On the decay rate for a stochastic delay differential equation with an unbounded delay

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Abstract

How does the delay function affect its decay rate for a stable stochastic delay differential equation with an unbounded delay? Under suitable Khasminskii-type conditions, an existence-and-uniqueness theorem for an SDDE with a general unbounded time-varying delay will be firstly given. Its decay rate will be discussed when the equation is stable. Given the unbounded delay function, it will be shown that the decay rate can be directly expressed as a function of the delay.

Keywords: Khasminskii-type condition, stable, decay rate, stochastic differential delay equations

1. Introduction

Systems in many branches of sciences and industries depend not only on their current states, but also on their past states. Stochastic delay differential equations (SDDEs) are widely used for modeling such systems. Consider following SDDE defined on $t \geq t_0$ with time-varying delay $\delta(t)$:

$$dx(t) = f(x(t), x(t-\delta(t)), t)dt + g(x(t), x(t-\delta(t)), t)dB(t), \tag{1.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n\times m}$ are Borel-measurable functions. Details on equation (1.1) will be introduced in next section. In this article , we focus on the influence of $\delta(t)$ on the stability and decay rate of the equation.

Among existing researches, three categories of delay functions have mostly been discussed. The first category consists of bounded functions, including constant and differentiable time-varying delays. This is the equation where many theories are initially developed. We can find plentiful results on asymptotic stability of such equations, primarily based on Lyapunov methods and other analytical techniques. Here we only mention two comprehensive books on theories for SDDEs, [1] and [2]. The second category comprises of bounded but non-differentiable delay functions. After [3] initially proposed a new Lebesgue-measure based argument, we have seen significant progresses on stability analyses for equations with such non-differentiable delays([4],[5]). The third category is proportional, say $\delta(t) = qt$ with $q \in (0, 1)$. Compared to exponential decay rates for stable SDDEs with bounded delays, those equations with proportional delays only have polynomial decay rates (see [6], [7], [8] and references therein).

Mathematically, $\delta(t)$ can be other unbounded functions than proportional ones. Now there comes a question that what the decay rate will be when such an equation is stable? On such equations, [9] has given results for a linear scalar SDDE based on related analyses for ordinary differential equations. To the best of authors' knowledge, there are no general results on decay rates for such equations with unbounded

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time-varying delays based on Lyapunov methods. This article aims to close the gap. Recent researches on multi-dimensional SDDEs(e.g. [10], [11]) have primarily focused on the highly nonlinear characteristics of the equation coefficients. We will focus on investigating the influence of unbounded time-varying delays on the decay rate of the equation. Our main contributions include:

(1) proposing a framework for analyzing an SDDE with a general unbounded delay based on Lyapunov argument and Khasminskii-type conditions;

(2) showing that the decay rate can be expressed as a function of the given unbounded delay.

This article is arranged as follows. Preliminaries and notations on the unbounded delay and the decay rate will be introduced in section 2. In section 3, Khasminskii-type conditions are proposed to guarantee the existence-and-uniqueness for the equation. Then its stability and decay rate are discussed. An illustrative example will given in section 4 to verify our theories. The conclusion and comparison with proportional delays are made in the last section.

2. Preliminaries and notations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $B(t)$ be an m-dimensional Brownian motion defined on the probability space. If A is a vector or matrix, its transpose is denoted by A^T. For an $x \in \mathbb{R}^n$, $|x|$ is its Euclidean norm. For a function $\xi(t)$ defined on $[a, b]$, denote $\|\xi\| = \sup_{t\in[a,b]} |\xi(t)|$.

Consider an SDDE (1.1) satisfying following two assumptions.

Assumption 2.1. *Both f and g satisfy the local Lipschitz condition: for any integer $l \geq 0$, there exists a constant H_l such that for any $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq l$ and $t \geq t_0$,*

$$|f(x, y, t) - f(\bar{x}, \bar{y}, t)|^2 \bigvee |g(x, y, t) - g(\bar{x}, \bar{y}, t)|^2 \leq H_l(|x - \bar{x}|^2 + |y - \bar{y}|^2). \tag{2.1}$$

Assumption 2.2. *$\delta(t_0) > 0$. For any $t \geq t_0$, $\delta(t)$ is differentiable with $0 < \delta'(t) \leq \bar{\delta} < 1$.*

On the base of Assumption 2.2, a suitable initial condition for equation(1.1) can be given as $\{x(t)|t_0 - \delta(t_0) \leq t \leq t_0\} = \xi(t)$ for some $\xi(t)$ defined on $[t_0 - \delta(t_0), t_0]$.

Define $\psi(t) = t - \delta(t)$ on $[t_0, +\infty)$. Obviously, $\psi(t)$ is strictly increasing and has its inverse $w(t)$. Then define $w^{(n+1)}(t) = w\Big(w^{(n)}(t)\Big)$ for $n \geq 1$.

Lemma 2.3. *Let Assumption 2.2 hold. The sequence $\{w^{(n)}(t_0)\}_{n\geq 1}$ is increasing to ∞ monotonously.*

Proof. Assumption 2.2 gives $1 - \bar{\delta} \leq \psi'(t) < 1$, so that $w(t)$ is increasing and $w'(t) \in (1, (1 - \bar{\delta})^{-1}]$. By $\psi(t) < t$, we have $t = w(\psi(t)) < w(t)$, which then proves $w^{(n+1)}(t) > w^{(n)}(t)$ for any $n \geq 1$.

By Lagrange's theorem, there is $\kappa_n \in (w^{(n-1)}(t_0), w^{(n)}(t_0))$ such that

$$w^{(n+1)}(t_0) - w^{(n)}(t_0) = w'(\kappa_n)(w^{(n)}(t_0) - w^{(n-1)}(t_0)) > w^{(n)}(t_0) - w^{(n-1)}(t_0)$$

and subsequently, $w^{(n)}(t_0) > n(w(t_0) - t_0) + t_0$ giving $\lim_{n\to\infty} w^{(n)}(t_0) = \infty$. □

Given the delay function $\delta(t)$, define

$$v^\delta(t) = \exp\left\{\int_{t_0}^{t} \frac{1}{\delta(s)} ds\right\}. \tag{2.2}$$

Lemma 2.4. *On $v^\delta(t)$, we have two useful properties:*

(1) For any $t > t_0$,

$$e \leq \frac{v^\delta(t)}{v^\delta(t - \delta(t))} \leq \exp\{(1 - \bar{\delta})^{-1}\}. \tag{2.3}$$

(2) For any constant $a > 0$ and $t > t_0$,

$$\int_{t_0}^{t} \left(v^\delta(s)\right)^a ds \le \frac{1}{a}\delta(t)\left[\left(v^\delta(t)\right)^a - 1\right] \tag{2.4}$$

Proof. (1) From the definition of $v^\delta(t)$, we see $\frac{v^\delta(t)}{v^\delta(t-\delta(t))} = \exp\left\{\int_{t-\delta(t)}^{t} \frac{1}{\delta(s)} ds\right\}$. Because $\delta(t)$ is increasing on t, it will be true that $\int_{t-\delta(t)}^{t} \frac{1}{\delta(s)} ds \le \frac{\delta(t)}{\delta(t-\delta(t))}$. By the Lagrange theorem again, there exists a κ_t lying between $\delta(t)$ and $t-\delta(t)$, such that

$$\delta(t) = \delta(t-\delta(t)) + \delta'(\kappa_t)\delta(t) \le \delta(t-\delta(t)) + \bar{\delta}\cdot\delta(t),$$

which then derives $\delta(t) \le \frac{1}{1-\bar{\delta}}\delta(t-\delta(t))$ and subsequently, (2.3) holds.

(2) Obviously, we have $\frac{d}{dt}\left(v^\delta(t)\right) = \delta^{-1}(t)v^\delta(t)$, so that

$$\int_{t_0}^{t} \left(v^\delta(s)\right)^a ds = \int_{t_0}^{t} \frac{\delta(s)}{a} d\left(\left(v^\delta(s)\right)^a\right) \le \frac{\delta(t)}{a}\left(\left(v^\delta(t)\right)^a - \left(v^\delta(t_0)\right)^a\right). \tag{2.5}$$

While we see $v^\delta(t_0) = 1$, (2.4) is then held. ☐

In this article, we will show that, under suitable conditions, equation (1.1) will be stable with decay rate $v^\delta(t)$ as defined in (2.2). Due to the page limit, we only focus on its moment decay rate.

Definition 2.5. *The equation (1.1) is stable in the p-th moment sense with decay rate $v^\delta(t)$, if there exists a positive number $\varepsilon_0 > 0$ such that*

$$\limsup_{t\to\infty} \frac{\ln\left(E|x(t)|^p\right)}{\ln v^\delta(t)} \le -\varepsilon_0. \tag{2.6}$$

On the existence and uniqueness of the solution for equation (1.1), besides Assumption 2.1, we should impose additional conditions.

Denote by $C^{2,1}(\mathbb{R}^n \times [t_0,\infty); \mathbb{R}_+)$ the family of all continuous non-negative functions $V(x,t)$ defined on $\mathbb{R}^n \times [t_0,\infty)$ such that they are continuously differentiable twice in x and once in t. Given such a $V(x,t)$, define $LV : \mathbb{R}^n \times \mathbb{R}^n \times [t_0,\infty) \to \mathbb{R}$ by

$$LV(x,y,t) = V_t(x,t) + V_x^T(x,t)f(x,y,t) + \frac{1}{2}\mathrm{trace}[g^T(x,y,t)V_{xx}(x,t)g(x,y,t)],$$

where

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t}, \quad V_x(x,t) = \left(\frac{\partial V(x,t)}{\partial x_1}, \cdots, \frac{\partial V(x,t)}{\partial x_n}\right)^T, \text{ and } V_{xx}(x,t) = \left(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

3. Khasminskii-type conditions for stability

Assumption 3.1. *There exists a function $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$, with*

$$\lim_{k\to\infty}\left(\inf_{|x|\ge k, t\ge t_0} V(x,t)\right) = \infty, \tag{3.1}$$

and a positive constant λ_0 such that for any (x,y,t),

$$LV(x,y,,t) \le \lambda_0[1 + V(x,t) + V(y,t-\delta(t))]. \tag{3.2}$$

Theorem 3.2. *Let Assumptions 2.1 and 3.1 hold. For any given initial data ξ, there exists a unique global solution $x(t)$ to (1.1). Moreover, for any $t \geq t_0$, it holds that*

$$\mathbb{E}V(x(t), t) < \infty. \tag{3.3}$$

Proof. On the base of Assumption 2.1, for any given initial data ξ, there is a maximal local solution $x(t)$ for equation (1.1) on $t \in [t_0, \sigma_\infty)$, where σ_∞ is the explosion time. Let $k_0 > 0$ be sufficiently large for $\|\xi\| < k_0$. For each integer $k \geq k_0$, define stopping time $\tau_k = \inf\{t \in [t_0, \sigma_\infty) : |x(t)| \geq k\}$. Clearly, $\{\tau_k\}$ is increasing and define $\tau_\infty = \lim_{k\to\infty} \tau_k$ a.s.. Note that if we can show $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s.. We will get this by showing that for any n, $\tau_\infty > w^{(n)}(t_0)$ a.s.

By the Itô formula and Assumption 3.1, we can show that, for any $k \geq k_0$ and $t \geq t_0$,

$$\begin{aligned}
&\mathbb{E}V(x(\tau_k \wedge t), \tau_k \wedge t) \\
\leq &\mathbb{E}V(x(t_0), t_0) + \lambda_0(t - t_0) + \lambda_0 \mathbb{E}\int_{t_0}^{\tau_k \wedge t} V\Big(x(s - \delta(s)), s - \delta(s)\Big)ds + \lambda_0 \mathbb{E}\int_{t_0}^{\tau_k \wedge t} V(x(s), s)ds.
\end{aligned}$$

Firstly, let us restrict $t \in [t_0, w(t_0)]$. By Assumption 3.1, we get

$$\mathbb{E}V(x(\tau_k \wedge t), \tau_k \wedge t) \leq H_1 + \lambda_0 \mathbb{E}\int_{t_0}^{\tau_k \wedge t} V(x(s), s)ds = H_1 + \lambda_0 \mathbb{E}\int_{t_0}^{t} V(x(\tau_k \wedge s), \tau_k \wedge s)ds \tag{3.4}$$

where $H_1 = \mathbb{E}V(x(t_0), t_0) + \lambda_0(w(t_0) - t_0) + \frac{\lambda_0}{1-\bar{\delta}} \int_{t_0-\delta(t_0)}^{t_0} V(\xi(s), s)ds < \infty$. And then by the Gronwall inequality, we have

$$\mathbb{E}V(x(\tau_k \wedge t), \tau_k \wedge t) \leq H_1 \exp\left(\lambda_0(w(t_0) - t_0)\right).$$

Denote $\mu_k = \inf_{|x|\geq k, t\geq t_0} V(x, t)$. By (3.1), we see $\lim_{k\to\infty} \mu_k = \infty$. Obviously, we have

$$\mu_k \mathbb{P}(\tau_k \leq w(t_0)) \leq H_1 \exp\left(\lambda_0(w(t_0) - t_0)\right).$$

Letting $k \to \infty$, we hence obtain that $\mathbb{P}(\tau_\infty \leq w(t_0)) = 0$, namely $\mathbb{P}(\tau_\infty > w(t_0)) = 1$ and then (3.3) holds for $t \in [t_0, w(t_0)]$.

Secondly, for $t \in [w(t_0), w^{(2)}(t_0)]$, it follows from (3.4) that

$$\mathbb{E}V(x(\tau_k \wedge t), \tau_k \wedge t) \leq H_2 + \lambda_0 \mathbb{E}\int_{t_0}^{t} V(x(\tau_k \wedge s), \tau_k \wedge s)ds \tag{3.5}$$

where

$$H_2 = H_1 + \lambda_0(w^{(2)}(t_0) - w(t_0)) + \frac{\lambda_0}{1-\bar{\delta}} \int_{t_0}^{w(t_0)} \mathbb{E}V(x(s), s)ds < \infty.$$

We then show that for any $t \in [w(t_0), w^{(2)}(t_0)]$, $\mathbb{E}V(x(\tau_k \wedge t), \tau_k \wedge t) \leq H_2 \exp\left(\lambda_0(w^{(2)}(t_0) - t_0)\right)$. Now using the same argument as in the first step, we have $\mathbb{P}(\tau_\infty > w^{(2)}(t_0)) = 1$ and (3.3) holds for any $t \in [w(t_0), w^{(2)}(t_0)]$.

Finally, repeating this procedure and noticing that for any $l \geq 1$,

$$\int_{w^{(l)}(t_0)}^{w^{(l+1)}(t_0)} V(x(s - \delta(s)), s - \delta(s))ds \leq \frac{1}{1-\bar{\delta}} \int_{w^{(l-1)}(t_0)}^{w^{(l)}(t_0)} V(x(s), s))ds,$$

we can show that, for any n, $\mathbb{P}(\tau_\infty > w^{(n)}(t_0)) = 1$, and the assertion (3.3) holds for any $t \in [t_0, w^{(n)}(t_0)]$. By Lemma 2.3, $w^{(n)}(t) \to \infty$ as $n \to \infty$, we then prove the theorem. □

Theorem 3.3. *If there exists a function $V(x, t)$, $\alpha_0 \geq 0$ and four positive constants c_0, p, α_1 and α_2 with*

$\alpha_2 < \alpha_1(1-\bar{\delta})$, such that $V(x,t) \geq c_0|x|^p$ and for any (x,y,t),

$$LV(x,y,t) \leq \alpha_0 - \alpha_1 V(x,t) + \alpha_2 V(y, t-\delta(t)), \tag{3.6}$$

we will have

(1) The solution $x(t)$ obeys

$$\limsup_{t\to\infty} \frac{\mathbb{E}|x(t)|^p}{\delta(t)} \leq \frac{\alpha_0}{c_0\varepsilon_0} \tag{3.7}$$

(2) If, in addition, $\alpha_0 = 0$, then the solution has the properties that

$$\limsup_{t\to\infty} \frac{\ln \mathbb{E}|x(t)|^p}{\ln v^\delta(t)} \leq -\varepsilon_0, \tag{3.8}$$

where ε_0 is the unique positive root for following equation on u

$$\alpha_1 - \frac{u}{\delta(t_0)} - \frac{\alpha_2}{1-\bar{\delta}} \exp\left(\frac{u}{1-\bar{\delta}}\right) = 0. \tag{3.9}$$

Proof. Denote $h(u) = \alpha_1 - (\delta(t_0))^{-1}u - \alpha_2(1-\bar{\delta})^{-1}\exp\left((1-\bar{\delta})^{-1}u\right)$. Obviously, $h(u)$ is continuous and strictly decreasing satisfying $h(0) = \alpha_1 - \alpha_2(1-\bar{\delta})^{-1} > 0$ and $\lim_{u\to+\infty} h(u) = -\infty$, which shows that such ε_0 exists and is unique.

For any $0 < \varepsilon < \varepsilon_0$, $h(\varepsilon) > 0$ holds obviously. Applying the generalized Itô formula on $[v^\delta(t)]^\varepsilon V(x(t),t)$ and with the aid of the stopping time sequence $\{\tau_k\}$ defined in the proof of Theorem 3.2, we have

$$\begin{aligned}
&\mathbb{E}([v^\delta(\tau_k\wedge t)]^\varepsilon V(x(\tau_k\wedge t), \tau_k\wedge t)) - V(x(t_0),t_0) \\
=&\mathbb{E}\int_{t_0}^{\tau_k\wedge t} [v^\delta(s)]^\varepsilon \Big(\frac{\varepsilon}{\delta(s)} V(x(s),s) + LV(x(s), x(s-\delta(s)), s)\Big) ds.
\end{aligned}$$

Under condition (3.6), we then obtain

$$\begin{aligned}
&\mathbb{E}([v^\delta(\tau_k\wedge t)]^\varepsilon V(x(\tau_k\wedge t), \tau_k\wedge t)) - V(x(t_0),t_0) \\
\leq&\alpha_0 \int_{t_0}^{t} [v^\delta(s)]^\varepsilon ds + \mathbb{E}\int_{t_0}^{\tau_k\wedge t} [v^\delta(s)]^\varepsilon \Big(\frac{\varepsilon}{\delta(t_0)} - \alpha_1\Big) V(x(s),s) ds + \alpha_2 \mathbb{E}\int_{t_0}^{\tau_k\wedge t} [v^\delta(s)]^\varepsilon V(x(s-\delta(s)), s-\delta(s)) ds.
\end{aligned}$$

Simply integrating with transformation $u = \psi(s) = s - \delta(s)$ or equivalently, $s = w(u)$, we can derive

$$\begin{aligned}
&\mathbb{E}\int_{t_0}^{\tau_k\wedge t} [v^\delta(s)]^\varepsilon V(x(s-\delta(s)), s-\delta(s)) ds \\
=&\mathbb{E}\int_{\psi(t_0)}^{\psi(\tau_k\wedge t)} [v^\delta(w(u))]^\varepsilon V(x(u),u) \frac{1}{1-\delta'(w(u))} du \\
\leq&\frac{1}{1-\bar{\delta}} \int_{t_0-\delta(t_0)}^{t_0} V(\xi(s),s) ds + \frac{1}{1-\bar{\delta}} \mathbb{E}\int_{t_0}^{\tau_k\wedge t} \Big[\frac{v^\delta(w(u))}{v^\delta(u)}\Big]^\varepsilon [v^\delta(u)]^\varepsilon V(x(u),u) du.
\end{aligned}$$

Applying two properties in Lemma 2.4 and $h(\varepsilon) > 0$, we then calculate

$$\begin{aligned}
\mathbb{E}([v^\delta(\tau_k\wedge t)]^\varepsilon V(x(\tau_k\wedge t), \tau_k\wedge t)) \leq& C + \alpha_0 \int_{t_0}^{t} [v^\delta(s)]^\varepsilon ds - h(\varepsilon)\mathbb{E}\int_{t_0}^{\tau_k\wedge t} [v^\delta(s)]^\varepsilon V(x(s),s) ds \\
\leq& C + \frac{\alpha_0}{\varepsilon}\delta(t)([v^\delta(t)]^\varepsilon - 1),
\end{aligned}$$

where $C = V(x(t_0), t_0) + \frac{1}{1-\bar{\delta}} \int_{t_0-\delta(t_0)}^{t_0} V(\xi(s), s)ds$. Taking $k \to \infty$ and applying $V(x,t) \geq c_0|x|^p$, we get

$$\mathbb{E}|x(t)|^p \leq \frac{C}{c_0}[v^\delta(t)]^{-\varepsilon} + \frac{\alpha_0}{c_0\varepsilon}\delta(t)(1 - [v^\delta(t)]^{-\varepsilon}) \tag{3.10}$$

and then the first assertion (3.7) is verified by letting $\varepsilon \to \varepsilon_0$.

If $\alpha_0 = 0$, we see from (3.10) that $\mathbb{E}|x(t)|^p \leq \frac{C}{c_0}[v^\delta(t)]^{-\varepsilon}$, which means $\limsup\limits_{t\to\infty} \frac{\ln \mathbb{E}|x(t)|^p}{\ln v^\delta(t)} \leq -\varepsilon$. Then the second assertion (3.8) is also verified by letting $\varepsilon \to \varepsilon_0$. $\square$

4. An illustrative example

In order to verify our results, consider following nonlinear equation

$$dx(t) = \left(-2x(t) + \frac{t}{1+t}x\left(t - \sqrt{t}\right)\right)dt + \frac{1}{2}\sin\left(x(t-\sqrt{t})\right)dB_t \tag{4.1}$$

defined on $[1, +\infty)$ with the unbounded delay $\delta(t) = \sqrt{t}$. The initial condition is given by $x(s) = s, 0 \leq s \leq 1$.

Obviously, the equation satisfies Assumption 2.1. When $t \geq 1$, $\delta'(t) \in (0, \frac{1}{2}]$ gives $\bar{\delta} = \frac{1}{2}$. Taking $V(x) = x^2$ for analyses, we see $LV(x,y,t) = 2x(-2x + \frac{t}{1+t}y) + \frac{1}{4}y^2 \leq -3x^2 + \frac{5}{4}y^2$, so that conditions in Theorem 3.3 are satisfied with $\alpha_0 = 0, \alpha_1 = 3$ and $\alpha_2 = \frac{5}{4}$, and equation (4.1) is mean-square stable with decay rate $v^\delta(t) = e^{\sqrt{t}}$. We use MATLAB to simulate $\mathbb{E}|x(t)|^2$ for the solution of (4.1) as shown in Figure 1. The simulation curve depicted in the left subfigure, where the vertical axis represents $\mathbb{E}|x(t)|^2$, demonstrates the system's stability. In the right subfigure, with the vertical axis $\frac{\ln \mathbb{E}|x(t)|^2}{\sqrt{t}}$, the tendency of the simulation curve shows the decay rate $e^{\sqrt{t}}$ for $\mathbb{E}|x(t)|^2$, validating our theories.

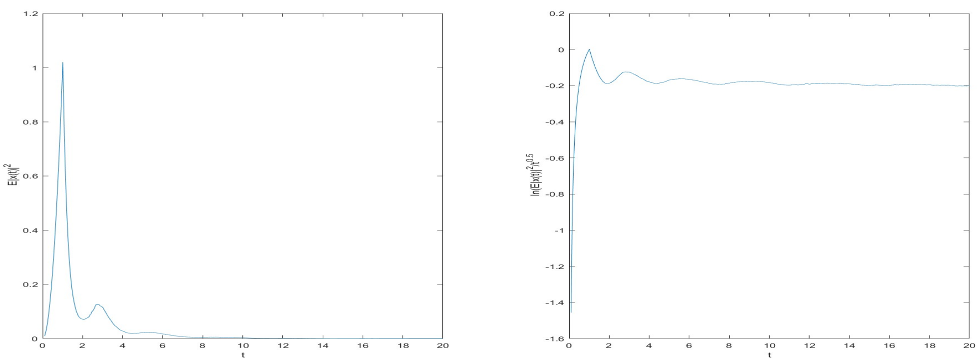

Fig. 1. Left: simulation curve of $\mathbb{E}|x(t)|^2$ as a function of t. Right: simulation curve of $\frac{\ln \mathbb{E}|x(t)|^2}{\sqrt{t}}$ as a function of t.

5. Conclusions

This article has focused on the decay rate for a stable SDDE with an unbounded time-varying delay function. Under Khasminskii-type conditions, the existence-and-uniqueness and stability theorems on the solution for such an SDDE are proposed. Given the unbounded delay function, a stable equation will have the decay rate dependent on the delay. Meanwhile, if we set $\delta(t) = qt$, $v^\delta(t)$ will be $t^{\frac{1}{q}}$, which just means a polynomial decay rate for an equation with a proportional delay. So our theories will cover related analyses for SDDEs with proportional delays. The results got in this article can be extended further to cover more equations and topics.

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Data availability

No data was used for the research described in the article.

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