assumption 2.1 can be readily validated, where $K_1 = 2$, $K_2 = 1$, $K_3 = 4$, $K_4 = 0$, $K_5 = 4$, $K_6 = 0$, $p = 4$, and $r = 3$. The coefficients the model (69) adhere to the more restrictive condition (62), wherein $\alpha_1 = 75.5$ and $\alpha_2 = 0.5$. For the numerical simulations, we choose $T = 5$ and a step size of $\Delta = 10^{-6}$. In Fig. 4, we depict the sample paths of the Markov chain $\Lambda(t)$ and the solution $Z(t)$ of model (69) obtained using the truncated Euler-Maruyama method. Notably, the graphical representation the instability of model (69).

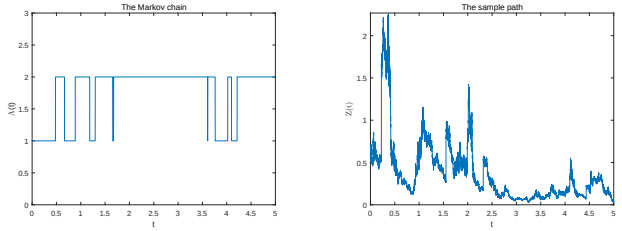

Fig. 4. Computer simulations for the sample paths of both the Markov chain and the model (69) with $\boldsymbol{T = 5}$ and $\boldsymbol{\Delta = 10^{-6}}$. (Example 4.2)

Next, we develop a feedback control strategy to stabilize model (69). Here we assign the control functions

$$u(z,1,t) = \begin{cases} -7.5z, & t \in [t_k, s_k), \\ 0, & t \in [s_k, t_{k+1}), \end{cases} \tag{70}$$

$$u(z,2,t) = \begin{cases} -6.25z, & t \in [t_k, s_k), \\ 0, & t \in [s_k, t_{k+1}). \end{cases} \tag{71}$$

Clearly, Assumption 2.3 is satisfied with $L = 7.5$. Select the control intervals as $\bigcup_{k=0}^{\infty}[t_k, s_k) = \bigcup_{k=0}^{\infty}([3k, 3k + 0.95)\bigcup[3k + 1, 3k + 2.9))$. Then, it can be obtained that average control rate $\rho = 0.95$. According to Theorem 3.2, the controlled model

$$\begin{aligned} dZ(t) =& \big[Z(t)(a(\Lambda(t)) - \sigma(\Lambda(t))Z^2(t)) \\ &+ u(Z(\eta(t)), \Lambda(\eta(t)), t)\big]dt \\ &+ \eta(\Lambda(t))Z(t)dB(t) + \theta(\Lambda(t))Z(t^-)d\tilde{N}(t), \end{aligned} \tag{72}$$

possesses a unique global solution on $t \in \mathbb{R}_+$ for any initial data $Z_0 \in \mathbb{R}$, $i_0 \in \mathbb{S}$ and the solution has the property that $\sup_{0 \leq t < \infty} \mathbb{E}|Z(t)|^p < \infty$.

Let us now examine Assumptions 3.1 and 3.2. For $(z, i) \in \mathbb{R} \times \mathbb{S}$, we can compute

$$\begin{aligned} & z(za(i) - \sigma(i)z^3) + \frac{c^2(i)}{2}|z|^2 + \theta^2(i)|z|^2 \\ \leq & \begin{cases} 6.5|z|^2 - 0.5|z|^4, & i = 1, \\ 4.25|z|^2 - |z|^4, & i = 2, \end{cases} \end{aligned}$$

and

$$z^\top \bar{u}(z,i,t) = \begin{cases} -7.5|z|^2, & i = 1, \\ -6.25|z|^2, & i = 2. \end{cases}$$

Through explicit computation, we have

$$\beta_1(1) = 6.5,\ \beta_2(1) = 0.5,\ \beta_1(2) = 4.25,\ \beta_2(2) = 1,$$
$$G_1 = 7.5,\ G_2 = 6.25,\ \mathcal{B}_1 = \begin{pmatrix} 7 & -5 \\ -3 & 7 \end{pmatrix},$$
$$c_1 = 0.3529, c_2 = 0.2941, \psi_1 = -1.417, \psi_2 = 10.8,$$

where $\mathcal{B}_1$ is non-singular M-matrices. Then (41) transforms into

$$\begin{aligned} \Theta = \{(b_1, b_2, \varepsilon, \rho) : & b_1 > 2 + \frac{8}{\varepsilon}, b_2 > 1.5, \\ & 0 < \varepsilon < 0.094, \rho = 0.95\} \end{aligned}$$

and in accordance with (42), we deduce that

$$\begin{aligned} \delta^* = \sup_{(b_1,b_2,\varepsilon,\rho)\in\Theta} \Big\{ & \frac{0.7056(b_1 - 2 - \frac{8}{\varepsilon})}{b_1} \wedge \frac{0.706(b_2 - 1.5)}{b_2} \\ & \wedge 0.0134 \wedge \frac{0.5 - 5.294\varepsilon}{3.75 + 2.647b_1 + 44.99} \\ & \wedge \frac{0.2941}{2.647b_2 + 3.529} \wedge \frac{1}{7.5} \Big\}. \end{aligned}$$

Consider the values $\varepsilon = 0.08$, $b_1 = 103$, and $b_2 = 2.5$. This yields $\delta^* = 0.00024$. Following Theorems 3.1 and 3.3, the controlled model (72) is not only the moment exponentially stable but also the almost sure exponentially stable for any $\tau + \delta < \delta^*$.

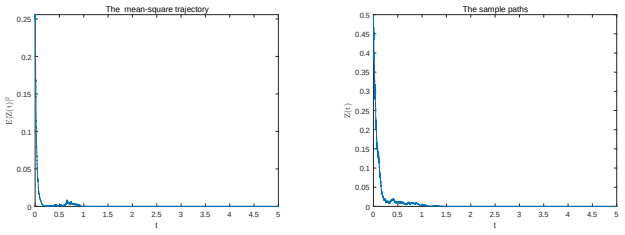

Fig. 5. Computer simulations of the mean-square trajectory and the sample path of the controlled model (72) with $\boldsymbol{T = 5}$ and $\boldsymbol{\Delta = 10^{-6}}$. (Example 4.2)

We conducted a computer simulation using initial data $Z_0 = 0.5$, $\Lambda(0) = 1$, $T = 5$, $\delta = 0.00002$, $\tau = 0.0002$, and a step size of $\Delta = 10^{-6}$. In Fig. 5, the mean-square trajectory $\mathbb{E}|Z(t)|^2$ for 100 sample points, along with the sample path $Z(t)$ of the controlled system (72), are plotted. Fig. 5 illustrates the stability of the controlled system (72) in both mean-square and almost sure senses. The simulation results consistently align with our theoretical findings, providing clear empirical support for our results.

Remark 4.1: To better show the advantages in terms of both cost-effectiveness and efficiency enhancement of our theory, we have compared the aperiodic delay intermittent control with continuous delay feedback control in Example 4.2. We consider a continuous delay feedback control function of the form $u(z, 1, t) = -7.5z$ and $u(z, 2, t) = -6.25z$ to stabilize model (69). In this regard, it can be easily checked that the controlled system (72) can achieve exponentially stability, and we will not elaborate further due to the limitation of space.

Taking the same initial conditions with ADIC, we also show the corresponding simulation results with delay feedback control functions in Fig. 6. It is easy to see that the controlled system (72) can achieve exponentially stability. Although both of these controllers can achieve stability, such a delay feedback control requires the continuous observations of the state, which is of course expensive and sometimes not possible. The controllers in Example 4.2 only are implemented in the discrete-time sequence $[t/\delta]\delta$ of the control interval, whose average control rate is $\rho = 0.95$ to be controlled, as opposed to the continuous control technique. This greatly saves the cost of control and the amount of information transmission.

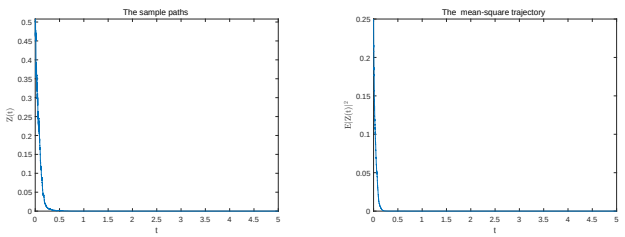

Fig. 6. Computer simulations of the mean-square trajectory and the sample path of the controlled model (72) with delay feedback control.

V. Conclusions

We have addressed the stabilization problem of HNSHSs-LN using ADIC based on the discrete-time observations of the states and modes in this paper. This finding extends its applicability to a broader range of stochastic systems, offering notable advantages in terms of both cost-effectiveness and efficiency enhancement. The mean-square exponential stability of a controlled HNSHSs-LN was established. With a more restrictive condition, we obtained the almost sure exponential stability of a controlled HNSHSs-LN. Remarkably, we introduced an innovative method for assessing the difference between the current state (or mode) and the discrete-time state (or mode). This modification leads to less strict requirements for both the foundational system and the control function. Additionally, we specified the average time rate range for ADIC as well as the duration separating two consecutive observations. Furthermore, we devised a straightforward and efficacious Lyapunov function, constituting a pivotal technical methodology in this paper.

The current study lays the foundation for several potential extensions:

- Variable and non-differentiable time delays in stochastic hybrid systems.
  Stochastic hybrid systems intricately depend on both the present state and specific past states. Addressing the stabilization challenges for the following highly nonlinear stochastic delay hybrid systems driven by Lévy noise
  $$\begin{aligned} dZ(t) &= f(Z(t), Z(t-\tau(t)), \Lambda(t), t)dt \\ &\quad + g(Z(t), Z(t-\tau(t)), \Lambda(t), t)dB(t) \\ &\quad + \int_X h(Z(t^-), Z((t-\tau(t))^-), \Lambda(t), t, x)\tilde{N}(dt, dx) \end{aligned}$$
  is a logical next step. While existing research primarily focuses on the examination of stochastic hybrid systems characterized by a fixed time delay, practical applications often involve time delays that exhibit variability over time. Consequently, investigating stochastic hybrid systems characterized by variable and non-differentiable time delays holds the potential to augment the faithfulness of mathematical models to real-world dynamics. This, in turn, facilitates a more precise comprehension of system behavior.
- Polynomial stability in stochastic hybrid systems.
  This paper primarily explores the exponential stability of the controlled systems. However, certain systems may manifest intricate nonlinear dynamics that pose challenges for a thorough characterization solely through the lens of exponential stability. Specifically, when a system exhibits pronounced nonlinear characteristics, the feasibility and precision of exponential stability analysis may be constrained. In the context of highly nonlinear systems, polynomial stability analysis emerges as a more adaptable methodology. By contemplating the polynomial representation of nonlinear systems, a more holistic comprehension

of the system's stability can be attained. Nonetheless, polynomial stability analysis commonly entails high-order polynomials, leading to relatively intricate mathematical expressions. This intricacy heightens the analytical challenge.

The aforementioned issues carry profound significance and, concurrently, pose substantial challenges. Tackling these problems will contribute to the enhancement of the stabilization theory of hybrid systems, bringing it into closer alignment with real-life scenarios. This holds paramount importance in propelling advancements in pertinent domains. We are enthusiastic about participating in extensive research and discourse with our readers, anticipating the reporting of progress on these issues in future publications.

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