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Application of Evolutionary Algorithms in Bayesian Multi-objective Reliability-Based Design Optimization

Abstract: Reliability-Based Design Optimization (RBDO) methods are well known in engineering design. However, these approaches usually require uncertainties to be modelled by statistical distributions. Hence, samples of uncertainty variables with enough size are necessary, so that these variables can be fitted by probabilistic distributions known as aleatory uncertainty.

In realistic engineering design, there is a lack of information about design variables or parameters and only a reduced set of samples are available (e.g. physical tests are very expensive). Therefore, design is carried out with incomplete information about input variables and parameters known as epistemic uncertainty.

Both types of uncertainties need to be considered in engineering design problems. However, epistemic uncertainty cause the reliability of the system or component to be also a random variable. Therefore, reliability constraints are imposed with a level of confidence specified by the designer posing a significant computational challenge for the design.

This paper proposes two effective multi-objective evolutionary algorithms to solve problems of design under uncertainty with incomplete information. The proposed approaches consider the cost and reliability as objective functions. The result is a Pareto front with a trade-off between cost and reliability for different levels of confidence. An analytical example and a structural problem are solved to show the applicability of the approach and how epistemic uncertainty may affect the results.

1 Introdution

Design optimization in engineering tries to obtain optimal solutions and reliable products. In order to achieve this result, it is necessary to consider the unavoidable variability of parameters characterizing geometry, loads and mechanical properties of a product.
Early methods in optimal design under uncertainty consider that uncertainties are characterized as random variables following probability distributions with known parameters. That is, there is such quantity of information about uncertain variables and parameters that we can apply a goodness of fit test and assign the probability distributions obtained in design optimization. Two optimal design methods under complete information of uncertainties are: Robust Design Optimization (RDO) and Reliability Based Design Optimization (RBDO).
In a classic RBDO problem, a single objective function is optimized subject to reliability constrains. Researchers have proposed several methods to solve this type of problems. These methods combine an optimization algorithm with reliability analysis methods.
RBDO methods are classified in three groups: double loop methods, single loop methods and decoupled methods. With respect to reliability analysis, two kinds of methods can be used: approximated methods (e.g. FORM, SORM) and Monte Carlo based simulation methods. Simulation methods often request prohibitive computational effort while approximate methods demand less computing resources but it is well known that FORM and SORM experiment convergence difficulties, especially when performance functions are highly nonlinear and random variables are not normal.
Usually, the designer or decision maker is interested in the trade-off between cost and reliability. In other cases, two or more objectives are optimised subject to reliability constraints. These problems are multi-objective optimization problems subject to reliability constraints. And the method to solve them is named Multi-Objective Reliability Based Design Optimization (MORBDO) [1]. Multi-objective Optimization Evolutionary Algorithms (MOEAs) such as Non-dominated Sorting Genetic Algorithm (NSGA-II) [2] and Multi-objective Particle Swarm Optimization (MOPSO) [3] are the most frequent MOEAs considered in the literature to solve MORBDO problems since these methods can handle constraints efficiently [4].
Firstly, it is necessary to make a classification of the uncertainties. Subsequently, depending on the type of uncertainty, different methods could be applied to determine optimal designs. Researchers have proposed several classifications of uncertainty. This one is well known:
Aleatory uncertainty or variability:
It is the inherent uncertainty of a physical variable. Variables with random uncertainty are generally described by probability distributions.
Epistemic uncertainty or ignorance:
This type of uncertainty is due to lack of knowledge about the model or about the uncertain variables involved, due to lack of data. In practice applications, the only information about uncertainties that designer can use is a set of samples or an interval. Samples are obtained experimentally through tests. For practical or budgetary reasons, it is only possible to carry out a small number of tests. Therefore there is not enough data available to determine the probabilistic distribution of the uncertain variables and parameters [5].
Recently, researchers have made great efforts in the field of quantification of uncertainty, reliability analysis and optimal design under uncertainty to deal with problems in which there is not enough information on uncertainties and, therefore, exact probability distributions cannot be assumed to model these uncertainties [6].
Probabilistic distributions adjusted with such limited data would cause erroneous and unsafe design if were considered in a RBDO or MORBDO problem.
Other authors have proposed other classifications. For example, Toft-Christensen and Murotsu [7], described three types of uncertainties in the field of structural reliability analysis:

physical uncertainty, statistical uncertainty, and simulation model uncertainty. Physical uncertainty is the randomness inherent to physical observations, which can be described in terms of probability distributions. Statistical uncertainty corresponds to the uncertainty caused by lack of statistical information or limited sample size and considered epistemic uncertainty. The uncertainty of the simulation model occurs because of errors and idealizations done in the mathematical model. The uncertainty of the model can also be considered as an epistemic uncertainty. In this work, the uncertainty of the simulation model is not considered, and only statistical uncertainty is taken account as epistemic uncertainty. Figure 1 represents different types of uncertainty according to the level of information.

Researchers have proposed different methods to compute optimal designs depending on the representation of epistemic uncertainty, e.g. [8]. Du *et al* [9], proposed Possibility Based Design Optimization (PBDO). They applied fuzzy sets to quantify uncertainties. Epistemic variables are modelled as membership functions. Sample size is not considered in uncertainty quantification and as consequence the approach is usually too much conservative. Optimal design methods have also been applied to interval variables [10], [11]. The disadvantage of these methods is that the information contained in the samples is not used. Mourelatos and Zhou apply Evidence Theory in design optimization and propose Evidence Based Design Optimization [12].

This paper proposes the use of Bayesian Inference because they can deal with different representation of the information available. The paper is organised as follows. A short review of Bayesian Inference theory in presented in section 2. Section 3 describes Bayesian RBDO method. Section 4 studies the Bayesian Multi-objective Reliability-Based Design Optimization (Bayesian MORBDO) problem, where two objective function are optimised: cost and reliability. Two state of the art Multi-objective Evolutionary Algorithms (MOEAS) are applied to solve Bayesian MORBDO problems: Non-dominated Sorting Genetic Algorithms (NSGA-II) and Multi-objective Particle Swarm Optimization (MOPSO). Although NSGA-II has already been applied to solve Bayesian Multi-objective Reliability-Based Design Optimization problem [13], the MOPSO algorithm is applied for the first time to solve this type of problem according to the authors' knowledge. Section 4 describe two examples to show how Bayesian MORBDO works. Finally, section 5 includes the conclusions.

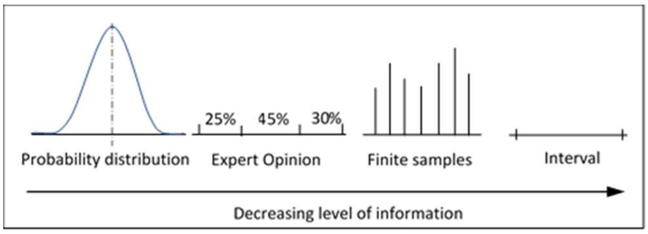

Fig 1 Types of uncertainty.

2 Bayesian Inference Methods

Researches have proposed methods applying Bayesian inference to solve optimal design problems under epistemic uncertainty [13], [14]. This section describes how to compute the reliability for a probabilistic constraint when there exist aleatory and epistemic random variables.

We can partition the vectors of uncertain variables X and parameters P in two sub vectors each one of them. That is, $X = [X_t, X_s]$ and $P = [P_t, P_s]$. The vectors X_t and P_t are aleatory variables and parameters whose probability density functions (PDFs) are known. In addition, the vectors X_s y P_s are epistemic random variables. Only a reduced set of samples are known for these epistemic variables.

Suppose that we want to compute the reliability for the j^{th} reliability constraint, that is,

$$R_j = Pr\left[g_j(X,P) > 0\right]$$

A priori distribution for the reliability of this performance function is proposed. If no previous information is available about this reliability, non-informative priori can be considered. Therefore, we assume that reliability R_j follows a uniform distribution, $U(0,1)$. Then, an updated value for R_j can be computed using Bayes' rule. It is supposed that a set of samples is available for epistemic uncertainties. We compute the reliability for each sample of epistemic variables and parameters:

$$Pr\left[g_j(X_t,P_t) > 0 | (X_s,P_s)_k\right] \ with\ k = 1,..,N,$$

where N is the sample size and $g_j(X,P) \leq 0$ is the failure region. Repeating this computation for the N samples available about the epistemic variables and parameters, we can obtain $E_j(r)$, the expected number of safety realizations is [13]:

$$E_j(r) = \sum_{k=1}^{N} Pr\left[g_j(X_t,P_t) > 0 | (X_s,P_s)_k\right]$$

The priori distribution is updated with the information given by $E_j(r)$ and we obtain a posteriori distribution for the reliability. This posteriori is a Beta distribution with parameters α and β, where $\alpha = E_j(r) + 1$ and $\beta = N - E_j(r) + 1$. That is:

$$R_j \sim Beta\left(r_j\, , E_j(r) + 1, N - E_j(r) + 1\ \right)$$

This posteriori distribution can be updated again if more samples become available.

The reliability R_j is a beta distribution and not anymore a crisp value. The confidence for a design μ_X with respect to j^{th} reliability constraint is defined as the probability that R_j, will exceed the target reliability:

$$\zeta_j(\mu_X) = Pr\left[g_j(X_t,P_t) > 0\big|_{\mu_X} \geq R_j^{target}\right] \qquad j = 1,..,J$$

Therefore, $\zeta_j(\mu_X) = 1 - \Phi_{Beta_j}(R_j)$, where $\Phi_{Beta_j}(\cdot)$ is the CDF of j^{th} Beta distribution. Also, the reliability can also be written in terms of the confidence:

$$R_j(\mu_X) = \Phi^{-1}_{Be\ \ j}\left(1 - \zeta_j(\mu_X)\right)$$

And the probability of failure is:

$$P_{f_j}(\mu_X) = 1 - \Phi^{-1}_{Beta_j}\left(1 - \zeta_j(\mu_X)\right)$$

It is important to note that when the sample size is small, it is not possible to obtain the design for a required value of reliability for a confidence established by the designer. For example, with $N = 50$ and $\zeta_j = 0.90$, if we suppose that all reliability values computed are 1, that is, $E_j(r) = 50$, and therefore, $\alpha = 51$ and $\beta = 1$ and the minimum value of P_{f_j} that we can obtain is 0.0441, as is computed below:

$$P_{f_j} = 1 - \Phi^{-1}_{Beta\ (51,1)}\left(1 - \zeta_j(\mu_X)\right) = 0.0441$$

If we want to determine an optimal design subject to reliability constraints for a specified confidence level, we need enough sample size. Also, this sample size depends on the confidence required by the designer [15].

3 Bayesian RBDO

Conventional problem in Reliability-Based Design Optimization (RBDO) consists of computing a design that minimises a cost function subject to reliability constraints. These constraints are formulated as component level reliabilities or as a system-level reliability. Complete information for the uncertainties is considered in conventional RBDO.

Under insufficient quantity of samples for the epistemic random variables, an optimal and reliable design can be computed applying Bayesian inference. This method is named, Bayesian RBDO. The formulation of Bayesian RBDO method is:

$$\begin{gathered}
\min_{\mathbf{d},\boldsymbol{\mu_X}} Cost(\mathbf{d}, \boldsymbol{\mu_X}, \boldsymbol{\mu_P}) \\
s.t.\ P_{f_j}(\mu_X) \leq P_{f_{j,target}} \qquad j = 1..nr \\
h_k(\boldsymbol{d}) \geq 0, \qquad k = 1,2,\dots,K \\
\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \boldsymbol{\mu}_X^L \leq \boldsymbol{\mu_X} \leq \boldsymbol{\mu}_X^U
\end{gathered}$$

where $P_{fj} = P\left(G_j(\mathbf{d}, \mathbf{X}, \mathbf{P}) \leq 0\right)$ and $\mathbf{X} = [\mathbf{X}_t, \mathbf{X}_s]$ and $\mathbf{P} = [\mathbf{P}_t, \mathbf{P}_s]$ and $G_j(\mathbf{d}, \mathbf{X}, \mathbf{P}) \leq 0$ is defined as the failure region. nr is the quantity of reliability constraints. As stated above $P_{f_j}(\mu_X)$ is computed for each design and depend of the confidence requested by the designer.

$$P_{f_j}(\mu_X) = 1 - \Phi^{-1}_{Beta_j}\left(1 - \zeta_j(\mu_X)\right)$$

Methods to solve this problem depend of the methods used to solve the optimization and the reliability analysis. When the sample size increases the optimum tends to the "exact" optimum obtained by RBDO under complete information.

4 Bayesian Multi-objective Reliability-Based Design Optimization

In realistic practice, designers and decision makers will prefer to know the various optimal designs for different values of probability of failure, for a determined confidence level established previously by the designer. Therefore, after setting the level of trust, a set of optimal solutions can be established, in which there is a compromise between cost and reliability. This front is the solution to a multi-objective optimization problem. This helps the selection of a design in a more practical way. Thus, the designer can see how much the cost increases if more reliable design is required. In the same way that in Bayesian RBDO, established a high value of confidence, an enormous sample size for the epistemic uncertainties is required to verify a very low value of probability of failure.

The formulation of the MORBDO problem is:

$$\begin{gathered}
\min_{\mathbf{d},\boldsymbol{\mu_X}} Cost(\mathbf{d}, \boldsymbol{\mu_X}, \boldsymbol{\mu_P}) \\
\min_{\mathbf{d},\boldsymbol{\mu_X}} P_{fS}(\mathbf{d}, \boldsymbol{\mu_X}, \boldsymbol{P}) \\
s.t.\ P^l_{fS} \leq P_{f_S} < P^u_{fS} \\
h_k(\boldsymbol{d}) \geq 0, \qquad k = 1,2,\dots,K \\
\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \boldsymbol{\mu}_X^L \leq \boldsymbol{\mu_X} \leq \boldsymbol{\mu}_X^U
\end{gathered}$$

where, $\mathbf{d}$, is the vector of deterministic design variables, $\boldsymbol{\mu_X}$ is the vector of uncertain design variables. P_{f_S} is the probability of system failure for a confidence established by the designer. $\boldsymbol{\mu}_X^L$ and $\boldsymbol{\mu}_X^U$ are lower and upper bounds for the mean values of uncertain design variables. P^l_{fS} and P^u_{fS} are bounds for the probability of system failure. All reliability constraints are combined in a unique system reliability constraint to formulate a bi-objective optimization problem.

P_{f_S} is computed as: $P_{f_S} = 1 - R_S$, where R_S is the reliability of the system. In this work, R_S has been computed as the minimum of the values of the reliabilities of the constraints. That is,

$$R_S = \min_{j=1,..,nr} R_j$$

where R_j, is the reliability of constraint j^{th} and nr is the number of constraints. However, more accurate values can be computed considering the configuration of the system (serial, parallel, mixed, etc) and taking in account the correlation between different failure modes [16].

The best methods to solve the optimization phase of MORBDO problems are based in multi-objective evolutionary algorithms. Rupesh Srivastava and Kalyanmoy Deb [13] applied Non-dominated Sorting Genetic Algorithm (NSGAII). We have applied other evolutionary algorithm method, named Multi-objective Particle Swarm Optimization (MOPSO) and compare the results obtained with both methods. Also, reliability analysis is carried out by FORM by computing the reliability index and then estimate the corresponding probability of failure of the constraints. It is well known that FORM cannot converge to the most probable point in case of non-linear performance function or when random variables have non-normally distributed. In order to address this point, the implemented algorithms assign large values to the objective functions when reliability analysis does not converge. This strategy prevents that these designs are continuously considered by the evolutionary optimization tools.

Both optimization methods request a large computational effort. In fact, each population can contain, for example, 100 individuals or design vectors. For each design vector, it is necessary to carry out so many reliability analyses as the sample size of epistemic variables. And this is repeated for each generation.
Some strategies have been implemented to reduce this computational cost. The first one is to use FORM in reliability analysis. The use of simulation methods would be prohibitively expensive. The second one is to discard a design immediately after that reliability analysis does not converge for some vector of the epistemic sample.

4.1 Analytical example

The first numerical example considers only two objectives and two design variables that permits a graphical representation of the Pareto front in the objective function space and the Pareto set in the space of design variables. The formulation of the Bayesian MORBDO problem is:

$$\min_{\boldsymbol{\mu_X}} f(\boldsymbol{\mu_X}) = \mu_{X_1} + \mu_{X_2}$$
$$\min_{\boldsymbol{\mu_X}} \; P_{f_S}(X_1, X_2, X_3)$$
$$s.t. \quad 0.0001 \le P_{fs} \le 0.1$$
$$s.t. \quad 0 \le \mu_{X_1} \le 10 \,;\; 0 \le \mu_{X_2} \le 10$$

The first objective is the cost function and the second objective represents the probability of the system failure computed as $P_{f_S} = 1 - R_S$. R_S is the reliability of the system. Lower and upper bounds are set as 0.0001 and 0.1 to obtain the optimal design and the Pareto front in the range of interest for the designer.

The performance functions are defined as:

$$g_1(\boldsymbol{X}) = X_1^2 X_2 X_3/20 - 1$$
$$g_2(\boldsymbol{X}) = \frac{(X_1 + X_2 + X_3 - 5)^2}{30} + \frac{(X_1 - X_2 - X_3 - 12)^2}{120} - 1$$
$$g_3(\boldsymbol{X}) = 80/(X_1^2 + 8X_2X_3 + 5) - 1$$

In this example, adapted from [15], there are two aleatory design variables: X_1 and X_2. The third variable, X_3, is a parameter with epistemic uncertainty and only a small sample size is available. It is not considered a design variable since none of its statistical properties is known. The design variables, X_1 and X_2 are distributed according a normal distribution:

$$X_1 \sim Normal\left(\mu_{X_1}, CoV = 0.12\right)$$
$$X_2 \sim Normal\left(\mu_{X_2}, CoV = 0.12\right)$$

The samples for X_3 are randomly generated from a normal distribution with $X_3 \sim Normal\left(\mu_{X_3} = 1.0, \sigma_{X_3} = 0.1\right)$.
The problem is solved using the NSGA-II and MOPSO algorithms, respectively. The problem has been solved for different sample size of the available information about the epistemic uncertainty $(i.e, N = 50, 100, 200)$ and different confidence level $(\zeta_j = 0.8, 0.9)$. The population size in NSGAII and the swarm size in MOPSO are equal to 100. The number of generations or evolutions is also set to 100 for both approaches. As reference solution, the MORBDO problem is also solved considering complete information.
Figures 2 and 3 show the Pareto fronts in the space of objective functions for different samples size as well for the case of complete information. The figures show how the Bayesian fronts tends to the front obtained with complete information as the number of samples increases.
The sample size for the epistemic uncertaintyX_3, constraints the minimum value of the probability of system failure that one could obtain with a specified confidence. For example, for $N = 200$ and with confidence $\zeta = 0.9$, it is impossible to obtain a design with P_{f_S} below 0.01 even though the lower bound of the probability of failure is 0.0001.
Table 1 shows the minimum values for the probability of system failure that can be found applying Bayesian MORBDO to the analytical example for various values of sample sizes and confidence. These minimum values decrease when sample size increases. Also, they decrease when confidence decreases. We have checked that the lower bound of probability of failure is searched if complete information is available about probability distribution of uncertain variables. Differences between NSGA-II and MOPSO results are practically negligible.

Table 1. Values of Maximum probability of system failure searchable.

	Confidence = 0,9		Confidence = 0,8	
Sample Size	NSGA-II	MOPSO	NSGA-II	MOPSO
N = 50	0,044394	0,044466	0,031446	0,031430
N = 100	0,022820	0,022832	0,016024	0,016044
N = 200	0,011707	0,011677	0,008210	0,008239
COMPLETE INF.	0,000158	0,000170	0,000158	0,000170

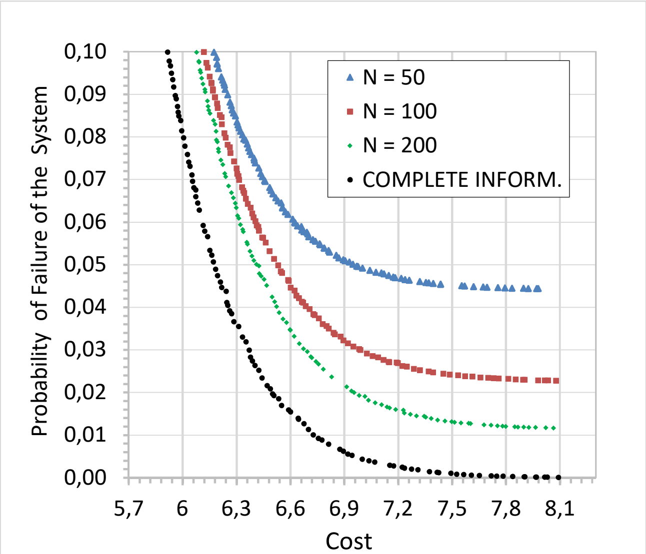

Fig 2. Pareto fronts obtained by NSGA-II for the Analytical example for different sample sizes with confidence 0.9.

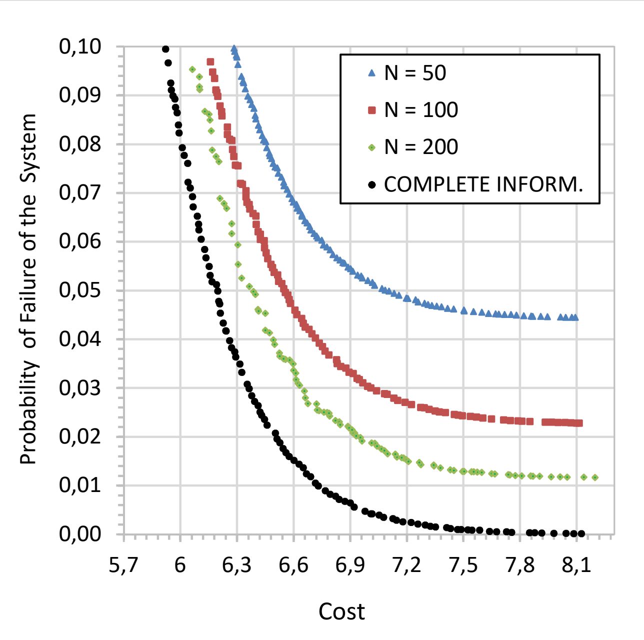

Fig 3. Pareto fronts obtained by MOPSO for the Analytical example for different sample sizes with confidence 0.9.

4.2 Ten bar truss example

The second example consider a ten bar truss as shown in figure 4. The formulation of the Bayesian MORBDO is as follow:

$$\min_{\mathbf{d},\boldsymbol{\mu}_X} Volume(\mathbf{d}, \boldsymbol{\mu}_X, \boldsymbol{\mu}_P)$$
$$\min_{\mathbf{d},\boldsymbol{\mu}_X} P_{f_S}(\mathbf{d}, \boldsymbol{\mu}_X, \boldsymbol{P})$$
$$s.t. \quad 0.001 \leq P_{f_{SYS}} \leq 0.1$$
$$4 \leq \mu_{X_j} \leq 75, \quad j = 1,2,3.$$

Two objective functions are considered. The first objective is the steel volume of the truss. As the steel density is a constant, to minimise the steel volume is equivalent to minimise steel mass. The second objective function represents the probability of system failure. Also, bounds are stated for the probability of system failure to find design solutions in the range of interest. Only one displacement constraint is imposed: the vertical displacement of the node 2 must be below 2 cm.

$$G_1(\mathbf{d}, \mathbf{X}, \mathbf{P}) = 1 - \left|q_{2,V}(\mathbf{d}, \mathbf{X}, \mathbf{P})\right| / q^a$$

The value of the displacement limit is $q^a = 2\ cm$.

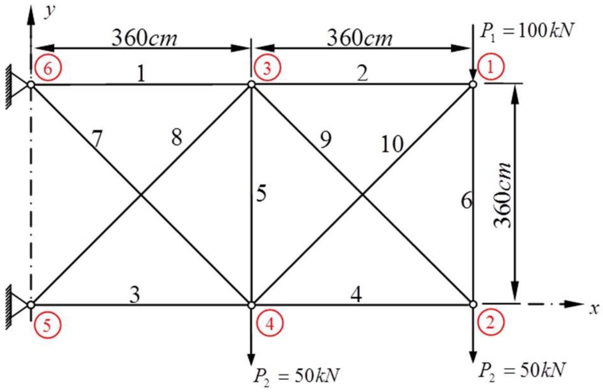

Fig 4. Ten bars truss.

Bars are grouped in three groups. Group 1 contains horizontal bars, group 2 contains vertical bars and group 3 contains diagonal bars. Bars in the same group have the same cross sectional area. Mean values of these areas are the design variables of the problem: $\mu_{X_1}, \mu_{X_2}, \mu_{X_j}$. Therefore, three normal aleatory design variables are assigned to these areas. Loads P_1 and P_2 are random parameters with normal distribution and are applied in nodes 1, 2 and 4. Also, the elastic modulus E is an epistemic random parameter and only a limited sample is available. The sample has been obtained from a normal distribution, $N(\mu = 21000\ kN/cm^2,\ \sigma = 210\ kN/cm^2)$. The uncertain variables and parameters of the problem are shown in the Table 2.

Table 2. Random variables in the ten bar truss example.

Random variable	Distribution	Mean Value	CoV (%) or σ^2
$X_1 \equiv A_1$	N	μ_{X_1}	5%
$X_2 \equiv A_2$	N	μ_{X_2}	5%
$X_3 \equiv A_3$	N	μ_{X_3}	5%
$X_4 \equiv P_1$	N	$100.0\ kN$	$20\ kN$
$X_5 \equiv P_2$	N	$50.0\ kN$	$2.5\ kN$
E	Epistemic		

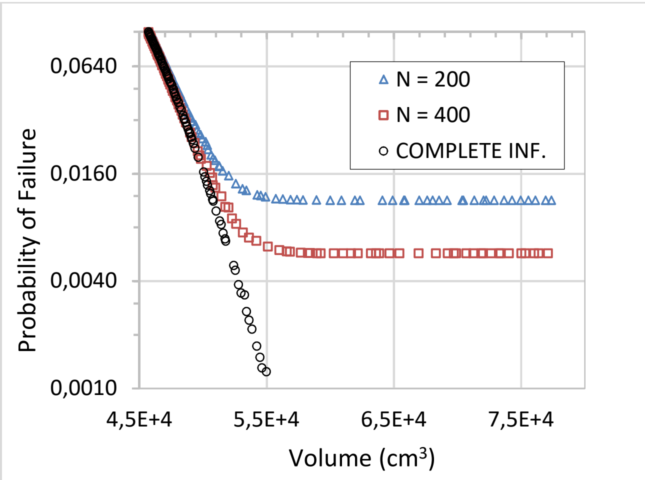

Fig 5. Pareto fronts obtained by NSGAII for the ten bars truss example for different sample sizes with confidence 0.9.

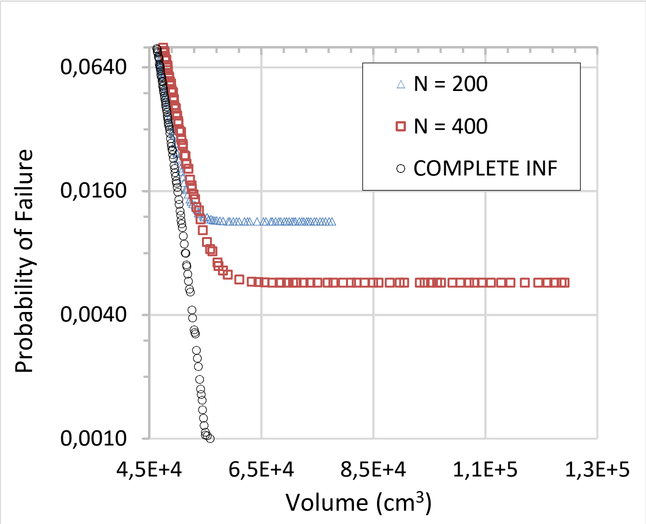

Fig 6. Pareto fronts obtained by MOPSO for the ten bars truss example for different sample sizes with confidence 0.9.

In the example of the structure of 10 bars the same thing happens as in the analytic case. For small sample sizes and for a large value of confidence (0.9), no matter how much the cross section increases, there is a limit value of failure probability that can be reached. This fact is of great importance for the decision maker since it means that a higher cost does not imply greater reliability. The figures 5 and 6 show that Pareto fronts become a horizontal line for these limit values of probability of failure.

5 Conclusions

Optimal designs under epistemic uncertainty and with a limited sample size can be obtained adopting Bayesian MORBDO. These methods allow to obtain different reliable and optimum designs for a range of admissible probability of system failure. However, these methods have an important disadvantage: the large computational effort required.

Here an approach that allows to reduce significantly the computation cost of the analysis is presented. The reliability of the system is updated using Bayes rule and the samples available for the epistemic parameters. Non-informative priors are used. Using these conjugate distributions has a major drawback. That is, there is a lower bound in the value of the failure probability that a design could reach. This lower bound depends on specified value for the confidence and the sample size for epistemic variables.

Then two state of the art optimisation methods (i.e. NSGA-II and MOPSO) have been applied to solve two examples. Both methods give similar results and have the same efficiency in term of calculation time. The examples presented have shown that large sample sizes are required to compute designs that produce reduced values of probability of system failure using a predefined level of confidence. Therefore, Bayesian RBDO with a target probability of failure less than this lower bound will not provide a reliable optimal design.

Some improvements could be proposed in the Bayesian MORBDO field such as implementing a more efficient and accurate reliability analysis. Here, the reliability of the system has been approximated by the minimum value of the reliabilities of the performance functions. However, more accurate results could be computed applying Dilevsen' bounds.

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