This is a peer-reviewed, accepted author manuscript of the following conference paper: Weiss, S, Schlecht, SJ & Moonen, M 2025, Polynomial Procrustes method for randomly perturbed near-paraunitary systems. in 2025 IEEE Statistical Signal Processing Workshop (SSP), IEEE, Piscataway, NJ.

Polynomial Procrustes Method for Randomly Perturbed Near-Paraunitary Systems

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***Abstract*—We want to recover paraunitary matrices under small random perturbations. The polynomial Procrustes method, based on the analytic singular value decomposition of the perturbed system, in principle solves this. For small random perturbations, where the analytic singular values are close to unity, we propose a simplified polynomial Procrustes method that exploits this property, but show that the support of the solution is generally increased compared to the perturbed matrix. We therefore embed the simplified Procrustes method into an iterative truncation scheme, which can reduce the support while ensuring that a paraunitary approximation remains within a perimeter that is equivalent to the level of perturbation.**

I. Introduction

In some cases a system $\boldsymbol{A}(z)$ of transfer functions that are analytic in $z \in \mathbb{C}$ is expected to be lossless or paraunitary. This includes, for example, physical processes on a microscopic level [1], the Kelly-Lochbaum vocal tract model for speech coding [2], as well as reverberation filters and scattering matrices in multichannel audio [3]–[5]. Modelling imprecisions and estimation errors, e.g. incurred during system identification [6], [7], will result in deviations of the measured system $\boldsymbol{A}(z)$ from a paraunitary one.

In order to estimate the closest paraunitary system to $\boldsymbol{A}(z)$ in a least squares sense, we have previously suggested a polynomial Procrustes method [8]. This is based on an analytic singular value decomposition (SVD) of $\boldsymbol{A}(z)$ [12]. To admit an SVD with analytic factors, the singular values of $\boldsymbol{A}(z)$ cannot be constrained to be non-negative. In order to detect such zero crossings, the singular values have to be extracted, before the Procrustes solution is assembled from the left- and right-singular vectors.

Paradoxically, while the method in [8] works well for systems with distinct non-zero singular values, it is ill-conditioned if $\boldsymbol{A}(z)$ is already close to a paraunitary system, because all its singular values are close to unity and hence can be very similar. Irrespective, a randomly perturbed matrix $\boldsymbol{A}(z)$ will possess singular values that are strictly positive on the unit circle [13] with probability one. Therefore in this paper, we suggest a simplified Procrustes method that avoids the recovery of singular values and exploits the cancellation of ambiguities in the left- and right-singular vectors when constructing a DFT-domain solution. This is then equivalent to a previous method in [9].

The temporal support of the simplified Procrustes method can be significant, and simulation results presented here indicate that such support increases with the level of perturbation. While analytic functions can be arbitrarily closely matched by polynomials of sufficient length via shifts and truncations [10], [11], the support impacts on cost, complexity, and latency of the polynomial realisation of the Procrustes solution. Therefore, we propose a modified scheme, whereby the precise Procrustes solution to $\boldsymbol{A}(z)$ is replaced by one that remains within the levels explainable by the incurred perturbation, but offers a shorter support while satisfying paraunitarity. A proposed scheme, operating an iterated truncation and recalculation of the Procrustes solution, is therefore investigated in this paper.

We reflect on measuring the level of perturbation from $\boldsymbol{A}(z)$ in Sec. II, review the polynomial Procrustes solution and an algorithm in Secs. III and IV. An iterative scheme to limit the support is proposed in Sec. V and explored in simulations in Sec. VI, before Sec. VII draws conclusions.

II. Random Perturbation of a Paraunitary Matrix

A. Randomly Perturbed Paraunitary Matrix

Assume that a paraunitary system $\boldsymbol{Q}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ is perturbed,

$$\boldsymbol{A}(z) = \boldsymbol{Q}(z) + \boldsymbol{E}(z) \; , \tag{1}$$

and we measure a noisy matrix $\boldsymbol{A}(z)$ instead. The perturbation matrix can arise in a system identification context, where the ground truth system is paraunitary or very nearly so. The elements of the random perturbation matrix $\mathbf{E}[n] \circ\!\!-\!\!\bullet \boldsymbol{E}(z) = \sum_n \mathbf{E}[n]z^{-n}$ are assumed to be temporally and spatially uncorrelated complex circularly-symmetric Gaussian with variance σ_e^2. If $\boldsymbol{E}(z)$ is of temporal support L, and with the expectation operator $\mathcal{E}\{\cdot\}$, we have $\mathcal{E}\{\mathbf{E}[n]\mathbf{E}^{\rm H}[n-\tau]\} = M\sigma_e^2\mathbf{I}\delta[\tau]$.

While for $\boldsymbol{Q}(z)$, paraunitarity implies $\boldsymbol{Q}(z)\boldsymbol{Q}^{\rm P}(z) = \boldsymbol{Q}^{\rm P}(z)\boldsymbol{Q}(z) = \mathbf{I}$ with the parahermitian transpose $\boldsymbol{Q}^{\rm P}(z) = \{\boldsymbol{Q}(1/z^*)\}^{\rm H}$, the same no longer holds for the perturbed $\boldsymbol{A}(z)$, and generally we find that

$$\boldsymbol{A}(z)\boldsymbol{A}^{\rm P}(z) \neq \mathbf{I} \; . \tag{2}$$

Unless $\boldsymbol{A}(z)$ is parahermitian such that $\boldsymbol{A}(z) = \boldsymbol{A}^{\rm P}(z)$, generally (2) also implies $\boldsymbol{A}(z)\boldsymbol{A}^{\rm P}(z) \neq \boldsymbol{A}^{\rm P}(z)\boldsymbol{A}(z)$.

B. Error in Paraunitarity

In many cases, we do not know the ground truth $\boldsymbol{Q}(z)$ of the system $\boldsymbol{A}(z)$ perturbed according to (1), but we can easily assess the error in paraunitarity based on (2) via

$$\xi = d\{\boldsymbol{A}(z)\boldsymbol{A}^{\rm P}(z), \mathbf{I}\} \; , \tag{3}$$

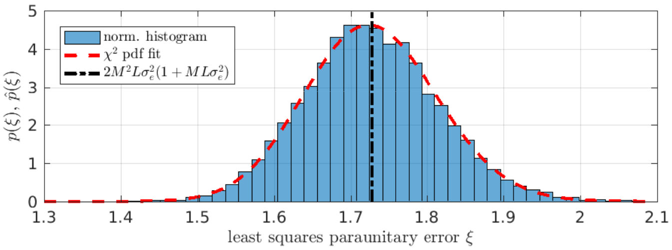

Fig. 1. Normalised histogram of ensemble results as the estimated probability density function $\hat{p}(\xi)$ of the paraunitary error ξ, and a χ^2 distribution $p(\xi)$ of degree M^2L and with the mean in (5).

where the distance $d\{\boldsymbol{B}(z), \boldsymbol{C}(z)\}$ between two analytic matrices $\boldsymbol{B}(z)$ and $\boldsymbol{C}(z)$ can be measured on the unit circle as

$$d\{\boldsymbol{B}(z), \boldsymbol{C}(z)\} = \oint_{|z|=1} \|\boldsymbol{B}(z) - \boldsymbol{C}(z)\|_{\mathrm{F}}^2 \frac{\mathrm{d}z}{z}. \quad (4)$$

In order to take the stochastic nature of the perturbation into account, we assess the mean squared value of the paraunitarity error. Using matrix trace rules, the Isserlis formula for fourth order cumulants [14], and the fact that for Gaussian processes complementary cross-correlations are zero [15], it can be shown that

$$\mathcal{E}\{\xi\} = 2LM^2\sigma_e^2(1 + LM\sigma_e^2) , \quad (5)$$

based on the parameters of the perturbation assumed in Sec. II-A.

Example 1: For a system with spatial dimension $M = 10$, both $\boldsymbol{Q}(z)$ and $\boldsymbol{E}(z)$ of temporal support $L = 8$, and with $\sigma_e^2 = 10^{-3}$, the mean square paraunitarity error is measured over an ensemble of $2 \cdot 10^4$ random perturbations. Since the elements of $\mathbf{E}[n]$ are Gaussian distributed, ξ should follow a χ^2-distribution. Fig. 1 shows the normalised histogram for the paraunitarity error in (3), together with a χ^2 distribution with M^2L degrees of freedom — the number of elements in $\mathbf{E}[n]$ — and mean 1.728 as given by (5), which closely matches the ensemble results. △

III. Polynomial Procrustes Solution and Impact of Near-Paraunitarity

A. Analytic Singular Value Decomposition

In most cases, a matrix $\boldsymbol{A}(z)$ that is analytic in z admits an analytic SVD [12]

$$\boldsymbol{A}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\boldsymbol{V}^{\mathrm{P}}(z) , \quad (6)$$

such that the matrices of left- and right-singular vectors $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$ are paraunitary and analytic. The diagonal matrix $\boldsymbol{\Sigma}(z) = \mathrm{diag}\{\sigma_1(z), \ldots, \sigma_M(z)\}$ contains the corresponding analytic singular values. On the unit circle, these satisfy $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$. While the analytic SVD has unique singular values, if $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$ are matrices holding valid left- and right-singular vectors, then so are $\boldsymbol{U}(z)\boldsymbol{\Psi}(z)$ and $\boldsymbol{V}(z)\boldsymbol{\Psi}(z)$ where $\boldsymbol{\Psi}(z)$ is a block-diagonal paraunitary matrix whose subblock dimensions are determined by the multiplicities of the analytic singular values [12]. For distinct singular values, $\boldsymbol{\Psi}(z)$ is a diagonal matrix containing allpass functions.

An analytic SVD as in (6) is denied if $\boldsymbol{A}(z)$ is connected to block filtering [16], and also in the case that singular values possess an odd number of zero crossings. The former case can be excluded by design. Regarding the latter case, for the near-paraunitary systems considered in this paper, we assume that singular values are close to unity, satisfying

$$|\sigma_m(\mathrm{e}^{\mathrm{j}\Omega}) - 1|^2 < \epsilon \quad (7)$$

for some threshold $0 < \epsilon \ll 1$. As a result, analytic singular values in the context of this paper do not possess zero-crossings, and the decomposition in (6) exists.

B. Polynomial Procrustes Problem

In [9], we have established a polynomial Procrustes problem of finding a paraunitary matrix $\boldsymbol{Q}_*(z)$ that is closest to a matrix $\boldsymbol{A}(z)$ in the least squares sense. Using this distance definition in (4), the polynomial Procrustes problem can be formulated as

$$\boldsymbol{Q}_*(z) = \underset{\boldsymbol{\Pi}(z)}{\mathrm{argmin}}\, d\{\boldsymbol{\Pi}(z), \boldsymbol{A}(z)\} \qquad \text{s.t.} \quad \boldsymbol{\Pi}(z)\boldsymbol{\Pi}^{\mathrm{P}}(z) = \mathbf{I} \, . \quad (8)$$

For (6) with non-negative singular values $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega}) \geq 0$, $m = 1, \ldots, M$, (8) is solved by [9]

$$\boldsymbol{Q}_*(z) = \boldsymbol{U}(z)\boldsymbol{V}^{\mathrm{P}}(z) , \quad (9)$$

which is a simple extension of the standard Procrustes method [17] to matrices of analytic functions. The allpass ambiguity that is shared between both $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$ cancels in (9).

Example 2: As a simple example for $M = 1$, consider $A(z) = 1 + \frac{1}{4}z^{-1} + \frac{1}{64}z^{-2} = (1 + \frac{1}{8}z^{-1})^2$. By expanding with an allpass filter,

$$A(z) = \frac{(1 + \frac{1}{8}z^{-1})}{1 + \frac{1}{8}z} \cdot (1 + \tfrac{1}{8}z)(1 + \tfrac{1}{8}z^{-1}) = \underbrace{z^{-1}\frac{\frac{1}{8} + z}{1 + \frac{1}{8}z}}_{U(z)} \cdot \underbrace{(\tfrac{1}{8}z + \tfrac{65}{64} + \tfrac{1}{8}z^{-1})}_{\Sigma(z)} \cdot \underbrace{1}_{V^{\mathrm{P}}(z)} \quad (10)$$

we obtain an analytic SVD. The singular value is symmetric and hence positive on the unit circle, with a deviation from a unit gain according to (7) by $\epsilon = (\frac{17}{64})^2$. Despite the simplicity of $A(z)$, the polynomial Procrustes solution

$$Q_*(z) = U(z)V^{\mathrm{P}}(z) = z^{-1}\frac{\frac{1}{8} + z}{1 + \frac{1}{8}z} \quad (11)$$

is of infinite temporal support and non-causal. Nonetheless the solution decays quickly as shown in Fig. 2, and can be made finite and causal by shift and truncation operations. △

While Example 2 featured a degenerate 1-d matrix without random perturbation, it highlights that for a matrix $\boldsymbol{A}(z)$ of finite support, the support of the polynomial Procrustes solution may no longer be finite. More so, the analytic SVD can lead to factors that, while analytic in z, are algebraic or even transcendental functions [12]. Therefore, the support of the Procrustes solution can be expected to often exceed the support of $\boldsymbol{A}(z)$.

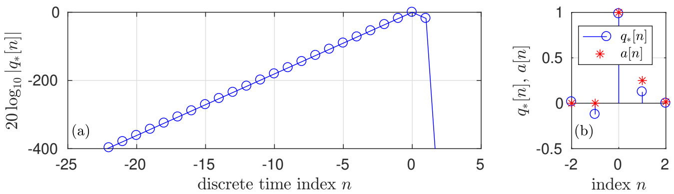

Fig. 2. (a) Time domain support of the polynomial Procrustes solution $q_*[n] \circ\!\!-\!\!\bullet\ Q_*(z)$ as given in (11) and (b) comparison between $a[n] \circ\!\!-\!\!\bullet\ A(z)$ and $q_*[n]$.

IV. DFT-Domain Simplified Procrustes Method

A. Bin-Wise SVD and Ambiguities

For the square matrix $\boldsymbol{A}(z)$, we evaluate bin-wise SVDs in K frequency bins $\Omega_k = 2\pi\frac{k}{K}$, $k = 0, \ldots, (K-1)$, such that in the kth bin

$$\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{U}_k \boldsymbol{\Sigma}_k \mathbf{V}_k^{\mathrm{H}} \ . \tag{12}$$

Under assumption (7), there are no zero singular values in any bin, and hence ambiguities between the left- and right-singular vectors are strictly coupled [12]: based on Sec. III-A, with an appropriately selected unitary matrix $\boldsymbol{\Psi}_k$, $\mathbf{U}'_k = \mathbf{U}_k\boldsymbol{\Psi}_k$ and $\mathbf{U}'_k = \mathbf{U}_k\boldsymbol{\Psi}_k$ are also valid matrices of left- and right-singular vectors w.r.t. (12). Therefore in every bin these ambiguities drop out from the product $\mathbf{U}'_k\mathbf{V}'^{\mathrm{H}}_k = \mathbf{U}_k\mathbf{V}^{\mathrm{H}}_k$, and the polynomial Procrustes solution in bin k is determined via

$$\hat{\boldsymbol{Q}}_{*,k} = \mathbf{U}_k\mathbf{V}_k^{\mathrm{H}} \ . \tag{13}$$

The reconstruction

$$\hat{\mathbf{Q}}_*^{(K)}[n] = \frac{1}{2\pi}\sum_{k=0}^{K-1}\hat{\boldsymbol{Q}}_{*,k}\mathrm{e}^{\mathrm{j}\Omega_k n} \tag{14}$$

yields an length K approximation of the polynomial Procrustes solution, which can be arbitrarily accurate for a sufficiently large DFT size K.

B. DFT Size

If K is too small, the reconstruction via (14) will suffer from time-domain aliasing, resulting in an error in paraunitarity. To measure the latter,

$$\chi_K = \sum_n \left\| \sum_\nu \hat{\mathbf{Q}}_*^{(K)}[\nu]\hat{\mathbf{Q}}_*^{(K)\mathrm{H}}[\nu-n] - \delta[n]\mathbf{I}\right\|_{\mathrm{F}}^2 \tag{15}$$

can be evaluated. Thus, the DFT size can be iteratively increased until χ_K falls below some sufficiently small threshold χ_0. If K is increased in powers of two, the DFT of $\mathbf{A}[n] \circ\!\!-\!\!\bullet\ \boldsymbol{A}(z)$ required for the l.h.s. of (12) can be computed efficiently using fast Fourier transform algorithms, and any bin-wise SVD calculations evaluated in (12) can be reused in the next iteration.

V. Iteratively Truncated Simplified Procrustes Method

A. Paraunitary Solution within the Range of Perturbation

Using the distance measure in (4), for the Procrustes solution $\boldsymbol{Q}_*(z)$, we will always have that

$$d\{\boldsymbol{A}(z), \boldsymbol{Q}_*(z)\} \leq d\{\boldsymbol{A}(z), \boldsymbol{Q}(z)\} = d\{\boldsymbol{E}(z), \mathbf{0}\} \ . \tag{16}$$

We therefore want to relax the strict solution of the Procrustes method to find a paraunitary matrix $\boldsymbol{Q}_{i,*}(z)$ that approximately satisfies $d\{\boldsymbol{A}(z), \boldsymbol{Q}_{i,*}(z)\} \leq d\{\boldsymbol{E}(z), \mathbf{0}\}$, and uses the additional design freedom to shorten the support of $\boldsymbol{Q}_{i,*}(z)$.

In order to estimate $d\{\boldsymbol{E}(z), \mathbf{0}\}$, we can estimate the paraunitarity error of $\boldsymbol{A}(z)$ as defined in (3) in conjunction with (5). The latter related the mean of the paraunitarity error to the variance σ_e^2 of the perturbation term $\boldsymbol{E}(z)$, but its inversion, bearing in mind the positivity constraint $\sigma_e^2 > 0$,

$$\sigma_e^2 = \frac{1}{2LM}\left(\sqrt{1 + 2\frac{M}{L}\mathcal{E}\{\xi\}} - 1\right) \ , \tag{17}$$

can still provide a useful bound towards the metric in (16).

B. Iterative Truncation: Chop Chop!

Without immediate guarantee of convergence but within the spirit of (the criminal rather than the algorithm) Procrustes, we trim or 'chop' the support of the solution from (14) after convergence, and reiterate the polynomial Procrustes method on this shortened version and no longer paraunitary matrix.

For the iterative truncation algorithm, we commence with an initial setting $\boldsymbol{A}_0(z) = \boldsymbol{A}(z)$, calculate the $\hat{\boldsymbol{Q}}_{0,*}(z)$ of the simplified Procrustes method according to Sec. IV, and measure its distance $\chi_0 = d\{\boldsymbol{A}(z), \boldsymbol{Q}_{0,*}(z)\}$. At the ith iteration, $i > 0$, we truncate the previous solution to form a new near-paraunitary matrix $\boldsymbol{A}_i(z) = f(\hat{\boldsymbol{Q}}_{i-1,*}(z))$, where $f(\cdot)$ represents a suitable truncation operation such as in [18]. For this, we define an element-wise energy term $p_i[n]$,

$$p_i[n] = \|\hat{\mathbf{Q}}_{i-1,*}[n]\|_{\mathrm{F}}^2 \tag{18}$$

with $\hat{\mathbf{Q}}_{i-1,*}[n] \circ\!\!-\!\!\bullet\ \hat{\boldsymbol{Q}}_{i-1,*}(z)$, such that $f(\cdot)$ trims all outer components from $\hat{\mathbf{Q}}_{i-1,*}[n]$ whose energy falls below some threshold. Using the method in Sec. IV then find the closest paraunitary matrix $\hat{\boldsymbol{Q}}_{i,*}(z)$. After measuring the distance $\chi_i = d\{\boldsymbol{A}(z), \hat{\boldsymbol{Q}}_{i,*}(z)\}$, we iterate until either the support has sufficiently shortened, until a maximum number of iterations is reached, or until χ_i exceeds the limit of the assumed perturbation due to (17). The later condition prevents the algorithm from diverging, and retains a paraunitary solution with a distance from $\boldsymbol{A}(z)$ equivalent to its perturbation.

Example 3: A paraunitary matrix $\boldsymbol{Q}(z) : \mathbb{C} \rightarrow \mathbb{C}^{4\times4}$ with temporal support $L = 6$ is perturbed by an equally dimensioned matrix $\boldsymbol{E}(z)$ with $\sigma_e^2 = 10^{-3}$. Running the Procrustes algorithm once results in a solution $\boldsymbol{Q}_{1,*}(z)$ of support $\Delta = 120$ that is equivalent to the approach in [9]. The iterated truncation to the original support $L = 6$ and application of the Procrustes algorithm leads to a reduction of the support Δ as indicated by $p_i[n]$ in Fig. 3. Fig. 4 shows the least squares mismatches to both the ground truth paraunitary $\boldsymbol{Q}(z)$ and the original matrix $\boldsymbol{A}(z)$; note that the latter remains below the assumed perturbation bound given by (16) and (17), as otherwise the algorithm would terminate. Overall, with 10 iterations, the support is more than halved. The convergence slows, and 3000 iterations the support has been reduced to one third of the solution achievable by a single iteration, i.e. by the method in [9].

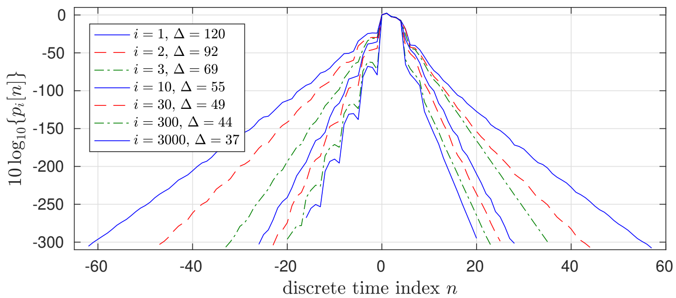

Fig. 3. Energy $p_i[n]$ in coefficients of $\mathbf{Q}_{i,*}[n]$, as defined in (18) for the iteratively truncated Procrustes solution in Example 3, and the support Δ of the different paraunitary approximations.

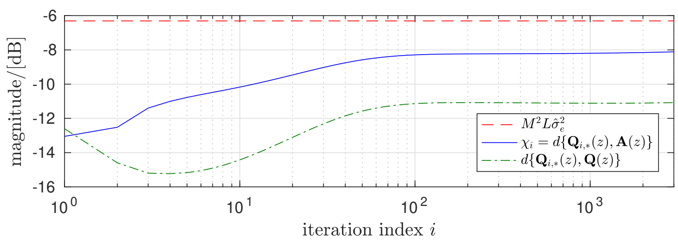

Fig. 4. Least squares mismatches of the iterated solution $\boldsymbol{Q}_{i,*}(z)$ and $\boldsymbol{A}(z)$, which remains below the estimated perturbation energy $d\{\boldsymbol{A}(z), \boldsymbol{Q}(z)\} = M^2 L\hat{\sigma}_e^2$, and the mismatch between $\boldsymbol{Q}_{i,*}(z)$ and $\boldsymbol{Q}(z)\}$ for Example 3.

The above Example 3 highlights that while the paraunitarity error remains within the range of the perturbation, the support of the solution can be significantly reduced.

VI. Ensemble Simulation

We provide a wider-ranging simulation scenario, in which we generate an ensemble of 10^4 random paraunitary matrices $\boldsymbol{Q}(z) : \mathbb{C} \rightarrow \mathbb{C}^{4\times 4}$ of temporal support $L = 6$ constructed from elementary paraunitary operations [19]. Each matrix is perturbed by a matrix $\boldsymbol{E}(z)$ with complex Gaussian elements of variance $\sigma_e^2 \in \{10^{-3}, 10^{-4}, 10^{-5}\}$. If ratios between the energies in $\boldsymbol{Q}(z)$ and $\boldsymbol{E}(z)$ are considered, then these values for σ_e^2 correspond to SNRs of 16.2, 26.2, and 36.2 dB.

We run the proposed iteratively truncated simplified Procrustes method with a maximum of 30 iterations; this may not fully exploit the possibilities highlighted in Example 3, but can hopefully provide a reasonable support reduction while maintaining a sensible number of iterations. For the first iteration of the polynomial Procrustes approach, equivalent to the method in [9], we record the initial support as Δ. Thereafter, the proposed algorithm is permitted to iterate until it terminates with a new support Δ'.

The measured distribution for the ratio $\gamma = \Delta/\Delta'$, i.e. the factor by which the proposed algorithm manages to shorten the support, is shown in Fig. 5. In some cases, the algorithm is not able to provide an improvement, and already stops at the first iteration — this is in 26% of the cases for $\sigma_e^2 = 10^{-5}$, 11% for $\sigma_e^2 = 10^{-4}$, and less than 2% for $\sigma_e^2 = 10^{-5}$. This results in Dirac impulses in the distributions shown in Fig. 5 for $\gamma = 1$, where no shortening is possible. It is clear that higher support reductions are possible for larger random perturbations.

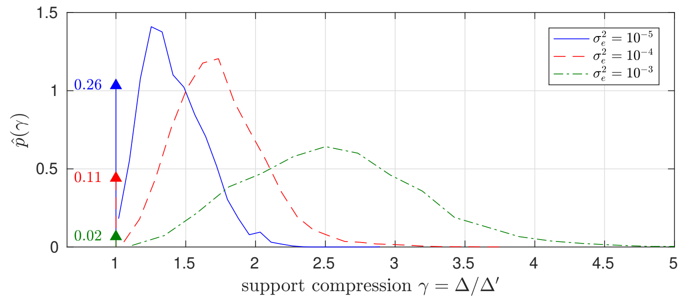

Fig. 5. Estimated distributions of support improvement γ based on ensemble simulations for different values of σ_e^2.

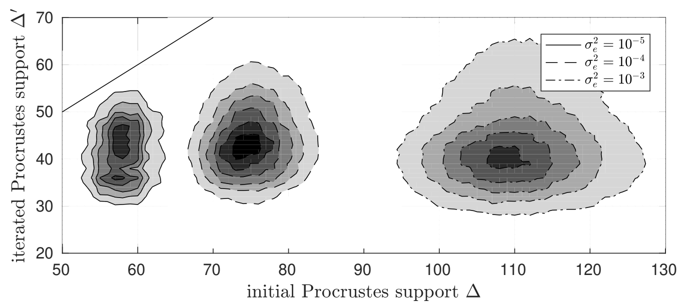

Fig. 6. Estimated joint distributions of initial and iterated supports Δ and Δ' for different levels of perturbation σ_e^2.

To highlight the absolute support values Δ and Δ', Fig. 6 visualises their estimated joint distributions; the cases $\gamma = 1$ lie on the line bisecting the angle, and this part of the distributions is shown in its entirety. The distributions for the three values of σ_e^2 are well separated, and indicate that higher perturbations result in a significant increase in the support Δ for the initial Procrustes solution according to [9]. Larger values of σ_e^2 in turn permit a larger range within which subsequent iterations can operate, and the reduction in support is such that, with the exception of those cases with $\gamma = 1$ — which are increasingly rare as σ_e^2 grows —, the proposed method appears capable of reducing the support Δ' to values that are almost independent of the level of perturbation of the original paraunitary system.

VII. Conclusions

We have proposed a simplified polynomial Procrustes method for finding the paraunitary matrix that is closest to a randomly perturbed one in the least squares sense. Because of the perturbation, this solution typically deviates from the ground truth one, also resulting in an increased support length. The proposed algorithm, through an iterated truncation and re-application of the proposed simplified Procrustes method, is able to shorten this support in the majority of cases and particularly where it matters most: for higher perturbations that otherwise result in large support when using an existing approach. By reducing the support length, the benefit of the proposed method directly translates to lower numerical complexity, computational cost, and latency for implementations.

References

[1] H. Sandberg, J.-C. Delvenne, and J.C. Doyle, "On lossless approximations, the fluctuation- dissipation theorem, and limitations of measurements," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 293–308, 2011.

[2] J.L. Flanagan and L.R. Rabiner, Eds., *Speech Synthesis*. Stroudsburg, PA: Dowden, Hutchinson, and Ross, Inc., 1973.

[3] S.J. Schlecht and E.A.P. Habets, "Scattering in feedback delay networks," *IEEE/ACM Transactions on Audio, Speech, and Language Processing*, vol. 28, pp. 1915–1924, 2020.

[4] S.J. Schlecht, "Allpass feedback delay networks," *IEEE Transactions on Signal Processing*, vol. 69, pp. 1028–1038, 2021.

[5] O. Das, S.J. Schlecht, and E. De Sena, "Grouped feedback delay networks with frequency-dependent coupling," *IEEE/ACM Transactions on Audio, Speech, and Language Processing*, vol. 31, pp. 2004–2015, May 2023.

[6] B. Widrow and S.D. Stearns, *Adaptive Signal Processing*. Englewood Cliffs, New York: Prentice Hall, 1985.

[7] S. Haykin, *Adaptive Filter Theory*, 2nd ed. Englewood Cliffs: Prentice Hall, 1991.

[8] S. Weiss, S.J. Schlecht, M. Moonen, "Best least squares paraunitary approximation of matrices of analytic functions," to be submitted, 2025.

[9] S. Weiss, S.J. Schlecht, O. Das, and E. De Sena, "Polynomial Procrustes problem: Paraunitary approximation of matrices of analytic functions," in *31st European Signal Processing Conference*, Helsinki, Finland, 2023, pp. 1629–1633.

[10] S. Weiss, I.K. Proudler, F.K. Coutts, and F.A. Khattak, "Eigenvalue decomposition of a parahermitian matrix: extraction of analytic eigenvectors," *IEEE Transactions on Signal Processing*, vol. 71, pp. 1642–1656, Apr. 2023.

[11] V. Neo, S. Redif, J.G. McWhirter, J. Pestana, I.K. Proudler, S. Weiss, and P.A. Naylor, "Polynomial eigenvalue decomposition for multichannel broadband signal processing: A mathematical technique offering new insights and solutions," *IEEE Signal Processing Magazine*, vol. 40, no. 7, pp. 18–37, Nov. 2023.

[12] S. Weiss, I.K. Proudler, G. Barbarino, J. Pestana, and J.G. McWhirter, "On properties and structure of the analytic singular value decomposition," *IEEE Transactions on Signal Processing*, vol. 72, pp. 2260–2275, 2024.

[13] M. Bakhit, F.A. Khattak, I.K. Proulder, and S. Weiss, "Impact of estimation errors of a matrix of transfer functions onto its analytic singular values and their potential algorithmic extraction," in *IEEE High Performance Extreme Computing Conference*. Boston, MA: IEEE, Sep. 2024, pp. 1–7.

[14] D.R. Brillinger, *Time series: data analysis and theory*. Society for Industrial and Applied Mathematics, 2001.

[15] P.J. Schreier and L.L. Scharf, *Statistical Signal Processing of Complex-Valued Data. The Theory of Improper and Non-Circular Signals*. Cambridge University Press, 2010.

[16] X.-g. Xia and B. Suter, "Multirate filter banks with block sampling," *IEEE Transactions on Signal Processing*, vol. 44, no. 3, pp. 484–496, March 1996.

[17] G.H. Golub and C.F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore, Maryland: John Hopkins University Press, 1996.

[18] J. Corr, K. Thompson, S. Weiss, I.K. Proudler, and J.G. McWhirter, "Row-shift corrected truncation of paraunitary matrices for PEVD algorithms," in *23rd European Signal Processing Conference*, Nice, France, August/September 2015, pp. 849–853.

[19] P.P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs: Prentice Hall, 1993.