



A variational theory for soft shells

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ABSTRACT

Three general modes are distinguished in the deformation of a thin shell; these are *stretching*, *drilling*, and *bending*. Of these, the drilling mode is the one more likely to emerge in a *soft matter* shell (as compared to a hard, structural one), as it is ignited by a swerve of material fibers about the local normal. We propose a hyperelastic theory for soft shells, based on a separation criterion that envisages the strain-energy density as the sum of three independent pure measures of stretching, drilling, and bending. Each individual measure is prescribed to vanish on all other companion modes. The result is a direct, second-grade theory featuring a bending energy *quartic* in an invariant strain descriptor that stems from the polar rotation hidden in the deformation gradient (although quadratic energies are also appropriate in special cases). The proposed energy functional has a multi-well character, which fosters cases of *soft elasticity* (with a continuum of ground states) related to minimal surfaces.

1. Introduction

The theories for elastic shells are plentiful. They fall into two broad categories: *direct* and *derived*. The former envision a shell as a material surface in three-dimensional space subject to the balance laws of continuum mechanics as fit for a low-dimensional body. The latter, on the other hand, envision a shell as a *thin* three-dimensional body described by the standard balance laws of continuum mechanics, which by the unbalance of sizes can be written in various approximate forms.¹

A bridge between these realms can be established by the theory of the Cosserat brothers (Cosserat and Cosserat, 1908, 1909),² whose use of *directors*³ for low-dimensional bodies can also be regarded as a means to recover information about the lost dimensions, as shown in the classical essay (Naghdi, 1972).

Writing yet another paper proposing a theory for the equilibrium of shells may feel like an *exercise in style*.⁴ However, rather than merely retelling an old story, here we pursue a line of reasoning that can be traced back to the work of Acharya (2000) (and was further developed in Wood and Hanna 2019, Vitral and Hanna 2023, Virga 2024, Acharya 2024) aiming at identifying *pure*

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¹ The reader is referred to Sects. 212 and 213 of Truesdell and Toupin (1960) for a classical summary of these approaches. We also recommend reading Sects. 13–15 of Chapt. XIV in Antman's book (Antman, 1995).

² As appropriately pointed out in Truesdell and Noll (1992, p.389), “For continua of one, two, and three dimensions, they developed their theory with elegance, precision, and thoroughness. It seems to have attracted slight attention.” The last assertion does no longer apply, especially in view of the conspicuous body of recent literature (Ghiba et al., 2020a,b, 2021a,b; Ghiba and Neff, 2023; Ghiba et al., 2024).

³ The terminology of *directors* was first introduced in the paper (Ericksen and Truesdell, 1957) on a general treatment of ordered media.

⁴ Having in mind Queneau's book (Queneau, 1981), where the same story is retold 99 times in different styles, an endeavour that was also recently emulated in mathematics by the book of Ording (Ording, 2019), where a single, simple statement is given 99 different proofs (see also the witty review Watkins, 2020).

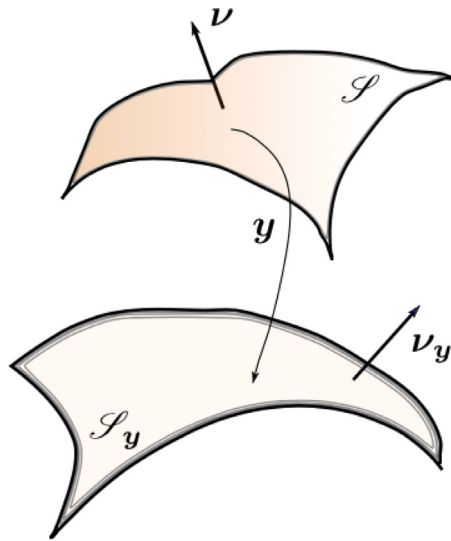


Fig. 1. In its reference configuration, a shell is represented by a smooth surface $\mathcal{S}$ oriented by the unit normal ν . The deformation y brings $\mathcal{S}$ into the surface $\mathcal{S}_y = y(\mathcal{S})$ oriented by the unit normal ν_y .

measures of deformations for shells (and plates).⁵ Our approach is direct: we shall regard a shell as a smooth material surface $\mathcal{S}$ in three-dimensional space.

By the customary polar decomposition of the deformation gradient, we first identify a stretching deformation mode and then, by the unique decomposition of the polar rotation⁶ described in Sonnet and Virga (2024), we identify *two* further deformation modes, *drilling* and *bending*. The former involves rotations about the local normal to $\mathcal{S}$ and the latter rotations about an axis on the local tangent plane.⁷

A *separation* criterion will guide our quest for pure measures of drilling and bending: for each deformation mode, we shall identify an invariant scalar that also vanishes on all local realizations of the companion mode. To accomplish this task, we start from local averages of two frame-dependent kinematic contents. The averaging process, which is reminiscent of the way order parameters are defined in soft matter physics, delivers invariant measures depending on the polar rotation gradient, which makes our theory effectively *second-grade*, although in an indirect way. It is not too much of a stretch to say that the averages from which pure drilling and bending measures originate are special incarnations of the Cosserat directors.

If anything, perhaps our theory differs the most from other direct theories in the attention paid to the drilling mode, which we reckon should be the more excitable the softer is the material constituting the shell, as drilling rotations would then be less hampered. This intuition makes us believe that our theory should be especially fit for *soft* shells.

The paper is organized as follows. In Section 2, we collect a number of kinematic preliminaries with the dual intent to make the paper self-contained and to accustom the reader to the notation employed here. Section 3 presents the theory: it starts from the definition of the local averages and ends with the pure measures and their geometric interpretations, which aid to give the proposed strain-energy density a form more directly connected with both stretching and curvature tensors. Finally, in Section 4, we summarize the conclusions of our study and reflect on its possible further evolution. The paper is closed by five appendices with a number of ancillary results. In particular, Appendix D illustrates explicit cases of *soft elasticity*, where a class of deformations inhabits the shell's ground state as a consequence of the *multiple* wells present in the strain energy.

2. Kinematics

In our theory, a *soft* shell is represented as a *material surface* with reference configuration $\mathcal{S}$, which is assumed to be a *smooth* orientable surface in three-dimensional Euclidean space $\mathcal{E}$ endowed with a translation space $\mathcal{V}$.⁸ We shall use an intrinsic approach to surface calculus, generally inspired by the work of Weatherburn (Weatherburn, 2016a,b), whose essential features are also outlined in Sonnet and Virga (2024). We refer the reader to Appendix A for a brief summary of useful definitions and properties that should make our development self-contained.

⁵ The detailed commentary in Ghiba et al. (2024) is especially recommended in this regard.

⁶ This is the rotation extracted from the polar decomposition of the deformation gradient.

⁷ We could trace back the definition of *drilling* rotation to the paper (Hughes and Brezzi, 1989).

⁸ In particular, here $\mathcal{S}$ will be assumed to be at least of class C^2 . Our notation for $\mathcal{E}$ and $\mathcal{V}$ is the same as in Truesdell (1991, p.324), where these geometric structures are further illuminated.

As shown in Fig. 1, a deformed configuration $\mathcal{S}_y$ of the shell $\mathcal{S}$ is represented by a mapping $\mathbf{y} : \mathcal{S} \rightarrow \mathcal{E}$, which we assume twice-differentiable, so that $\mathcal{S}_y = \mathbf{y}(\mathcal{S})$. Extending to the present setting results well-known from three-dimensional kinematics (see, for example, Chapt. 6 of Gurtin et al. 2010), we can represent the (surface) deformation gradient $\nabla_s \mathbf{y}$ as

$$\nabla_s \mathbf{y} = \lambda_1 \mathbf{v}_1 \otimes \mathbf{u}_1 + \lambda_2 \mathbf{v}_2 \otimes \mathbf{u}_2, \quad (1)$$

where the positive scalars λ_1, λ_2 are the *principal stretches*, while $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{v}_1, \mathbf{v}_2$ are the *right* and *left* principal *directions of stretching*, characterized as the (normalized) eigenvectors of the stretching tensors

$$\mathbf{U} := \sqrt{\mathbf{C}} = \lambda_1 \mathbf{u}_1 \otimes \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 \otimes \mathbf{u}_2, \quad (2a)$$

$$\mathbf{V} := \sqrt{\mathbf{B}} = \lambda_1 \mathbf{v}_1 \otimes \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \otimes \mathbf{v}_2, \quad (2b)$$

where

$$\mathbf{C} := (\nabla_s \mathbf{y})^\top (\nabla_s \mathbf{y}) \quad \text{and} \quad \mathbf{B} := (\nabla_s \mathbf{y})(\nabla_s \mathbf{y})^\top \quad (3)$$

are the *right* and *left* Cauchy-Green tensors, respectively. It should be noted that, for a point on $\mathcal{S}$ represented by the position vector $\mathbf{x}$, $\mathbf{U}(\mathbf{x})$ is a linear mapping of $\mathcal{T}_x$ onto itself, that is, $\mathbf{U}(\mathbf{x}) \in \text{GL}(\mathcal{T}_x)$, whereas $\mathbf{V}(\mathbf{y}(\mathbf{x}))$ is a linear mapping of $\mathcal{T}_{\mathbf{y}(\mathbf{x})}$ onto itself, that is, $\mathbf{V}(\mathbf{y}(\mathbf{x})) \in \text{GL}(\mathcal{T}_{\mathbf{y}(\mathbf{x})})$.⁹

By the polar decomposition theorem for the deformation of surfaces proved in Man and Cohen (1986) in the spirit of the general coordinate-free theory of material surfaces of Gurtin and Murdoch (1975a,b), which we also embrace, the deformation gradient $\nabla_s \mathbf{y}$ can also be written as

$$\nabla_s \mathbf{y} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \quad (4)$$

where the rotation $\mathbf{R}$ is an element of the special orthogonal group $\text{SO}(3)$ in three-dimensional translation space $\mathcal{V}$; it will also be called the *polar* rotation. It readily follows from (1) and (4) that $\mathbf{v}_i = \mathbf{R}\mathbf{u}_i$ for $i = 1, 2$.

As also recalled in Sonnet and Virga (2024), a rotation $\mathbf{R} \in \text{SO}(3)$ can be represented in terms of a vector $\mathbf{a} \in \mathcal{V}$ through Rodrigues' formula (Altmann, 1989),

$$\mathbf{R}(\mathbf{a}) = \frac{1}{1 + a^2} \{ (1 - a^2)\mathbf{I} + 2\mathbf{a} \otimes \mathbf{a} + 2\mathbf{W}(\mathbf{a}) \}, \quad (5)$$

where $a^2 := \mathbf{a} \cdot \mathbf{a}$, $\mathbf{I}$ is the second-rank identity tensor, and $\mathbf{W}(\mathbf{a})$ is the skew-symmetric tensor associated with $\mathbf{a}$.¹⁰ The connection between (5) and Euler's classical representation,

$$\mathbf{R} = \mathbf{I} + \sin \alpha \mathbf{W}(\mathbf{e}) + (1 - \cos \alpha) \mathbf{W}(\mathbf{e})^2, \quad (6)$$

where $\mathbf{e} \in \mathbb{S}^2$ is a unit vector designating the *axis* of rotation and $\alpha \in [0, \pi]$ is the *angle* of rotation, is established by setting

$$\mathbf{a} = \tan\left(\frac{\alpha}{2}\right) \mathbf{e}. \quad (7)$$

Thus, a π -turn, which is represented by (6) for $\alpha = \pi$, is represented by (5) for $a \rightarrow \infty$. Eq. (5) can easily be inverted, as the vector $\mathbf{a}$ representing $\mathbf{R}$ is identified by

$$\mathbf{W}(\mathbf{a}) = \frac{\mathbf{R} - \mathbf{R}^\top}{1 + \text{tr} \mathbf{R}}, \quad (8)$$

from which it follows that

$$a^2 = \frac{3 - \text{tr} \mathbf{R}}{1 + \text{tr} \mathbf{R}}, \quad (9)$$

which shows that all π -turns are characterized by having $\text{tr} \mathbf{R} = -1$.¹¹

By the decomposition formula of Rodrigues (1840), for every rotation $\mathbf{R}(\mathbf{a})$ and a given direction in space there is a single pair of vectors $(\mathbf{a}_1, \mathbf{a}_2)$, with $\mathbf{a}_1$ along the given direction and $\mathbf{a}_2$ orthogonal to it, such that $\mathbf{R}(\mathbf{a}) = \mathbf{R}(\mathbf{a}_2)\mathbf{R}(\mathbf{a}_1)$ (see Sonnet and Virga 2024, where a variational interpretation of this decomposition is also provided). Thus, at any point $\mathbf{x} \in \mathcal{S}$, choosing $\mathbf{a}_1$ along $\mathbf{v}$, we can decompose the rotation $\mathbf{R}$ in (4) as

$$\mathbf{R}(\mathbf{a}) = \mathbf{R}(\mathbf{b})\mathbf{R}(\mathbf{d}), \quad (10)$$

where

$$\mathbf{d} = a_v \mathbf{v}, \quad \mathbf{b} = \frac{1}{1 + a_v^2} (\mathbf{I} + a_v \mathbf{W}(\mathbf{v})) \mathbf{P}(\mathbf{v}) \mathbf{a}, \quad \text{with} \quad a_v := \mathbf{a} \cdot \mathbf{v}, \quad (11)$$

and $\mathbf{P}(\mathbf{v}) := \mathbf{I} - \mathbf{v} \otimes \mathbf{v}$ is the projection onto the tangent plane. We call $\mathbf{R}(\mathbf{d})$ the *drilling* rotation and $\mathbf{R}(\mathbf{b})$ the *bending* rotation.

⁹ Here $\mathcal{T}_x$ is the plane tangent to $\mathcal{S}$ at $\mathbf{x}$ and $\mathcal{T}_{\mathbf{y}(\mathbf{x})}$ is the plane tangent to $\mathcal{S}_y$ at $\mathbf{y}(\mathbf{x})$, see also Appendix A.

¹⁰ The action of $\mathbf{W}(\mathbf{a})$ on any vector $\mathbf{v} \in \mathcal{V}$ is given by $\mathbf{W}(\mathbf{a})\mathbf{v} = \mathbf{a} \times \mathbf{v}$, where $\times$ denotes the vector product.

¹¹ Eq. (9) is a consequence of the identity $\text{tr} \mathbf{R}^2 = (\text{tr} \mathbf{R})^2 - 2 \text{tr} \mathbf{R}$, valid for all $\mathbf{R} \in \text{SO}(3)$; the latter follows from the Cayley–Hamilton theorem applied to $\mathbf{R}$.

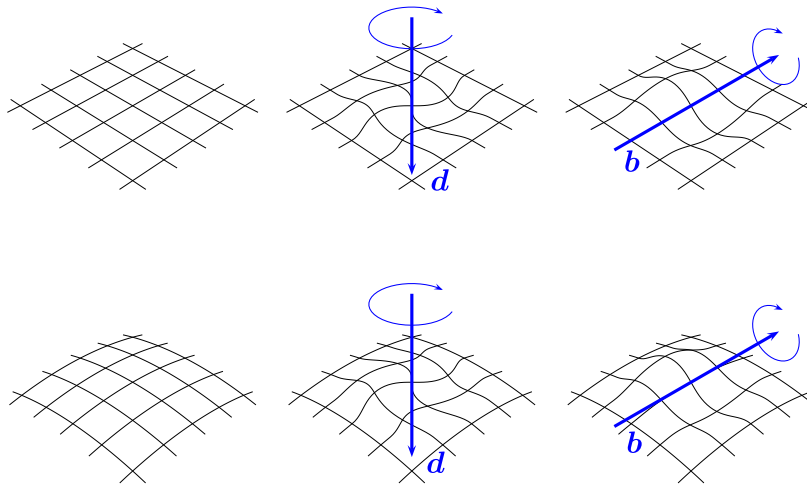


Fig. 2. Pictorial representation of local pure drilling and bending deformations. The upper and lower rows show how a flat and a curved surface are correspondingly affected by the two independent deformation modes. Depicted are the images of the material lines shown in the first column after undergoing local drilling and bending. Note that drilling does not actually alter the shape of the surface, and that both drilling and bending are *pure* only on the shown axes. The drilling and bending contents, d and b , are represented alongside the corresponding drilling and bending angles, α_d and α_b , defined by (12).

We see from (11) that whenever $\mathbf{R}(\mathbf{a})$ is a π -turn, so is also $\mathbf{R}(\mathbf{d})$, unless $a_v = 0$, in which case $\mathbf{d} = \mathbf{0}$ and $\mathbf{b} = \mathbf{a}$. For a given deformation $\mathbf{y}$, both $\mathbf{d}$ and $\mathbf{b}$ are vector fields on $\mathcal{S}$; we shall call them drilling and bending *contents* of $\mathbf{y}$, respectively. Each of them can be associated with a rotation angle, denoted α_d and α_b , respectively, which in accordance with (7) are defined as

$$\alpha_d := 2 \arctan d \quad \text{and} \quad \alpha_b := 2 \arctan b, \quad (12)$$

where $d := \sqrt{\mathbf{d} \cdot \mathbf{d}}$ and $b := \sqrt{\mathbf{b} \cdot \mathbf{b}}$. We call the angles α_d and α_b the *drilling* and *bending* angles, respectively.

The vectors $\mathbf{d}$ and $\mathbf{b}$ constitute the *non-metric* kinematic contents of the deformation $\mathbf{y}$ in a given frame, while the stretching tensors $\mathbf{U}$ and $\mathbf{V} = \mathbf{P}(\mathbf{v})\mathbf{R}\mathbf{U}\mathbf{R}^T$ constitute its *metric* kinematic contents (in the same frame). Thus, an *isometric* deformation occurs whenever $\mathbf{U} = \mathbf{P}(\mathbf{v})$, and correspondingly $\mathbf{V} = \mathbf{P}(\mathbf{v}_y)$, where $\mathbf{v}_y$ is the unit normal that orients $\mathcal{S}_y = \mathbf{y}(\mathcal{S})$. A deformation $\mathbf{y}$ for which there is a frame where $\mathbf{b}(\mathbf{x}) = \mathbf{0}$ at a point $\mathbf{x} \in \mathcal{S}$ is locally a *pure drilling*; it may also be called a *bending-neutral* deformation because when composed with another deformation it leaves the bending content of the latter unchanged (a formal proof of this property is given in Appendix B). More restrictively, a deformation $\mathbf{y}$ for which there is a frame where $\mathbf{d}(\mathbf{x}) = \mathbf{0}$ is a *pure bending*, if the direction of $\mathbf{b}(\mathbf{x})$ does not swerve (at least locally) on the tangent plane of $\mathcal{S}$.¹²

Fig. 2 represents pictorially how a pure drilling and a pure bending may affect locally a surface originally flat or curved.

Pure measures of bending were defined in Virga (2024) as strain measures invariant under all bending-neutral deformations. It was proved in Sonnet and Virga (2024) that bending-neutral deformations actually exist in the large. They preserve *minimal surfaces*, in the sense that a minimal surface is mapped into another minimal surface by a bending-neutral deformation, and for any given pair of such surfaces with the same Gauss map,¹³ there is a bending-neutral deformation mapping one into the other.

The positive scalars d^2 and b^2 (along with the corresponding angles α_d and α_b) will play a special role in our development. They are related to the vector $\mathbf{a}$ that represents $\mathbf{R}$ through the equations (see also Sonnet and Virga, 2024)

$$d^2 = a_v^2 \quad \text{and} \quad b^2 = \frac{a^2 - a_v^2}{1 + a_v^2}. \quad (13)$$

By use of (8) and (9), these can also be given the following form¹⁴

$$d^2 = \left(\frac{\mathbf{R} \cdot \mathbf{W}(\mathbf{v})}{1 + \text{tr } \mathbf{R}} \right)^2 \quad \text{and} \quad b^2 = \frac{(3 - \text{tr } \mathbf{R})(1 + \text{tr } \mathbf{R}) - (\mathbf{R} \cdot \mathbf{W}(\mathbf{v}))^2}{(1 + \text{tr } \mathbf{R})^2 + (\mathbf{R} \cdot \mathbf{W}(\mathbf{v}))^2}. \quad (14)$$

To relate both d^2 and b^2 directly to the deformation gradient $\nabla_s \mathbf{y}$, we need to extract the polar rotation $\mathbf{R}$ from the latter. To this end, we remark that $\mathbf{R}$ can be represented as follows,

$$\mathbf{R} = \mathbf{v}_1 \otimes \mathbf{u}_1 + \mathbf{v}_2 \otimes \mathbf{u}_2 + \mathbf{v}_y \otimes \mathbf{v} \quad (15)$$

¹² This requirement is enforced by setting $\nabla_s \hat{\mathbf{b}}(\mathbf{x}) = \mathbf{0}$ for the unit vector $\hat{\mathbf{b}}$ of $\mathbf{b}$. We shall discuss in Appendix D the consequences of a less restrictive definition of pure bending.

¹³ That is, which have one and the same system of normals (possibly, to within a uniform rigid rotation).

¹⁴ Here the inner product of two second-rank tensors, $\mathbf{A}$ and $\mathbf{B}$, is defined as $\mathbf{A} \cdot \mathbf{B} := \text{tr}(\mathbf{A}^T \mathbf{B}) = \text{tr}(\mathbf{B}^T \mathbf{A})$.

and that $\mathbf{v}_y = \mathbf{v}_1 \times \mathbf{v}_2$, if we have oriented $\mathcal{S}$ according to the choice $\mathbf{v} = \mathbf{u}_1 \times \mathbf{u}_2$. Since, by (1) and (2),

$$(\nabla_s \mathbf{y})\mathbf{U}^{-1} = \mathbf{v}_1 \otimes \mathbf{u}_1 + \mathbf{v}_2 \otimes \mathbf{u}_2, \quad (16)$$

we have that

$$\mathbf{v}_y = (\nabla_s \mathbf{y})\mathbf{U}^{-1} \mathbf{u}_1 \times (\nabla_s \mathbf{y})\mathbf{U}^{-1} \mathbf{u}_2 = \text{co}((\nabla_s \mathbf{y})\mathbf{U}^{-1}) \mathbf{v}, \quad (17)$$

where $\text{co}(\cdot)$ denotes the cofactor of a tensor (as defined, for example, in Sect. 2.11 of Gurtin et al. 2010). By combining (15), (16), and (17), we arrive at

$$\mathbf{R} = (\nabla_s \mathbf{y})\mathbf{U}^{-1} + \text{co}((\nabla_s \mathbf{y})\mathbf{U}^{-1}), \quad (18)$$

where we have extended the tensor $(\nabla_s \mathbf{y})\mathbf{U}^{-1}$ to be an element of $\text{GL}(3)$ by requiring that $(\nabla_s \mathbf{y})\mathbf{U}^{-1} \mathbf{v} = \mathbf{0}$.¹⁵

The non-metric kinematic contents, $\mathbf{d}$ and $\mathbf{b}$, of the deformation $\mathbf{y}$ express independent degrees of freedom coexisting in the polar rotation $\mathbf{R}$. While the bending content $\mathbf{b}$ is expected to be significant in the deformation of all shells, irrespective of the material that constitutes them, the drilling content $\mathbf{d}$ is expected to be significant especially in the deformation of shells constituted by materials sufficiently *soft* to allow for surface fibers to be wrung in the tangent plane.

Following Gurtin et al. (2010, Chapt. 20) we consider the surface $\mathcal{S}$ as living in a *reference* space of its own (also called the *material* space), thus remaining unaffected by a change of frame that transforms the deformation $\mathbf{y}$ into

$$\mathbf{y}^*(\mathbf{x}) = \mathbf{y}_0(t) + \mathbf{Q}(t)\mathbf{y}(\mathbf{x}), \quad (19)$$

where $\mathbf{y}_0(t) \in \mathcal{V}$ is a translation and $\mathbf{Q}(t) \in \text{O}(3)$ is an orthogonal tensor, both possibly depending on time t .¹⁶

Under the action of a change of frame, the principal directions $\mathbf{u}_i$ remain unchanged, as do all material vectors, whereas the principal directions $\mathbf{v}_i$ are transformed into $\mathbf{v}_i^* = \mathbf{Q}\mathbf{v}_i$. Thus, the stretching tensor $\mathbf{U}$ is *frame-invariant*, as $\mathbf{U}^* = \mathbf{U}$, while the stretching tensor $\mathbf{V}$ is *frame-indifferent*, as $\mathbf{V}^* = \mathbf{Q}\mathbf{V}\mathbf{Q}^T$.

On the other hand, the polar rotation $\mathbf{R}$ transforms into $\mathbf{R}^* = \mathbf{Q}\mathbf{R}$, and so it is neither frame-invariant nor frame-indifferent. Accordingly, it follows from (8) and (9) that

$$\mathbf{W}(\mathbf{a}^*) = \frac{\mathbf{Q}\mathbf{R} - \mathbf{R}^T \mathbf{Q}^T}{1 + \text{tr}(\mathbf{Q}\mathbf{R})} \quad \text{and} \quad a^{*2} = \frac{3 - \text{tr}(\mathbf{Q}\mathbf{R})}{1 + \text{tr}(\mathbf{Q}\mathbf{R})}, \quad (20)$$

which are not neatly related to $\mathbf{W}(\mathbf{a})$ and a^2 . Even more intricate transformation laws would follow from (14) for both d^2 and b^2 .

Despite a naive intuition which would suggest to take d^2 and b^2 as scalar *measures* of the two independent components of $\mathbf{R}$, neither of them could be one, as they are frame-dependent. We find it instructive to illustrate in Appendix C how complicated this dependence could be, even in a simple, innocent-looking case.

In the following section, we shall extract from d^2 and b^2 appropriate frame-invariant measures of drilling and bending.

3. Theory

In this section, we attempt to build a theory for shells based upon an elastic stored energy where the stretching degrees of freedom of the material surface $\mathcal{S}$ representing the mechanical system are combined on equal footing with the *non-stretching* degrees of freedom. Among these, our kinematic analysis has distinguished two independent components, that is, drilling and bending, the former perhaps more distinctive of soft shells than the latter.

It is quite clear how the stretching of $\mathcal{S}$ can be measured for a material (transversely) isotropic about the normal $\mathbf{v}$: a natural pure stretching measure w_s is the mismatch between the (two-dimensional) stretching tensor $\mathbf{U}$ and the projection $\mathbf{P}(\mathbf{v})$, which would correspond to an isometric deformation,¹⁷

$$w_s = |\mathbf{U} - \mathbf{P}(\mathbf{v})|^2 = |\mathbf{V} - \mathbf{P}(\mathbf{v}_y)|^2. \quad (21)$$

To define similar, appropriate pure measures of *drilling* and *bending* w_d and w_b , we start from d^2 and b^2 , respectively. We need to infer from the latter a content unaffected by a uniform rotation: this is achieved by taking *local* averages over *relative* rotations, as illustrated below.

¹⁵ For such an extension, it is a simple matter to prove that $\text{co}((\nabla_s \mathbf{y})\mathbf{U}^{-1})\mathbf{u}_i = \mathbf{0}$, for $i = 1, 2$.

¹⁶ At any time t , the change of frame corresponds to an isometry of Euclidean space, which is given by a translation and an orthogonal transformation, see Truesdell (1991, p. 325).

¹⁷ Here, for a second-rank tensor $\mathbf{A}$, $|\mathbf{A}|^2 := \mathbf{A} \cdot \mathbf{A}$. Moreover, as customary in Koiter's model (Koiter, 1960), one could also set

$$w_s = |\mathbf{C} - \mathbf{P}(\mathbf{v})|^2 = |\mathbf{B} - \mathbf{P}(\mathbf{v}_y)|^2,$$

which would make w_s *quartic* in the principal stretches. However, stretching measures are not our primary interest in this paper.

3.1. Relative rotation local averaging

To suppress any uniform background rotation, which is the source of frame-dependence of d^2 and b^2 , we take (transversely isotropic) averages over relative rotations in a small disk on the tangent plane. We thus obtain intrinsic measures of drilling and bending, at least for materials isotropic about the shell's normal. The origin of these measures is formally akin to the definition of scalar *order parameters* in soft matter physics; they play in our theory a role similar to that played by *directors* in the classical Cosserat theory.

For $\mathbf{x} \in \mathcal{S}$, let $\mathbb{D}_\ell$ be a (small) disk of radius ℓ and centre $\mathbf{x}$ on the tangent plane $\mathcal{T}_\mathbf{x}$. The points $\mathbf{x}_\varepsilon$ of $\mathbb{D}_\ell$ are described by

$$\mathbf{x}_\varepsilon = \mathbf{x} + \varepsilon \mathbf{u}, \quad (22)$$

where ε ranges in the interval $[0, \ell]$ and $\mathbf{u} \in \mathbb{S}^1$ is a unit vector on $\mathcal{T}_\mathbf{x}$. By lifting onto $\mathbb{D}_\ell$ along the normal $\mathbf{v}$ at $\mathbf{x}$ the neighborhood of $\mathbf{x}$ intercepted on $\mathcal{S}$ by the cylinder based on $\mathbb{D}_\ell$ (following the method introduced in [Gurtin and Murdoch 1975a](#)),¹⁸ we can define on $\mathbb{D}_\ell$ both fields $\mathbf{R}_\varepsilon := \mathbf{R}(\mathbf{x}_\varepsilon)$ and $\mathbf{v}_\varepsilon := \mathbf{v}(\mathbf{x}_\varepsilon)$.

For given $\mathbf{u} \in \mathbb{S}^1$ such that $\mathbf{v} \cdot \mathbf{u} = 0$, we expand both $\mathbf{R}_\varepsilon$ and $\mathbf{v}_\varepsilon$ up to the lowest order in ε ,

$$\mathbf{R}_\varepsilon = \mathbf{R} + \varepsilon(\nabla_s \mathbf{R})\mathbf{u} + o(\varepsilon), \quad (23a)$$

$$\mathbf{v}_\varepsilon = \mathbf{v} + \varepsilon(\nabla_s \mathbf{v})\mathbf{u} + o(\varepsilon), \quad (23b)$$

where $(\nabla_s \mathbf{R})\mathbf{u} \in \text{GL}(3)$ is a second-rank tensor (obtained by contracting the third leg of $\nabla_s \mathbf{R}$ with $\mathbf{u}$). It readily follows from (23a) that the *relative* rotation $\mathbf{R}'_\varepsilon$ is represented on $\mathbb{D}_\ell$ as

$$\mathbf{R}'_\varepsilon := \mathbf{R}^\top \mathbf{R}_\varepsilon = \mathbf{I} + \varepsilon \mathbf{W}_\mathbf{u} + o(\varepsilon), \quad (24a)$$

where $\mathbf{W}_\mathbf{u} := \mathbf{R}^\top (\nabla_s \mathbf{R})\mathbf{u}$ is a skew-symmetric tensor (depending linearly on $\mathbf{u}$), and from (23b) it follows that

$$\mathbf{W}(\mathbf{v}_\varepsilon) = \mathbf{W}(\mathbf{v}) + \varepsilon \mathbf{W}((\nabla_s \mathbf{v})\mathbf{u}) + o(\varepsilon). \quad (24b)$$

With the aid of (14), the *relative average* drilling content is then defined as

$$\langle d^2 \rangle := \frac{1}{A(\mathbb{D}_\ell)} \int_0^\ell \varepsilon d\varepsilon \int_{\mathbb{S}^1} \left(\frac{\mathbf{R}'_\varepsilon \cdot \mathbf{W}(\mathbf{v}_\varepsilon)}{1 + \text{tr } \mathbf{R}'_\varepsilon} \right)^2 d\mathbf{u}, \quad (25)$$

where A denotes the area measure. By use of Eqs. (24), we also give (25) the form

$$\langle d^2 \rangle = \frac{\ell^2}{32} \langle (\mathbf{W}_\mathbf{u} \cdot \mathbf{W}(\mathbf{v}))^2 \rangle_{\mathbf{u} \in \mathbb{S}^1} + o(\ell^2) = \frac{\ell^2}{64} |\mathbf{W}(\mathbf{v}) \circ \mathbf{H}|^2 + o(\ell^2), \quad (26)$$

where we have set

$$\mathbf{H} := \mathbf{R}^\top \nabla_s \mathbf{R}, \quad (27)$$

a third-rank tensor,¹⁹ which in a generic Cartesian frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ would have components

$$H_{ijk} = R_{hi} R_{hj;k}, \quad (28)$$

if R_{hi} and $R_{hj;k}$ denote the components (in the same frame) of $\mathbf{R}$ and $\nabla_s \mathbf{R}$, respectively.²⁰ In (26), $\mathbf{W}(\mathbf{v}) \circ \mathbf{H}$ is a vector of $\mathcal{V}$, as for a generic second-rank tensor $\mathbf{A}$ and any triadic component $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}$ of $\mathbf{H}$,

$$\mathbf{A} \circ \mathbf{H} := (\mathbf{a} \cdot \mathbf{A})\mathbf{b}\mathbf{c}. \quad (29)$$

More generally, it follows from (27) that, for any orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ on the tangent plane $\mathcal{T}_\mathbf{x}$ to $\mathcal{S}$, $\mathbf{H}(\mathbf{x})$ can be represented as

$$\mathbf{H}(\mathbf{x}) = \mathbf{W}_1 \otimes \mathbf{e}_1 + \mathbf{W}_2 \otimes \mathbf{e}_2, \quad (30)$$

where both $\mathbf{W}_1$ and $\mathbf{W}_2$ are skew-symmetric tensors. Thus, combining (29) and (30), we can also write

$$\mathbf{W}(\mathbf{v}) \circ \mathbf{H} = (\mathbf{W}_1 \cdot \mathbf{W}(\mathbf{v}))\mathbf{e}_1 + (\mathbf{W}_2 \cdot \mathbf{W}(\mathbf{v}))\mathbf{e}_2 = 2((\mathbf{w}_1 \cdot \mathbf{v})\mathbf{e}_1 + (\mathbf{w}_2 \cdot \mathbf{v})\mathbf{e}_2), \quad (31)$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are the axial vectors of $\mathbf{W}_1$ and $\mathbf{W}_2$, respectively. Equivalently, by introducing a movable orthonormal frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{v})$ gliding on $\mathcal{S}$, $\mathbf{H}$ can also be represented in the form

$$\mathbf{H} = \mathbf{W}(\mathbf{e}_1) \otimes \mathbf{a}_1 + \mathbf{W}(\mathbf{e}_2) \otimes \mathbf{a}_2 + \mathbf{W}(\mathbf{v}) \otimes \mathbf{a}_3, \quad (32)$$

¹⁸ An alternative method, introduced in [Arroyo and Belytschko \(2002\)](#), makes use of the *exponential map* for surfaces (see also p. 77 of [Morgan 1993](#).)

¹⁹ $\mathbf{H}$ is somewhat reminiscent of the third-rank tensor $\mathbf{F}^\top \nabla_s \mathbf{F}$, where $\mathbf{F} := \nabla_s \mathbf{y}$, that [Murdoch \(1978\)](#) employed for his direct second-grade hyperelastic theory of shells and that has recently received a full kinematic characterization ([Tiwari and Fried, 2025](#)).

²⁰ Here and below, we shall use the usual convention of summing over repeated indices. Moreover, as also recalled in [Appendix A](#), a semicolon denotes surface differentiation.

where $\mathbf{W}(e_1)$ and $\mathbf{W}(e_2)$ are the skew-symmetric tensors associated with e_1 and e_2 , respectively, and a_1 , a_2 , and a_3 are vectors such that $a_i \cdot v = 0$.

It readily follows from (31) that

$$|\mathbf{W}(v) \circ \mathbf{H}|^2 = 4((w_1 \cdot v)^2 + (w_2 \cdot v)^2). \quad (33)$$

Similarly, again by (14), we arrive at the following expression for the *relative average* bending content,

$$\begin{aligned} \langle b^2 \rangle &:= \frac{1}{A(\mathbb{D}_\ell)} \int_0^\ell \varepsilon d\varepsilon \int_{\mathbb{S}^1} \frac{(3 - \text{tr } \mathbf{R}'_\varepsilon)(1 + \text{tr } \mathbf{R}'_\varepsilon) - (\mathbf{R}'_\varepsilon \cdot \mathbf{W}(v_\varepsilon))^2}{(1 + \text{tr } \mathbf{R}'_\varepsilon)^2 + (\mathbf{R}'_\varepsilon \cdot \mathbf{W}(v_\varepsilon))^2} d\mathbf{u}, \\ &= \frac{\ell^2}{32} \langle 2\mathbf{W}_u \cdot \mathbf{W}_u - (\mathbf{W}_u \cdot \mathbf{W}(v))^2 \rangle_{u \in \mathbb{S}^1} + o(\ell^2), \\ &= \frac{\ell^2}{32} \left\{ |\mathbf{H}|^2 - \frac{1}{2} |\mathbf{W}(v) \circ \mathbf{H}|^2 \right\} + o(\ell^2), \end{aligned} \quad (34)$$

where, by (30),

$$|\mathbf{H}|^2 = |\mathbf{W}_1|^2 + |\mathbf{W}_2|^2 = 2(w_1^2 + w_2^2). \quad (35)$$

The detailed computations that prove (26) and (34) are an easy exercise presented in Appendix A; here we only note that they make use of the identity

$$\langle u \otimes u \rangle_{u \in \mathbb{S}^1} = \frac{1}{2} \mathbf{P}(v), \quad (36)$$

which embodies our notion of transverse isotropy.

Since in a change of frame $\mathbf{R}^* = \mathbf{Q}\mathbf{R}$ and $\nabla_s \mathbf{R} = \mathbf{Q}\nabla_s \mathbf{R}$, it is a simple matter to see from (27) that $\mathbf{H}^* = \mathbf{H}$, and so both $\langle d^2 \rangle$ and $\langle b^2 \rangle$ are frame-invariant, as expected. Moreover, depending only on $\mathbf{R}$, they are plainly invariant under a superimposed *pure extension*, in which $\mathbf{U}$ is replaced by $\lambda \mathbf{U}$ with $\lambda > 0$ (and $\mathbf{V}$ by $\lambda \mathbf{V}$). Finally, they both vanish if $\nabla_s \mathbf{R}$ does, that is, if $\mathbf{R}$ is locally *uniform*.

The scene is now set to extract from $\langle d^2 \rangle$ and $\langle b^2 \rangle$ pure measures of drilling and bending, w_d and w_b , akin to the natural measure of stretching w_s in (21). Precisely as w_s vanishes on isometries, we wish w_d and w_b to vanish on pure bending and drilling deformation modes, respectively.²¹ We start by building w_b .

3.2. Pure bending measure

A pure drilling mode is embodied by a bending-neutral deformation, for which $\mathbf{R} = \mathbf{R}_d$, where $\mathbf{R}_d$ is a rotation about v . Here we shall not be concerned with the existence in the large of a bending-neutral deformation; it will suffice to assume that the algebraic condition that makes it compatible with the surface is met locally (see Virga 2024, Sonnet and Virga 2024). A pure measure of bending w_b is thus constructed from $\langle b^2 \rangle$ (at the leading order in ℓ stripped of its scaling factor) by requiring that it vanishes locally on a generic bending-neutral deformation of $\mathcal{S}$.

To this end, we recall that a movable frame (e_1, e_2, v) glides over $\mathcal{S}$ according to the laws

$$\begin{cases} \nabla_s e_1 = e_2 \otimes c + v \otimes d_1, \\ \nabla_s e_2 = -e_1 \otimes c + v \otimes d_2, \\ \nabla_s v = -e_1 \otimes d_1 - e_2 \otimes d_2, \end{cases} \quad (37)$$

where the vector fields (c, d_1, d_2) are everywhere tangent to $\mathcal{S}$; these are the *connectors* of the movable frame: more precisely, c is the *spin* connector and d_1, d_2 are the *curvature* connectors. Since the curvature tensor $\nabla_s v$ is symmetric, the curvature connectors must obey the identity

$$d_1 \cdot e_2 = d_2 \cdot e_1. \quad (38)$$

By (6), $\mathbf{R}_d$ can also be represented in the frame (e_1, e_2, v) in the form

$$\mathbf{R}_d = \cos \alpha (e_1 \otimes e_1 + e_2 \otimes e_2) + \sin \alpha (e_2 \otimes e_1 - e_1 \otimes e_2) + v \otimes v, \quad (39)$$

where α is a scalar field.

Repeated use of (37) in (39) leads us to express $\mathbf{H}_d := \mathbf{R}_d^T \nabla_s \mathbf{R}_d$ as

$$\mathbf{H}_d = -\mathbf{W}(e_1) \otimes ((1 - \cos \alpha) d_2 + \sin \alpha d_1) + \mathbf{W}(e_2) \otimes ((1 - \cos \alpha) d_1 - \sin \alpha d_2) + \mathbf{W}(v) \otimes \nabla_s \alpha, \quad (40)$$

which specializes (32). By direct computation, we then obtain that

$$|\mathbf{H}_d|^2 - \frac{1}{2} |\mathbf{W}(v) \circ \mathbf{H}_d|^2 = 4(1 - \cos \alpha)(d_1^2 + d_2^2) = 4(1 - \cos \alpha) |\nabla_s v|^2 = 4v \cdot \mathbf{H}_d \circ \nabla_s v, \quad (41)$$

²¹ In addition to vanish on uniform rotations.

where $\mathbf{H}_d \circ \nabla_s \mathbf{v}$ is a vector defined according to the convention that for a generic triadic component $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}$ of $\mathbf{H}$ and a generic second-rank tensor $\mathbf{A}$ sets

$$\mathbf{H} \circ \mathbf{A} := (\mathbf{b} \cdot \mathbf{A} \mathbf{c}) \mathbf{a}. \quad (42)$$

Thus the simplest invariant pure (positive) measure of bending can be defined as²²

$$w_b := \left(|\mathbf{H}|^2 - \frac{1}{2} |\mathbf{W}(\mathbf{v}) \circ \mathbf{H}|^2 - 4 \mathbf{v} \cdot \mathbf{H} \circ \nabla_s \mathbf{v} \right)^2. \quad (43)$$

This measure is *quartic* in $\mathbf{H}$ and it vanishes for both $\mathbf{H} = \mathbf{0}$ and $\mathbf{H} = \mathbf{H}_d$. We may say that w_b as defined by (43) has turned $\langle b^2 \rangle$ from a measure of bending into a pure measure of bending.

3.3. Pure drilling measure

We proceed in a similar way to extract from $\langle d^2 \rangle$ a pure measure of drilling. In analogy with (39), letting $\mathbf{e}_2$ be the direction of bending in a local pure bending deformation,²³ we write the rotation component $\mathbf{R}_b$ of the deformation gradient as

$$\mathbf{R}_b = \cos \beta (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{v} \otimes \mathbf{v}) + \mathbf{e}_2 \otimes \mathbf{e}_2 + \sin \beta (\mathbf{v} \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{v}), \quad (44)$$

where β is a scalar field. Resorting again to (37), we readily arrive at

$$\mathbf{H}_b := \mathbf{R}_b^T \nabla_s \mathbf{R}_b = \mathbf{W}(\mathbf{e}_1) \otimes ((\cos \beta - 1) \mathbf{d}_2 + \sin \beta \mathbf{c}) - \mathbf{W}(\mathbf{e}_2) \otimes \nabla_s \beta + \mathbf{W}(\mathbf{v}) \otimes ((\cos \beta - 1) \mathbf{c} - \sin \beta \mathbf{d}_2), \quad (45)$$

whence it follows that

$$\mathbf{W}(\mathbf{v}) \circ \mathbf{H}_b = -2((1 - \cos \beta) \mathbf{c} + \sin \beta \mathbf{d}_2). \quad (46)$$

By (37), requiring that $\nabla_s \mathbf{e}_2 = \mathbf{0}$ amounts to require that both connectors $\mathbf{c}$ and $\mathbf{d}_2$ vanish (possibly only locally), and so also does $\mathbf{W}(\mathbf{v}) \circ \mathbf{H}_b$. This shows that $\langle d^2 \rangle$ (again at leading order in ℓ and stripped of its scaling factor) is itself a pure measure of drilling,

$$w_d := |\mathbf{W}(\mathbf{v}) \circ \mathbf{H}|^2. \quad (47)$$

The reader is referred to Appendix D to see how the definition of w_d would change when in a pure bending deformation $\nabla_s \mathbf{e}_2$ is not required to vanish. In Appendix E, we study instead the case where all admissible deformations are pure bending deformations with $\nabla_s \mathbf{e}_2$ vanishing everywhere.

3.4. Energetics

The pure measures for the different independent degrees of deformation introduced above suggest to define a general elastic stored energy functional in the form

$$\mathcal{F}[\mathbf{y}] := \int_{\mathcal{S}} W(w_s, w_d, w_b) dA, \quad (48)$$

where W is a positive-definite function of all three measures. The functional $\mathcal{F}$ is properly invariant under a change of frame and keeps separate all independent mechanisms that can deform $\mathcal{S}$.

The easiest form of W would be the following,

$$W = \frac{1}{2} \mu_s w_s + \frac{1}{2} \mu_d w_d + \frac{1}{4} \mu_b w_b, \quad (49)$$

where the positive scalars μ_s , μ_d , and μ_b are the *stretching*, *drilling*, and *bending* moduli, respectively. While w_s is quadratic in the stretching tensor $\mathbf{U}$ and w_d is quadratic in $\mathbf{H}$, w_b is quartic in the latter. This is the price to pay to enforce our separation criterion, which requires w_d and w_b to vanish each on the whole class of deformations that selectively activate the other.

By construction, W is a *multi-well* energy, as it is the sum of potentials with multiple local minima, corresponding to individual local ground states. W can also exhibit cases of *soft elasticity*,²⁴ where it vanishes on a whole class of deformations, as shown in Appendix D, for a class of deformations generating a family of minimal surfaces.

While μ_s in (49) has the physical dimension of energy per unit area, μ_d has that of energy, and μ_b of energy times area. Our definition of $\langle d^2 \rangle$ and $\langle b^2 \rangle$ suggest that μ_d scales like $\mu_s \ell^2$, whereas μ_b scales like $\mu_s \ell^4$, where ℓ is the size of the averaging disc, totally unrelated to the thickness of the shell (which in our direct theory vanishes by definition).

As shown in Appendix E, in the presence of an appropriate constraint on the admissible deformations, also w_b can be given a quadratic form (one that had actually been known since the work of Antman (1968) on elastic rods).

²² It is a simple matter to show that failing to square the right hand side of (43) would fail to make w_b positive definite.

²³ Apart from the case studied in Appendix E, pure bending deformations are not known to exist in the large. This is why our considerations here are purely local.

²⁴ The reader is referred to Chapt. 7 of Warner and Terentjev (2003) for a general discussion of soft elasticity in nematic elastomers (see also Warner et al., 1994 for an earlier specific contribution).

Another notable case where the energy density W need not be quartic in $\mathbf{H}$ is that of plates, for which $\nabla_s \mathbf{v} \equiv \mathbf{0}$, so that $\mathbf{H}_d \circ \nabla_s \mathbf{v}$ also vanishes in (41) and our separation criterion can be enforced by simply taking

$$\tilde{w}_b := |\mathbf{H}|^2 - \frac{1}{2} |\mathbf{W}(\mathbf{v}) \circ \mathbf{H}|^2 \quad (50)$$

as a pure bending measure. Thus, W in (49) can be replaced by

$$W = \frac{1}{2} \mu_s w_s + \frac{1}{2} \mu_d w_d + \frac{1}{2} \tilde{\mu}_b \tilde{w}_b, \quad (51)$$

which is entirely quadratic. Comparing (47) and (50), we see that (51) can be given the following simpler form

$$W = \frac{1}{2} \mu_s w_s + \mu_d |\mathbf{H}|^2, \quad (52)$$

whenever $\tilde{\mu}_b = 2\mu_d$, a special instance that defines a useful simplification of the theory.

3.5. Geometric representation

Both w_d and w_b can be given a more telling representation in terms of mismatches between tensorial quantities with an intrinsic geometric meaning. The formulas that we shall obtain generalize those found in Appendix E for cylindrical shells and are based on similar calculations, although admittedly more complicated.

We start from the systems of connectors for the movable frames $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v})$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_y)$ on surfaces $\mathcal{S}$ and $\mathcal{S}_y$, respectively. According to (37), these are the fields $(c, \mathbf{d}_1, \mathbf{d}_2)$ and $(c^*, \mathbf{d}_1^*, \mathbf{d}_2^*)$, tangent to $\mathcal{S}$ and $\mathcal{S}_y$, that obey the equations

$$\begin{cases} \nabla_s \mathbf{u}_1 = \mathbf{u}_2 \otimes c + \mathbf{v} \otimes \mathbf{d}_1, \\ \nabla_s \mathbf{u}_2 = -\mathbf{u}_1 \otimes c + \mathbf{v} \otimes \mathbf{d}_2, \\ \nabla_s \mathbf{v} = -\mathbf{u}_1 \otimes \mathbf{d}_1 - \mathbf{u}_2 \otimes \mathbf{d}_2, \end{cases} \quad \begin{cases} \nabla_s^* \mathbf{v}_1 = \mathbf{v}_2 \otimes c^* + \mathbf{v}_y \otimes \mathbf{d}_1^*, \\ \nabla_s^* \mathbf{v}_2 = -\mathbf{v}_1 \otimes c^* + \mathbf{v}_y \otimes \mathbf{d}_2^*, \\ \nabla_s^* \mathbf{v}_y = -\mathbf{v}_1 \otimes \mathbf{d}_1^* - \mathbf{v}_2 \otimes \mathbf{d}_2^*, \end{cases} \quad (53)$$

where ∇_s^* denotes the surface gradient on $\mathcal{S}_y$. As in (38), the curvature connectors must obey the identities

$$\mathbf{d}_1 \cdot \mathbf{u}_2 = \mathbf{d}_2 \cdot \mathbf{u}_1 \quad \text{and} \quad \mathbf{d}_1^* \cdot \mathbf{v}_2 = \mathbf{d}_2^* \cdot \mathbf{v}_1. \quad (54)$$

Eqs. (53) are instrumental to computing the tensor $\mathbf{H}$ starting from the following representation of $\mathbf{R}$,

$$\mathbf{R} = \mathbf{v}_1 \otimes \mathbf{u}_1 + \mathbf{v}_2 \otimes \mathbf{u}_2 + \mathbf{v}_y \otimes \mathbf{v}, \quad (55)$$

which is (15) recalled. To achieve this goal, we also need to combine (53) with the equations

$$\nabla_s \mathbf{v}_i = (\nabla_s^* \mathbf{v}_i) \nabla_s \mathbf{y}, \quad \nabla_s \mathbf{v}_y = (\nabla_s^* \mathbf{v}_y) \nabla_s \mathbf{y}, \quad (56)$$

which are mere applications of the chain rule. Moreover, we shall use the following representations for the connectors,

$$\begin{cases} c = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2, \\ \mathbf{d}_1 = d_{11} \mathbf{u}_1 + d_{12} \mathbf{u}_2, \\ \mathbf{d}_2 = d_{12} \mathbf{u}_1 + d_{22} \mathbf{u}_2, \end{cases} \quad \begin{cases} c^* = c_1^* \mathbf{v}_1 + c_2^* \mathbf{v}_2, \\ \mathbf{d}_1^* = d_{11}^* \mathbf{v}_1 + d_{12}^* \mathbf{v}_2, \\ \mathbf{d}_2^* = d_{12}^* \mathbf{v}_1 + d_{22}^* \mathbf{v}_2, \end{cases} \quad (57)$$

where (54) has also been used. By employing repeatedly (53), (54), (56), and (57), recalling that $\mathbf{v}_i = \mathbf{R} \mathbf{u}_i$, we arrive at the following form of (32),

$$\mathbf{H} = \mathbf{W}_1 \otimes \mathbf{u}_1 + \mathbf{W}_2 \otimes \mathbf{u}_2, \quad (58)$$

where $\mathbf{W}_i$ are skew-symmetric tensors defined by

$$\begin{aligned} \mathbf{W}_i &:= (\lambda_i c_i^* - c_i) \mathbf{W}(\mathbf{v}) - (\lambda_i d_{1i}^* - d_{1i}) \mathbf{W}(\mathbf{u}_2) + (\lambda_i d_{2i}^* - d_{2i}) \mathbf{W}(\mathbf{u}_1) \\ &= (\mathbf{V} c^* - \mathbf{R} c) \cdot \mathbf{v}_i \mathbf{W}(\mathbf{v}) - (\mathbf{V} \mathbf{d}_1^* - \mathbf{R} \mathbf{d}_1) \cdot \mathbf{v}_i \mathbf{W}(\mathbf{u}_2) + (\mathbf{V} \mathbf{d}_2^* - \mathbf{R} \mathbf{d}_2) \cdot \mathbf{v}_i \mathbf{W}(\mathbf{u}_1), \end{aligned} \quad (59)$$

and $\mathbf{W}(\mathbf{u}_i)$ is the skew-symmetric tensor associated with $\mathbf{u}_i$.

It follows from (53) and (58) that

$$\mathbf{W}(\mathbf{v}) \circ \mathbf{H} = (\mathbf{W}(\mathbf{v}) \cdot \mathbf{W}_1) \mathbf{u}_1 + (\mathbf{W}(\mathbf{v}) \cdot \mathbf{W}_2) \mathbf{u}_2, \quad (60a)$$

$$|\mathbf{H}|^2 = \mathbf{W}_1 \cdot \mathbf{W}_1 + \mathbf{W}_2 \cdot \mathbf{W}_2, \quad (60b)$$

$$\mathbf{H} \circ \nabla_s \mathbf{v} = -(\mathbf{W}_1 \mathbf{d}_1 + \mathbf{W}_2 \mathbf{d}_2). \quad (60c)$$

Thus, by (59), we easily arrive at

$$|\mathbf{W}(\mathbf{v}) \circ \mathbf{H}|^2 = 4 |\mathbf{V} c^* - \mathbf{R} c|^2, \quad (61a)$$

$$|\mathbf{H}|^2 - \frac{1}{2} |\mathbf{W}(\mathbf{v}) \circ \mathbf{H}|^2 = 2 \{ |\mathbf{V} \mathbf{d}_1^* - \mathbf{R} \mathbf{d}_1|^2 + |\mathbf{V} \mathbf{d}_2^* - \mathbf{R} \mathbf{d}_2|^2 \}, \quad (61b)$$

$$\mathbf{v} \cdot \mathbf{H} \circ \nabla_s \mathbf{v} = d_1^2 + d_2^2 - \mathbf{V} \mathbf{d}_1^* \cdot \mathbf{R} \mathbf{d}_1 - \mathbf{V} \mathbf{d}_2^* \cdot \mathbf{R} \mathbf{d}_2. \quad (61c)$$

Eq. (61b) can also be given a possibly more telling form involving the mismatch between stretched and rotated curvature tensors. Since both curvature tensors $\nabla_s \mathbf{v}$ and $\nabla_s^* \mathbf{v}_y$ are symmetric, they can also be written as

$$\nabla_s \mathbf{v} = -\mathbf{d}_1 \otimes \mathbf{u}_1 - \mathbf{d}_2 \otimes \mathbf{u}_2, \quad \nabla_s^* \mathbf{v}_y = -\mathbf{d}_1^* \otimes \mathbf{v}_1 - \mathbf{d}_2^* \otimes \mathbf{v}_2, \quad (62)$$

and so

$$\mathbf{V} \mathbf{d}_i^* - \mathbf{R} \mathbf{d}_i = [\mathbf{R}(\nabla_s \mathbf{v}) \mathbf{R}^\top - \mathbf{V}(\nabla_s^* \mathbf{v}_y)] \mathbf{v}_i, \quad (63)$$

which, once combined with (61b), delivers

$$|\mathbf{H}|^2 - \frac{1}{2} |\mathbf{W}(\mathbf{v}) \circ \mathbf{H}|^2 = 2 |\mathbf{V}(\nabla_s^* \mathbf{v}_y) - \mathbf{R}(\nabla_s \mathbf{v}) \mathbf{R}^\top|^2. \quad (64)$$

Similarly, we arrive at

$$\mathbf{v} \cdot \mathbf{H} \circ \nabla_s \mathbf{v} = |\nabla_s \mathbf{v}|^2 - \mathbf{V}(\nabla_s^* \mathbf{v}_y) \cdot \mathbf{R}(\nabla_s \mathbf{v}) \mathbf{R}^\top. \quad (65)$$

By (61a), (64), and (65) we can finally write the pure measures of drilling and bending envisioned in our theory as

$$w_d = 4 |\mathbf{V} \mathbf{c}^* - \mathbf{R} \mathbf{c}|^2 \quad \text{and} \quad w_b = 4 (|\mathbf{V}(\nabla_s^* \mathbf{v}_y)|^2 - |\nabla_s \mathbf{v}|^2)^2. \quad (66)$$

They vanish whenever, correspondingly,

$$\mathbf{V} \mathbf{c}^* = \mathbf{R} \mathbf{c} \quad \text{or} \quad |\mathbf{V}(\nabla_s^* \mathbf{v}_y)| = |\nabla_s \mathbf{v}|. \quad (67)$$

Informally, we may say that the former condition means that the stretched present spin connector equals the *rotated* reference spin connector, while the latter means that the stretched present curvature tensor has the same strength as the reference curvature tensor. It is worth noting that, by (64), the bending measure $\tilde{w}_b$ in (50) vanishes under a more restrictive requirement,

$$\mathbf{V}(\nabla_s^* \mathbf{v}_y) = \mathbf{R}(\nabla_s \mathbf{v}) \mathbf{R}^\top, \quad (68)$$

which says that the stretched present curvature tensor equals the rotated reference curvature tensor.²⁵

It is a simple matter to show that if $\mathbf{y}$ is a uniform rotation in space, $\mathbf{y}(\mathbf{x}) = \mathbf{Q}\mathbf{x}$ with $\mathbf{Q} \in \text{SO}(3)$ independent of $\mathbf{x}$, then the equalities in both (67) and (68) are identically satisfied, as $\mathbf{V} = \mathbf{P}(\mathbf{v}_y)$, $\mathbf{c}^* = \mathbf{Q}\mathbf{c}$, and $\nabla_s^* \mathbf{v}_y = \mathbf{Q}(\nabla_s \mathbf{v}) \mathbf{Q}^\top$.

4. Conclusion

We have proposed a direct hyperelastic theory for soft shells, based upon a separation criterion that requires the strain-energy density (per unit area in the reference configuration) to be the sum of *three* independent contributions, each penalizing a single deformation mode. Alongside the traditional stretching deformation mode, we have identified and described two non-stretching modes, that is, *drilling* and *bending*. They originate from the distinct, uniquely identified rotations embedded in the deformation gradient that have axes along the local normal to the reference surface and on the tangent plane, respectively. Invariant averages were extracted out of these frame-dependent kinematic contents by a method applicable when the material is transversely isotropic about the normal. These averages turned out to depend on the second deformation gradient via an invariant third-order tensor $\mathbf{H}$ constructed from the surface gradient of the polar rotation $\mathbf{R}$ (extracted from the polar decomposition of the first deformation gradient).

We identified local pure drilling and pure bending deformations upon which pure bending and pure drilling measures derived from the corresponding invariant averages were required to vanish (in that order) to enforce our separation criterion. We found a pure measure of drilling quadratic in $\mathbf{H}$, but a pure measure of bending must in general be at least *quartic*, although it could be taken quadratic in special cases (notably, for plates and cylindrical shells).

Although our pure measures emerged from a formal reasoning, originally divorced from geometry, they were also shown to be related to mismatches between geometric quantities evaluated in the deformed and undeformed configurations of the shell, including the curvature tensor.

The stretching tensor $\mathbf{V}$, which is the primary player in pure stretching measures, also features in our pure measures of drilling and bending: this indeed turns out to be necessarily the case for a pure measure to be such in our theory, as was already in the elastic theory of Antman for curved rods in Antman (1968).

Of the local pure drilling and bending deformations that were instrumental to the definition of the ground state for the strain energy, the former are also known to exist in the large; they are the *bending-neutral* deformations studied in Sonnet and Virga (2024). On the other hand, pure bending deformations are not known to exist in the large (apart from the special case of cylindrical shells).

We also encountered a peculiar case of *soft elasticity* predicted by our theory, where all pure measures vanish on a family of deformations which produced a class of isometric minimal surfaces (see Appendix D). One may wonder whether this is indeed the sign of a more general property and ask: are all isometric bending-neutral deformations of a minimal surface cases of soft elasticity for our theory?

²⁵ Were the rotation $\mathbf{R}$ replaced in (68) by the full deformation gradient $\nabla_s \mathbf{y}$, (68) would require the stretched present curvature tensor to coincide with the *pushforward* of the reference curvature tensor, in complete analogy with the classical theory of Koiter (1966).

We have called *soft* the shells described by our theory to highlight the role played here by the drilling deformation mode, which is expected to remain silent for *hard* structural shells.

Material symmetry was not addressed here. Apart from the requirement of transverse isotropy, which we believe is implicit in our averaging process, systematic considerations of material symmetry, in the same spirit as those expounded for other direct second-grade theories in Steigmann (2018), Murdoch and Cohen (1979), Davini (2020) were not entertained. Presumably, different material symmetries would affect our averaging process, making $\langle d^2 \rangle$ and $\langle b^2 \rangle$ more openly akin to order parameters.

CRedit authorship contribution statement

André M. Sonnet: Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization. **Epifanio G. Virga:** Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Surface calculus: Definitions and calculations

Here, we collect a few definitions and properties that apply to calculus on a smooth surface $\mathcal{S}$.

Let $\varphi : \mathcal{S} \rightarrow \mathbb{R}$ be a scalar field; we say that φ is differentiable at a point identified on $\mathcal{S}$ by the position vector $\mathbf{x}$, if there is a vector $\nabla_s \varphi$ on the tangent plane $\mathcal{T}_{\mathbf{x}}$ to $\mathcal{S}$ at $\mathbf{x}$ such that for every curve $c : t \mapsto c(t) \in \mathcal{S}$ designating a trajectory on $\mathcal{S}$ for which $c(t_0) = \mathbf{x}$,

$$\left. \frac{d}{dt} \varphi(c(t)) \right|_{t=t_0} = \nabla_s \varphi \cdot \mathbf{t}, \quad (\text{A.1})$$

where $\mathbf{t}$ is the tangent vector to c at $\mathbf{x}$. We call $\nabla_s \varphi$ the surface gradient of φ . Similarly, a vector field $\mathbf{h} : \mathcal{S} \rightarrow \mathcal{V}$ is differentiable at $\mathbf{x} \in \mathcal{S}$, if

$$\left. \frac{d}{dt} \mathbf{h}(c(t)) \right|_{t=t_0} = (\nabla_s \mathbf{h}) \mathbf{t}, \quad (\text{A.2})$$

where the surface gradient $\nabla_s \mathbf{h}$ is a second-rank tensor mapping $\mathcal{T}_{\mathbf{x}}$ into $\mathcal{V}$. We can also differentiate a second-rank tensor field $\mathbf{F} : \mathcal{S} \rightarrow \text{GL}(3)$, where $\text{GL}(3)$ is the *linear group* on the three-dimensional translation space $\mathcal{V}$. The surface gradient $\nabla_s \mathbf{F}$, defined by

$$\left. \frac{d}{dt} \mathbf{F}(c(t)) \right|_{t=t_0} = (\nabla_s \mathbf{F}) \mathbf{t}, \quad (\text{A.3})$$

is a third-rank tensor mapping $\mathcal{T}_{\mathbf{x}}$ into $\text{GL}(3)$.

Letting $\mathbf{v}$ denote a unit normal field orienting $\mathcal{S}$, if $\nabla_s \mathbf{h}$ is *dyadic*, that is, $\nabla_s \mathbf{h} = \mathbf{a}_1 \otimes \mathbf{a}_2$, then $\mathbf{a}_2 \cdot \mathbf{v} = 0$ and $(\nabla_s \mathbf{h}) \mathbf{t} = (\mathbf{a}_2 \cdot \mathbf{t}) \mathbf{a}_1$; similarly, if $\nabla_s \mathbf{F}$ is *triadic*, that is, $\nabla_s \mathbf{F} = \mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3$, then $\mathbf{a}_3 \cdot \mathbf{v} = 0$ and $(\nabla_s \mathbf{F}) \mathbf{t} = (\mathbf{a}_3 \cdot \mathbf{t}) \mathbf{a}_1 \otimes \mathbf{a}_2$.

The surface gradients $\nabla_s \mathbf{h}$ and $\nabla_s \mathbf{F}$ that we have encountered in the main text can always be seen as linear combinations of dyadic and triadic tensors. In particular, given a Cartesian frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, we can write

$$\mathbf{F} = F_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad \text{and} \quad \nabla_s \mathbf{F} = F_{ij;k} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \quad (\text{A.4})$$

where a semicolon denotes differentiation with respect to tangential components.²⁶ Similarly, we can decompose the third-rank tensor $\mathbf{H}$ defined by (27) as a linear combination,

$$\mathbf{H} = H_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \quad (\text{A.5})$$

where the scalar components H_{ijk} obey the conditions

$$H_{ijk} = -H_{jik} \quad \text{and} \quad H_{ijk} \mathbf{v}_k = 0, \quad (\text{A.6})$$

the latter following from (27) when denoting by $\mathbf{v}_k$ the components of $\mathbf{v}$ in the given frame. Similarly, the skew-symmetric tensor $\mathbf{W}_{\mathbf{u}} := \mathbf{H} \mathbf{u}$, where $\mathbf{u}$ is a unit vector such that

$$\mathbf{u} \cdot \mathbf{v} = 0, \quad (\text{A.7})$$

has the following representation,

$$\mathbf{W}_{\mathbf{u}} = W_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad \text{with} \quad W_{ij} := H_{ijk} u_k, \quad (\text{A.8})$$

where u_k are the components of $\mathbf{u}$.

²⁶ The convention of summing over repeated indices is adopted whenever components are used.

Letting any second-rank tensor $\mathbf{A}$ be represented as

$$\mathbf{A} = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \quad (\text{A.9})$$

we can write

$$\mathbf{W}_u \cdot \mathbf{A} = W_{ij} A_{ij} = A_{ij} H_{ijk} u_k, \quad (\text{A.10})$$

so that

$$\langle (\mathbf{W}_u \cdot \mathbf{A})^2 \rangle_{u \in \mathbb{S}^1} = A_{ij} H_{ijk} A_{lm} H_{lmh} \langle u_k u_h \rangle_{u \in \mathbb{S}^1} = \frac{1}{2} A_{ij} H_{ijk} A_{lm} H_{lmk}, \quad (\text{A.11})$$

where we have used (A.7) and the following form of the identity (36),

$$\langle u_k u_h \rangle_{u \in \mathbb{S}^1} = \frac{1}{2} (\delta_{kh} - v_k v_h), \quad (\text{A.12})$$

where δ_{kh} is Kronecker's symbol. Eq. (26) in the main text is then an immediate consequence of (A.11) and definition (29).

To prove (34), we only need to remark that

$$\mathbf{W}_u \cdot \mathbf{W}_u = H_{ijk} H_{ijh} u_k u_h \quad \text{and} \quad |\mathbf{H}|^2 = H_{ijk} H_{ijk}, \quad (\text{A.13})$$

and apply again (A.12).

Appendix B. Bending-neutrality

Here, we prove that applying a pure drilling deformation to an already deformed surface leaves its bending content unchanged.

Assume that a surface $\mathcal{S}$ has been deformed by $\mathbf{y}^*$ into the surface $\mathcal{S}^*$, which is further deformed by a pure drilling deformation $\mathbf{y}'$ into the surface $\mathcal{S}' = \mathbf{y}'(\mathcal{S}^*)$. We denote by $\mathbf{R}^*$ the polar rotation associated with $\nabla_s \mathbf{y}^*$ as in (4), and by $\mathbf{R}'$ the polar rotation associated with $\nabla_s \mathbf{y}'$, where ∇_s^* is the surface gradient on $\mathcal{S}^*$. Moreover, $\mathbf{v}^*$ and $\mathbf{v}'$ are the unit normal fields on $\mathcal{S}^*$ and $\mathcal{S}'$, respectively.

Let $\mathbf{y} := \mathbf{y}^* \circ \mathbf{y}'$ be the composed deformation that changes $\mathcal{S}$ into $\mathcal{S}'$ in a single stroke. Since in the present frame $\mathbf{y}'$ preserves the normal to $\mathcal{S}^*$,

$$\mathbf{v}' = \mathbf{R}' \mathbf{v}^* = \mathbf{v}^* = \mathbf{R}^* \mathbf{v}, \quad (\text{B.1})$$

where $\mathbf{v}$ is the unit normal field to $\mathcal{S}$. Letting $\mathbf{R}$ denote the polar rotation extracted from $\nabla_s \mathbf{y}$, (B.1) can also be rewritten as

$$\mathbf{R} \mathbf{v} = \mathbf{R}^* \mathbf{v}, \quad (\text{B.2})$$

from which, decomposing both $\mathbf{R}$ and $\mathbf{R}^*$ in their bending and drilling components as in (10), here denoted $\mathbf{R}_b$, $\mathbf{R}_b^*$ and $\mathbf{R}_d$, $\mathbf{R}_d^*$, respectively, we obtain that

$$\mathbf{R}_b \mathbf{R}_d \mathbf{v} = \mathbf{R}_b \mathbf{v} = \mathbf{R}_b^* \mathbf{R}_d^* \mathbf{v} = \mathbf{R}_b^* \mathbf{v}. \quad (\text{B.3})$$

Calling $\mathbf{b}$ and $\mathbf{b}^*$ the bending contents of $\mathbf{y}$ and $\mathbf{y}^*$, respectively, and using (5) in (B.3), since both $\mathbf{b}$ and $\mathbf{b}^*$ are vectors tangent to $\mathcal{S}$, we arrive at

$$\frac{1}{1+b^2} ((1-b^2)\mathbf{v} + 2\mathbf{b} \times \mathbf{v}) = \frac{1}{1+b^{*2}} ((1-b^{*2})\mathbf{v} + 2\mathbf{b}^* \times \mathbf{v}). \quad (\text{B.4})$$

Projecting both sides of this equation along $\mathbf{v}$ and on the plane tangent to $\mathcal{S}$ delivers

$$\mathbf{b} = \mathbf{b}^*, \quad (\text{B.5})$$

which is the desired result.

Appendix C. Contents' frame-dependence

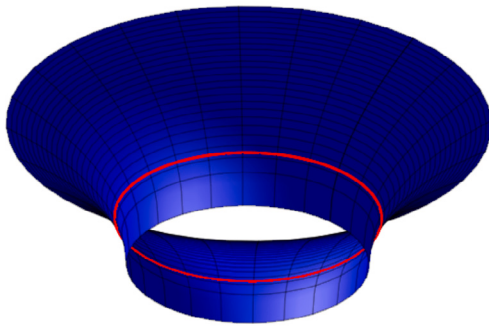
Since in a change of frame the polar rotation $\mathbf{R}$ is transformed into $\mathbf{R}^* = \mathbf{Q}\mathbf{R}$, neither of the vectors $\mathbf{d}$ or $\mathbf{b}$ is frame-invariant or frame-indifferent. Here, we explore by example how the vectors' lengths a , d , and b transform under a change of frame. In particular, we shall show how the angles α , α_d , and α_b appear in a family of frames parameterized by a single rotation angle.

Let $\mathcal{S}$ be the half-catenoid depicted in Fig. 3(a); in a Cartesian frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, it can be represented as (see also Chen et al., 2022),

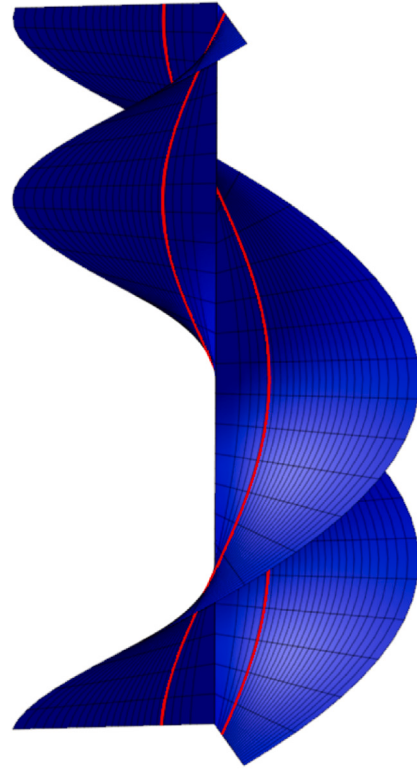
$$\mathbf{x}(r, \vartheta) = r \cos \vartheta \mathbf{e}_1 + r \sin \vartheta \mathbf{e}_2 + R \operatorname{arccosh} \left(\frac{r}{R} \right) \mathbf{e}_3, \quad (\text{C.1})$$

where $0 \leq \vartheta \leq 2\pi$, $r \geq R$, and $R > 0$ is the radius of the neck. We scale all lengths to R and introduce the dimensionless variable $\xi := R/r$, which ranges in the interval $[0, 1]$. The surface $\mathcal{S}_y$ is the half-helicoid delivered by the deformation $\mathbf{y} : \mathbf{x} \mapsto \mathbf{y}(\mathbf{x})$ represented in the same parametric form as $\mathbf{x}$ by

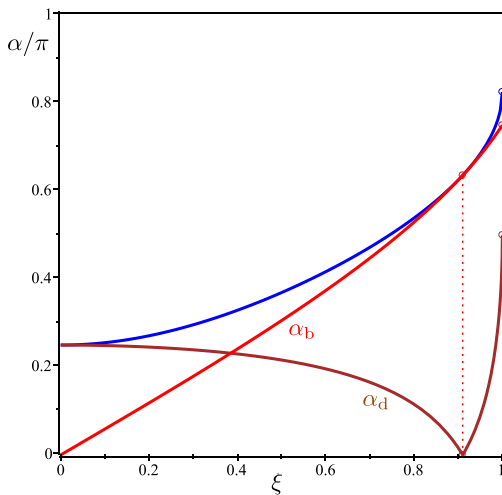
$$\mathbf{y}(r, \vartheta) = R \left\{ \frac{1}{\xi} \sqrt{1-\xi^2} \cos(\vartheta + \vartheta_0) \mathbf{e}_1 + \frac{1}{\xi} \sqrt{1-\xi^2} \sin(\vartheta + \vartheta_0) \mathbf{e}_2 + \vartheta \mathbf{e}_3 \right\}, \quad (\text{C.2})$$



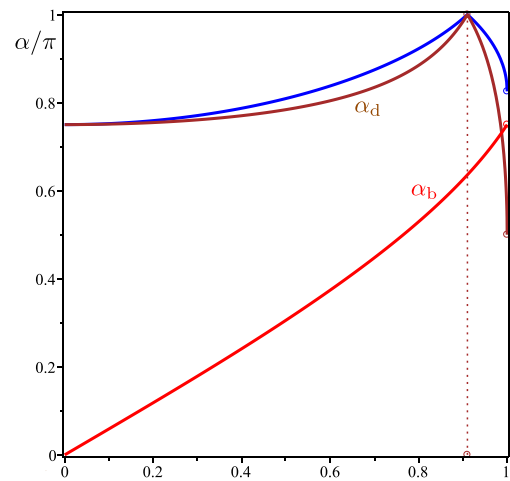
(a) The half-catenoid described by (C.1) for $0 \leq \vartheta \leq 2\pi$ and $R \leq r \leq 2R$. The neck has radius R , while the red circle has radius $r \doteq 1.09R$.



(b) Half-helicoids obtained by deforming the half-catenoid in Fig. 3(a) through y in (C.2) with $\vartheta_0 = \pi/4$ and $\vartheta_0 = 3\pi/4$. The red helices are images of the red circle in Fig. 3(a). The two surfaces differ only by a rotation of angle $\pi/2$ about the helicoids' common axis.



(c) Plots of α , α_d , and α_b extracted from (C.10) as functions of $\xi = R/r$ for $\vartheta_0 = \pi/4$. The drilling angle α_d vanishes for $\xi \doteq 0.91$, corresponding to the red circle in Fig. 3(a).



(d) Plots of α , α_d , and α_b extracted from (C.10) as functions of $\xi = R/r$ for $\vartheta_0 = 3\pi/4$. The drilling angle α_d equals π for $\xi \doteq 0.91$, corresponding to the red circle in Fig. 3(a).

Fig. 3. An overall rotation about the axis e_3 affects dramatically the drilling and bending angles. Only for $\vartheta_0 = -\pi/2$ do these reduce to $\alpha_d = \pi/2$ and $\alpha_b = 0$, indicating that y in (C.2) is indeed a bending-neutral deformation corresponding to the classical Bonnet transformation of a catenoid into a helicoid.

where ϑ_0 is a parameter chosen in the interval $[-\pi, \pi]$. All surfaces $\mathcal{S}_y$ represented by (C.2) differ from one another by the action of a rigid (counter-clockwise) rotation $\mathbf{Q}$ by angle ϑ_0 about the axis $\mathbf{e}_3$.

For a given ϑ_0 , we now compute $\nabla_s \mathbf{y}$. This task is better accomplished in the mobile frame $(\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{v})$ on $\mathcal{S}$, where $\mathbf{v} := \mathbf{e}_r \times \mathbf{e}_\vartheta$ and the unit vectors $(\mathbf{e}_r, \mathbf{e}_\vartheta)$ are associated with the variables (r, ϑ) so that, for any curve $t \mapsto \mathbf{x}(t)$ on $\mathcal{S}$, we can decompose $\dot{\mathbf{x}}$ as

$$\dot{\mathbf{x}} = \frac{\dot{r}}{\sqrt{1-\xi^2}} \mathbf{e}_r + R \frac{\dot{\vartheta}}{\xi} \mathbf{e}_\vartheta. \quad (\text{C.3})$$

Simple calculations show that

$$\begin{cases} \mathbf{e}_r = \sqrt{1-\xi^2}(\cos \vartheta \mathbf{e}_1 + \sin \vartheta \mathbf{e}_2) + \xi \mathbf{e}_3, \\ \mathbf{e}_\vartheta = -\sin \vartheta \mathbf{e}_1 + \cos \vartheta \mathbf{e}_2, \\ \mathbf{v} = -\xi(\cos \vartheta \mathbf{e}_1 + \sin \vartheta \mathbf{e}_2) + \sqrt{1-\xi^2} \mathbf{e}_3. \end{cases} \quad (\text{C.4})$$

By use of (C.2) and (C.4), we easily see that the curve $t \mapsto \mathbf{x}(t)$ on $\mathcal{S}$ is transformed into a curve $t \mapsto \mathbf{y}(t)$ on $\mathcal{S}_y$, for which

$$\begin{aligned} \dot{\mathbf{y}} = R \Big\{ & \left(\left(\xi - \frac{1-\xi^2}{\xi} \sin \vartheta_0 \right) \dot{\vartheta} + \cos \vartheta_0 \frac{\dot{r}}{R} \right) \mathbf{e}_r + \left(\frac{\sqrt{1-\xi^2}}{\xi} \cos \vartheta_0 \dot{\vartheta} + \frac{1}{\sqrt{1-\xi^2}} \sin \vartheta_0 \frac{\dot{r}}{R} \right) \mathbf{e}_\vartheta \\ & + \left(\sqrt{1-\xi^2}(1 + \sin \vartheta_0) \dot{\vartheta} - \frac{\xi}{\sqrt{1-\xi^2}} \cos \vartheta_0 \frac{\dot{r}}{R} \right) \mathbf{v} \Big\}. \end{aligned} \quad (\text{C.5})$$

By requiring that $\dot{\mathbf{y}} = (\nabla_s \mathbf{y}) \dot{\mathbf{x}}$ for all possible curves $\mathbf{x}(t)$, we obtain that

$$\begin{aligned} \nabla_s \mathbf{y} = & \sqrt{1-\xi^2} \cos \vartheta_0 \mathbf{e}_r \otimes \mathbf{e}_r + (\xi^2 + (\xi^2 - 1) \sin \vartheta_0) \mathbf{e}_r \otimes \mathbf{e}_\vartheta + \sin \vartheta_0 \mathbf{e}_\vartheta \otimes \mathbf{e}_r \\ & + \sqrt{1-\xi^2} \cos \vartheta_0 \mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta - \xi \cos \vartheta_0 \mathbf{v} \otimes \mathbf{e}_r + \xi \sqrt{1-\xi^2}(1 + \sin \vartheta_0) \mathbf{v} \otimes \mathbf{e}_\vartheta, \end{aligned} \quad (\text{C.6})$$

whence it follows that $\mathbf{U} = \mathbf{P}(\mathbf{v})$, so that the deformation $\mathbf{y}$ in (C.2) is an isometry for all ϑ_0 , and

$$\text{co}((\nabla_s \mathbf{y}) \mathbf{P}(\mathbf{v})) = \xi \sqrt{1-\xi^2}(1 + \sin \vartheta_0) \mathbf{e}_r \otimes \mathbf{v} - \xi \cos \vartheta_0 \mathbf{e}_\vartheta \otimes \mathbf{v} + (1 - \xi^2(1 + \sin \vartheta_0)) \mathbf{v} \otimes \mathbf{v}. \quad (\text{C.7})$$

In a similar way, it follows from (C.3) and (C.4) that

$$\begin{cases} \nabla_s \mathbf{e}_r = \frac{1}{R} \left(\xi \sqrt{1-\xi^2} \mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta - \xi^2 \mathbf{v} \otimes \mathbf{e}_r \right), \\ \nabla_s \mathbf{e}_\vartheta = \frac{1}{R} \left(-\xi \sqrt{1-\xi^2} \mathbf{e}_r \otimes \mathbf{e}_\vartheta + \xi^2 \mathbf{v} \otimes \mathbf{e}_\vartheta \right), \\ \nabla_s \mathbf{v} = \frac{1}{R} \left(\xi^2 \mathbf{e}_r \otimes \mathbf{e}_r - \xi^2 \mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta \right), \end{cases} \quad (\text{C.8})$$

so that the connectors of the movable frame $(\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{v})$ of a catenoid are

$$\mathbf{c} = \frac{\xi}{R} \sqrt{1-\xi^2} \mathbf{e}_\vartheta, \quad \mathbf{d}_1 = -\frac{\xi^2}{R} \mathbf{e}_r, \quad \mathbf{d}_2 = \frac{\xi^2}{R} \mathbf{e}_\vartheta. \quad (\text{C.9})$$

Inserting both (C.6) and (C.7) in (18), by use of (9) and (14), we arrive at

$$a^2 = \frac{2(1 - \sqrt{1-\xi^2} \cos \vartheta_0) + \xi^2(1 + \sin \vartheta_0)}{2(1 + \sqrt{1-\xi^2} \cos \vartheta_0) - \xi^2(1 + \sin \vartheta_0)}, \quad (\text{C.10a})$$

$$d^2 = \left(\frac{2 \sin \vartheta_0 - \xi^2(1 + \sin \vartheta_0)}{2(1 + \sqrt{1-\xi^2} \cos \vartheta_0) - \xi^2(1 + \sin \vartheta_0)} \right)^2, \quad (\text{C.10b})$$

$$b^2 = \frac{\xi^2(1 + \sin \vartheta_0)}{2 - \xi^2(1 + \sin \vartheta_0)}, \quad (\text{C.10c})$$

which clearly depend on the change of frame embodied by the rotation $\mathbf{Q}$ by angle ϑ_0 about $\mathbf{e}_3$. As an illustration of these equations, we plot in Figs. 3(c) and 3(d) the angles α , α_d , and α_b as functions of ξ for both $\vartheta_0 = \pi/4$ and $\vartheta_0 = 3\pi/4$. In particular, one sees how in a frame the drilling angle could vanish along a circle of $\mathcal{S}$, precisely the same where it attains its maximum in a different frame, rotated relative to the other by angle $\pi/2$ about $\mathbf{e}_3$ (see Fig. 3(b)).

There is one special frame where $b^2 = 0$ and $d^2 = a^2$, so that $\alpha_b = 0$ and $\alpha = \alpha_d = \pi/2$: it is obtained for $\vartheta_0 = -\pi/2$. This indicates that $\mathbf{y}$ in (C.2) is indeed a bending-neutral deformation corresponding to the classical Bonnet transformation of a catenoid (see also Sonnet and Virga, 2024).

Clearly, neither d^2 nor b^2 is an invariant measure of drilling or bending.

Appendix D. Soft elasticity

Had we not prescribed a pure bending deformation to have the direction of bending locally constant, the definition of w_d in (47) would not be acceptable as a measure of pure drilling, as it would not vanish on all pure bending deformations. Here we want to

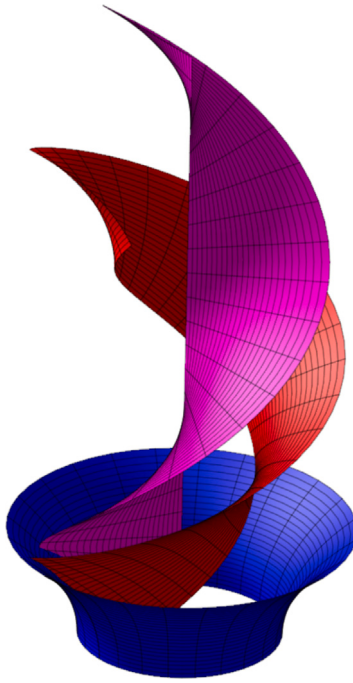


Fig. 4. The surface $y_\alpha(\mathcal{S})$ represented by (D.4), with ξ ranging in $[\frac{1}{2}, 1]$ and ϑ in $[0, 2\pi]$, for $\alpha = 0$ (blue), $\alpha = \frac{\pi}{4}$ (red), and $\alpha = \frac{\pi}{2}$ (magenta). The half-catenoid corresponding to $\alpha = 0$ is the same as in Fig. 3(a).

define an alternative measure of pure drilling $\tilde{w}_d$ that vanishes on all unconstrained pure bending deformations. To this end, we remark that, by (45),

$$\mathbf{P}(\mathbf{v})(\mathbf{H} \circ \mathbf{W}(\mathbf{v})) = (1 - \cos \beta) \mathbf{c} + \sin \beta \mathbf{d}_2, \quad (\text{D.1})$$

so that, by (46) and in tune with (43), a natural definition for $\tilde{w}_d$ would be

$$\tilde{w}_d := |\mathbf{W}(\mathbf{v}) \circ \mathbf{H}|^2 - 4|\mathbf{P}(\mathbf{v})(\mathbf{H} \circ \mathbf{W}(\mathbf{v}))|^2. \quad (\text{D.2})$$

However, it follows from the general representation for $\mathbf{H}$ in (32) that

$$\mathbf{P}(\mathbf{v})(\mathbf{H} \circ \mathbf{W}(\mathbf{v})) = -\frac{1}{2} \mathbf{W}(\mathbf{v}) \circ \mathbf{H}, \quad (\text{D.3})$$

and so $\tilde{w}_d$ vanishes identically.

Thus, according to our theory, enlarging the class of pure bending deformations would entail that no pure measure of drilling could exist. This would mean that no energy cost could ever be associated with a pure drilling mode, which would then be universally *soft*. Since we are not prepared to accept such a general conclusion, we have adopted w_d in (47) as a pure measure of drilling. However, even with this choice for w_d , special instances of *soft elasticity* still remain possible: these are deformations that make W in (49) vanish identically.

Consider again as a reference surface $\mathcal{S}$ the half-catenoid represented by (C.1) and subject it to the deformation described by

$$\mathbf{y}_\alpha(r, \vartheta) := R \left\{ \cos \alpha \left(\frac{1}{\xi} \cos \vartheta \mathbf{e}_1 + \frac{1}{\xi} \sin \vartheta \mathbf{e}_2 + \operatorname{arccosh} \left(\frac{1}{\xi} \right) \mathbf{e}_3 \right) + \sin \alpha \left(\frac{1}{\xi} \sqrt{1 - \xi^2} \sin \vartheta \mathbf{e}_1 - \frac{1}{\xi} \sqrt{1 - \xi^2} \cos \vartheta \mathbf{e}_2 + \vartheta \mathbf{e}_3 \right) \right\}, \quad (\text{D.4})$$

where $\xi = R/r \leq 1$ and α is a constant ranging in $[0, \frac{\pi}{2}]$. For $\alpha = 0$, $\mathbf{y}_\alpha$ is the identity, and so $\mathbf{y}_\alpha(\mathcal{S})$ is the catenoid $\mathcal{S}$ itself, whereas for $\alpha = \frac{\pi}{2}$, $\mathbf{y}_\alpha$ is the same as the deformation defined by (C.2) for $\vartheta_0 = -\frac{\pi}{2}$, and so $\mathbf{y}_\alpha(\mathcal{S})$ is a half-helicoid. For $0 < \alpha < \frac{\pi}{2}$, $\mathbf{y}_\alpha(\mathcal{S})$ is also a minimal surface, precisely, it is *Scherk's second surface* (see, for example, p. 148 of Dierkes et al. 2010). Fig. 4 illustrates the surface $\mathbf{y}_\alpha(\mathcal{S})$ for three values of α .

By use of (C.6) with $\vartheta_0 = -\frac{\pi}{2}$, it is a simple matter to show that

$$\nabla_s \mathbf{y}_\alpha = \cos \alpha \mathbf{P}(\mathbf{v}) - \sin \alpha \mathbf{W}(\mathbf{v}), \quad (\text{D.5})$$

whence it follows that $(\nabla_s \mathbf{y}_\alpha)^T (\nabla_s \mathbf{y}_\alpha) = \mathbf{P}(\mathbf{v})$, so that $\mathbf{y}_\alpha$ is an isometry for all values of α and w_s in (21) vanishes. Then we extract from $\nabla_s \mathbf{y}_\alpha$ its polar rotation,

$$\mathbf{R}_\alpha = \cos \alpha \mathbf{I} + (1 - \cos \alpha) \mathbf{v} \otimes \mathbf{v} - \sin \alpha \mathbf{W}(\mathbf{v}), \quad (\text{D.6})$$

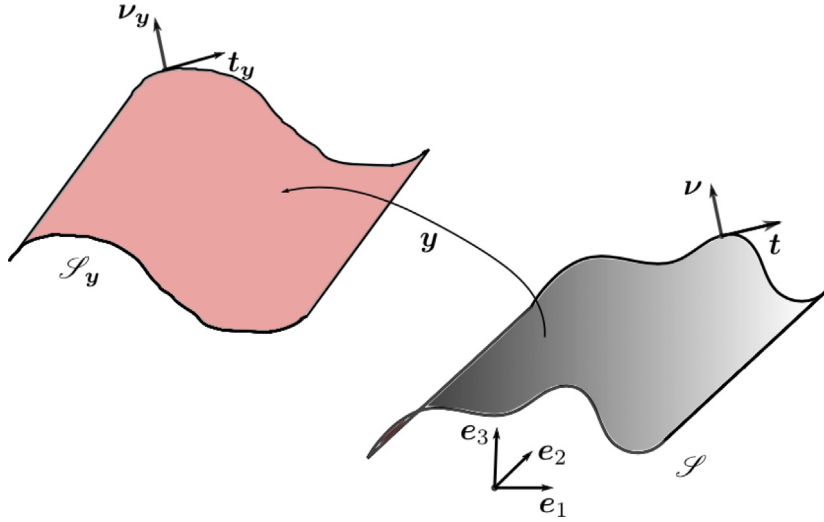


Fig. 5. Deformation of cylindrical shells. Both $\mathcal{S}$ and $\mathcal{S}_y$ are generated by the translation of a planar curve along e_2 ; (t, v) and (t_y, v_y) , both in the (e_1, e_3) plane, are the unit tangent and normal vectors to the curves generating $\mathcal{S}$ and $\mathcal{S}_y$, respectively.

which, by (6), represents a *clockwise* rotation by angle α about v . A simple calculation that uses (C.8) leads us to

$$\mathbf{H}_\alpha := \mathbf{R}_\alpha^T \nabla_s \mathbf{R}_\alpha = \frac{\xi^2}{R} \left\{ - \left((1 - \cos \alpha) \mathbf{W}(e_\theta) + \sin \alpha \mathbf{W}(e_r) \right) \otimes e_r + \left(\sin \alpha \mathbf{W}(e_\theta) - (1 - \cos \alpha) \mathbf{W}(e_r) \right) \otimes e_\theta \right\}, \quad (\text{D.7})$$

from which it follows that

$$|\mathbf{H}_\alpha|^2 = \frac{8\xi^4}{R^2} (1 - \cos \alpha), \quad \mathbf{W}(v) \circ \mathbf{H}_\alpha = \mathbf{0}, \quad \mathbf{H}_\alpha \circ \nabla_s v = \frac{2\xi^4}{R^2} (1 - \cos \alpha) v, \quad (\text{D.8})$$

so that both w_d and w_b , as given by (47), and (43), respectively, vanish.

Thus we conclude that within the theory presented in this paper the special class of deformations y_α in (D.4) represents a case of *soft elasticity* for the energy density W in (49).

Appendix E. Cylindrical shells

Here we consider the special class of *cylindrical* shells, for which both $\mathcal{S}$ and $\mathcal{S}_y$ are generated by the translation in space of *planar* curves along the direction orthogonal to their common plane: actually, it will suffice to know how y deforms the curve generating $\mathcal{S}$ to obtain $\mathcal{S}_y$. Fig. 5 shows both $\mathcal{S}$ and $\mathcal{S}_y$ in a Cartesian frame (e_1, e_2, e_3) , where e_2 is the direction of translation common to both surfaces; t is the unit tangent to the curve generating $\mathcal{S}$, while t_y is the unit tangent to the curve generating $\mathcal{S}_y$; v and v_y are the corresponding unit normals.

According to (1),

$$\nabla_s y = \lambda t_y \otimes t + e_2 \otimes e_2, \quad (\text{E.1})$$

where $\lambda > 0$ is the stretch of the generating curve. It readily follows from (E.1) that

$$\mathbf{U} = \lambda t \otimes t + e_2 \otimes e_2 \quad \text{and} \quad \mathbf{R} = t_y \otimes t + v_y \otimes v + e_2 \otimes e_2. \quad (\text{E.2})$$

Moreover, letting κ be the (signed) curvature of the curve generating $\mathcal{S}$ and κ_y that of the curve generating $\mathcal{S}_y$, we have that

$$\nabla_s t = \kappa v \otimes t, \quad \nabla_s v = -\kappa t \otimes t, \quad (\text{E.3a})$$

$$\nabla_s^* t_y = \kappa_y v_y \otimes t_y, \quad \nabla_s^* v_y = -\kappa_y t_y \otimes t_y, \quad (\text{E.3b})$$

where, as already in (53), ∇_s^* denotes the surface gradient on $\mathcal{S}_y$. By the chain rule, Eqs. (E.3b) imply that

$$\nabla_s t_y = (\nabla_s^* t_y) \nabla_s y = \lambda \kappa_y v_y \otimes t, \quad \nabla_s v_y = (\nabla_s^* v_y) \nabla_s y = -\lambda \kappa_y t_y \otimes t. \quad (\text{E.4})$$

Making use of (E.4) in (E.2), we obtain the representation for $\mathbf{H}$ in the form (32),

$$\mathbf{H} = (\kappa - \lambda \kappa_y) \mathbf{W}(e_2) \otimes t, \quad (\text{E.5})$$

where $\mathbf{W}(e_2)$ is the skew-symmetric tensor associated with e_2 . It follows from (E.5) that

$$\mathbf{W}(v) \circ \mathbf{H} = (\kappa - \lambda \kappa_y) (\mathbf{W}(v) \cdot \mathbf{W}(e_2)) t = 2(\kappa - \lambda \kappa_y) (v \cdot e_2) t = \mathbf{0}, \quad (\text{E.6a})$$

$$|\mathbf{H}|^2 = 2(\lambda\kappa_y - \kappa)^2. \quad (\text{E.6b})$$

Thus, by (47), the pure measure of drilling w_d vanishes identically. This comes as no surprise, as drilling is inhibited in the class of cylindrical shells. For the same reason, the pure measure of bending w_b need not vanish on the non-existent class of pure drilling deformations, and so it can be simply taken to be quadratic in $\mathbf{H}$,

$$\tilde{w}_b = |\mathbf{H}|^2, \quad (\text{E.7})$$

so that the energy density W in (49) can be replaced by

$$W = \frac{1}{2}\mu_s w_s + \frac{1}{2}\tilde{\mu}_b \tilde{w}_b = \frac{1}{2}\mu_s(\lambda - 1)^2 + \tilde{\mu}_b(\lambda\kappa_y - \kappa)^2, \quad (\text{E.8})$$

which is just the elastic energy posited by Antman (1968) for an elastic rod. It was already remarked in Virga (2024) that $\tilde{w}_b$ in (E.7) is a pure measure of bending for cylindrical plates (where $\kappa = 0$); thus it also appears to be for cylindrical shells. The interested reader is also referred to the abundant bibliography cited in Hanna and Vitral (2025) for sources for an ampler discussion.

Data availability

No data was used for the research described in the article.

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