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Polynomial eigenvalue decomposition for eigenvalues with unmajorised ground truth – Reconstructing analytic dinosaurs

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ABSTRACT

When estimated space-time covariance matrices from finite data, any intersections of ground truth eigenvalues will be obscured, and the exact eigenvalues become spectrally majorised with probability one. In this paper, we propose a novel method for accurately extracting the ground truth analytic eigenvalues from such estimated space-time covariance matrices. The approach operates in the discrete Fourier transform (DFT) domain and groups sufficiently eigenvalues over a frequency interval into segments that belong to analytic functions and then solves a permutation problem to align these segments. Utilising an inverse partial DFT and a linear assignment algorithm, the proposed EigenBone method retrieves analytic eigenvalues efficiently and accurately. Experimental results demonstrate the effectiveness of this approach in reconstructing eigenvalues from noisy estimates. Overall, the proposed method offers a robust solution for approximating analytic eigenvalues in scenarios where state-of-the-art methods may fail.

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Figures and tables

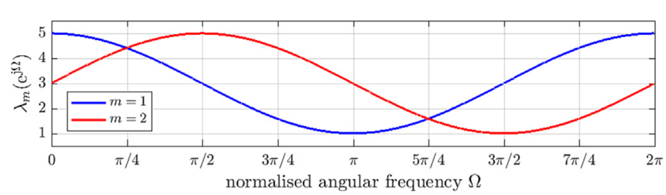

Fig. 1. Example of two analytic eigenvalues $\lambda_m(e^{j\Omega})$, $m = 1, 2$, of a parahermitian matrix $R(z)$ [1–4] derived from a source model consisting of innovation filters [5] that model the power spectral densities of source signals and of causal and stable transfer function matrices that form the convolutive mixing system [6,7]. For such parahermitian matrices, an analyti eigenvalue decomposition exists in almost all cases [4,8,9], with factors that are analytic functions in the complex variable$z$or of the normalised angular frequency $\Omega$when evaluated on the unit circle $z = e^{j\Omega}$.

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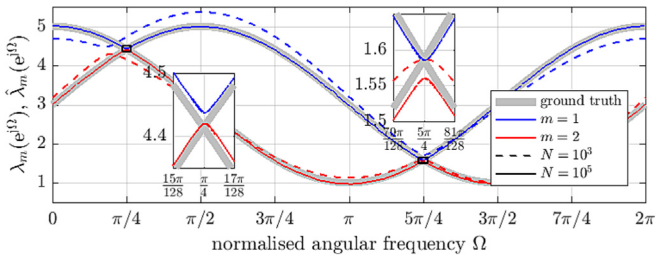

Fig. 2. If $R(z)$is not exactly known but estimated from N snapshots of data [10–13], the resulting perturbed matrix $\widehat{R}(z)$ has analytic eigenvalues $\widehat{\lambda_m}(e^{j\Omega})$, $m = 1, 2$. For $N < \infty$, with probability one eigenvalues $\widehat{\lambda_m}(e^{j\Omega})$ will be spectrally majorised, i.e. will have lost any zero crossings compared to $\lambda_m(e^{j\Omega})$ [14]. Existing analytic EVD algorithms [15–22] will extract $\widehat{\lambda_m}(e^{j\Omega})$, which will require much higher time domain support and hence incur higher computational complexity than any processing derived from the ground truth EVD factors. Hence the extraction of the unmajorised ground truth eigenvalues is important for applications involving the construction of paraunitary systems [23–26] or subspace projections [27–30].

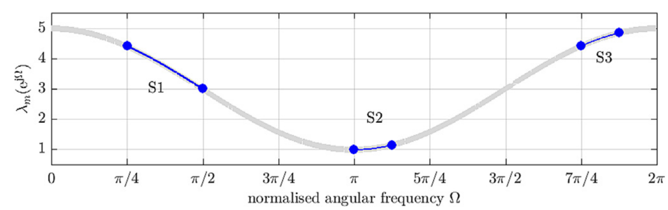

Fig. 3. Visualisation of how an analytic function, here $\lambda_m(e^{j\Omega})$, $m = 1$, can be reconstructed from any segment, here S1, S2, or S3. This is due to the definition of an analytic function as being identical to its own Taylor series expansion in any point, and the uniqueness theorem of analytic functions [31].

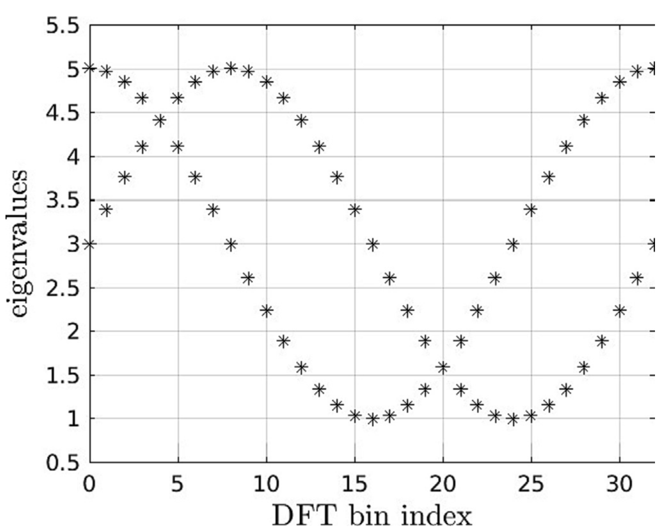

Fig. 4. The proposed EigenBone method [32] operates in the DFT domain, where in – here 32 – frequency bins standard EVDs [33,34] are computed.

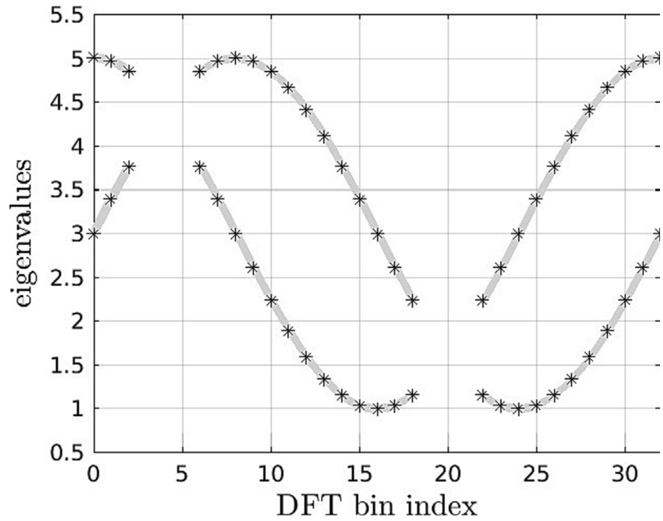

Fig. 5. EigenBone segmentation step: segments are formed on sufficiently long frequency intervals where the minimum eigenvalue distance exceeds a preset threshold. Here, segments are formed on 2 frequency intervals — note the wrap-around due to 2π-periodicity of the eigenvalues.

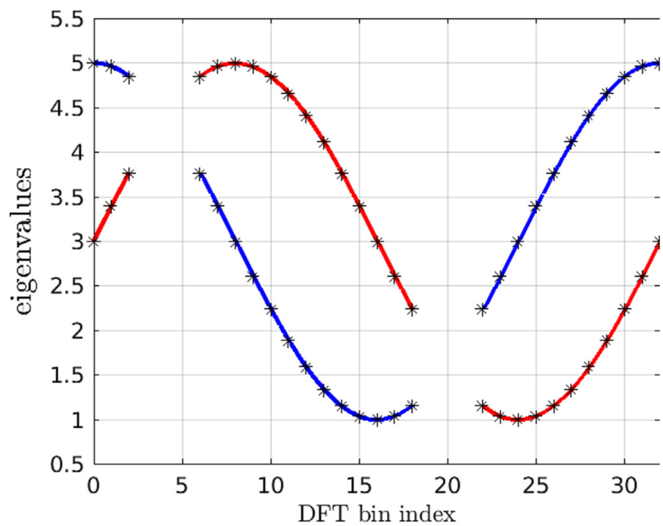

Fig. 6. EigenBone alignment step: segments are associated across different frequency intervals; this is performed via a partial reconstruction, exploiting the "Dinosaur Bone" theorem.

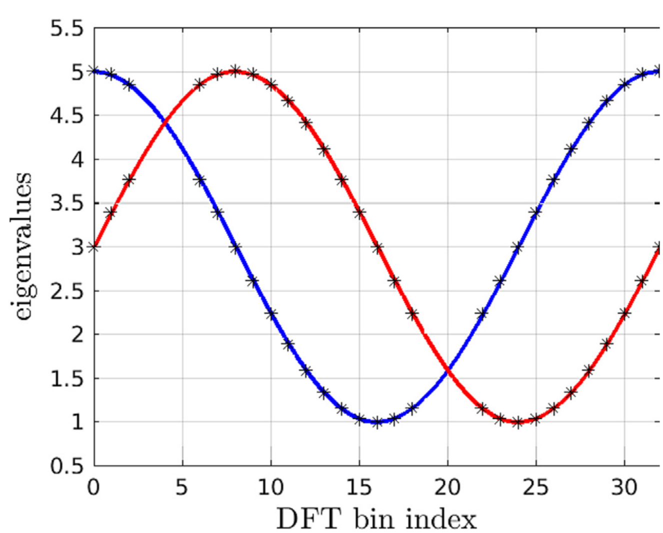

Fig. 7. EigenBone retrieval step: the analytic ground truth eigenvalues are recovered via a reconstruction from segment-weighted average.

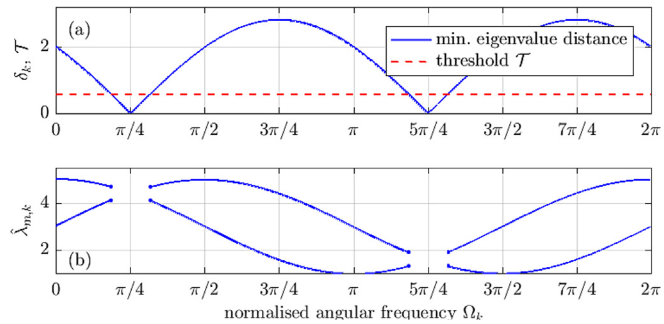

Fig. 8. EigenBone Segmentation step example with (a) the minimum eigenvalue distance for the example eigenvalues of Fig. 1, and (b) the resulting segmentation.

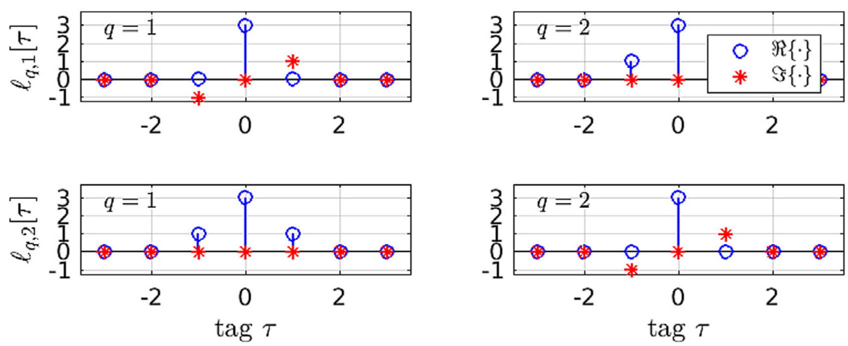

Fig. 9. Partial reconstruction via a least squares fit of time domain functions to the 4 segments in Fig. 8. The first column is for the wrapped-around frequency interval in Fig. 8, the second column for the second interval in Fig. 8. In order to associate the functions correctly, the segments in the 2nd column must be permuted. This can be identified via a number of algorithms [35–39], with the Hungarian algorithm [38] being our method of choice.

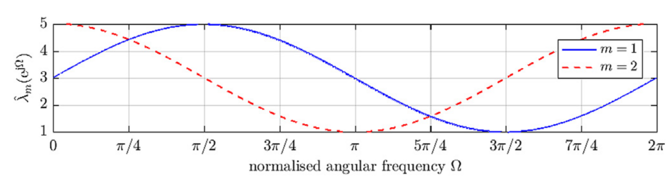

Fig. 10. Correctly retrieved ground truth eigenvalues via a reconstruction from segment-weighted averages.

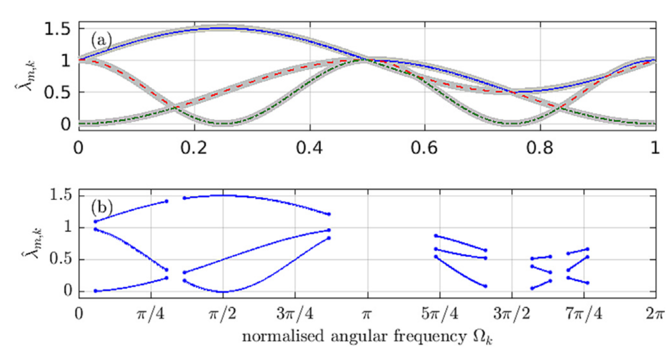

Fig. 11. Numerical example for a 3 × 3 parahermitian matrix with (a) challenging intersections of eigenvalues [18], and (b) the segmentation performed by the EigenBone method.

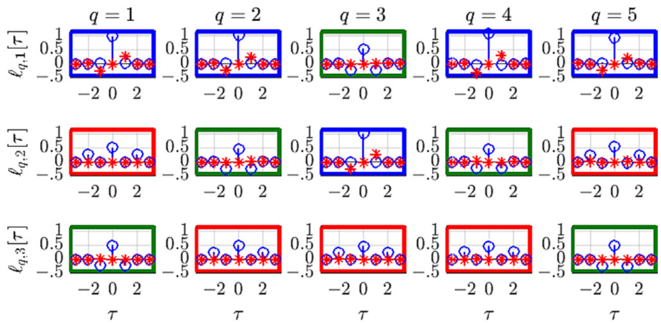

Fig. 12. Partial reconstructions akin to the simpler case in Fig. 9. Each response represents a time domain best least square fit to a segment, with real (blue circles) and imaginary parts (red asterisks). The association by the Hungarian algorithm is not difficult, and colour-coded for the three analytic eigenvalues. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

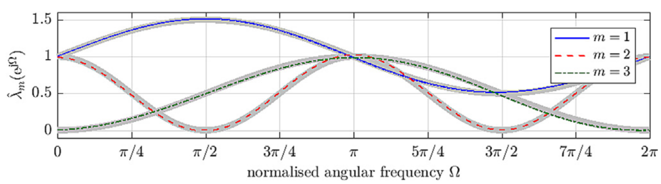

Fig. 13. Retrieved analytic eigenvalues based on a reconstruction from the segment-weighted average. The shaded curves represent the ground truth, which is closely matched by the proposed approach. Although demonstrated here for analytic eigenvalues, the method can similarly be applied to recover analytic singular values [40,41], where a similar loss of intersections occurs when addressing estimated transfer function matrices [42].

CRediT authorship contribution statement

Sebastian J. Schlecht: Conceptualization, Formal analysis, Software, Writing – original draft, Writing – review & editing. **Stephan Weiss:** Conceptualization, Formal analysis, Software, Writing – original draft, Writing – review & editing.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Data availability

Data will be made available on request.

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