Almost sure stabilization of hybrid systems by intermittent stochastic noise with jumps ⋆

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Abstract

This paper is concerned with the almost sure stabilization of hybrid systems by introducing intermittent stochastic noise with jumps. By means of the Lyapunov function method and the ergodic property of Markov chain, sufficient conditions are given for stability of hybrid stochastic systems with intermittent stochastic noise. With the aid of the comparison method and M-matrix theory, it is shown that hybrid systems can be stabilized by discrete-time intermittent stochastic feedback control provided the duration between two consecutive observations is sufficient small. Finally, two examples are presented to illustrate our results.

Key words: Hybrid stochastic systems, Poisson random measure, Almost sure stochastic stabilization, Intermittent stochastic feedback control, Discrete-time state observation.

1 Introduction

In our life, uncertainty is ubiquitous and pervasive, which is often encountered and regarded as the resource of instability of systems. However, it is noted that noise can also be used to stabilise a given unstable system or to make a system even more stable when it is stable. The study of stochastic stabilization was initiated by Hasminskii (1980) who used two white noise sources to stabilize a system. Since then, stabilization of differential system by noise have been studied by many authors and we here mention (Arnold et al.,1983; Appleby et al.,2008; Mao,1994; Caraballo et al.,2003; Wu and Hu,2009). In particular, stabilization theory was further established by Deng et al.(2012), Mao et al.(2007), Yuan et al.(2005), Yin et al.(2012), Zhao et al.(2020) for hybrid stochastic systems. Mao et al. (2007) considered the following unstable hybrid system

$$\dot{x}(t) = f(x(t), r(t)) \tag{1.1}$$

and stabilized this system by introducing the noise feedback control $g(x(t), r(t))dw(t)$, which is equivalent to examining the stability of the following stochastic hybrid system

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dw(t). \tag{1.2}$$

In these systems, the underlying process is a two-component process $(x(t), r(t))$, in which $x(t)$ describes the state and $r(t)$ is a continuous-time Markov chain in a finite state space $S = \{1, 2, \cdots, N\}$ and describes the random environment. Actually, (1.2) can also be regarded as a random switching system among the following N stochastic differential equation (SDE)

$$dx(t) = f(x(t), i)dt + g(x(t), i)dw(t), i = 1, 2, \cdots, N, \tag{1.3}$$

according to the Markov chain $r(t)$. Recently, Chao et al.(2017), Yin and Xi(2010), Zong et al.(2014), Tran and Tien(2022) have extended the stability results of Mao et al.(2007) to (1.2) and showed that even if some subsystems are not stable, the overall hybrid system (1.2) may still be stable, that is, Markov chain can serve as a stabilizing factor. Chao et al.(2017), Yin and Xi(2010), Zong et al.(2014), Tran and Tien (2022) revealed that jump diffusion noise can also play stabilizing roles.

It can be noted that the above noise as the feedback control needs continuous observation for the state $x(t)$, which is not easy to realize in practice. Mao(2013, 2016) introduced the feedback control based on the discrete-time state observations, and developed the corresponding theory in Hagiwara and Araki (1988) from deterministic systems to stochastic systems, which may save observing costs. That is, introduce a discrete-time stochastic feedback control $g(x(\delta(t)), r(t))dw(t)$, where $\delta(t) = t_0 + \lfloor (t - t_0)/\tau \rfloor \tau$, and $\lfloor (t - t_0)/\tau \rfloor$ is the integer part of

⋆ This paper was not presented at any IFAC meeting. This work is entirely theoretical and the results can be reproduced using the methods described in this paper. Corresponding author F. Wu. E-mail: wufuke@hust.edu.cn

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$(t-t_0)/\tau$, such that the controlled system

$$dx(t) = f(x(t), r(t))dt + g(x(\delta(t)), r(t))dw(t), \quad t \geq t_0 \tag{1.4}$$

becomes stable. Following this work, some interesting results on stochastic stabilization with discrete-time feedback control were established by Dong(2018), Song et al.(2018), Wan et al.(2023) and Liu et al.(2024). Indeed, the feedback control $g(x(\delta(t)), r(t))dw(t)$ is designed based on the discrete-time observations of the state $x(t)$ at times $t_0 + n\tau$ for $n \in \mathbb{N}$. Thus this strategy can not only save cost but also be easily implemented. However, the existing results (Fei et al.,2019; Li et al.,2020; Shao,2017; You et al.,2015) indicate that the original system (1.4) can be stabilized by feedback control only when the duration τ is sufficiently small. It means that the feedback controller $g(x(\delta(t)), r(t))dw(t)$ is almost continuously observing without any rest time. This paper will introduce a class of intermittent control strategy that allows the feedback controller to have obvious rest time, which implies that this strategy is more economic.

As a non-continuous control strategy, intermittent control includes working time and rest time. During the working time, the feedback controller is in running state, while during the rest time, it is in off state. Therefore, the controlled system can be viewed as a switching between a random system and a deterministic system. In comparison with continuous-time feedback control, this strategy can achieve the stability of system with less control time, and has wider application potential in stochastic stabilization theory. Zhang et al.(2018) applied the intermittent stochastic noise to stabilize nonlinear systems, and establish a class of theories on stabilization by intermittent stochastic perturbation. Liu and Wu(2020) showed that an unstable system can be stabilized by the intermittent stochastic feedback based on delay and discrete time observation. Mao et al.(2022) studied the stabilization of hybrid systems via periodically intermittent control and established a sufficient stability criterion on intermittent hybrid SDEs. Yin and Cao(2020) investigated the almost sure exponential stabilization of differential system by intermittent stochastic perturbation with jumps. For more intermittent control results of stochastic system, please refer to Liu et al. 2019, Liu et al. 2020, Wu et al. 2019,2022, Mao et al. 2023; Wan et al. 2024.

This work will focus on the almost sure exponential stabilization of nonlinear hybrid systems by intermittent stochastic noise with jumps. Given the unstable hybrid system (1.1), this paper aims to introduce the intermittent stochastic feedback control driven by Brownian motion $w(t)$ and the compensated Poisson random measure $\tilde{N}(dt, dv)$ to make the controlled system

$$\begin{aligned} dx(t) = {} & f(x(t), r(t))dt + g(x(\delta(t)), r(t))I(t)dw(t) \\ & + \int_{0<|v|<c} h(x(\delta(t)^-), r(t^-), v)I(t)\tilde{N}(dt, dv) \end{aligned} \tag{1.5}$$

to become almost surely exponentially stable, where $I : [t_0, \infty) \to \{0, 1\}$ is defined by

$$I(t) = \sum_{n=0}^{\infty} I_{[t_n, t_n+\theta_n)}(t), \quad t \geq t_0,$$

to show the control time and $[t_n, t_{n+1})$ is called the n-th control period. By the comparison method and M-matrix theory, it can be shown that the hybrid system (1.1) can be stabilized by intermittent stochastic feedback control.

The main contributions of this paper are highlighted as follows:

(1) This paper exhibits significant differences in the control of stochastic noise. Specifically, the control of stochastic noise adopts an intermittent mode, rather than the continuous control approach used in previous studies (Chao et al.,2017; Yin and Xi,2010; Zong et al.,2014). Different from the above mentioned works, we need to overcome the complex switching mechanism where stochastic noise and intermittent control occur simultaneously, and the derived stability criteria not only depend on the intensity of stochastic noise $w(\cdot)$, $\tilde{N}(\cdot, \cdot)$, but also on the intermittent control parameter θ. Furthermore, Theorem 3.10 reveals that intermittent stochastic perturbation induced by the compensation Poisson random jumps serves as a stabilizing factor, which is illustrated by Example 5.1.

(2) This paper focuses on the non-periodic intermittent control strategy for hybrid stochastic systems. By removing the quasi-periodic condition and the minimum control ratio condition in the existing works (Liu and Wu, 2020; Wu et al., 2022) and introducing the average noise control ratio, this control strategy becomes more flexible and efficient, which is also verified in Example 5.2. Meanwhile, it is mentioned that if $t_{n+1} - t_n = T$ and $\theta_n = \varrho T$, $\varrho \in (0, 1)$, the non-periodic intermittent control strategy adopted in this paper can degenerate into a periodic case, which is consistent with the studies of Zhang et al. (2018), Mao et al. (2022), Wu et al. (2019) and Wan et al. (2024).

(3) When $I(t) = 1$, (1.5) degenerates into the hybrid stochastic systems with the classical feedback control based on discrete time observation, namely, the state need to be observed at discrete times $t_0 + n\tau$ for $n \in \mathbb{N}$. In comparison with the existing feedback control base on discrete time observation in Mao (2013), Dong (2018), Song et al. (2018), Fei et al. (2019), Wan et al. (2023) and Liu et al. (2024), the intermittent control of this paper implies that the state is only observed at discrete times in the control width, say $\{t_0 + n\tau, n \in \mathbb{N}\} \bigcap \{\bigcup_{n\in\mathbb{N}}[t_n, t_n + \theta_n)\}$. In other words, the intermittent control strategy proposed in this paper only requires discrete-time observations within the working time of intermittent control period. This significantly reduces unnecessary observation operations, avoids the monitoring of the system state during non critical periods, and thus further reduces the control cost.

2 Preliminaries

Throughout this paper, unless otherwise specified, we use the following notation. Let $|.|$ denote the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by $A^\top$. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^\top A)}$ while its operator norm is defined by $\|A\| = \sup\{|Ax| : |x| = 1\}$. If A is a symmetric matrix, denote by $\Lambda_{max}(A)$ and $\Lambda_{min}(A)$ its

largest and smallest eigenvalue, respectively. The function $f: R_+ \to R^n$ is said to be cadlag if it is right continuous and left limits with $f(t^-) = \lim_{s\uparrow t} f(s)$. Let $w(t) = (w_1(t), \cdots, w_m(t))^\top$ be an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, $\mathcal{F}_0$ contains all P-null sets). Let $L(t)$ be an $\mathcal{F}_t$-adapted Lévy process with Lévy measure $\pi(.)$. Define

$$N(t, Z) := \sum_{t_0\leq s\leq t} I_Z(\Delta L(s)) = \sum_{t_0\leq s\leq t} I_Z[L(s) - L(s^-)],$$

where $N(\cdot,\cdot)$ defined on $[t_0, \infty) \times R_0^n$ is an $\mathcal{F}_t$-adapted Poisson random measure, and Z is a Borel subset of $R_0^n = R^n\backslash\{0\}$. The compensator $\tilde{N}$ of N is denoted by

$$\tilde{N}(dt, dv) = N(dt, dv) - \pi(dv)dt,$$

where π is a Lévy measure satisfying $\int_{R_0^n}(1\wedge|v|^2)\pi(dv) < \infty$.

Let $r(t), t \geq t_0$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2 \ldots N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. As a standing hypothesis, we assume that the Markov chain $r(t)$ is irreducible. Under this condition, we have that $r(t)$ has a unique stationary distribution $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_N) \in R^{1\times N}$ satisfying the linear equation

$$\tilde{\pi}\Gamma = 0 \quad \text{subject to } \sum_{i=1}^{N}\tilde{\pi}_i = 1$$

and $\tilde{\pi}_i > 0$ for each $i \in S$. Assume that the Markov chain $r(t)$, the Brownian motion $w(t)$ and the Poisson random measure $\tilde{N}(dt, dv)$ are independent. Let us consider the following hybrid stochastic systems by aperiodically intermittent feedback control based on discrete-time state observations

$$\begin{aligned} dx(t) &= f(x(t), r(t))dt + g(x(\delta(t)), r(t))I(t)dw(t) \\ &\quad + \int_{0<|v|<c} h(x(\delta(t)^-), r(t^-), v)I(t)\tilde{N}(dt, dv), \end{aligned} \tag{2.1}$$

on $t \geq t_0$, with the initial data $x(t_0) = x_0 \in R^n$, $r(t_0) = r_0 \in S$. Here $f: R^n \times S \to R^n$, $g: R^n \times S \to R^{n\times m}$ and $h: R^n \times S \times R_0^n \to R^n$. Let $\tau > 0$, $\delta(t) = t_0 + \lfloor(t - t_0)/\tau\rfloor\tau$, in which $\lfloor(t - t_0)/\tau\rfloor$ is the integer part of $(t - t_0)/\tau$. Recall that $I: [t_0, \infty) \to \{0, 1\}$ is defined as

$$I(t) = \sum_{n=0}^{\infty} I_{[t_n, t_n+\theta_n)}(t), \quad t \geq t_0, \tag{2.2}$$

where $[t_n, t_{n+1})$ is called the n-th control period, $0 < \theta_n \leq t_{n+1} - t_n$ is the n-th control width and $\lim_{n\to\infty} t_n = \infty$ such that the average noise control ratio θ is defined by

$$\theta := \liminf_{n\to\infty} \frac{1}{(t_n - t_0)} \sum_{k=0}^{n-1} \theta_k \in (0, 1). \tag{2.3}$$

Clearly, when $\tau = 0$, the discrete-time state feedback controlled system (1.5) degenerates into the continuous-time controlled SDEs

$$\begin{aligned} dx(t) &= f(x(t), r(t))dt + g(x(t), r(t))I(t)dw(t) \\ &\quad + \int_{0<|v|<c} h(x(t^-), r(t^-), v)I(t)\tilde{N}(dt, dv). \end{aligned} \tag{2.4}$$

By the strong ergodicity of Markov chains, this paper also utilizes the Lyapunov function method to establish sufficient criteria for almost sure exponential stability of hybrid stochastic systems (2.4).

Denote by $C^2(R^n \times S; R_+)$ the family of all real-valued nonegative functions $V(y, i)$ on $R^n \times S$, which are continuously twice differentiable in y. For any $V(y, i) \in C^2(R^n \times S; R_+)$, define $\mathcal{G}V: R^n \times S \times [t_0, \infty) \to R$ as

$$\begin{aligned} &\mathcal{G}V(y, i, t) \\ &= \mathcal{L}V(y, i, t) + \int_{0<|v|<c}[V(y + h(y, i, v)I(t), i) - V(y, i) \\ &-V_y(y, i)h(y, i, v)I(t)]\pi(dv) + (\Gamma V)(y, i), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \mathcal{L}V(y, i, t) &= V_y(y, i)f(y, i) \\ &\quad + \frac{1}{2}\text{trace}[g^\top(y, i)V_{yy}(y, i)g(y, i)I(t)], \\ (\Gamma V)(y, i) &= \sum_{j\in S}\gamma_{ij}V(y, j) = \sum_{j\in S, j\neq i}\gamma_{ij}(V(y, j) - V(x, i)) \end{aligned}$$

and $V_y(\cdot,\cdot)$, $V_{yy}(\cdot,\cdot)$ denote the gradient and Hessian matrix of $V(\cdot,\cdot)$, respectively.

Remark 2.1 Eq. (2.1) can also be expressed as follows:

$$\begin{cases} dx(t) = f(x(t), r(t))dt + g(x(\delta(t)), r(t))dw(t) \\ \quad + \displaystyle\int_{0<|v|<c} h(x(\delta(t)^-), r(t^-), v)\tilde{N}(dt, dv), \\ \qquad\qquad t \in [t_n, t_n + \theta_n), \\ dx(t) = f(x(t), r(t))dt, \ t \in [t_n + \theta_n, t_{n+1}). \end{cases} \tag{2.6}$$

When $t \in [t_n, t_n + \theta_n)$, (2.6) is a switching diffusion process with jumps. when $t \in [t_n, t_n + \theta_n)$, (2.6) is a switching process. In other words, system (2.6) will switch back and forth between these two classes of systems over time. This means that these three classes of noise, as noise feedback controls, are combined with intermittent control strategies to form a special switching mechanism.

In Liu and Wu(2020), the authors added the following quasi-periodic condition: $\inf_n\{\theta_n\} = a$ and $\sup_n\{t_{n+1} - t_n\} = b$, where $0 < a < b < \infty$, or the minimum control ratio condition: $c = \limsup_{n\to\infty} \frac{t_{n+1}-t_n-\theta_n}{t_{n+1}-t_n} \in (0, 1)$ to the aperiodic intermittent control strategy. However, it should be noted that these two conditions are still conservative and may not satisfy the wind power generation (see Liu and Chen (2015)). In fact, these two conditions do not guarantee that the solution of Eq. (5.6) is pth moment exponentially stable. This paper uses the average noise control ratio defined by (2.3) to characterize

the aperiodic intermittent control. By the average noise control ratio θ, this paper can estimate the lower bound on the length of the control interval on (t_0, t), i.e., for any $t \geq t_0$, there exists a positive constant $L_0 \geq 0$

$$L(t_0, t) \geq \theta(t - t_0) - L_0 \tag{2.7}$$

denotes the total control interval length in (t_0, t).

Remark 2.2 If $t_{n+1} - t_n = T$ and $\theta_n = \varrho T$, $\varrho \in (0, 1)$, then

$$I(t) := \tilde{I}(t) = \sum_{n=0}^{\infty} I_{[t_0+nT, t_0+nT+\varrho T)}(t). \tag{2.8}$$

This strategy is called as the periodic intermittent control with the control period T and the control width ϱT. Moreover, it can be observed that the average noise control ratio is $\theta = \varrho$.

In this paper, the following assumption is imposed on the coefficients f, g and h.

Assumption 2.3 *Assume that there exist three nonnegative constants k_1, k_2 and k_3 such that*

$$|f(x,i)-f(y,i)| \leq k_1|x-y|, |g(x,i)-g(y,i)| \leq k_2|x-y|$$
$$\int_{0<|v|<c} |h(x,i,v) - h(y,i,v)|^2 \pi(dv) \leq k_3|x-y|^2$$

for all $x, y \in R^n$ and $i \in S$. Moreover, assume that $f(0,i) = 0$, $g(0,i) = 0$ and $h(0,i,v) = 0$ for all $i \in S$.

Theorem 2.4 *Under Assumption 2.3, Eq. (2.1) has a unique global solution $x(t)$ on $t \geq t_0$.*

The proof is omitted since it is similar to Yin and Xi (2010).

Remark 2.5 When k_1, k_2 and k_3 are dependent on the state i, that is, k_1, k_2 and k_3 are replaced by k_{1i}, k_{2i} and k_{3i} in Assumption 2.4, the result of this theorem still holds and the proof process is not changed by letting $k_j = \max_{i \in S} k_{ji}$, $j = 1, 2, 3$. Of course, in this theorem, the global Lipschitz condition in this assumption can be replaced by the local Lipschitz condition and the linear growth condition.

3 Intermittent continuous-time stochastic stabilization

In this section, the stabilization problem of Eq. (3.1) via the continuous-time intermittent control is investigated. It is also shown that the hybrid system (1.1) can be stabilized by the linear feedback control.

3.1 Almost sure exponential stabilization

This subsection mainly aims to establish some sufficient stability criteria on intermittently hybrid stochastic systems with jumps. Consider the following continuous-time controlled SDEs

$$dy(t) = f(y(t), r(t))dt + g(y(t), r(t))I(t)dw(t) + \int_{0<|v|<c} h(y(t^-), r(t^-), v)I(t)\tilde{N}(dt, dv) \tag{3.1}$$

on $t \geq t_0$ with the initial value $y(t_0) = y_0$, $r(t_0) = r_0$.

Similar to Theorem 2.4, it can be proved that Eq. (3.1) has a unique global solution on $t \geq t_0$. Assumption 2.3 also indicate that $y(t) = 0$ is the trivial solution of Eq. (3.1). To study the stability of Eq. (3.1), let us propose the following non-zero property, which means that almost all sample paths of all solutions to (3.1) starting from the non-zero initial value will never reach the origin.

Assumption 3.1 *Assume that there exist two constants $h_i > 0$ and $k_4 \geq 0$ such that $|x + h(x,i,v)| \geq h_i|x|$ and $\int_{0<|v|<c} |x \cdot h(x,i,v)|\pi(dv) \leq k_4|x|^2$ for all $x \in R^n$, $i \in S$ and $v \in R_0^n$.*

Lemma 3.2 *Let Assumptions 2.3 and 3.1 hold. Then for any $y_0 \in R_0^n$ and $i \in S$,*

$$\mathbb{P}_{y_0,i}(y(t) \neq 0 \text{ for all } t \geq t_0) = 1. \tag{3.2}$$

Proof. For $y \neq 0$ and $i \in S$, define $V(y,i) = |y|^{-1}$. It is easily observed that

$$V_y(y,i) = -|y|^{-3}y, \; V_{yy}(y,i) = -|y|^{-3}I + 3|y|^{-5}yy^\top.$$

Then we obtain

$$\begin{aligned}\mathcal{G}V(y,i,t) = &-|y|^{-3}x^\top f(y,i) + \frac{1}{2}\Big(- |y|^{-3}|g(y,i)|^2 \\ &+3|y|^{-5}|y^\top g(y,i)|^2\Big)I(t) \\ &+ \int_{0<|v|<c} \Big[|y + h(y,i,v)I(t)|^{-1} - |y|^{-1} \\ &+|y|^{-3}y^\top h(y,i,v)I(t)\Big]\pi(dv).\end{aligned} \tag{3.3}$$

Noting that $|y + h(y,i,v)| > 0$, and by Lemma 3.1 in Applebaum and Siakalli(2009)

$$\begin{aligned}&|y + h(y,i,v)I(t)|^{-1} - |y|^{-1} + |y|^{-3}y^\top h(y,i,v)I(t) \\ &\leq \frac{[|h(y,i,v)|^2 + 2|y^\top \cdot h(y,i,v)|]I(t)}{|y|^2(|y + h(y,i,v)I(t)|)}.\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) yields

$$\begin{aligned}&\mathcal{G}V(y,i,t) \\ &= -|y|^{-3}y^\top f(y,i) + \frac{1}{2}(-|y|^{-3}|g(y,i)|^2 \\ &+3|y|^{-5}|y^\top g(y,i)|^2)I(t) \\ &+ \int_{0<|v|<c} \frac{[|h(y,i,v)|^2 + 2|y^\top \cdot h(y,i,v)|]I(t)}{|y|^2(|y + h(y,i,v)I(t)|)}\pi(dv).\end{aligned}$$

If $I(t) = 0$, then by Assumption 2.3,

$$\begin{aligned}\mathcal{G}V(y,i,t) &= -|y|^{-3}y^\top f(y,i) \\ &\leq |y|^{-2}|f(y,i)| \leq k_1|y|^{-1}.\end{aligned} \tag{3.5}$$

If $I(t)=1$,

$$\begin{aligned}\mathcal{G}V(y,i,t) &\le |y|^{-2}|f(y,i)| + |y|^{-3}|g(y,i)|^2 \\ &\quad + \frac{1}{h_i|y|^3}\int_{0<|v|<c}[|h(y,i,v)|^2 \\ &\quad + 2|y^\top \cdot h(y,i,v)|]\pi(dv) \\ &\le (k_1 + k_2^2 + \frac{k_3}{h_i} + \frac{2k_4}{h_i})|y|^{-1}. \end{aligned} \tag{3.6}$$

Combing (3.5) with (3.6) gives $\mathcal{G}V(y,i,t) \le (k_1 + k_2^2 + \frac{k_3}{h_i} + \frac{2k_4}{h_i})|y|^{-1}$. For the initial values $y(t_0) = y_0$ and $r(t_0) = i$, let $0 < \varepsilon < |y_0| < d$ and define the stopping time $\tau_\varepsilon := \inf\{t \ge t_0 : |y(t)| \le \varepsilon\}$ and $\tau_d := \inf\{t \ge t_0 : |y(t)| \ge d\}$. By the Itô formula, we have

$$\begin{aligned}&\mathbb{E}\Big[e^{-(k_1+k_2^2+\frac{k_3}{h_i}+\frac{2k_4}{h_i})(\tau_\varepsilon\wedge\tau_d\wedge t)}|y(\tau_\varepsilon\wedge\tau_d\wedge t)|^{-1}\Big] \\ &= |y_0|^{-1}e^{-(k_1+k_2^2+\frac{k_3}{h_i}+\frac{2k_4}{h_i})t_0} \\ &+\mathbb{E}\int_{t_0}^{\tau_\varepsilon\wedge\tau_d\wedge t} e^{-(k_1+k_2^2+\frac{k_3}{h_i}+\frac{2k_4}{h_i})s} \\ &\times\Big[-(k_1+k_2^2+\frac{k_3}{h_i}+\frac{2k_4}{h_i})|y(s)|^{-1}+\mathcal{G}V(y(s),r(s),s)\Big]ds \\ &\le |y_0|^{-1}e^{-(k_1+k_2^2+\frac{k_3}{h_i}+\frac{2k_4}{h_i})t_0}. \end{aligned} \tag{3.7}$$

Noting that $\tau_\varepsilon < \tau_d \wedge t$, we have $|y(\tau_\varepsilon\wedge\tau_d\wedge t)|^{-1} > \varepsilon^{-1}$. Then from (3.7),

$$\begin{aligned}&e^{-(k_1+k_2^2+\frac{k_3}{h_i}+\frac{2k_4}{h_i})t}\varepsilon^{-1}\mathbb{P}(\tau_\varepsilon < \tau_d\wedge t) \\ &\le \mathbb{E}\Big[e^{-(k_1+k_2^2+\frac{k_3}{h_i}+\frac{2k_4}{h_i})(\tau_\varepsilon\wedge\tau_d\wedge t)}|y(\tau_\varepsilon\wedge\tau_d\wedge t)|^{-1}\Big] \\ &\le |y_0|^{-1}e^{-(k_1+k_2^2+\frac{k_3}{h_i}+\frac{2k_4}{h_i})t_0}. \end{aligned}$$

By Assumption 2.3, Eq. (3.1) has a unique global solution. This implies that hybrid jump-diffusion process (3.1) has no finite explosion time, that is, $\tau_d \to \infty$ almost surely as $d \to \infty$. Therefore, for any $t \ge t_0$, $\mathbb{P}(\tau_\varepsilon < t) \le |y_0|^{-1}\varepsilon e^{(k_1+k_2^2+\frac{k_3}{h_i}+\frac{2k_4}{h_i})(t-t_0)}$. Now letting $\varepsilon \to \infty$ gives $\mathbb{P}(\tau_0 < t) = 0$ for any $t \ge t_0$, where $\tau_0 := \inf\{t \ge t_0 : |y(t)| = 0\}$. This completes the proof. □

Now, let us show that (3.1) is almost surely exponentially stable.

Assumption 3.3 *Assume that there exist constants $\alpha_i \in R$, $\beta_i \ge 0$ and $\lambda_i \ge 0$ such that*

$$x^\top f(x,i) \le \alpha_i|x|^2, |g(x,i)| \le \beta_i|x|, |x^\top g(x,i)| \ge \lambda_i|x|^2$$

for all $x \in R^n$ and $i \in S$.

Assumption 3.4 *Assume that there exists a function $\bar{h}_i(v) \ge 0$ and a constant $\vartheta_i > 0$ such that $|h(x,i,v)| \le \bar{h}_i(v)|x|$ and $\int_{0<|v|<c} x^\top h(x,i,v)\pi(dv) \ge \vartheta_i|x|^2$ for all $x \in R^n$ and $i \in S$.*

Remark 3.5 By Lemma 3.2, it is known that Assumption 3.1 prevents the process $y(t)$ of (3.1) jumps from a nonzero state to zero, which can help us construct a Lyapunov function such as $V(x) = |x|^p$ for $p \in (0,1)$ to obtain stability of Eq. (3.1). However, the proof of Theorem 3.7 indicates that Assumptions 2.3 and 3.1 alone are not sufficient to prove that Eq. (3.1) is almost surely exponentially stable. To ensure the stability of the intermittent hybrid stochastic system with jumps (3.1), Assumptions 3.3 and 3.4 are needed.

Let us first give a strong law of large numbers for local martingales.

Lemma 3.6 *(see Lipster (1980)) Let $M(t)$, $t \ge t_0$ be a local martingale vanishing at $t = t_0$ and $\zeta(t)$ a continuous adapted non-negative increasing process such that $\lim_{t\to\infty}\zeta(t) = \infty$ and $\int_{t_0}^{\infty}\frac{d\langle M\rangle(s)}{(1+\zeta(s))^2} < \infty$ a.s., where $\langle M\rangle(t) := \langle M, M\rangle(t)$ is the quadratic variation process. Then $\lim_{t\to\infty}\frac{M(t)}{\zeta(t)} = 0$ a.s.*

Theorem 3.7 *Let $p \in (0,1)$ and Assumptions 2.3, 3.1, 3.3, 3.4 hold. If*

$$\max_{i\in S}\Big\{\int_{0<|v|<c}\Big([\log(1+\bar{h}_i^p(v))]^2 + \bar{h}_i^p(v)\Big)\pi(dv)\Big\} < \infty,$$

Eq. (3.1) satisfies

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log(|y(t)|) &\le \sum_{i\in S}\tilde{\pi}_i\alpha_i + \theta\max_{i\in S}\Big(0.5\beta_i^2 \\ &\quad -\lambda_i^2 - \vartheta_i + \frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)\Big)\ a.s. \end{aligned} \tag{3.8}$$

In particular, if $\sum_{i\in S}\tilde{\pi}_i\alpha_i + \theta\max_{i\in S}\Big(0.5\beta_i^2 - \lambda_i^2 - \vartheta_i + \frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)\Big) < 0$, then the trivial solution of Eq. (3.1) is almost surely exponentially stable.

Proof. Fix $y_0 \ne 0$ and $r_0 \in S$ arbitrary, and rewrite $y(t;t_0,y_0,r_0) = y(t)$. By Lemma 3.2, $y(t) \ne 0$ for all $t \ge t_0$ almost surely. Define $V(x,i) = \log|x|^p$ for all $i \in S$ and $p \in (0,1)$. Applying the generalized Itô formula gives

$$\begin{aligned}&\log|y(t)|^p = \log|y_0|^p \\ &+\int_{t_0}^{t}\frac{1}{|y(s)|^p}\Big[p|y(s)|^{p-2}y^\top(s)f(y(s),r(s)) \\ &+\Big(\frac{p}{2}|y(s)|^{p-2}trace[g^\top(y(s),r(s))g(y(s),r(s))] \\ &+\frac{p(p-2)}{2}|y(s)|^{p-4}|y^\top(s)g(y(s),r(s))|^2\Big)I(s)\Big]ds \\ &-\int_{t_0}^{t}\frac{1}{2|y(s)|^{2p}}p^2|y(s)|^{2p-4}|y^\top(s)g(x(s),r(s))|^2I(s)ds \\ &+\int_{t_0}^{t}\int_{0<|v|<c}\Big\{\log(|y(s)+h(y(s),r(s),v)I(s)|^p) \\ &-\log(|y(s)|^p) \\ &-\frac{p|y(s)|^{p-2}y^\top(s)h(y(s),r(s),v)I(s)}{|y(s)|^p}\Big\}\pi(dv)ds \\ &+M_1(t)+M_2(t), \end{aligned} \tag{3.9}$$

where $M_1(t) = \int_{t_0}^t \frac{py^\top(s)g(y(s),r(s))I(s)}{|y(t)|^2}dw(s)$ and

$$M_2(t) = \int_{t_0}^t \int_{0<|v|<c} \log\Big[|y(s^-) + h(y(s^-),r(s^-),v)I(s)|^p\Big] - \log\Big[|y(s^-)|^p\Big]\tilde{N}(ds,dv)$$

are two local martingales vanishing at $t = t_0$. By Assumption 3.4 and the elementary inequality $\log(1+x) \le x$ for $x > 0$,

$$\begin{aligned}
&\int_{t_0}^t \int_{0<|v|<c} \Big\{ \log(|y(s) + h(y(s),r(s),v)I(s)|^p) \\
&\qquad - \log(|y(s)|^p) \\
&- \frac{p|y(s)|^{p-2}y^\top(s)h(y(s),r(s),v)I(s)}{|y(s)|^p}\Big\}\pi(dv)ds \\
&= \int_{t_0}^t \int_{0<|v|<c} \Big\{ \log\Big[\frac{|y(s)+h(y(s),r(s),v)I(s)|^p}{|y(s)|^p}\Big] \\
&\qquad - \frac{py^\top(s)h(y(s),r(s),v)I(s)}{|y(s)|^2}\Big\}\pi(dv)ds \\
&= \int_{t_0}^t \int_{0<|v|<c} \sum_{i=1}^N \log\Big[\frac{|y(s)+h(y(s),r(s),v)I(s)|^p}{|y(s)|^p}\Big] \\
&\qquad \times I_{\{r(s)=i\}}\pi(dv)ds \\
&- \int_{t_0}^t \int_{0<|v|<c} \sum_{i=1}^N \frac{py^\top(s)h(y(s),r(s),v)I(s)}{|y(s)|^2} \\
&\qquad \times I_{\{r(s)=i\}}\pi(dv)ds \\
&\le \int_{t_0}^t \int_{0<|v|<c} \sum_{i=1}^N \log\Big[1+\bar{h}_i^p(v)I(s)\Big] I_{\{r(s)=i\}}\pi(dv)ds \\
&\qquad - \int_{t_0}^t \sum_{i=1}^N p\vartheta_i I_{\{r(s)=i\}}I(s)ds \\
&\le \int_{t_0}^t \int_{0<|v|<c} \bar{h}_{r(s)}^p(v)I(s)\pi(dv)ds - \int_{t_0}^t p\vartheta_{r(s)}I(s)ds. \quad (3.10)
\end{aligned}$$

By Assumption 3.3, substituting (3.10) into (3.9) gives

$$\begin{aligned}
\log|y(t)|^p \le \log|y_0|^p + \int_{t_0}^t \Big(&p\alpha_{r(s)} + [0.5p\beta_{r(s)}^2 \\
&- p\vartheta_{r(s)} + \int_{0<|v|<c} \bar{h}_{r(s)}^p(v)\pi(dv)]I(s)\Big)ds \\
&- \int_0^t \frac{p}{|y(t)|^4}|y^\top(s)g(y(s),r(s))|^2 I(s)ds \\
&+ M_1(t) + M_2(t). \quad (3.11)
\end{aligned}$$

The quadratic variation of the martingale $M_1(t)$ is

$$\langle M_1(t), M_1(t)\rangle = \int_{t_0}^t \frac{p^2|y^\top(s)g(y(s),r(s))|^2 I(s)}{|y(s)|^4}ds.$$

For arbitrary $\varepsilon \in (0,1)$ and $k = 1, 2, \cdots$, the exponential martingale inequality (Applebaum and Siakalli (2009)) gives

$$\mathbb{P}\Big\{\sup_{t_0\le t\le n}\Big[M_1(t) - \frac{\varepsilon}{2}\langle M_1(t), M_1(t)\rangle\Big] > \frac{2}{\varepsilon}\log n\Big\} \le \frac{1}{n^2}.$$

By the Borel–Cantelli lemma, for almost all $\omega \in \Omega$, there is an integer $n_0 = n_0(\omega)$ such that if $n \ge n_0$,

$$M_1(t) \le \frac{2}{\varepsilon}\log n + \frac{\varepsilon}{2}\langle M_1(t), M_1(t)\rangle \quad (3.12)$$

for all $t_0 \le t \le n$. Substituting (3.12) into (3.11), and then using Assumption 3.3 yields

$$\begin{aligned}
\log|y(t)|^p \le \log|y_0|^p + \frac{2}{\varepsilon}\log n + \int_{t_0}^t p\alpha_{r(s)}ds + M_2(t) \\
+ \int_{t_0}^t [0.5p\beta_{r(s)}^2 - p\vartheta_{r(s)} + \int_{0<|v|<c} \bar{h}_{r(s)}^p(v)\pi(dv) \\
- (p - \frac{p^2}{2}\varepsilon)\lambda_{r(s)}^2]I(s)ds
\end{aligned}$$

for all $t_0 \le t \le n,\ n \ge n_0$ almost surely. Hence for almost all $\omega \in \Omega$, if $t_0 + n - 1 \le t \le t_0 + n$ and $n \ge n_0$,

$$\begin{aligned}
&\frac{1}{t}\log|y(t)| \\
&\le \frac{1}{p(t_0+n-1)}\Big(\log|y_0|^p + \frac{2}{\varepsilon}\log n + M_2(t)\Big) \\
&+ \frac{1}{t}\int_{t_0}^t \alpha_{r(s)}ds + \frac{1}{t}\int_{t_0}^t \Big\{[0.5\beta_{r(s)}^2 - \vartheta_{r(s)} \\
&+ \frac{1}{p}\int_{0<|v|<c} \bar{h}_{r(s)}^p(v)\pi(dv) - (1 - \frac{p}{2}\varepsilon)\lambda_{r(s)}^2]I(s)\Big\}ds.
\end{aligned}$$

On the other hand, by Assumption 3.4,

$$\begin{aligned}
&\langle M_2(t), M_2(t)\rangle \\
&= \int_{t_0}^t \int_{0<|v|<c} \Big[\log(|y(s^-) + h(y(s^-),r(s^-),v) \\
&\qquad \times I(s)|^p) - \log(|y(s^-)|^p)\Big]^2\pi(dv)ds \\
&\le \int_{t_0}^t \int_{0<|v|<c} \max_{i\in S}[\log(1+\bar{h}_i^p(v)I(s))]^2\pi(dv)ds \\
&\le C(t - t_0),
\end{aligned}$$

where $C = \max_{i\in S}\int_{0<|v|<c}[\log(1+\bar{h}_i^p(v))]^2\pi(dv) < \infty$. Then, by Lemma 3.6, $\lim_{t\to\infty}\frac{M_2(t)}{t} = 0$ $a.s.$, which implies that

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log(|y(t)|) \\
&\le \lim_{t\to\infty}\frac{1}{t}\int_{t_0}^t \alpha_{r(s)}ds + \lim_{t\to\infty}\frac{1}{t}\int_{t_0}^t \Big\{[0.5\beta_{r(s)}^2 - \vartheta_{r(s)} \\
&+ \frac{1}{p}\int_{0<|v|<c} \bar{h}_{r(s)}^p(v)\pi(dv) - (1 - \frac{p}{2}\varepsilon)\lambda_{r(s)}^2]I(s)\Big\}ds. \quad (3.13)
\end{aligned}$$

Since the Markov chain is exponentially ergodic for the finite states, $\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^t I_{\{r(s)=i\}}ds = \tilde{\pi}_i$ $a.s$ and $\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^t \alpha_{r(s)}ds = \lim_{t\to\infty}\sum_{i\in S}\alpha_i\frac{1}{t}\int_{t_0}^t I_{\{r(s)=i\}}ds =$

$\sum_{i\in S}\tilde{\pi}_i\alpha_i$ *a.s.* Thus,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log(|y(t)|)\\
&\le\sum_{i\in S}\tilde{\pi}_i\alpha_i+\max_{i\in S}\Big(0.5\beta_i^2-\vartheta_i+\frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)\\
&\qquad-(1-\frac{p}{2}\varepsilon)\lambda_i^2\Big)\lim_{t\to\infty}\frac{1}{t-t_0}\int_{t_0}^{t}I(s)ds\\
&\le\sum_{i\in S}\tilde{\pi}_i\alpha_i+\theta\max_{i\in S}\Big(0.5\beta_i^2-\vartheta_i\\
&\qquad+\frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)-(1-\frac{p}{2}\varepsilon)\lambda_i^2\Big). \quad (3.14)
\end{aligned}$$

Letting $\varepsilon\to 0$ yields the desired assertion (3.8). □

Remark 3.8 The emergence of $I(t)$ has changed the original characteristics of the hybrid system in Yin and Xi(2010), Chao et al.(2017). As a result, the derived stability results are no longer applicable to the intermittent stochastic system. Therefore, we utilize stochastic analysis and inequality techniques to address the problem of the complex switching mechanism that arises when noise feedback is combined with intermittent control. In addition, Theorem 3.7 shows that the trivial solution of Eq. (3.1) is almost surely exponentially stable if and only if $\theta\in(\frac{\sum_{i\in S}\tilde{\pi}_i\alpha_i}{\min_{i\in S}\{\lambda_i^2-0.5\beta_i^2+\vartheta_i-\frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)\}},1)$, whose convergent speed not only depends on $\alpha_i,\beta_i,\lambda_i,\vartheta_i$ and $\bar{h}_i(v)$, but also on the parameter θ.

Remark 3.9 Based on our assumption, the deterministic system

$$\dot{y}(t)=f(y(t),r(t)) \quad (3.15)$$

is unstable, that is $\sum_{i\in S}\tilde{\pi}_i\alpha_i>0$. Therefore, the right hand side of equation (3.8), $\max_{i\in S}\theta\Big(0.5\beta_i^2-\lambda_i^2-\vartheta_i+\frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)\Big)<0$ implies that for all $i\in S$, the deterministic system (3.15) can be stabilized by noise, which in turn leads to the stability of the entire hybrid system.

Let us return to Remark 2.2. If $I(t)$ is defined as (2.8), stochastic system (3.1) degenerates the hybrid SDEs with periodically intermittent stochastic noise

$$\begin{aligned}
&dy(t)=f(y(t),r(t))dt+g(y(t),r(t))\tilde{I}(t)dw(t)\\
&+\int_{0<|v|<c}h(y(t^-),r(t^-),v)\tilde{I}(t)\tilde{N}(dt,dv). \quad (3.16)
\end{aligned}$$

Unlike $I(t)$, we shall use the periodicity of the function $\tilde{I}(t)$ to obtain a better stabilization results than Theorem 3.7. By improving the intermittent parameter ϱ, the following stability theorem follows.

Theorem 3.10 *Let conditions of Theorem 3.7 hold. The trivial solution $y(t)$ of Eq. (3.16) satisfies*

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}\log(|y(t)|)&\le\sum_{i\in S}\tilde{\pi}_i\Big(\alpha_i+\varrho[0.5\beta_i^2-\lambda_i^2\\
&\quad-\vartheta_i+\frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)]\Big)\ a.s. \quad (3.17)
\end{aligned}$$

In particular, if ϱ is in

$$\Big(\frac{\sum_{i\in S}\tilde{\pi}_i\alpha_i}{\sum_{i\in S}\tilde{\pi}_i[\lambda_i^2-0.5\beta_i^2+\vartheta_i-\frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)]},1\Big),$$

then the trivial solution $y(t)$ of Eq. (3.16) is almost surely exponentially stable.

To prove Theorem 3.10, the following lemma is need (also see Song et al.(2018)).

Lemma 3.11 *For any $t\ge 0$, $h>0$ and $i\in S$,*

$$\mathbb{P}(r(s)\ne i\ \text{for some}\ s\in[t,t+h]|r(t)=i)\le 1-e^{-\tilde{\gamma}h},$$

where $\tilde{\gamma}=\max_{i\in S}(-\gamma_{ii})$.

Proof of Theorem 3.10. Similar to the proof of (3.13),

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log(|y(t)|)\le\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\Big(\alpha_{r(s)}+[0.5\beta_{r(s)}^2\\
&-\lambda_{r(s)}^2-\vartheta_{r(s)}+\frac{1}{p}\int_{0<|v|<c}\bar{h}_{r(s)}^p(v)\pi(dv)]\tilde{I}(s)\Big)ds.
\end{aligned}$$

By the ergodic property of Markov chain $r(t)$,

$$\lim_{t\to\infty}\frac{1}{t}\log(|y(t)|)\le\sum_{i\in S}\tilde{\pi}_i\alpha_i+\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\psi_{r(s)}\tilde{I}(s)ds, \quad (3.18)$$

where

$$\psi_{r(s)}=0.5\beta_{r(s)}^2-\lambda_{r(s)}^2-\vartheta_{r(s)}+\frac{1}{p}\int_{0<|v|<c}\bar{h}_{r(s)}^p(v)\pi(dv).$$

Let us claim that

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\psi_{r(s)}\tilde{I}(s)ds=\varrho\sum_{i\in S}\tilde{\pi}_i\psi_i\ \ a.s. \quad (3.19)$$

To prove (3.19), it is sufficient to prove

$$\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\psi_{r(s)}\tilde{I}(s)ds\ge\varrho\sum_{i\in S}\tilde{\pi}_i\psi_i \quad (3.20)$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\psi_{r(s)}\tilde{I}(s)ds\le\varrho\sum_{i\in S}\tilde{\pi}_i\psi_i. \quad (3.21)$$

Let $\delta \in (0,1)$ be arbitrary and $\varpi = \varrho T/m$ for a sufficiently large integer m such that $\varpi < \delta$. It follows that

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \psi_{r(s)}\tilde{I}(s)ds$$
$$= \liminf_{n\to\infty} \frac{1}{(n+1)T}\sum_{j=0}^{n}\int_{t_0+jT}^{t_0+jT+\varrho T} \psi_{r(s)}ds \geq \sum_{v=0}^{m-1} Q_v, \quad (3.22)$$

where $Q_v = \liminf_{n\to\infty} \frac{1}{(n+1)T}\sum_{j=0}^{n}\int_{t_0+jT+v\varpi}^{t_0+jT+(v+1)\varpi} \psi_{r(s)}ds.$

Let us first show

$$Q_0 = \liminf_{n\to\infty} \frac{1}{(n+1)T}\sum_{j=0}^{n}\int_{t_0+jT}^{t_0+jT+\varpi} \psi_{r(s)}ds.$$

For each $i \in S$, define $\tau_0^i = \inf\{j \geq 0 : r(t_0+jT) = i\}$ and $\tau_n^i = \inf\{j \geq \tau_{n-1}^i : r(t_0+jT) = i\}$ for $n \geq 1$. Then τ_n^i are finite stopping times such that $0 \leq \tau_0^i < \cdots < \tau_n^i \uparrow \infty$ a.s. Set $S_n^i = \inf\{j \geq 0 : \tau_j^i \leq n\}$ and let T_n^i represent the number of the elements in set S_n^i. By the ergodic property of the Markov chain, we have $\lim_{n\to\infty} \frac{T_n^i}{n+1} = \tilde{\pi}_i$. Thus,

$$Q_0 = \liminf_{n\to\infty} \frac{1}{(n+1)T}\sum_{i\in S}\sum_{j\in S_n^i}\int_{t_0+jT}^{t_0+jT+\varpi} \psi_{r(s)}ds$$
$$= \liminf_{n\to\infty}\Big\{\sum_{i\in S}\frac{T_n^i}{(n+1)T}\Big(\frac{1}{T_n^i}\sum_{j\in S_n^i}\int_{t_0+jT}^{t_0+jT+\varpi} \psi_{r(s)}ds\Big)\Big\}$$
$$\geq \sum_{i\in S}\frac{1}{T}\liminf_{n\to\infty}\frac{T_n^i}{(n+1)}\Big(\liminf_{n\to\infty}\frac{1}{T_n^i} \times \sum_{j\in S_n^i}\int_{t_0+jT}^{t_0+jT+\varpi} \psi_{r(s)}ds\Big)$$
$$= \sum_{i\in S}\frac{\tilde{\pi}_i}{T}\liminf_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\int_{\tau_j^iT}^{\tau_j^iT+\varpi} \psi_{r(s)}ds. \quad (3.23)$$

By the strong Markov property, $\{r(\tau_j^iT+t)\}_{t\geq t_0}$ forms a Markov chain with the same generator Γ which starts from i. Then we have that $\{\int_{\tau_j^iT}^{\tau_j^iT+\varpi} \psi_{r(s)}ds\}_{j\geq 0}$ are independent identically distributed with mean value $\mathbb{E}(\int_{\tau_0^iT}^{\tau_0^iT+\varpi} \psi_{r(s)}ds)$. By the large number law,

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\int_{\tau_j^iT}^{\tau_j^iT+\varpi} \psi_{r(s)}ds = \mathbb{E}\int_{\tau_0^iT}^{\tau_0^iT+\varpi} \psi_{r(s)}ds.$$

Note that $r(\tau_0^iT) = i$ and $\varpi < \delta$. Letting $\Omega_0 = \{\omega \in \Omega : r(s) \neq i \text{ for some s} \in [\tau_0^iT, \tau_0^iT+\varpi]\}$ and by Lemma 3.11, we get $\mathbb{P}(\Omega_0) \leq 1 - e^{-\tilde{\gamma}\varpi} \leq \tilde{\gamma}\varpi$. Hence,

$$\mathbb{E}\int_{\tau_0^iT}^{\tau_0^iT+\varpi} \psi_{r(s)}ds$$
$$= \mathbb{E}\Big(I_{\Omega_0}\int_{\tau_0^iT}^{\tau_0^iT+\varpi} \psi_{r(s)}ds\Big) + \mathbb{E}\Big(I_{\bar{\Omega}_0}\int_{\tau_0^iT}^{\tau_0^iT+\varpi} \psi_{r(s)}ds\Big)$$
$$\geq \mathbb{E}\Big(I_{\Omega_0}\int_{\tau_0^iT}^{\tau_0^iT+\varpi} \min_{j\neq i}\psi_j ds\Big) + \mathbb{E}\Big(I_{\bar{\Omega}_0}\int_{\tau_0^iT}^{\tau_0^iT+\varpi} \psi_i ds\Big)$$
$$\geq \mathbb{E}\Big(I_{\Omega_0}\int_{0}^{\varpi} \min_{j\in S}\psi_j ds\Big) + \mathbb{E}\Big(I_{\bar{\Omega}_0}\int_{0}^{\varpi} \psi_i ds\Big)$$
$$= \min_{i\in S}\psi_i\varpi\mathbb{P}(\Omega_0) + \psi_i\varpi\mathbb{P}(\bar{\Omega}_0)$$
$$\geq \min_{i\in S}\psi_i\tilde{\gamma}\varpi^2 + \psi_i\varpi - \psi_i\tilde{\gamma}\varpi^2$$
$$\geq \psi_i\varpi - (\max_{i\in S}\psi_i - \min_{i\in S}\psi_i)\tilde{\gamma}\varpi^2.$$

Substituting this into (3.23) gives $Q_0 \geq \frac{\varpi}{T}\sum_{i\in S}\tilde{\pi}_i\psi_i - (\max_{i\in S}\psi_i - \min_{i\in S}\psi_i)\tilde{\gamma}\frac{\varpi^2}{T}$. Similarly, we can show $Q_v \geq \frac{\varpi}{T}\sum_{i\in S}\tilde{\pi}_i\psi_i - (\max_{i\in S}\psi_i - \min_{i\in S}\psi_i)\tilde{\gamma}\frac{\varpi^2}{T}$ for $v = 1, 2, \cdots, m-1$. Combing these with (3.22) yields

$$\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\psi_{r(s)}\tilde{I}(s)ds$$
$$\geq m[\frac{\varpi}{T}\sum_{i\in S}\tilde{\pi}_i\psi_i - (\max_{i\in S}\psi_i - \min_{i\in S}\psi_i)\tilde{\gamma}\frac{\varpi^2}{T}]$$
$$\geq \varrho\sum_{i\in S}\tilde{\pi}_i\psi_i - (\max_{i\in S}\psi_i - \min_{i\in S}\psi_i)\rho\tilde{\gamma}\delta.$$

Similarly,

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\psi_{r(s)}\tilde{I}(s)ds \leq m[\frac{\varpi}{T}\sum_{i\in S}\tilde{\pi}_i\psi_i + \max_{i\in S}\psi_i\tilde{\gamma}\frac{\varpi^2}{T}]$$
$$\leq \varrho\sum_{i\in S}\tilde{\pi}_i\psi_i + \max_{i\in S}\psi_i\rho\tilde{\gamma}\delta.$$

Since $\delta > 0$ is arbitrary, $\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\psi_{r(s)}\tilde{I}(s)ds \geq \varrho\sum_{i\in S}\tilde{\pi}_i\psi_i$, $\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\psi_{r(s)}\tilde{I}(s)ds \leq \varrho\sum_{i\in S}\tilde{\pi}_i\psi_i$. The required assertion (3.17) follows from (3.18) and (3.19). The proof is therefore completed. □

Remark 3.12 The proof of Theorem 3.10 is highly technical. In this process, not only the transition probability inequality is employed, but also the periodic characteristics of $\tilde{I}(t)$ and the ergodic property of the Markov chain are utilized. Furthermore, under the periodic control strategy, the criteria for determining stability are much more lenient. In Theorem 3.10, when the right hand side of equation (3.17) is less than 0, it implies that the hybrid system is stable. However, the right hand side of equation (3.17),

$$\sum_{i\in S}\tilde{\pi}_i\Big(\varrho[0.5\beta_i^2 - \lambda_i^2 - \vartheta_i + \frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)]\Big) < 0$$

does not require that for all $i \in S$, all deterministic systems (3.15) can be stabilized by noise. In other words,

this theorem allows that the subsystems are unstable under some modes i. By using the ergodicity of the Markov chain, we can cleverly ensure the stability of the whole hybrid system and effectively make up for the instability of subsystems in some modes.

Remark 3.13 In Theorems 3.7 and 3.10, λ_i and ϑ_i reveal the stabilization contribution of Brownian motion $w(t)$, the compensated Poisson noise $\tilde{N}$ and Markov chain $r(t)$. For the periodic intermittent control, when the control period T is fixed, we can enhance or weaken the noise intensity to shorten or increase the control time. Hence, the intermittent control strategy in this paper is more flexible. For example, as the two parameters λ_i and ϑ_i increases, the lower bound $\frac{\sum_{i\in S}\tilde{\pi}_i\alpha_i}{\sum_{i\in S}\tilde{\pi}_i[\lambda_i^2-0.5\beta_i^2+\vartheta_i-\frac{1}{p}\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)]}$ of the intermittent parameter ϱ can be taken smaller, which means that the control time can be shortened and the control cost can be reduced accordingly.

3.2 Design of linear feedback control with intermittent jump-diffusion noise

Suppose that f satisfies Assumptions 2.3 and 3.3. Let us show that the hybrid system (1.1) can be stabilized by linear feedback control with intermittent Brownian motion and the compensated Poisson random measures. Let $g(x,i)=(G_{1,i}x,G_{2,i}x,\cdots,G_{m,i}x)$ and $h(x,i,v)=H_i(v)x$ where $G_{k,i}$ and $H(i,v)$ are all $n\times n$ matrices. Then the controlled hybrid stochastic systems becomes

$$dy(t)=f(y(t),r(t))dt+\sum_{k=1}^{m}G_{k,r(t)}y(t)I(t)dw(t)+\int_{0<|v|<c}H_{r(t^-)}(v)y(t^-)I(t)\tilde{N}(dt,dv),\quad(3.24)$$

where $I(t)$ is defined by (2.2).

Assumption 3.14 *For any $i\in S$, assume that $\int_{0<|v|<c}|H_i(v)|\pi(dv)<\infty$ and that $H_i(v)$ does not have any eigenvalue equal to $-\pi$ almost surely.*

It is obvious that Eq. (3.24) has a unique solution on $t\geq t_0$. By Theorem 3.7, we shall obtain some sufficient conditions on the coefficients of (3.24) for the trivial solution to be almost surely exponentially stable.

Theorem 3.15 *Let Assumptions 2.3 and 3.1 hold. Assume that for any $x\in R^n$ and $i\in S$, there are nonnegative constants $\bar{\beta}_i$, $\bar{\lambda}_i$ and $\bar{\vartheta}_i$ such that $\sum_{k=1}^m|G_{k,i}x|^2\leq\bar{\beta}_i|x|^2$, $\sum_{k=1}^m|x^\top G_{k,i}x|^2\geq\bar{\lambda}_i|x|^4$, and $\int_{0<|v|<c}x^\top H_i(v)x\pi(dv)\geq\bar{\vartheta}_i|x|^2$. Then the trivial solution $y(t)$ of Eq. (3.24) satisfies*

$$\limsup_{t\to\infty}\frac{1}{t}\log(|y(t)|)\leq\sum_{i\in S}\tilde{\pi}_i\alpha_i+\theta\max_{i\in S}\Big(0.5\bar{\beta}_i-\bar{\lambda}_i-\bar{\vartheta}_i+\frac{1}{p}\int_{0<|v|<c}|H_i(v)|^p\pi(dv)\Big)\ a.s.$$

In particular, if $\sum_{i\in S}\tilde{\pi}_i\alpha_i+\theta\max_{i\in S}\Big(0.5\bar{\beta}_i-\bar{\lambda}_i-\bar{\vartheta}_i+\frac{1}{p}\int_{0<|v|<c}|H_i(v)|^p\pi(dv)\Big)<0$, then the trivial solution of Eq. (3.24) is almost surely exponentially stable.

Proof. Let $\beta_i=\sqrt{\bar{\beta}_i}$, $\lambda_i=\sqrt{\bar{\lambda}_i}$ and $\vartheta_i=\bar{\vartheta}_i$. It follows that the coefficients of Eq. (3.24) satisfy Assumptions 3.3 and 3.4. By Theorem 3.7, Eq. (3.24) is almost surely exponentially stable if

$$\sum_{i\in S}\tilde{\pi}_i\alpha_i+\theta\max_{i\in S}\Big(0.5\bar{\beta}_i-\bar{\lambda}_i-\bar{\vartheta}_i+\frac{1}{p}\int_{0<|v|<c}|H_i(v)|^p\pi(dv)\Big)<0.$$

If $I(t)$ of Eq. (3.24) is changed to $\tilde{I}(t)$, then we have the following hybrid stochastic systems

$$dy(t)=f(y(t),r(t))dt+\sum_{k=1}^{m}G_{k,r(t)}y(t)\tilde{I}(t)dw(t)+\int_{0<|v|<c}H_{r(t^-)}(v)y(t^-)\tilde{I}(t)\tilde{N}(dt,dv).\quad(3.25)$$

Theorem 3.16 *Let conditions of Theorem 3.15 hold. The trivial solution of Eq. (3.25) satisfies*

$$\limsup_{t\to\infty}\frac{1}{t}\log(|y(t)|)\leq\sum_{i\in S}\tilde{\pi}_i\Big(\alpha_i+\varrho[0.5\bar{\beta}_i-\bar{\lambda}_i-\bar{\vartheta}_i+\frac{1}{p}\int_{0<|v|<c}|H_i(v)|^p\pi(dv)]\Big)a.s.\quad(3.26)$$

If

$$\varrho\in\Big(\frac{\sum_{i\in S}\tilde{\pi}_i\alpha_i}{\sum_{i\in S}\tilde{\pi}_i[\bar{\lambda}_i-0.5\bar{\beta}_i+\bar{\vartheta}_i-\frac{1}{p}\int_{0<|v|<c}|H_i(v)|^p\pi(dv)]},1\Big),$$

Eq. (3.25) is almost surely exponentially stable.

Actually, there are many choices for the matrices $G_{k,i}$ and $H_i(v)$ in order to stabilize the given hybrid systems (1.1). For example, in (3.25), let $G_{k,i}=\sigma_{k,i}I$ for $1\leq k\leq m$, $i\in S$ and $H_i(v)=h_i(v)I$, where $\sigma_{k,i}$ are constants and I is the $n\times n$ identity matrix. Then the hybrid stochastic system (3.25) can be rewritten as

$$dy(t)=f(y(t),r(t))dt+\sum_{k=1}^{m}\sigma_{k,r(t)}y(t)\tilde{I}(t)dw(t)+\int_{0<|v|<c}h_{r(t^-)}(v)y(t^-)\tilde{I}(t)\tilde{N}(dt,dv).\quad(3.27)$$

Note that for each $i\in S$, $\sum_{k=1}^m|G_{k,i}x|^2=\Big(\sum_{k=1}^m\sigma_{k,i}^2\Big)|x|^2$, $\sum_{k=1}^m|x^\top G_{k,i}x|^2=\Big(\sum_{k=1}^m\sigma_{k,i}^2\Big)|x|^4$ and $\int_{0<|v|<c}x^\top H_i(v)x\pi(dv)=\int_{0<|v|<c}h_i(v)\pi(dv)|x|^2$ for all $x\in R^n$. By Theorem 3.16, Eq. (3.27) satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log(|y(t)|)\leq\sum_{i\in S}\tilde{\pi}_i\Big(\alpha_i+\varrho[C_{h_i}^p-0.5\sum_{k=1}^m\sigma_{k,i}^2]\Big)$$

almost surely, where

$$C_{h_i}^p = \frac{1}{p}\int_{0<|v|<c}|h_i(v)|^p\pi(dv) - \int_{0<|v|<c}h_i(v)\pi(dv).$$

If $\varrho \in \Big(\frac{\sum_{i\in S}\tilde{\pi}_i\alpha_i}{\sum_{i\in S}\tilde{\pi}_i[0.5\sum_{k=1}^m\sigma_{k,i}^2 - C_{h_i}^p]}, 1\Big)$, then the hybrid stochastic system (3.27) is almost surely exponentially stable.

4 Intermittent discrete-time stochastic stabilization

This section will establish a sufficient stability criterion on hybrid stochastic systems by intermittent feedback control based on discrete-time observations. In fact, Eq. (2.1) can be rewritten as the following hybrid stochastic delay differential equations (SDDEs)

$$\begin{aligned} dx(t) &= f(x(t), r(t))dt + g(x(t-\zeta(t)), r(t))I(t)dw(t) \\ &+ \int_{0<|v|<c} h(x((t-\zeta(t))^-), r(t^-), v)I(t)\tilde{N}(dt, dv), \end{aligned} \tag{4.1}$$

where $\zeta : [t_0, \infty) \to [0, \tau]$ is defined by $\zeta(t) = t - t_0 - k\tau$ for $t_0 + k\tau \le t < t_0 + (k+1)\tau$, $k = 0, 1, 2, \cdots$. For delay system, Lemma 3.2 is invalid, so the Itô formula to $\log|x(t)|^p$ for $p \in (0, 1)$ cannot be used. This section will adopt a comparative method to study the almost sure exponential stability of the solution to Eq. (4.1). It will be shown that if auxiliary hybrid stochastic system with continuous-time intermittent feedback control (3.1) is pth moment exponentially stable, then so is the hybrid stochastic systems by discrete-time intermittent feedback control (4.1) for sufficiently small τ. Let us use the M-matrix theory to study the pth moment exponential stability of the system (3.1).

Assumption 4.1 *There exists $p \in (0, 1)$ such that*

$$\mathcal{A}_p := \mathrm{diag}(\rho_1(p), \cdots, \rho_N(p)) - \Gamma \tag{4.2}$$

is a nonsingular M-matrix, where $\rho_i(p) = 0.5p[(2-p)\lambda_i^2 - \beta_i^2] + p\vartheta_i - \int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv) - p\alpha_i$.

Remark 4.2 Assumption 4.1 is proposed to ensure the exponential stability of Eq. (3.1), which is the key to obtain the exponential stability of Eq. (4.1) by using comparative method. Moreover, this assumption is easy to be checked by the M-matrix theory.

Since $\mathcal{A}_p$ is a nonsingular M-matrix. By Lemma 4.2 of Mao(1999), $\mathcal{A}_p^{-1}$ is well defined with $\mathcal{A}_p^{-1} \ge 0$. Setting $(\varsigma_1, \cdots, \varsigma_N)^\top := \mathcal{A}_p^{-1}\overrightarrow{1}$, where $\overrightarrow{1} = (1, \cdots, 1)^\top$, we can obtain that $\varsigma_i > 0$, $i \in S$. For simplicity, define $\hat{\varsigma} = \min_{i\in S}\varsigma_i$, $\check{\varsigma} = \max_{i\in S}\varsigma_i$, $\bar{\varsigma} = \max_{i\in S}(p\alpha_i + \sum_{j=1}^N\gamma_{ij}\frac{\varsigma_j}{\varsigma_i})$.

Theorem 4.3 *Let Assumptions 2.3, 3.1, 3.3, 3.4 and 4.1 hold. There exists constant $T > 0$ such that for any $t \ge T$*

$$\mathbb{E}|y(t)|^p \le \frac{\check{\varsigma}}{\hat{\varsigma}}|y_0|^p e^{-\gamma(t-t_0)}, \tag{4.3}$$

where $\gamma = (\check{\varsigma}^{-1} + \bar{\varsigma})\theta - \bar{\varsigma}$. The trivial solution of Eq. (3.1) is pth moment exponentially stable if $\theta \in (\frac{\bar{\varsigma}}{\check{\varsigma}^{-1}+\bar{\varsigma}}, 1)$.

Proof. If $y_0 = 0$, $y(t; t_0, 0, r_0) \equiv 0$, the assertion (4.3) is trivial. Fix $y_0 \ne 0$ and $r_0 \in S$ arbitrary. Define $V(y, i) = \varsigma_i|y|^p$. Clearly, $\hat{\varsigma}|y|^p \le V(y, i) \le \check{\varsigma}|y|^p$. By (2.5),

$$\begin{aligned} &\mathcal{G}V(y, i, t) \\ &= \Big\{p\frac{x^\top f(y,i)}{|y|^2} + \frac{p}{2}\Big(\frac{\mathrm{trace}[g^\top(y,i)g(y,i)]}{|y|^2} \\ &+(p-2)\frac{|y^\top g(y,i)|^2}{|y|^4}\Big)I(t) + \int_{0<|v|<c}[\frac{|y+h(y,i,v)I(t)|^p}{|y|^p} \\ &-1 - p\frac{y^\top h(y,i,v)I(t)}{|y|^2}]\pi(dv) + \sum_{j=1}^N\gamma_{ij}\frac{\varsigma_j}{\varsigma_i}\Big\}\varsigma_i|y|^p. \end{aligned} \tag{4.4}$$

By Assumptions 3.4,

$$\begin{aligned} &\int_{0<|v|<c}\Big[\frac{|y+h(y,i,v)I(t)|^p}{|y|^p} - 1 - p\frac{y^\top h(y,i,v)I(t)}{|y|^2}\Big]\pi(dv) \\ &\le \int_{0<|v|<c}[\frac{|y|^p + |h(y,i,v)|^pI(t)}{|y|^p} - 1]\pi(dv) - p\vartheta_iI(t) \\ &\le \int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv)I(t) - p\vartheta_iI(t). \end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.4) gives

$$\begin{aligned} &\mathcal{G}V(y, i, t) \\ &\le \Big\{\Big(p\alpha_i + 0.5p[\beta_i^2 - (2-p)\lambda_i^2] + \int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv) - p\vartheta_i\Big) \\ &+\sum_{j=1}^N\gamma_{ij}\frac{\varsigma_j}{\varsigma_i}\Big\}I(t)\varsigma_i|y|^p + \Big(p\alpha_i + \sum_{j=1}^N\gamma_{ij}\frac{\varsigma_j}{\varsigma_i}\Big)(1-I(t))\varsigma_i|y|^p \\ &= \frac{1}{\varsigma_i}\Big\{\Big(p\alpha_i + 0.5p[\beta_i^2 - (2-p)\lambda_i^2] \\ &+\int_{0<|v|<c}\bar{h}_i^p(v)\pi(dv) - p\vartheta_i\Big)\varsigma_i + \sum_{j=1}^N\gamma_{ij}\varsigma_j\Big\}I(t)V(y, i) \\ &+\Big(p\alpha_i + \sum_{j=1}^N\gamma_{ij}\frac{\varsigma_j}{\varsigma_i}\Big)(1 - I(t))V(y, i). \end{aligned}$$

Then, by Assumption 4.1, we have

$$\begin{aligned} \mathcal{G}V(y, i, t) &= -\frac{1}{\varsigma_i}I(t)V(y, i) \\ &+\Big(p\alpha_i + \sum_{j=1}^N\gamma_{ij}\frac{\varsigma_j}{\varsigma_i}\Big)(1 - I(t))V(y, i) \\ &\le [\bar{\varsigma} - (\check{\varsigma}^{-1} + \bar{\varsigma})I(t)]V(y, i). \end{aligned} \tag{4.6}$$

When $t_0 \le t < t_0 + \theta_0$, the generalized Itô formula gives

$$\begin{aligned} &\mathbb{E}\Big[V(y(t), r(t))e^{\check{\varsigma}^{-1}(t-t_0)}\Big] - V(y_0, r_0) \\ &= \mathbb{E}\int_{t_0}^t e^{\check{\varsigma}^{-1}(s-t_0)}\Big(\mathcal{G}V(y(s), r(s), s) + \check{\varsigma}^{-1}V(y(s), r(s))\Big)ds. \end{aligned}$$

By (4.6), we have $\mathcal{G}V(y, i, t) \le -\check{\varsigma}^{-1}V(y, i)$ for $t_0 \le t < t_0 + \theta_0$, which implies that $\mathbb{E}V(y(t), r(t)) \le$

$V(y_0,r_0)e^{-\check{\varsigma}^{-1}(t-t_0)}$. Similarly, when $t_0+\theta_0\le t<t_1$,

$$\begin{aligned}&\mathbb{E}\Big[V(y(t),r(t))e^{-\bar{\varsigma}(t-t_0-\theta_0)}\Big]-\mathbb{E}V(y(t_0+\theta_0),r(t_0+\theta_0))\\&=\mathbb{E}\int_{t_0}^{t}e^{-\bar{\varsigma}(s-t_0-\theta_0)}\Big(\mathcal{G}V(y(s),r(s),s)-\bar{\varsigma}V(y(s),r(s))\Big)ds.\end{aligned}$$

Therefore, when $t_0+\theta_0\le t<t_1$, $\mathcal{G}V(y,i,t)\le\bar{\varsigma}V(y,i)$ and

$$\begin{aligned}\mathbb{E}V(y(t),r(t))&\le\mathbb{E}V(y(t_0+\theta_0),r(t_0+\theta_0))e^{\bar{\varsigma}(t-t_0-\theta_0)}\\&\le V(y_0,r_0)e^{[-\check{\varsigma}^{-1}\theta_0+\bar{\varsigma}(t-t_0-\theta_0)]}.\end{aligned}$$

By induction, when $t_n\le t<t_n+\theta_n$, we have

$$\begin{aligned}\mathbb{E}V(y(t),r(t))&\le\mathbb{E}V(y(t_n),r(t_n))e^{\bar{\varsigma}(t-t_n)}\\&\le V(y_0,r_0)e^{-\check{\varsigma}^{-1}[\sum_{k=0}^{n-1}\theta_k+(t-t_n)]}\\&\quad\times e^{\bar{\varsigma}\sum_{k=0}^{n-1}(t_{k+1}-t_k-\theta_k)}.\end{aligned}\tag{4.7}$$

And when $t_n+\theta_n\le t<t_{n+1}$, we have

$$\begin{aligned}&\mathbb{E}V(y(t),r(t))\\&\le\mathbb{E}V(y(t_n+\theta_n),r(t_n+\theta_n))e^{\bar{\varsigma}(t-t_n-\theta_n)}\\&\le V(y_0,r_0)e^{-\check{\varsigma}^{-1}\sum_{k=0}^{n}\theta_k}\\&\quad\times e^{\bar{\varsigma}[\sum_{k=0}^{n-1}(t_{k+1}-t_k-\theta_k)+(t-t_n-\theta_n)]}.\end{aligned}\tag{4.8}$$

(4.7) and (4.8) together yields that for any $t\ge t_0$, $\mathbb{E}V(y(t),r(t))\le V(y_0,r_0)e^{\int_{t_0}^{t}[\bar{\varsigma}-(\check{\varsigma}^{-1}+\bar{\varsigma})I(s)]ds}$. This implies

$$\mathbb{E}|y(t)|^p\le\frac{\check{\varsigma}}{\hat{\varsigma}}|y_0|^pe^{\int_{t_0}^{t}[\bar{\varsigma}-(\check{\varsigma}^{-1}+\bar{\varsigma})I(s)]ds}.\tag{4.9}$$

In fact, for any $t\in[t_n,t_n+\theta_n)$, we can derive that

$$\begin{aligned}&\int_{t_0}^{t}[\bar{\varsigma}-(\check{\varsigma}^{-1}+\bar{\varsigma})I(s)]ds\\&\le\bar{\varsigma}(t-t_0)-(\check{\varsigma}^{-1}+\bar{\varsigma})\int_{t_0}^{t_n}I(s)ds\\&=\bar{\varsigma}(t-t_0)-(\check{\varsigma}^{-1}+\bar{\varsigma})\mathcal{L}(t_0,t_n)\\&\le\bar{\varsigma}(t-t_0)-(\check{\varsigma}^{-1}+\bar{\varsigma})[\theta(t_n-t_0)-\mathcal{L}_0].\end{aligned}$$

While for any $t\in[t_n+\theta_n,t_{n+1})$, we have

$$\begin{aligned}&\int_{t_0}^{t}[\bar{\varsigma}-(\check{\varsigma}^{-1}+\bar{\varsigma})I(s)]ds\\&\le\bar{\varsigma}(t-t_0)-(\check{\varsigma}^{-1}+\bar{\varsigma})\int_{t_0}^{t_n+\theta_n}I(s)ds\\&=\bar{\varsigma}(t-t_0)-(\check{\varsigma}^{-1}+\bar{\varsigma})\mathcal{L}(t_0,t_n+\theta_n)\\&\le\bar{\varsigma}(t-t_0)-(\check{\varsigma}^{-1}+\bar{\varsigma})[\theta(t_n+\theta_n-t_0)-\mathcal{L}_0].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t-t_0}\int_{t_0}^{t}[\bar{\varsigma}-(\check{\varsigma}^{-1}+\bar{\varsigma})I(s)]ds\\&\le\limsup_{t\to\infty}\frac{1}{t-t_0}\Big(\bar{\varsigma}(t-t_0)-(\check{\varsigma}^{-1}+\bar{\varsigma})[\theta(t_n-t_0)-\mathcal{L}_0]\Big)\\&=\bar{\varsigma}-(\check{\varsigma}^{-1}+\bar{\varsigma})\theta.\end{aligned}\tag{4.10}$$

Consequently, (4.10) shows that there exists a $T>0$ such that for any $t>T$, $\int_{t_0}^{t}[\bar{\varsigma}-(\check{\varsigma}^{-1}+\bar{\varsigma})I(s)]ds\le[\bar{\varsigma}-(\check{\varsigma}^{-1}+\bar{\varsigma})\theta](t-t_0)$. Hence, it is concluded that $\mathbb{E}|y(t)|^p\le\frac{\check{\varsigma}}{\hat{\varsigma}}|y_0|^pe^{[\bar{\varsigma}-(\check{\varsigma}^{-1}+\bar{\varsigma})\theta](t-t_0)}$, as required. □

In fact, we are only interested in the case $\bar{\varsigma}\ge0$, otherwise $dy(t)/dt=f(y(t),r(t))$ is already exponentially stable in pth moment. In addition, Theorem 4.3 will play a crucial role in the proof of the following Theorem 4.6. In order to prove the main result, the following two lemmas are needed.

Lemma 4.4 *Let Assumptions 2.3, 3.3 hold and $p\in(0,1)$. Then, for any $t\ge t_0$,*

$$\mathbb{E}|x(t)|^p\le|x_0|^pe^{0.5pC_1(t-t_0)},\tag{4.11}$$

$$\mathbb{E}|x(t)-x(\delta(t))|^p\le|x_0|^pe^{0.5pC_1(t-t_0)}H_1(p,\tau),\tag{4.12}$$

where $\check{\alpha}=\max_{i\in S}\alpha_i$, $\check{\beta}=\max_{i\in S}\beta_i$, $C_1=[2\check{\alpha}+\check{\beta}^2+k_3]$, $H_1(p,\tau)=[3\tau(\tau k_1^2+k_2^2+k_3)]^{\frac{p}{2}}$.

Proof. Applying the Itô formula yields that for $t\ge t_0$,

$$\begin{aligned}|x(t)|^2&=|x_0|^2+\int_{t_0}^{t}\Big\{2x^\top(s)f(x(s),r(s))\\&\quad+|g(x(\delta(s)),r(s))|^2I(s)\Big\}ds\\&\quad+\int_{t_0}^{t}2x^\top(s)g(x(\delta(s)),r(s))I(s)dw(s)\\&\quad+\int_{t_0}^{t}\int_{0<|v|<c}(|x(s)+h(x(\delta(s)^-),r(s^-),v)I(s)|^2\\&\quad-|x(s)|^2-2x^\top(s)h(x(\delta(s)^-),r(s^-),v)I(s))\pi(dv)ds\\&\quad+\int_{t_0}^{t}\int_{0<|v|<c}(|x(s)+h(x(\delta(s)^-),r(s^-),v)I(s)|^2\\&\quad-|x(s)|^2)\tilde{N}(dt,dv).\end{aligned}$$

By Assumptions 2.3 and 3.3,

$$\begin{aligned}\mathbb{E}|x(t)|^2&\le|x_0|^2+(2\check{\alpha}+\check{\beta}^2)\int_{t_0}^{t}\mathbb{E}|x(s)|^2ds\\&\quad+k_3\int_{t_0}^{t}\mathbb{E}|x(\delta(s)^-)|^2ds\\&\le|x_0|^2+C_1\int_{t_0}^{t}\sup_{t_0\le u\le s}\mathbb{E}|x(u)|^2ds,\end{aligned}$$

where $C_1=[2\check{\alpha}+\check{\beta}^2+k_3]$. It is easy to observed that

$$\sup_{t_0\le u\le t}\mathbb{E}|x(u)|^2\le|x_0|^2+C_1\int_{t_0}^{t}\sup_{t_0\le u\le s}\mathbb{E}|x(u)|^2ds.$$

Consequently, the Gronwall inequality gives

$$\sup_{t_0\le u\le t}\mathbb{E}|x(u)|^2\le|x_0|^2e^{C_1(t-t_0)}.\tag{4.13}$$

By the Hölder inequality, we then have $\mathbb{E}|x(t)|^p\le\Big(\mathbb{E}|x(t)|^2\Big)^{\frac{p}{2}}\le|x_0|^pe^{0.5pC_1(t-t_0)}$. On the other hand, it

can be computed that

$$\begin{aligned}
&\mathbb{E}|x(t)-x(\delta(t))|^2\\
&\le 3\mathbb{E}\Big|\int_{\delta(t)}^{t} f(x(s),r(s))ds\Big|^2\\
&+3\mathbb{E}\Big|\int_{\delta(t)}^{t} g(x(\delta(s)),r(s))I(s)dw(s)\Big|^2\\
&+3\mathbb{E}\Big|\int_{\delta(t)}^{t}\int_{0<|v|<c} h(x(\delta(s)^-),r(s^-),v)I(s)\tilde{N}(dv,ds)\Big|^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality, the Itô martingale isometry and Assumption 2.3,

$$\begin{aligned}
&\mathbb{E}|x(t)-x(\delta(t))|^2\\
&\le 3\tau\mathbb{E}\int_{\delta(t)}^{t}|f(x(s),r(s))|^2ds\\
&+3\mathbb{E}\int_{\delta(t)}^{t}|g(x(\delta(s)),r(s))|^2I(s)ds\\
&+3\mathbb{E}\int_{\delta(t)}^{t}\int_{0<|v|<c}|h(x(\delta(s)^-),r(s^-),v)|^2I(s)\pi(dv)ds\\
&\le 3\tau k_1^2\int_{\delta(t)}^{t}\mathbb{E}|x(s)|^2ds+3k_2^2\int_{\delta(t)}^{t}\mathbb{E}|x(\delta(s))|^2ds\\
&+3k_3\int_{\delta(t)}^{t}\mathbb{E}|x(\delta(s)^-)|^2ds. \qquad (4.14)
\end{aligned}$$

Substituting (4.13) into (4.14) gives

$$\begin{aligned}
&\mathbb{E}|x(t)-x(\delta(t))|^2\\
&\le 3\Big(\tau k_1^2+k_2^2+k_3\Big)\int_{t-\tau}^{t}\sup_{t_0\le u\le s}\mathbb{E}|x(u)|^2ds\\
&\le 3\tau\Big(\tau k_1^2+k_2^2+k_3\Big)|x_0|^2e^{C_1(t-t_0)}.
\end{aligned}$$

Once again, by the Hölder inequality, we have $\mathbb{E}|x(t)-x(\delta(t))|^p\le|x_0|^pe^{0.5pC_1(t-t_0)}H_1(p,\tau)$, as required. □

Lemma 4.5 *Let Assumptions 2.3, 3.3 hold and $p\in(0,1)$. Then, for any $t\ge t_0$,*

$$\begin{aligned}
&\mathbb{E}|x(t)-y(t)|^p\\
&\le|x_0|^pH_2(p,\tau)(e^{C_1(t-t_0)}-1)^{\frac{p}{2}}e^{0.5pC_2(t-t_0)}, \qquad (4.15)
\end{aligned}$$

where $C_2=2k_1+2k_2^2+2k_3$ and $H_2(p,\tau)=\big[\frac{(2k_2^2+2k_3)H_1(2,\tau)}{C_1}\big]^{\frac{p}{2}}$.

Proof. By the Itô formula,

$$\begin{aligned}
&\mathbb{E}|x(t)-y(t)|^2\le 2k_1\mathbb{E}\int_{t_0}^{t}|x(s)-y(s)|^2ds\\
&+\mathbb{E}\int_{t_0}^{t}|g(x(\delta(s)),r(s))-g(y(s),r(s))|^2ds\\
&+\mathbb{E}\int_{t_0}^{t}\int_{0<|v|<c}|h(x(\delta(s)^-),r(s^-),v)\\
&\qquad -h(y(s^-),r(s^-),v)|^2\pi(dv)ds. \qquad (4.16)
\end{aligned}$$

By Assumption 2.3,

$$\begin{aligned}
&\mathbb{E}\int_{t_0}^{t}|g(x(\delta(s)),r(s))-g(y(s),r(s))|^2ds\\
&\le 2\mathbb{E}\int_{t_0}^{t}|g(x(\delta(s)),r(s))-g(x(s),r(s))|^2ds\\
&+2\mathbb{E}\int_{0}^{t}|g(x(s),r(s))-g(y(s),r(s))|^2ds\\
&\le 2k_2^2\mathbb{E}\int_{t_0}^{t}|x(\delta(s))-x(s)|^2ds\\
&\qquad +2k_2^2\mathbb{E}\int_{t_0}^{t}|x(s)-y(s)|^2ds \qquad (4.17)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\int_{t_0}^{t}\int_{0<|v|<c}|h(x(\delta(s)^-),r(s^-),v)\\
&\qquad -h(y(s^-),r(s^-),v)|^2\pi(dv)ds\\
&\le 2k_3\mathbb{E}\int_{t_0}^{t}|x(\delta(s)^-)-x(s^-)|^2ds\\
&\qquad +2k_3\mathbb{E}\int_{t_0}^{t}|x(s^-)-y(s^-)|^2ds. \qquad (4.18)
\end{aligned}$$

Because both $x(t)$ and $y(t)$ are *càdlàg* processes, they have at most countable jumps. Therefore, the jump processes $\triangle x(s)=x(s)-x(s^-)=0$ and $\triangle y(s)=y(s)-y(s^-)=0$ for almost every $s\in[t_0,t]$, and hence $\mathbb{E}\int_{t_0}^{t}|x(s^-)-y(s^-)|^2ds=\mathbb{E}\int_{t_0}^{t}|x(s)-y(s)|^2ds$. Similarly, we can have $\mathbb{E}\int_{t_0}^{t}|x(\delta(s)^-)-x(s^-)|^2ds=\mathbb{E}\int_{t_0}^{t}|x(\delta(s))-x(s)|^2ds$. By Lemma 4.4, substituting (4.17) and (4.18) into (4.16) gives

$$\begin{aligned}
&\mathbb{E}|x(t)-y(t)|^2\le C_2\mathbb{E}\int_{t_0}^{t}|x(s)-y(s)|^2ds\\
&+(2k_2^2+2k_3)\mathbb{E}\int_{t_0}^{t}|x(\delta(s))-x(s)|^2ds\\
&\le C_2\mathbb{E}\int_{t_0}^{t}|x(s)-y(s)|^2ds\\
&\quad+\frac{(2k_2^2+2k_3)H_1(2,\tau)|x_0|^2}{C_1}\Big(e^{C_1(t-t_0)}-1\Big),
\end{aligned}$$

where $C_2=2k_1+2k_2^2+2k_3$. Then, the Gronwall inequality implies that

$$\begin{aligned}
\mathbb{E}|x(t)-y(t)|^2&\le\frac{(2k_2^2+2k_3)H_1(2,\tau)|x_0|^2}{C_1}\\
&\quad\times\Big(e^{C_1(t-t_0)}-1\Big)e^{C_2(t-t_0)}.
\end{aligned}$$

Applying the Hölder inequality gives

$$\begin{aligned}
\mathbb{E}|x(t)-y(t)|^p&\le|x_0|^p\Big[\frac{(2k_2^2+2k_3)H_1(2,\tau)}{C_1}\Big]^{\frac{p}{2}}\\
&\quad\times\Big(e^{C_1(t-t_0)}-1\Big)^{\frac{p}{2}}e^{0.5pC_2(t-t_0)},
\end{aligned}$$

which is the required assertion. □

Now, let us establish the main results.

Theorem 4.6 *Let Assumptions 2.3, 3.1, 3.3, 3.4 and 4.1 hold. Choose a free parameter $\varepsilon \in (0,1)$. Let $\tau^* \geq 0$ be the unique root to the equation*

$$H_2(p,\tau)[e^{C_1(\tau+\frac{\log(\frac{\check{\zeta}}{\hat{\zeta}\varepsilon})}{\gamma})}-1]^{\frac{p}{2}}e^{0.5pC_2(\tau+\frac{\log(\frac{\check{\zeta}}{\hat{\zeta}\varepsilon})}{\gamma})}=1-\varepsilon. \tag{4.19}$$

Then, for any initial values $x_0 \in R^n$ and $r_0 \in S$, the solution of Eq. (2.1) satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^p)<0 \tag{4.20}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|)<0 \ \ a.s. \tag{4.21}$$

provided $\tau \in (0,\tau^*)$.

Proof. Let

$$F(\tau,p)=H_2(p,\tau)[e^{C_1(\tau+\frac{\log(\frac{\check{\zeta}}{\hat{\zeta}\varepsilon})}{\gamma})}-1]^{\frac{p}{2}}e^{0.5pC_2(\tau+\frac{\log(\frac{\check{\zeta}}{\hat{\zeta}\varepsilon})}{\gamma})}.$$

When p is fixed, $F(\cdot,p)$ is a continuously increasing function of $\tau \geq 0$. When $\tau = 0$, $F(\cdot,p)$ equals to zero, so Eq. (4.19) must have a unique root $\tau^* > 0$. Choose a positive integer $\tilde{k}$ such that

$$\frac{\log(\frac{\check{\zeta}}{\hat{\zeta}\varepsilon})}{\gamma\tau}\leq\tilde{k}<1+\frac{\log(\frac{\check{\zeta}}{\hat{\zeta}\varepsilon})}{\gamma\tau}. \tag{4.22}$$

This implies $\frac{\check{\zeta}}{\hat{\zeta}}e^{-\gamma\tilde{k}\tau}\leq\varepsilon$. Write $y(t_0+\tilde{k}\tau;x_0,r_0)=y_{\tilde{k}}$. By Theorem 4.3,

$$\mathbb{E}|y_{\tilde{k}}|^p\leq\frac{\check{\zeta}}{\hat{\zeta}}|x_0|^pe^{-\gamma\tilde{k}\tau}\leq\varepsilon|x_0|^p. \tag{4.23}$$

By the elementary inequality $(a+b)^p \leq a^p+b^p$ for any $a, b \geq 0$ and $0<p<1$, $\mathbb{E}|x_{\tilde{k}}|^p \leq \mathbb{E}|y_{\tilde{k}}|^p+\mathbb{E}|x_{\tilde{k}}-y_{\tilde{k}}|^p$. Using (4.23) and Lemma 4.5,

$$\mathbb{E}|x_{\tilde{k}}|^p\leq|x_0|^p\Big(\varepsilon+H_2(p,\tau)[e^{C_1\tilde{k}\tau}-1]^{\frac{p}{2}}e^{0.5pC_2\tilde{k}\tau}\Big). \tag{4.24}$$

By (4.22), we have

$$\begin{aligned}&\varepsilon+H_2(p,\tau)[e^{C_1\tilde{k}\tau}-1]^{\frac{p}{2}}e^{0.5pC_2\tilde{k}\tau}<\varepsilon+H_2(p,\tau)\\&\times[e^{C_1(\tau+\frac{\log(\frac{\check{\zeta}}{\hat{\zeta}\varepsilon})}{\gamma})}-1]^{\frac{p}{2}}e^{0.5pC_2(\tau+\frac{\log(\frac{\check{\zeta}}{\hat{\zeta}\varepsilon})}{\gamma})}\leq 1.\end{aligned}$$

Thus, we can choose $\lambda>0$ such that $\varepsilon+H_2(p,\tau)[e^{C_1\tilde{k}\tau}-1]^{\frac{p}{2}}e^{0.5pC_2\tilde{k}\tau}=e^{-\lambda\tilde{k}\tau}$. It then follows from (4.24) that $\mathbb{E}|x_{\tilde{k}}|^p \leq |x_0|^pe^{-\lambda\tilde{k}\tau}$. Let us now consider the solution $x(t)$ of Eq. (2.1) on $t \geq t_0+\tilde{k}\tau$. Since Eq. (2.1) is a SDDEs, the pair $(x(t),r(t))$ is not a time-homogeneous Markov process. However, due to the special feature of Eq. (2.1), the pair $\{(x(t_0+\tilde{k}\tau),r(t_0+\tilde{k}\tau))\}_{\tilde{k}\in\mathbb{N}_+}$ forms a discrete-time $R^n\times S$-valued time-homogeneous Markov process (for example, see Mao (2013)). Owing to the time-homogeneous property of Eq. (2.1), we can get $\mathbb{E}(|x_{2\tilde{k}}|^p|\mathcal{F}_{\tilde{k}\tau})\leq|x_{\tilde{k}}|^pe^{-\lambda\tilde{k}\tau}$. This implies $\mathbb{E}|x_{2\tilde{k}}|^p\leq\mathbb{E}|x_{\tilde{k}}|^pe^{-\lambda\tilde{k}\tau}\leq|x_0|^pe^{-2\lambda\tilde{k}\tau}$. Repeating this procedure gives

$$\mathbb{E}|x_{l\tilde{k}}|^p\leq\mathbb{E}|x_{(l-1)\tilde{k}}|^pe^{-\lambda\tilde{k}\tau}\leq|x_0|^pe^{-l\lambda\tilde{k}\tau},l=0,1,2,\cdots. \tag{4.25}$$

For any $t \geq t_0$, there is a unique $l \geq 0$ such that $t_0+l\tilde{k}\tau \leq t < t_0+(l+1)\tilde{k}\tau$. By Lemma 4.4 and (4.25),

$$\begin{aligned}\mathbb{E}|x(t)|^p&\leq\mathbb{E}|x_{l\tilde{k}}|^pe^{0.5pC_1(t-t_0-l\tilde{k}\tau)}\\&\leq|x_0|^pe^{-l\lambda\tilde{k}\tau}e^{0.5pC_1\tilde{k}\tau}\leq|x_0|^p\tilde{M}e^{-\lambda(t-t_0)},\end{aligned}$$

where $\tilde{M}=e^{(\lambda+0.5pC_1)\tilde{k}\tau}$. The assertion (4.20) follows. Finally, by Mao(2016), we can obtain another assertion (4.21). The proof is therefore completed. □

5 Examples

Example 5.1 Let $r(t)$ be a right-continuous Markov chain taking values in $S=\{1,2\}$ with the generator $\Gamma=(\gamma_{ij})_{2\times2}=\begin{bmatrix}-2 & 2\\ 1 & -1\end{bmatrix}$. Consider the following hybrid system

$$\dot{x}(t)=A_{r(t)}x(t),\ \ t\geq0 \tag{5.1}$$

where $A_1=\begin{pmatrix}-1 & 1\\ 1.5 & -1\end{pmatrix}$, $A_2=\begin{pmatrix}-0.5 & 1\\ 1 & -1.5\end{pmatrix}$. Obviously, the Markov chain $r(t)$ has the stationary distribution $\sum_{i\in S}\tilde{\pi}=(\tilde{\pi}_1,\tilde{\pi}_2)=(\frac{1}{3},\frac{2}{3})$. In fact, $f(x,i)=A_ix,(i=1,2)$ satisfies Assumption 3.3, where $\alpha_i=\Lambda_{\max}(\frac{A_i+A_i^\top}{2})$. Then, we have $\sum_{i=1}^2\tilde{\pi}_i\alpha_i=\sum_{i=1}^2\tilde{\pi}_i\Lambda_{\max}(\frac{A_i+A_i^\top}{2})=0.162$, which implies that the trivial solution of hybrid systems (5.1) is almost surely exponentially unstable. To make this hybrid systems (5.1) stable, let us introduce the period intermittent noise such that the intermittently hybrid SDEs with jumps can be described by

$$\begin{aligned}dx(t)&=A_{r(t)}x(t)dt+B_{r(t)}x(t)\tilde{I}(t)dw(t)\\&\quad+\int_{0<|v|<1}C_{r(t^-)}(v)x(t^-)\tilde{I}(t)\tilde{N}(dt,dv),\end{aligned} \tag{5.2}$$

where $\tilde{N}$ is a compensated Poisson random measure on $[0,\infty)\times R_0$ whose corresponding Lévy measure is given by $\pi(dv)=0.5e^{-|v|}dv$. Here $B_1=\begin{pmatrix}-0.5 & -0.4\\ 0.2 & 0.3\end{pmatrix}$, $B_2=\begin{pmatrix}0.1 & 0.6\\ -0.3 & -0.4\end{pmatrix}$, and $C_1(v)=\begin{pmatrix}v^2 & 0\\ 0 & 1.2v^2\end{pmatrix}$, $C_2(v)=\begin{pmatrix}1.5v^2 & 0\\ 0 & 1.5v^2\end{pmatrix}$, $\tilde{I}(t)=\sum_{n=0}^{\infty}I_{[nT,nT+\varrho T)}(t)$. Assume that $w(t)$, $\tilde{N}(dt,dv)$ and $r(t)$ are independent. It is easy to see

that $\bar{\beta}_1 = 0.7285, \bar{\lambda}_1 = 0.5258, \bar{\beta}_2 = 0.7659, \bar{\lambda}_2 = 0.3898$ and $\bar{\vartheta}_1 = 0.1606, \bar{\vartheta}_2 = 0.2409$. Then we get that

$$\sum_{i=1}^{2} \tilde{\pi}_i \Big(\alpha_i + \varrho[0.5\bar{\beta}_i - \bar{\lambda}_i - \bar{\vartheta}_i + 2 \int_{0<|v|<1} \sqrt{C_i(v)} \pi(dv)] \Big) = 0.162 - 0.2849\varrho. \tag{5.3}$$

By Theorem 3.16, if $\varrho \in (0.5686, 1)$, then Eq. (5.2) is almost surely exponentially stable. That is, hybrid systems (5.1) can be stabilized by intermittent stochastic perturbation $B_{r(t)}x(t)\tilde{I}(t)dw(t)$ and $\int_{0<|v|<1} C_{r(t^-)}(v)x(t^-)\tilde{I}(t)\tilde{N}(dt,dv)$ with $\varrho \in (0.5686, 1)$.

However, if the jumps are excluded from the hybrid stochastic systems (5.2), that is, if $C_i(v) = 0,\ i = 1, 2, 3$ or $\tilde{N} = 0$, we have $\sum_{i=1}^{2} \tilde{\pi}_i \Big(\alpha_i + \varrho[0.5\bar{\beta}_i - \bar{\lambda}_i] \Big) = 0.162 - 0.0842\varrho$. In particular, when $\varrho = 1$, then we get $\sum_{i=1}^{2} \tilde{\pi}_i \Big(\alpha_i + 0.5\bar{\beta}_i - \bar{\lambda}_i \Big) = 0.0778 > 0$, which implies that the following linear hybrid system

$$dx(t) = A_{r(t)}x(t)dt + B_{r(t)}x(t)dw(t) \tag{5.4}$$

is almost surely exponentially unstable. This example shows that even if $\varrho = 1$, the intermittent noise of only one Brownian motion cannot ensure the stability of the system (5.4), while the jump noise $\tilde{N}$ can stabilize stochastic system (5.4). Obviously, Eq. (5.3) becomes

$$\sum_{i=1}^{2} \tilde{\pi}_i \Big(\alpha_i + 0.5\bar{\beta}_i - \bar{\lambda}_i + \varrho[-\bar{\vartheta}_i + 2 \int_{0<|v|<1} \sqrt{C_i(v)} \pi(dv)] \Big) = 0.0778 - 0.2141\varrho.$$

By Theorem 3.10, if $\varrho \in (0.3634, 1)$, the following intermittently hybrid SDEs

$$dx(t) = A_{r(t)}x(t)dt + B_{r(t)}x(t)dw(t) + \int_{0<|v|<1} C_{r(t^-)}(v)x(t^-)\tilde{I}(t)\tilde{N}(dt,dv)$$

becomes almost surely exponentially stable. This implies that hybrid stochastic systems (5.4) can be stabilized by intermittent jump noise $\int_{0<|v|<1} C_{r(t^-)}(v)x(t^-)\tilde{I}(t)\tilde{N}(dt,dv)$ if $\varrho \in (0.3634, 1)$.

Example 5.2 Let $r(t)$ be a right-continuous Markov chain taking values $S = \{1, 2\}$ with the generator $\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$. Consider the scalar hybrid system

$$\dot{x}(t) = \mu_{r(t)}x(t), \quad t \geq 0, \tag{5.5}$$

where $\mu_1 = 0.4, \mu_2 = -0.2$. Obviously, the Markov chain $r(t)$ has the stationary distribution $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2) = (\frac{3}{4}, \frac{1}{4})$ with the average $\sum_{i=1}^{2} \tilde{\pi}_i \mu_i = 0.25$. It is obvious that hybrid systems (5.5) is unstable.

Now, let us design an aperiodic intermittent control to stabilize the hybrid systems (5.5). Consider the intermittently hybrid stochastic systems with jumps

$$dx(t) = \mu_{r(t)}x(t)dt + \sigma_{r(t)}x(t)I(t)dw(t) + \int_{0<|v|<1} h_{r(t^-)}(v)x(t^-)I(t)\tilde{N}(dt,dv), \tag{5.6}$$

where $\tilde{N}$ is a compensated Poisson random measure on $[0,\infty) \times R_0$ whose corresponding Lévy measure is given by $\pi(dv) = 0.5e^{-|v|}dv$. Here $\sigma_1 = \sqrt{2},\ \sigma_2 = 1,\ h_1(v) = -0.5v^2,\ h_2(v) = -0.4v^2$ and

$$I(t) = \sum_{k=0}^{\infty} I_{[k,k+0.15)\cup[k+0.3,k+\iota)}(t), \quad 0.3 < \iota < 1. \tag{5.7}$$

Assume $w(t)$, $\tilde{N}(dt,dv)$ and $r(t)$ are independent. By (5.7),

$$\lim_{n\to\infty} \frac{1}{t_{2n}} \sum_{k=0}^{2n-1} \theta_k = \lim_{n\to\infty} \frac{0.15n + (\iota - 0.3)n}{n} = \iota - 0.15$$

and

$$\lim_{n\to\infty} \frac{1}{t_{2n+1}} \sum_{k=0}^{2n} \theta_k = \lim_{n\to\infty} \frac{0.15n + (\iota - 0.3)n + 0.15}{n + 0.3} = \iota - 0.15.$$

Hence, we obtain the average control ratio $\theta = \iota - 0.15$. Choose $p = 0.5$ such that $\rho_1(0.5) = 0.089$ and $\rho_2(0.5) = 0.256$. The matrix $\mathcal{A}_{0.5}$ defined in (4.2) becomes $\mathcal{A}_{0.5} = \begin{bmatrix} 1.089 & -1 \\ -3 & 3.256 \end{bmatrix}$, which is a nonsingular M-matrix. Then, we can determine $\varsigma_1 = 7.795, \varsigma_2 = 7.489$ and hence, $\check{\varsigma}^{-1} = 0.1283$ and $\bar{\varsigma} = 0.1607$. By Theorem 4.3, if $\theta \in (0.556, 1)$, the intermittently hybrid stochastic system (5.6) has the property that $\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|x(t)|^p \leq 0.1607 - 0.289\theta$. This implies that hybrid systems (5.5) can be stabilized by intermittent stochastic perturbation $\sigma_{r(t)}x(t)I(t)dw(t)$ and $\int_{0<|v|<1} h_{r(t^-)}(v)x(t^-)I(t)\tilde{N}(dt,dv)$ if $\iota \in (0.706, 1)$.

Next, let us use the intermittent feedback control based on discrete-time state observations to stabilize (5.5)

$$dx(t) = \mu_{r(t)}x(t)dt + \sigma_{r(t)}x(\delta(t))I(t)dw(t) + \int_{0<|v|<1} h_{r(t^-)}(v)x(\delta(t)^-)I(t)\tilde{N}(dt,dv). \tag{5.8}$$

Choose $\theta = 0.75$, $\varepsilon = 0.9$ such that (4.19) becomes $H_2(0.5,\tau)[e^{(3.5577\tau+4.0116)} - 1]^{0.25}e^{(1.5788\tau+1.7803)} = 0.1$, which has the unique positive root $\tau^* = 1.161 \times 10^{-10}$ (which is about 3.66 milliseconds if the time unit is of year). By Theorem 4.6, the intermittently hybrid stochastic system (5.8) is almost surely exponentially stable if $\tau < 1.161 \times 10^{-10}$. Figure 1 shows the hybrid system (5.5) can be stabilized by intermittently feedback control based on discrete-time state observations when $\theta = 0.75$.

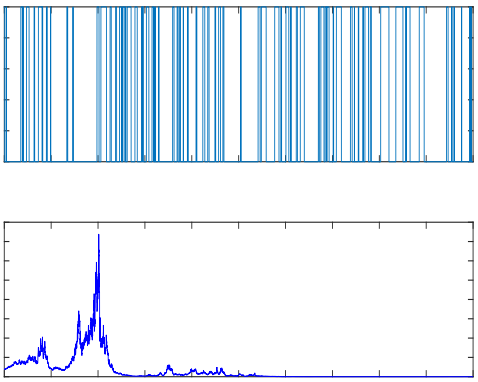

Fig. 1. The sample paths of intermittently hybrid stochastic system (5.8).

Remark 5.3 If the minimum control ratio c is employed in Example 5.2, then we have $c = 0.5$ and the average control ratio $\theta = 1 - c$, $\gamma = (\bar{\varsigma}^{-1} + \bar{\varsigma})\theta - \bar{\varsigma} = -0.0162 < 0$, this implies that stochastic controlled system (5.6) is pth moment exponentially unstable. In other words, the existing assumptions about the infimum of control time and the upper limit of control ratio cannot work.

Remark 5.4 It can be seen from the definition of $I(t)$ that the control cycle is not periodic and the control width is not unique. In fact, let $\iota = 0.9$, when n is odd, then $t_{n+1} - t_n = 0.3$ and $\theta_n = 0.15$, while when n is even, then $t_{n+1} - t_n = 0.7$ and $\theta_n = 0.6$. Therefore, Example 5.2 shows that the unstable system (5.5) can be stabilized by using the aperiodic intermittent control with (5.7).

Remark 5.5 The control strategy proposed in this paper can also be used to investigate the consensus problem of homogeneous multi-agent systems. For example, Li et al. (2014) considered the consensus control for a network of multi-agent systems, and utilize stochastic noise to achieve mean square and almost sure consensus of these systems. However, when communication bandwidth and channels are limited, it is unrealistic for multi-agent systems to perform continuous measurements to obtain control signals. Intermittent stochastic control in multi-agent systems can work well.

6 Conclusion

This paper is devoted to the stochastic stabilization of hybrid systems by intermittent stochastic feedback control. The Lyapunov function method and intermittent control strategy are used to establish sufficient stability criterion on intermittent hybrid SDEs with jumps. However, by comparison method and M-matrix theory, it is also shown that the hybrid system can be stabilized by the intermittent feedback control based on discrete-time observations as long as $\tau < \tau^*$. This paper mainly aims to investigage the time-triggered control strategy. The event-triggered control laws for hybrid systems is also intersting and will be considered in the future. In addition, the consensus of multi-agent systems with intermittent stochastic noise will be investigated.

Acknowledgments

We would like to thanks the editior and reviewers for their helpful comments and suggestions. Wei Mao would like to thank the Qing Lan Project of Jiangsu Province. Fuke Wu would like to thank the National Natural Science Foundation of China (Grant No. 62273158). Xuerong Mao would like to thank the Royal Society, United Kingdom (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society of Edinburgh, United Kingdom (RSE1832) for their financial support.

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