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Extracting analytic singular values from a polynomial matrix

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ABSTRACT

A matrix of transfer functions is, in most cases, known to admit an analytic singular value decomposition (SVD), with singular values that are real-valued but potentially negative on the unit circle. In this contribution, we propose an algorithm to retrieve such analytic singular values. We propose approach operates in the frequency domain, and first computes a standard SVD of the given polynomial matrix in each discrete Fourier transform (DFT) bin. Thereafter, in order to re-establish their association across bins, the bin-wise singular values are permuted by assessing the orthogonality of singular vectors in adjacent DFT bins. In addition, the proposed algorithm determines whether bin-wise singular value should become negative, which can be required for analyticity. The proposed algorithm is validated through an ensemble simulation involving polynomial matrices with known analytic SVD factors.

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Figures and tables

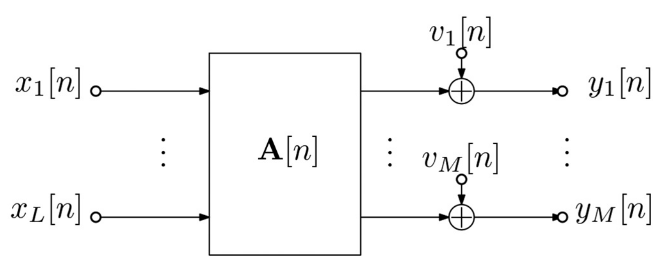

Fig. 1. Example of a system with a matrix of transfer functions [1,2], that can e.g. arise as a communications scenario with L transmit signals in $x[n] \in C^L$ and M receive signals in $y[n] \in C^M$, corrupted by noise $v[n] \in C^M$. The transmission channel is modelled by an $M \times L$ matrix $A[n]$of impulse responses. Its z-transform $A(z)$•—∘$A[n]$ is analytic in the complex parameter, provided that all system components in the transmission link as causal and stable [3–5]. In most cases, such a matrix admits an analytic singular value decomposition $A(z) = U(z)\Sigma(z)V^{\mathrm{P}}(z)$ [5–7] with paraunitary factors $U(z)$ and $V(z)$ containing the left- and right singular vectors, and a diagonal $\Sigma(z)$ containing the analytic singular values. The parahermitian operator $V^{\mathrm{P}}(z) = \{V(1/z^*)\}^{\mathrm{H}}$. Further, $\Sigma(z)$ is a parahermitian matrix such that the eigenvalues are real valued when evaluated on the unit circle.

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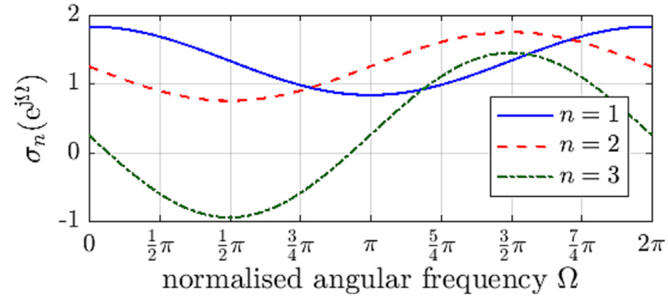

Fig. 2. Example for analytic singular values. In order to be analytic and real valued on the unit circle, different from ordinary singular values, analytic singular values must be permitted to take on negative values, such as third singular value in the diagram; a restriction to non-negative values would result in a non-differentiable singular value. The smoothness of such functions relates to the time domain support of any approximated decomposition factors, and therefore impacts on the implementation complexity of these factors [8]. Relaxing the constraint of non-negativity is analogous to the analytic singular value decomposition of a matrix that is analytic in a real parameter [9,10] for continuous-time control applications, and has been similarly established for the analytic SVD of $A(z)\bullet\!\!-\!\!\circ A[n]$ in the discrete time domain [6,7]. For the calculation of such analytic singular values, it is possible to utilise two analytic eigenvalue decompositions (EVD) [11–15]. An approximation of the analytic SVD with spectrally majorised singular values is possible via two polynomial EVDs [15,16] or by means of specific polynomial SVD time-domain algorithms [18,19].

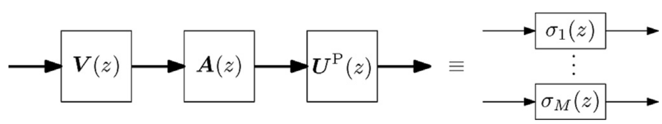

Fig. 3. System decoupling using an analytic SVD. To highlight the benefit of an analytic SVD, consider a multiple-input multiple-output communications application, with a precoder $V(z)$and an equaliser $U^{\mathrm{P}}(z)$. Analogous to narrowband optimal MIMO processing in [20], the resulting decoupled systems consists of independent, parallel transmission channels governed by the analytic singular values $\sigma_m(z)$. An analytic solution will ensure maximally smooth singular values as in Fig. 2, while a polynomial SVD will lead to potentially piece-wise analytic functions that are not necessarily differentiable at the interval margins, and hence require a higher approximation order than strictly necessary. Nonetheless, the polynomial SVD has been applied in a number of communications scenarios for decoupling [21–25].

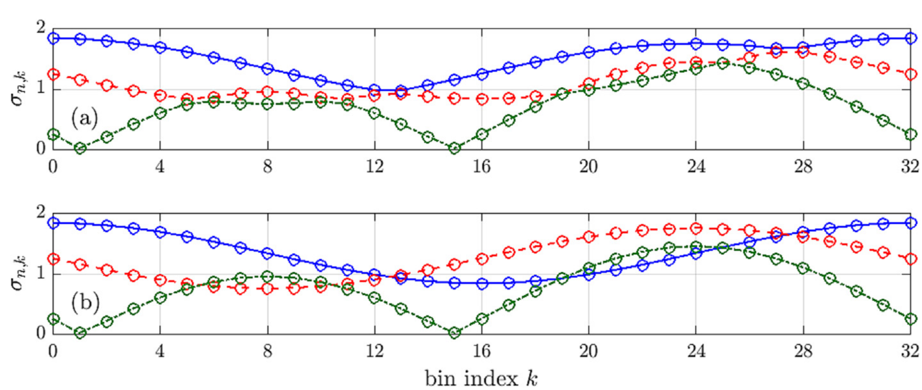

Fig. 4. Association of bin-wise singular values across frequency bins. The DFT domain method calculates a bin-wise SVD of the matrix $A(z)$. In (a), the spectrally majorised (i.e. ordered) singular values in $K = 32$ bins are shown for system characterised in Fig. 2. In (b), the correct associations is bins are shown based on a criterion derived from the left- and right-singular vectors. This yields correct associations, but leaves the negative part of the third singular value (in green) unresolved. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

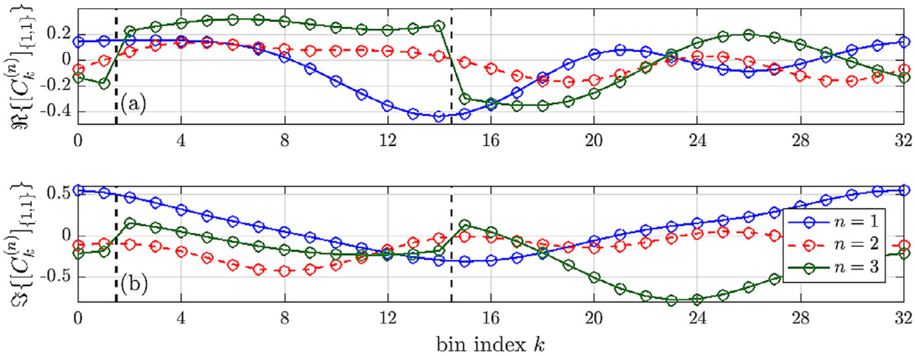

Fig. 5. Example for proposed orthogonality criterion. A missed sign change in the singular value will appear in the corresponding either left- or right-singular vector. Therefore assessing jumps in values of the outer product of the singular vectors can yield locations where sign change must be applied to the singular values, here for the third singular value (in green) between bins 2 and 14. Detecting such sign changes had previous not been possible for preliminary DFT domain approaches in [26,27]. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

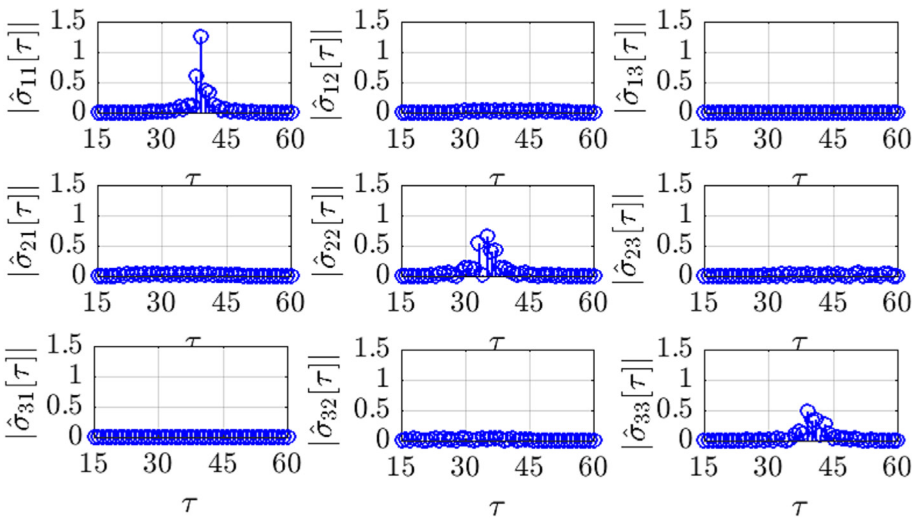

Fig. 6. Example for singular values estimated via two polynomial EVDs. The figure shows the matrix $\Sigma[\tau] \circ\!\!-\!\!\bullet \Sigma(z)$ obtained two polynomial EVDs via the sequential matrix diagonalisation (SMD) algorithm [17] for a matrix $A(z)$ with ground truth eigenvalues in Fig. 2. Time domain methods lead typically to an incomplete diagonalisation with remaining small off-diagonal components; the spectral majorisation encouraged by algorithms such the SMD [17] or similarly by the approaches in [16,18,19] will lead to a time domain support that is much longer then strictly necessary for an analytic solution; also, solutions are typically complex-valued rather than real-valued on the unit circle.

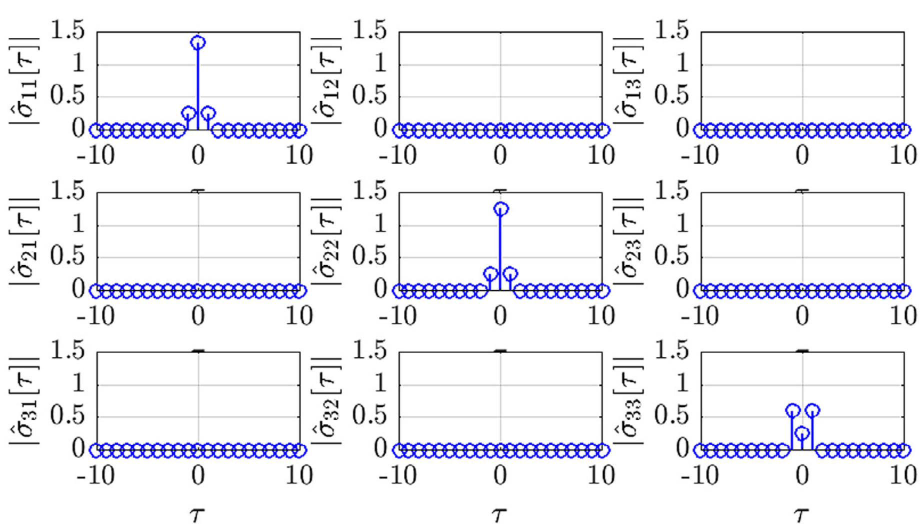

Fig. 7. Example for singular values extracted using the proposed approach. The extracted singular values by the proposed approach match the analytic ground truth of order 2. Compared to Fig. 6, the diagonalisation is complete, and the symmetry of the response indicates real-valuedness on the unit circle. When applied to system decoupling, the resulting decoupled components are therefore much more compact than those obtainable via [16–19]; the technique can also find wider use for applications where an extension of the standard SVD to polynomial matrices is considered [28,29].

CRediT authorship contribution statement

Faizan A. Khattak: Writing – review & editing, Writing – original draft, Software, Formal analysis, Conceptualization. **Mohammed Bakhit:** Writing – review & editing, Writing – original draft, Software. **Ian K. Proudler:** Writing – review & editing, Conceptualization. **Stephan Weiss:** Writing – review & editing, Conceptualization.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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