Optimal Numerical Convergence of Backward Euler Method for the Random Periodic Solution of Semilinear SDE

Yujia Guo, Xiaojie Wang

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Abstract

In this paper, we make use of backward Euler method to stuty the numercial approximation of random periodic solutions of semilinear SDE with additive noise. The existence and uniqueness of the random periodic solution are discussed as the limit of the pull-back flows of the SDE. We discretise the SDE using the backward Euler scheme which is proved to be strongly convergent with order 1 in the mean square sense. Numerical examples are presented to verify our theoretical analysis.

Keywords: Backward Euler method, Random periodic solution, Stochastic differential equations, Additive noise, Pull-back.

1 Introduction

Many phenomena in the real world would have both periodic and random nature, e.g. daily temperature, energy consumption, airline passenger volumes. Taking into account the influence of random factors, these laws of motion can often be modeled by stochastic differential equations, so it is crucial to study the periodic solution of stochastic differential equations. Although the qualitative theory of stochastic differential equations has been extensively studied, we refer the reader to [3, 4, 11]. For the deterministic case, the fixed point theorem is the most effective way to prove the existence of periodic solutions. However, almost all efficient fixed point theorems are not applicable to stochastic systems. Needless to say, for random dynamical systems, to study the pathwise random periodic solutions is of great interest and challenge.

Periodic solution has been a central concept in the theory of the deterministic dynamical systems since Poincaré's seminal work [15]. Zhao and Zheng [21] started to study the problem and gave a definition of the pathwise random periodic solutions for C^1-cocycles. This pioneering study boosts a series of work, including anticipating random solutions of SDEs with multiplicative linear noise [6], the existence for semi-flows generated by non-autonomous SPDEs with additive noise [7], periodic measures and ergodicity [8], etc.

As the random periodic solution is not explicitly constructible, it is useful to study the numerical approximation. In recent years, the research on numerical solution of stochastic differential equations has made rapid progress [2, 12–14, 16, 18]. It is worth mentioning here that this is a numericial approximamtion of an infinite time horizon problem. For a dissipative system with global Lipschitz condition, [5] is the first paper to approximate the random period trajectory by

Euler–Marymaya method and a modified Milstein method. Wei and Chen [19] generalized Euler method to the stochastic theta method (STM) and proved that this approximated solution converges to the exact one at the $1/4$ order of the time step in $L^2(\Omega)$ when initial time tends to $-\infty$. Wu [20] studied the existence and uniqueness of the random periodic solution for a stochastic differential equation with a one-sided Lipschitz condition and the convergence of its numerical approximation via the backward Euler method.

In this paper, we mainly consider the optimal convergence rate of the backward Euler method for the random periodic solution of semilinear SDE compared to [20]. The outline of the paper is as follows. In section 2, we persent some standard notation and assumptions that will be employed in our proofs. In section 3, we focus on the existence and uniqueness of the random periodic solution. Section 4 is about that the backward Euler method admits a unique random periodic solution with a strong oeder 1. Some numerical experiments are finally presented in section 5.

2 Assumptions and preliminary results

Let us recall the definition of the random periodic solution for stochastic semi-flows given in [9]. Let X be a separable Banach space. Denote by $\{\Omega, \mathcal{F}, \mathbb{P}, (\theta_s)_{s\in\mathbb{R}}\}$ a metric dynamical system and $\theta_s : \Omega \to \Omega$ is assumed to be a measurably invertible for all $s \in \mathbb{R}$. Denote $\Delta := \{(t, s) \in \mathbb{R}^2, s \leq t\}$. Consider a stochastic periodic semi-flow $u : \Delta\times\Omega\times X \to X$ of period τ, which satisfies the following standard condition

$$u(t, r, \omega) = u(t, s, \omega) \circ u(s, r, \omega), \tag{2.1}$$

and the periodic property

$$u(t + \tau, s + \tau, \omega) = u(t, s, \theta_\tau\omega), \tag{2.2}$$

for all $r \leq s \leq t, r, s, t \in \mathbb{R}$, for a.e. $\omega \in \Omega$.

Definition 2.1. *A random periodic solution of period $\tau > 0$ of a semi-flow $u : \Delta \times \Omega \times X \to X$ is an $\mathcal{F}$-measurable map $Y : \mathbb{R} \times \Omega \to X$ such that*

$$u(t + \tau, t, Y (t, \omega), \omega) = Y (t + \tau, \omega) = Y (t, \theta_\tau\omega). \tag{2.3}$$

for any $(t, s) \in \Delta$, $\omega \in \Omega$.

Throughout this paper the following notation is frequently used. For notational simplicity, the letter C is used to denote a generic positive constant independent of time step size and may vary for each appearance. Let $|\cdot|$ and $\langle\cdot, \cdot\rangle$ be the Euclidean norm and the inner product of vectors in $\mathbb{R}^d$, respectively. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we use $\mathbb{E}$ to mean expectation and $L^p(\Omega; \mathbb{R}^{d\times m})$, $r \in \mathbb{N}$, to denote the family of $\mathbb{R}^{d\times m}$-valued variables with the norm defined by $\|\xi\|_{L^p(\Omega;\mathbb{R}^d)} = (\mathbb{E}[\|\xi\|^p])^{\frac{1}{p}} < \infty$. Let $W : \mathbb{R}\times\Omega \to \mathbb{R}^d$ be a standard two-sided Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration is defined as follows: $\mathcal{F}^t_s := \sigma\{W_u-W_v : s \leq v \leq u \leq t\}$, $\mathcal{F}^t = \mathcal{F}^t_{-\infty} = \bigvee_{s\leq t} \mathcal{F}^t_s$.

In this paper we consider the following stochastic differential equation with additive noise

$$\begin{cases} \mathrm{d}X^{t_0}_t = [-\Lambda X^{t_0}_t + f(t,\, X^{t_0}_t)\,]\mathrm{d}t + g(t)\,\mathrm{d}W(t), \quad t \in (t_0, T], \\ \ \ X^{t_0}_{t_0} = \xi, \end{cases} \tag{2.4}$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R} \to \mathbb{R}$, Λ is a symmetric and positive-definite $d \times d$ matrix, the random initial conditon ξ is $\mathcal{F}^{t_0}$- measurable. By the variation of constant formula, the solution of (2.4) is determined by

$$X_t^{t_0}(\xi) = e^{-\Lambda(t-t_0)}\xi + \int_{t_0}^{t} e^{-\Lambda(t-s)} f(s, X_s^{t_0})\,\mathrm{d}s + \int_{t_0}^{t} e^{-\Lambda(t-s)} g(s)\,\mathrm{d}W_s. \tag{2.5}$$

Denote the standard $\mathbb{P}$-preserving ergodic Wiener shift by $\theta : \mathbb{R} \times \Omega \to \Omega$, $\theta_t(\omega)(s) := W_{t+s} - W_t$, $s, t \in \mathbb{R}$. We will show that when $k \to \infty$, the pull-back $X_t^{-k\tau}(\xi)$ has a limit X_t^* in $L^2(\Omega)$ and X_t^* is the random perdioc solution of (2.4), satisfying

$$X_t^* = \int_{-\infty}^{t} e^{-\Lambda(t-s)} f(s, X_s^*)\,\mathrm{d}s + \int_{-\infty}^{t} e^{-\Lambda(t-s)} g(s)\,\mathrm{d}W_s. \tag{2.6}$$

We assume the following conditions.

Assumption 2.2. *Λ is self-adjoint and positive definite operator. There exists a non-decreasing sequence $(\lambda_i)_{i\in[d]} \subset \mathbb{R}$ of positive real numbers and an orthonormal basis $(e_i)_{i\in[d]}$, such that*

$$\Lambda e_i = \lambda_i e_i \tag{2.7}$$

for every $i \in [d]$, where $[d] := \{1, ..., d\}$.

Assumption 2.3. *The drift coefficient functions f is continuous and $f(t, x) = f(t+\tau, x)$. There exists a constant $C_f > 0$ such that for any $x, y \in \mathbb{R}^d$ and $t \in [0, \tau)$*

$$\begin{aligned} \langle x - y, f(t, x) - f(t, y)\rangle &\le C_f |x - y|^2, \\ \langle x, f(t, x)\rangle &\le C_f (1 + |x|^2), \end{aligned} \tag{2.8}$$

where $C_f < \lambda_1$.

Assumption 2.4. *The diffusion coefficient functions g is continuous and $g(t) = g(t + \tau)$ and there exists a constant $C_g > 0$ such that*

$$\begin{aligned} |g(t_1) - g(t_2)| &\le C_g |t_2 - t_1|, \\ \sup_{s\in[0,\tau)} |g(s)| &\le C_g, \end{aligned} \tag{2.9}$$

for all $t_1, t_2 \in [0, \tau)$

Assumption 2.5. *There exists a constant $C^* > 0$ such that $\mathbb{E}|\xi|^2 \le C^*$.*

Assumption 2.6. *There exists a constant $\tilde{C}_f$ such that*

$$\left| f(t, x) - \frac{\langle f(t, x), x\rangle}{|x|^2} x \right| \le \tilde{C}_f (1 + |x|^2), \tag{2.10}$$

for $x \in \mathbb{R}^d, t \in [0, \tau)$.

Assumption 2.7. *There exists a constant $\gamma \in (1, \infty)$ and a positive L such that*

$$|f(t_1, x) - f(t_2, y)| \le L(1 + |x|^{\gamma-1} + |y|^{\gamma-1})|x - y|, \tag{2.11}$$

for $x, y \in \mathbb{R}^d$ and $t_1, t_2 \in [0, \tau)$.

Under Assumption 2.2-2.4 and Assumption 2.6 following a similar argument as in [17, Proposition 7.1], we get that SDE (2.4) admits a global semiflow.

3 Existence and uniqueness of random periodic solutions

In this section, we want to ensure the existence and uniqueness of random periodic solutions. Although [20] provided the uniform boundedness for the p-th moment of the SDE solution, our contribution is to overcome the restriction on γ_p in Asumption 13. To achieve this goal, we first get a generalized lemma based on [10, Lemma 8.1].

Lemma 3.1. *Let $m : [a,\infty) \to [0,\infty)$, $\zeta : [a,\infty) \to \mathbb{R}$ be continuous functions for $a \in \mathbb{R}$. If*

$$m(t) - m(s) \le -\delta \int_s^t m(u)\,du + \int_s^t \zeta(u)\,du \qquad (a \le s < t < \infty), \tag{3.1}$$

there exists a positive constant δ, then

$$m(t) \le m(a) + \int_a^t e^{-\delta(t-u)}\zeta(u)\,du. \tag{3.2}$$

Proof of Lemma 3.1. Denote $m_1(t) := m(a) + \int_a^t e^{-\delta(t-u)}\zeta(u)\,\mathrm{d}u$,

$$\begin{aligned} m_1'(t) &= -\delta \int_a^t e^{-\delta(t-u)}\zeta(u)\,\mathrm{d}u + \zeta(t) \\ &= -\delta(m_1(t) - m(a)) + \zeta(t). \end{aligned} \tag{3.3}$$

So we get

$$m_1(t) - m_1(s) = -\delta \int_s^t (m_1(u) - m(a))\,\mathrm{d}u + \int_s^t \zeta(u)\,\mathrm{d}u. \tag{3.4}$$

Set $m_2(t) = m(t) - m_1(t)$,

$$\begin{aligned} m_2(t) - m_2(s) &= m(t) - m(s) - (m_1(t) - m_1(s)) \\ &\le -\delta \int_s^t (m(u) - m_1(u) + m(a))\,\mathrm{d}u \\ &\le -\delta \int_s^t (m(u) - m_1(u))\,\mathrm{d}u \\ &= -\delta \int_s^t m_2(u)\,\mathrm{d}u. \end{aligned} \tag{3.5}$$

To prove (3.2), it is enough to prove that $m_2(t) \le 0$ for any $t \in \mathbb{R}$. If $m_2(t) > 0$, for some t, $m_2(a) = 0$ implies that there exists an interval $[s_1, t_1] \subset [a, \infty]$, $m_2(t_1) > m_2(s_1)$ and $m_2 > 0$ on $[s_1, t_1]$, which contradict (3.5).

We consider the boundedness of the exact solution in $L^p(\Omega)$.

Theorem 3.2. *Let Assumptions 2.2 to 2.5 be sastisfied, there exists a positive constant C, one has*

$$\mathbb{E}\big[|X_t^{-k\tau}|^{2p}\big] \le C\mathbb{E}\big[(1 + |\xi|^{2p})\big], \tag{3.6}$$

where ξ is the initial value when $t = -k\tau$.

Proof of Theorem 3.2. Applying the Itô formula to $\big(1+|X_t^{-k\tau}|^2\big)^p$,

$$\begin{aligned}\big(1+|X_t^{-k\tau}|^2\big)^p =&\big(1+|\xi|^2\big)^p + 2p\int_{-k\tau}^{t}\big(1+|X_s^{-k\tau}|^2\big)^{p-1}\langle X_s^{-k\tau}, -\Lambda X_s^{-k\tau}\rangle\,\mathrm{d}s\\ &+2p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\langle X_s^{-k\tau}, f(s,X_s^{-k\tau})\rangle\,\mathrm{d}s\\ &+2p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\langle X_s^{-k\tau}, g(s)\,\mathrm{d}W_s\rangle\\ &+p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}|g(s)|^2\,\mathrm{d}s\\ &+2p(p-1)\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-2}|(X_s^{-k\tau})^T g(s)|^2\,\mathrm{d}s,\end{aligned}\tag{3.7}$$

which straightforwardly gives

$$\begin{aligned}(1+|X_t^{-k\tau}|^2)^p \le&(1+|\xi|^2)^p + 2p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\langle X_s^{-k\tau}, -\Lambda X_s^{-k\tau}\rangle\,\mathrm{d}s\\ &+2p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\langle X_s^{-k\tau}, f(s,X_s^{-k\tau})\rangle\,\mathrm{d}s\\ &+2p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\langle X_s^{-k\tau}, g(s)\,\mathrm{d}W_s\rangle\\ &+p(2p-1)\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}|g(s)|^2\,\mathrm{d}s.\end{aligned}\tag{3.8}$$

Combining with Assumptions 2.2 to 2.4, we can get

$$\begin{aligned}(1+|X_t^{-k\tau}|^2)^p \le&(1+|\xi|^2)^p + 2p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}[-\lambda_1(1+|X_s^{-k\tau}|^2)]\,\mathrm{d}s\\ &+2pC_f\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p}\,\mathrm{d}s\\ &+2p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\langle X_s^{-k\tau}, g(s)\,\mathrm{d}W_s\rangle\\ &+p[(2p-1)C_g^2+2\lambda_1]\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\,\mathrm{d}s.\end{aligned}\tag{3.9}$$

Then

$$\begin{aligned}(1+|X_t^{-k\tau}|^2)^p \le&(1+|\xi|^2)^p - 2p(\lambda_1-C_f)\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p}\,\mathrm{d}s\\ &+2p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\langle X_s^{-k\tau}, g(s)\,\mathrm{d}W_s\rangle\\ &+p[(2p-1)C_g^2+2\lambda_1]\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\,\mathrm{d}s.\end{aligned}\tag{3.10}$$

By the Young inequality, $a^{p-1}b \leq \frac{p-1}{p}a^p + \frac{1}{p}b^p$, we get

$$
\begin{aligned}
(1+|X_t^{-k\tau}|^2)^p \leq &(1+|\xi|^2)^p - p(\lambda_1 - C_f)\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^p \,\mathrm{d}s \\
&+ \int_{-k\tau}^{t} \tfrac{((2p-1)C_g^2+2\lambda_1))^p}{(\lambda_1-C_f)^{p-1}} \,\mathrm{d}s \\
&+ 2p\int_{-k\tau}^{t}(1+|X_s^{-k\tau}|^2)^{p-1}\langle X_s^{-k\tau}, g(s)\,\mathrm{d}W_s\rangle.
\end{aligned}
\tag{3.11}
$$

For every intertal $n \geq 1$, define the stopping time

$$
\tau_n = \inf\{s \in [-k\tau, t] : |x(s)| \geq n\}. \tag{3.12}
$$

Clearly, $\tau_n \uparrow t$ a.s.. Moreover, it follows from (3.11) and the property of the Itô integral that

$$
\begin{aligned}
\mathbb{E}[(1+|X_{t\wedge\tau_n}^{-k\tau}|^2)^p] \leq \mathbb{E}[(1+|\xi|^2)^p] - p(\lambda_1 - C_f)\mathbb{E}\Big[\int_{-k\tau}^{t\wedge\tau_n}(1+|X_s^{-k\tau}|^2)^p \,\mathrm{d}s\Big] \\
+ \mathbb{E}\Big[\int_{-k\tau}^{t} \tfrac{((2p-1)C_g^2+2\lambda_1)^p}{(\lambda_1-C_f)^{p-1}} \,\mathrm{d}s\Big].
\end{aligned}
\tag{3.13}
$$

Letting $n \to \infty$ and by the Fatou lemma, we have

$$
\begin{aligned}
\mathbb{E}[(1+|X_t^{-k\tau}|^2)^p] \leq \mathbb{E}[(1+|\xi|^2)^p] - p(\lambda_1 - C_f)\int_{-k\tau}^{t}\mathbb{E}\Big[(1+|X_s^{-k\tau}|^2)^p\Big] \,\mathrm{d}s \\
+ \int_{-k\tau}^{t} \tfrac{((2p-1)C_g^2+2\lambda_1)^p}{(\lambda_1-C_f)^{p-1}} \,\mathrm{d}s.
\end{aligned}
\tag{3.14}
$$

Similiarly, according to Lemma 3.1, we take $\delta = p(\lambda_1 - C_f) > 0$, $\zeta = \frac{((2p-1)C_g^2+2\lambda_1)^p}{(\lambda_1-C_f)^{p-1}}$, it is easy to have

$$
\begin{aligned}
\mathbb{E}[(1+|X_t^{-k\tau}|^2)^p] &\leq \mathbb{E}[(1+|\xi|^2)^p] + \int_{-k\tau}^{t} e^{-p(\lambda_1-C_f)(t-u)}\tfrac{((2p-1)C_g^2+2\lambda_1)^p}{(\lambda_1-C_f)^{p-1}} \,\mathrm{d}u \\
&\leq \mathbb{E}[(1+|\xi|^2)^p] + \tfrac{((2p-1)C_g^2+2\lambda_1)^p}{p(\lambda_1-C_f)^p}(1-e^{-p(\lambda_1-C_f)(t+k\tau)}) \\
&\leq C\mathbb{E}[(1+|\xi|^{2p}].
\end{aligned}
\tag{3.15}
$$

In the next lemma, we consider the difference of the solutions under various initial values.

Lemma 3.3. *Let Assumptions 2.2 to 2.4 be satisfied, set by $X_t^{-k\tau}$ and $Y_t^{-k\tau}$ two solutions of SDE (2.4) with initial values ξ and η, respectively. Then for every $\epsilon > 0$, there exists a $t \geq -k\tau$ such that*

$$
\mathbb{E}[|X_t^{-k\tau} - Y_t^{-k\tau}|^2] \leq \epsilon. \tag{3.16}
$$

With the help of Theorem 3.2, Lemma 3.3 and Assumptions 2.6, we can identify the existence and uniqueness of the random periodic solution to (2.4). Then we get the following theorem.

Theorem 3.4. *Let Assumptions 2.2 to 2.7 be hold, there exists a unique random periodic solution $X_t^*(\cdot) \in L^2(\Omega)$ such that the solution of (2.4) satisfies*

$$
\lim_{k\to\infty}\mathbb{E}\big[|X_t^{-k\tau}(\xi) - X_t^*|^2\big] = 0. \tag{3.17}
$$

The proof of this theorem is essentially the same as that of [5, Theorem 2.4]. It is easy to show that the sequence $X_t^{-k\tau}$ is a Cauchy sequence and therefore has a limit.

4 Strong convergence rate of the backward Euler method

In this section, we analyze the strong convergence rate of the backward Euler approximation. Take an equidistant partition $\mathcal{T}^h := \{jh, j \in \mathbb{Z}\}$, such that $h \in (0,1)$. Note that $\mathcal{T}^h$ stretch along the real line because we are dealing with an infinite time horizon problem. The backward Euler method applied to SDE (2.4) takes the following form:

$$\tilde{X}^{-k\tau}_{-k\tau+(j+1)h} = \tilde{X}^{-k\tau}_{-k\tau+jh} - \Lambda h \tilde{X}^{-k\tau}_{-k\tau+(j+1)h} + hf\big((j+1)h, \tilde{X}^{-k\tau}_{-k\tau+(j+1)h}\big) + g(jh)\Delta W_{-k\tau+jh} \tag{4.1}$$

for all $j \in \mathbb{N}$, where $\Delta W_{-k\tau+jh} := W_{-k\tau+(j+1)h} - W_{-k\tau+jh}$, and the initial value $\tilde{X}^{-k\tau}_{-k\tau} = \xi$. According to the periodicity of f and g, we have that $f(-k\tau + jh, \tilde{X}^{-k\tau}_{-k\tau+jh}) = f(jh, \tilde{X}^{-k\tau}_{-k\tau+jh})$, $g(-k\tau + jh) = g(jh)$.

First of all, we provide uniform bounds for the second moment of the numerical approximation, already established in [20, Lemma 17].

Proposition 4.1. *Let Assumptions 2.2 to Assumptions 2.6 be satisfied. Then there exists $\tilde{C} > 0$ such that*

$$\sup_{k,j\in\mathbb{N}} \mathbb{E}[|\tilde{X}^{-k\tau}_{-k\tau+jh}|^2] \le \tilde{C}, \tag{4.2}$$

where $\{\tilde{X}^{-k\tau}_{-k\tau+jh}\}_{k,j\in\mathbb{N}}$ is given by (4.1).

The next lemma is the discrete analogue of Lemma 3.3. It shows the numercial solution dependence on different conditions.

Lemma 4.2. *Let Assumptions 2.2 to 2.5 be satisfied. Denote by $\tilde{X}^{-k\tau}_{-k\tau+jh}$ and $\tilde{Y}^{-k\tau}_{-k\tau+jh}$ two approximations of* (4.1) *with different initial values ξ and η. Then, for any positive ϵ, there exists a j^* such that for any $j \ge j^*$,*

$$\mathbb{E}\big[|\tilde{X}^{-k\tau}_{-k\tau+jh} - \tilde{Y}^{-k\tau}_{-k\tau+jh}|^2\big] \le \epsilon. \tag{4.3}$$

Theorem 4.3. *Let Assumptions 2.2 to 2.5 be satisfied, for $h \in (0,1)$, the time domain is divided as $\tau = jh$. The backward Euler method* (4.1) *admits a random period solution $\tilde{X}^* \in L^2(\Omega)$ such that*

$$\lim_{k\to\infty} \mathbb{E}\big[|\tilde{X}^{-k\tau}_{-k\tau+jh}(\xi) - \tilde{X}^*|^2\big] = 0. \tag{4.4}$$

With Proposition 4.1 and Lemma 4.2, the proof is similar to the proof of Theorem 3.4 in [5].

In order to establish a strong convergence rate of backward Euler method, we need the following conditions. Considering (2.4), we denote $F(X) := -\Lambda X + f(t, X)$.

Assumption 4.4. *Suppose $F : \mathbb{R} \to \mathbb{R}$ is twice differentiable, satisfying*

$$\begin{aligned} \langle x - y, F(x) - F(y)\rangle &\le L|x-y|^2, \quad \forall x, y \in \mathbb{R} \\ |F^{(i)}(x)| &\le C(1 + |x|^{\gamma-1}), \quad \forall x \in \mathbb{R}, \quad i \in 1, 2, \end{aligned} \tag{4.5}$$

where $F^{(i)}$, $i \in \mathbb{N}$ stands for the i-th derivative of the function F.

Following a similar argument as in Proposition 5.4 and Lemma 5.5 [1], we can easily have the following bounds for analysis later.

Proposition 4.5. *Let Assumptions 2.2 to 2.7 be satisfied. Then there exists a positive constant C which depends on q, d, Λ, C_f only, such that*

$$\|X_{t_1}^{-k\tau} - X_{t_2}^{-k\tau}\|_{L^2(\Omega;R^d)} \le C(1 + \sup_{k\in\mathbb{N}} \sup_{t\ge -k\tau} \|X_t^{-k\tau}\|^q_{L^{2q}(\Omega;R^d)})|t_2 - t_1|^{\frac{1}{2}}, \tag{4.6}$$

for all $t_1, t_2 \ge -k\tau$. Moreover

$$\begin{aligned}&\int_{t_1}^{t_2} \left\|\Lambda(X_s^{-k\tau} - X_{t_4}^{-k\tau}) + f(s, X_s^{-k\tau}) - f(t_3, X_{t_4}^{-k\tau})\right\|_{L^2(\Omega;R^d)} ds \\ &\le C(1 + \sup_{k\in\mathbb{N}} \sup_{t\ge -k\tau} \|X_t^{-k\tau}\|^{2q-1}_{L^{4q-2}(\Omega;R^d)})|t_2 - t_1|^{\frac{3}{2}},\end{aligned} \tag{4.7}$$

for all $t_3, t_4 \in [t_1, t_2]$.

In the following, we will prove the approximate solution converges to the exact solution in an infinite horizon.

Theorem 4.6. *Under Assumptions 2.2 to 2.5 and 2.7, for any $h \in (0,1)$, with $\tau = nh, j \in \mathbb{N}$. If $X_{-k\tau+jh}^{-k\tau}$ and $\tilde{X}_{-k\tau+jh}^{-k\tau}$ are the exact and the numercial solutions given by (2.4) and (4.1), respectively, then there exists a constant C that depends on the Λ, f, g and d such that*

$$\sup_{k,j} \|X_{-k\tau+jh}^{-k\tau} - \tilde{X}_{-k\tau+jh}^{-k\tau}\|_{L^2(\Omega;R^d)} \le Ch \tag{4.8}$$

Proof of Theorem 4.6. First note that

$$\begin{aligned}X_{-k\tau+(j+1)h}^{-k\tau} &= X_{-k\tau+jh}^{-k\tau} - \int_{-k\tau+jh}^{-k\tau+(j+1)h} \Lambda X_s^{-k\tau} \mathrm{d}s + \int_{-k\tau+jh}^{-k\tau+(j+1)h} f(s, X_s^{-k\tau})\mathrm{d}s + \int_{-k\tau+jh}^{-k\tau+(j+1)h} g(s)\mathrm{d}W_s \\ &= X_{-k\tau+jh}^{-k\tau} - \Lambda h X_{-k\tau+(j+1)h}^{-k\tau} + hf((j+1)h, X_{-k\tau+(j+1)h}^{-k\tau}) + g(jh)\Delta W_{-k\tau+jh} + R_{j+1},\end{aligned} \tag{4.9}$$

where:

$$\begin{aligned}R_{j+1} = &-\int_{-k\tau+jh}^{-k\tau+(j+1)h} \Lambda(X_s^{-k\tau} - X_{-k\tau+(j+1)h}^{-k\tau})\mathrm{d}s \\ &+ \int_{-k\tau+jh}^{-k\tau+(j+1)h} f(s, X_s^{-k\tau}) - f((j+1)h, X_{-k\tau+(j+1)h}^{-k\tau})\mathrm{d}s \\ &+ \int_{-k\tau+jh}^{-k\tau+(j+1)h} g(s) - g(jh)\mathrm{d}W_s.\end{aligned} \tag{4.10}$$

Substracting (4.1) from this yields

$$\begin{aligned}X_{-k\tau+(j+1)h}^{-k\tau} - \tilde{X}_{-k\tau+(j+1)h}^{-k\tau} = &X_{-k\tau+jh}^{-k\tau} - \tilde{X}_{-k\tau+jh}^{-k\tau} - \Lambda h(X_{-k\tau+(j+1)h}^{-k\tau} - \tilde{X}_{-k\tau+(j+1)h}^{-k\tau}) \\ &+ h[f((j+1)h, X_{-k\tau+(j+1)h}^{-k\tau}) - f((j+1)h, \tilde{X}_{-k\tau+(j+1)h}^{-k\tau})] \\ &+ g(jh)\Delta W_{-k\tau+jh} - g(jh)\Delta W_{-k\tau+jh} + R_{j+1}.\end{aligned} \tag{4.11}$$

For brevity we write

$$\begin{aligned} e_j &:= X^{-k\tau}_{-k\tau+jh} - \tilde{X}^{-k\tau}_{-k\tau+jh}, \\ \Delta f_j &:= f(jh, X^{-k\tau}_{-k\tau+jh}) - f(jh, \tilde{X}^{-k\tau}_{-k\tau+jh}). \end{aligned} \tag{4.12}$$

Using the short hand notation, we can rewrite (4.11) as

$$e_{j+1} = e_j - \Lambda h e_{j+1} + h\Delta f_{j+1} + R_{j+1}, \tag{4.13}$$

and thus

$$|e_{j+1} + \Lambda h e_{j+1} - h\Delta f_{j+1}|^2 = |e_j + R_{j+1}|^2 \tag{4.14}$$

$$\begin{aligned} LHS &= |e_{j+1}|^2 + 2h\langle e_{j+1}, \Lambda e_{j+1}\rangle - 2h\langle e_{j+1}, \Delta f_{j+1}\rangle + h^2|\Lambda e_{j+1} - \Delta f_{j+1}|^2 \\ &\geq |e_{j+1}|^2 + 2\lambda_1 h|e_{j+1}|^2 - 2C_f h|e_{j+1}|^2 \\ &= [1 + 2(\lambda_1 - C_f)h]|e_{j+1}|^2. \end{aligned} \tag{4.15}$$

Then,

$$RHS = |e_j|^2 + 2\langle e_j, R_{j+1}\rangle + |R_{j+1}|^2. \tag{4.16}$$

Since LHS=RHS, we get

$$[1 + 2(\lambda_1 - C_f)h]|e_{j+1}|^2 \leq |e_j|^2 + 2\langle e_j, R_{j+1}\rangle + |R_{j+1}|^2. \tag{4.17}$$

We set $v := \lambda_1 - C_f$, and taking expectation,

$$(1 + 2vh)\mathbb{E}[|e_{j+1}|^2] \leq \mathbb{E}[|e_j|^2] + 2\mathbb{E}[\langle e_j, R_{j+1}\rangle] + \mathbb{E}[|R_{j+1}|^2]. \tag{4.18}$$

Employing properties of the conditional expectation,

$$\mathbb{E}[\langle e_j, R_{j+1}\rangle] = \mathbb{E}\big(\mathbb{E}[\langle e, R_{j+1}\rangle|\mathcal{F}_{t_j}]\big). \tag{4.19}$$

Now by the Cauchy-Schwarz inequality $2ab \leq vha^2 + \frac{1}{vh}b^2$,

$$\begin{aligned} (1 + 2vh)\mathbb{E}[|e_{j+1}|^2] &\leq \mathbb{E}[|e_j|^2] + 2\mathbb{E}\big(\langle\sqrt{vh}e_j, \frac{1}{\sqrt{vh}}\mathbb{E}(R_{j+1}|\mathcal{F}_{t_j})\rangle\big) + \mathbb{E}[|R_{j+1}|^2] \\ &\leq (1 + vh)\mathbb{E}[|e_j|^2] + \mathbb{E}[|R_{j+1}|^2] + \frac{1}{vh}\mathbb{E}\big[|\mathbb{E}(R_{j+1}|\mathcal{F}_{t_j})|^2\big], \end{aligned} \tag{4.20}$$

hence,

$$\begin{aligned} \mathbb{E}[|e_{j+1}|^2] &\leq \frac{1 + vh}{1 + 2vh}\mathbb{E}[|e_j|^2] + \frac{1}{1 + 2vh}\mathbb{E}[|R_{j+1}|^2] + \frac{1}{(1 + 2vh)vh}\mathbb{E}\big[|\mathbb{E}(R_{j+1}|\mathcal{F}_{t_j})|^2\big] \\ &= (1 - \frac{v}{1 + 2vh}h)\mathbb{E}[|e_j|^2] + \frac{1}{1 + 2vh}\mathbb{E}[|R_{j+1}|^2] + \frac{1}{(1 + 2vh)vh}\mathbb{E}\big[|\mathbb{E}(R_{j+1}|\mathcal{F}_{t_j})|^2\big]. \end{aligned} \tag{4.21}$$

Therefore, we only need to estimate two error term $\mathbb{E}[|R_{j+1}|^2]$ and $\mathbb{E}\big[|\mathbb{E}(R_{j+1}|\mathcal{F}_{t_j})|^2\big]$. It follows from an elementary inequality that

$$\begin{aligned} \|R_{j+1}\|_{L^2(\Omega;R^d)} \leq &\left\| \int_{-k\tau+jh}^{-k\tau+(j+1)h} F(X^{-k\tau}_s) - F(X^{-k\tau}_{-k\tau+(j+1)h})\mathrm{d}s \right\|_{L^2(\Omega;R^d)} \\ &+ \left\| \int_{-k\tau+jh}^{-k\tau+(j+1)h} g(s) - g(jh)\mathrm{d}W_s \right\|_{L^2(\Omega;R^d)}. \end{aligned} \tag{4.22}$$

Next we estimate the first term in (4.22), using Proposition 4.5 shows

$$
\begin{aligned}
&\Big\| \int_{-k\tau+jh}^{-k\tau+(j+1)h} F(X_s^{-k\tau}) - F(X_{-k\tau+(j+1)h}^{-k\tau})\mathrm{d}s \Big\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq \int_{-k\tau+jh}^{-k\tau+(j+1)h} \|F(X_s^{-k\tau}) - F(X_{-k\tau+(j+1)h}^{-k\tau})\|_{L^2(\Omega;\mathbb{R}^d)}\mathrm{d}s \\
&\leq C(1+\sup_{k\in\mathbb{N}}\sup_{t\geq -k\tau}\|X_t^{-k\tau}\|_{4q-2}^{2q-1})_{L^2(\Omega;\mathbb{R}^d)} h^{\frac{3}{2}} \\
&\leq Ch^{\frac{3}{2}}.
\end{aligned}
\tag{4.23}
$$

In view of the Itô isomery, by Assumption 2.4

$$
\begin{aligned}
&\Big\| \int_{-k\tau+jh}^{-k\tau+(j+1)h} g(s) - g(jh)\mathrm{d}W_s \Big\|_{L^2(\Omega;\mathbb{R}^d)} \\
&= \Big(\int_{-k\tau+jh}^{-k\tau+(j+1)h} \mathbb{E}[\|g(s)-g(jh)\|^2]\mathrm{d}s \Big)^{\frac{1}{2}} \\
&\leq \Big(\int_{-k\tau+jh}^{-k\tau+(j+1)h} \mathbb{E}[C_g^2|s-jh|^2]\mathrm{d}s \Big)^{\frac{1}{2}} \\
&\leq \Big(\int_{-k\tau+jh}^{-k\tau+(j+1)h} C_g^2 h^2\mathrm{d}s \Big)^{\frac{1}{2}} \\
&\leq Ch^{\frac{3}{2}}.
\end{aligned}
\tag{4.24}
$$

Thus we get

$$
\mathbb{E}[|R_{j+1}|^2] \leq Ch^3. \tag{4.25}
$$

Next, we need to estimate $\mathbb{E}\big[|\mathbb{E}(R_{j+1}|\mathcal{F}_{t_j})|^2\big] \leq Ch^4$. Noting that the stochastic integral vanishes under the conditional expection, we arrive at

$$
\mathbb{E}\big[|\mathbb{E}(R_{j+1}|\mathcal{F}_{t_j})|^2\big] \leq \mathbb{E}[|\mathbb{E}(F(X_s^{-k\tau}) - F(X_{-k\tau+(j+1)h}^{-k\tau})\mathrm{d}s|\mathcal{F}_{t_j})|^2]. \tag{4.26}
$$

By means of the Itô formula we have, for $s \in [-k\tau+jh, -k\tau+(j+1)h]$,

$$
\begin{aligned}
F(X_s^{-k\tau}) - F(X_{-k\tau+(j+1)h}^{-k\tau}) = &\int_s^{-k\tau+(j+1)h} F'(X_r^{-k\tau})F(X_r^{-k\tau})\mathrm{d}r + \int_s^{-k\tau+(j+1)h} g(r)F'(X_r^{-k\tau})\mathrm{d}W_r \\
&+ \int_s^{-k\tau+(j+1)h} \frac{g^2(r)}{2}F''(X_r^{-k\tau})\mathrm{d}r.
\end{aligned}
\tag{4.27}
$$

Since $E[\int_s^{-k\tau+(j+1)h} g(r)F'(X_r^{-k\tau})\mathrm{d}W_r|\mathcal{F}_{t_j})]=0$,

$$\begin{aligned}
&\mathbb{E}\Big[\big|\mathbb{E}(F(X_s^{-k\tau})-F(X_{-k\tau+(j+1)h}^{-k\tau})\mathrm{d}s\big|\mathcal{F}_{t_j})\big|^2\Big]\\
&\leq \mathbb{E}\Bigg[\Bigg|\mathbb{E}\Bigg(\int_{-k\tau+jh}^{-k\tau+(j+1)h}\int_s^{-k\tau+(j+1)h} F'(X_r^{-k\tau})F(X_r^{-k\tau})\mathrm{d}r\mathrm{d}s\Bigg|\mathcal{F}_{t_j}\Bigg)\\
&\quad+\mathbb{E}\Bigg(\int_{-k\tau+jh}^{-k\tau+(j+1)h}\int_s^{-k\tau+(j+1)h}\frac{g^2(r)}{2}F''(X_r^{-k\tau})\mathrm{d}r\mathrm{d}s\Bigg|\mathcal{F}_{t_j}\Bigg)\Bigg|^2\Bigg]\\
&\leq 2\mathbb{E}\Bigg[\Bigg|\int_{-k\tau+jh}^{-k\tau+(j+1)h}\int_s^{-k\tau+(j+1)h} F'(X_r^{-k\tau})F(X_r^{-k\tau})\mathrm{d}r\mathrm{d}s\Bigg|^2\Bigg]\\
&\quad+\frac{C_g^4}{2}\mathbb{E}\Bigg[\Bigg|\int_{-k\tau+jh}^{-k\tau+(j+1)h}\int_s^{-k\tau+(j+1)h} F''(X_r^{-k\tau})\mathrm{d}r\mathrm{d}s\Bigg|^2\Bigg]\\
&\leq \Big(2\sup_{r\in[-k\tau,T]}\mathbb{E}[|F'(X_r^{-k\tau})F(X_r^{-k\tau})|^2]+\frac{C_g^4}{2}\sup_{r\in[-kt,T]}\mathbb{E}[|F''(X_r^{-k\tau})|^2]\Big)h^4\\
&\leq C\Big(1+\sup_{r\in[-k\tau,T]}\|X_t^{-k\tau}\|^2_{L^{4\gamma-2}(\Omega;\mathbb{R}^d)}\Big)h^4\\
&\leq Ch^4
\end{aligned}\tag{4.28}$$

and therefore

$$\mathbb{E}[|\mathbb{E}[R_{j+1}|\mathcal{F}_{t_j}]|^2]\leq Ch^4.\tag{4.29}$$

Considering (4.21), define $\hat{v}:=\frac{v}{1+2vh}$, we easily get

$$\begin{aligned}
\mathbb{E}[|e_{j+1}|^2]&\leq(1-\hat{v}h)\mathbb{E}[|e_j|^2]+\frac{1}{1+2vh}Ch^3\\
&\leq(1-\hat{v}h)^{j+1}\mathbb{E}[|e_0|^2]+\big[1+(1-\hat{v}h)+...+(1-\hat{v}h)^j\big]Ch^3\\
&=(1-\hat{v}h)^{j+1}\mathbb{E}[|e_0|^2]+\frac{1-(1-\hat{v}h)^{j+1}}{\hat{v}h}Ch^3
\end{aligned}\tag{4.30}$$

By observing $e_0=0$,

$$\mathbb{E}[|e_{j+1}|^2]\leq Ch^2\tag{4.31}$$

Then the assertion follows.

Corollary 4.7. *Under Assumptions 2.2 to 2.7. Let X_t^* be the random periodic solution of SDE* (2.4) *and $\tilde{X}_t^*$ be the random periodic solution of the backward Euler numerical approximation. Then there exists a constant C that depends on q,Λ,f,g and d such that*

$$\sup_{t\in\mathcal{T}^h}\mathbb{E}([|X_t^*-\tilde{X}_t^*|^2])^{1/2}\leq Ch\tag{4.32}$$

Proof of Corollary 4.7. Due to

$$|X_t^*-\tilde{X}_t^*|^2\leq\limsup_k\big[|X_t^*-X_t^{-k\tau}|^2+|X_t^{-k\tau}-\tilde{X}_t^{-k\tau}|^2+|\tilde{X}_t^{-k\tau}-\tilde{X}_t^*|^2\big],\tag{4.33}$$

thus the conclusion can be obtained by Theorem 3.4, Theorem 4.3 and Theorem 4.6. Corollary 4.7 implies that the optimal order of convergence can be achieved 1.

5 Numercial experiments

Some numerical experiments are performed to illustrate the previous theoretical findings. In this section, we consider a specific SDE [20]

$$\mathrm{d}X_t^{t_0} = -10\pi X_t^{t_0}\mathrm{d}t + \sin(2\pi t)\mathrm{d}t + 0.05\mathrm{d}W_t, \tag{5.1}$$

that is

$$\begin{aligned} \Lambda &= -10\pi, \\ f(t, X_t^{t_0}) &= \sin(2\pi t), \\ g(t) &= 0.05. \end{aligned} \tag{5.2}$$

It is easy to check that the associated period is 1 and Assumptions 2.2 to 2.4 are fulfilled with $\lambda_1 = 10$, $C_f = 2$, $C_g = 0.05$. According to Theorem 3.4, (5.1) has a random periodic solution. By Theorem 4.3, its backward Euler simulation also has a random periodic path.

First of all, we want to show that the numerical approximation converges to its random periodic path regardless its initial value. We select the time grid between $t_0 = -20$ and $t = 0$, and set two initial values to be 0.5 and -0.3. Based on the above conditions, two simulated paths can be depicted in Fig. 1 by applying the backward Euler method. From Fig. 1, one can clearly observe that two paths coincide shortly after the start.

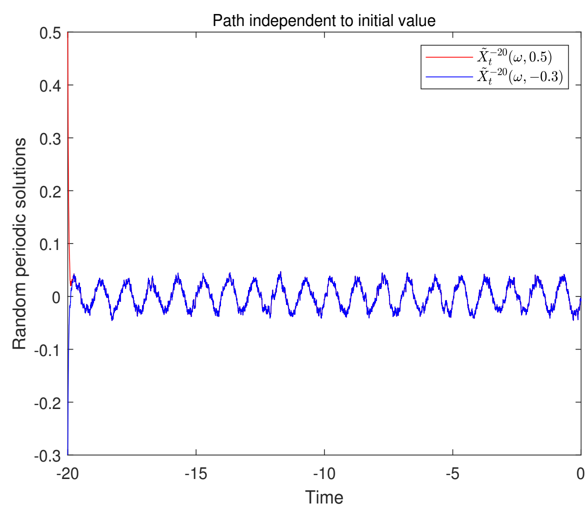

Figure 1: Two paths by backward Euler method from different initial value.

Then, to see the "periodicity" numerically, we choose to simulate the process $\tilde{X}_t^*(\omega) = \tilde{X}_t^{-5}(\omega, 0.3)$, $-5 \le t \le -1$ and $\tilde{X}_t^*(\theta_{-1}\omega) = \tilde{X}_t^{-5}(\theta_{-1}\omega, 0.3)$,$-5 \le t \le 0$ with the same ω. We can observe that the two segmented processes are identical in Fig. 2, thus illustrating the periodicity of path.

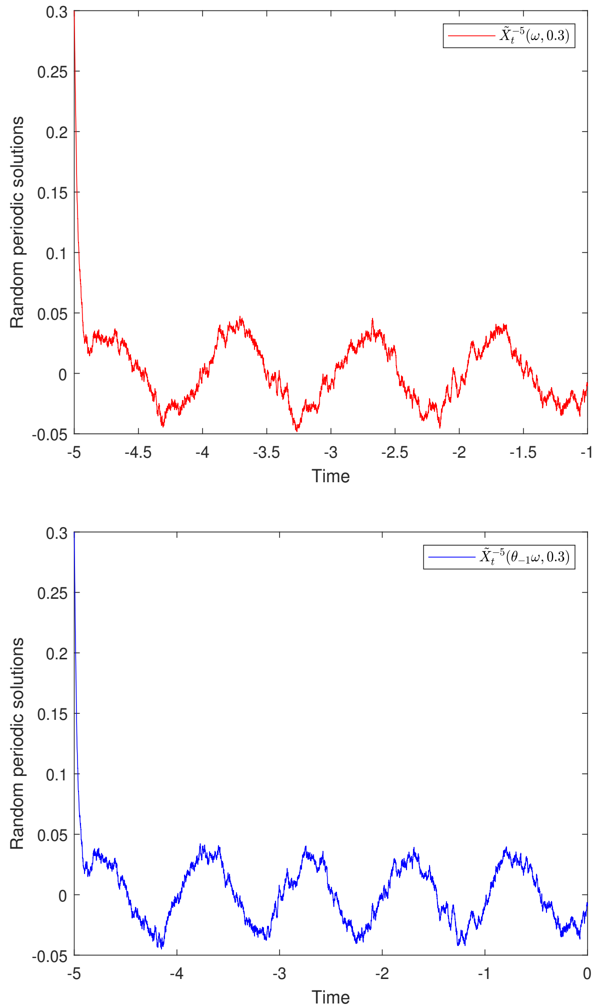

Figure 2: Simulations of the processse $\{\tilde{X}_t^*(\omega), -5 \leq t \leq -1\}$ and $\{\tilde{X}_t^*(\theta_{-1}\omega), -5 \leq t \leq 0\}$.

Finally, to test the optimal convergence rate of backward Euler method, we simulate the random solution of (5.1) with 5000 different noise realisations. We plot in Fig. 3 for seven different stepsizes $h = 2^{-i}$, $i = 0, 2, 3, 4, 5, 6, 7$. From Fig 3, one can clearly observe that the resulting error decrease at slope close to 1, which is consistent with the predicted convergence order.

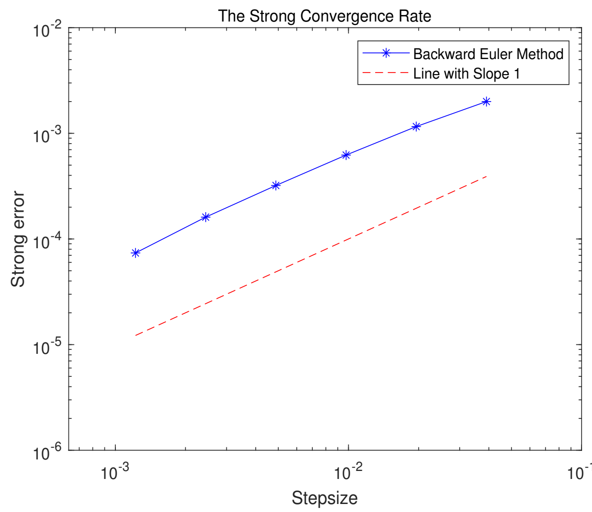

Figure 3: Numercial experiment for simulating the random periodic solution of SDE (5.1):Step sizes versus L^2 error.

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