

On (joint) equidistributions of mesh patterns 123 and 132 with symmetric shadings

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Abstract. A notable problem within permutation patterns that has attracted considerable attention in literature since 1973 is the search for a bijective proof demonstrating that 123-avoiding and 132-avoiding permutations are equinumerous, both counted by the Catalan numbers. Despite this equivalence, the distributions of occurrences of the patterns 123 and 132 are distinct. When considering 123 and 132 as mesh patterns and selectively shading boxes, similar scenarios arise, even when avoidance is defined by the Bell numbers or other sequences, rather than the Catalan numbers.

However, computer experiments suggest that mesh patterns 123 and 132 may indeed be jointly equidistributed. Furthermore, by considering symmetric shadings relative to the diagonal, a maximum of 93 equidistributed pairs can potentially exist. This paper establishes 75 joint equidistributions, leaving the justification of the remaining cases as open problems. As a by-product, we also prove 36 relevant non-symmetric joint equidistributions. All our proofs are bijective and involve swapping occurrences of the patterns in question, thereby demonstrating their joint equidistribution. Our findings are a continuation of the systematic study of distributions of short-length mesh patterns initiated by Kitaev and Zhang in 2019.

Keywords: mesh pattern, equidistribution, joint equidistribution, bijection

AMS Subject Classifications: 05A05, 05A15.

1 Introduction

A permutation of length n , or n -permutation, is a rearrangement of the set $[n] := \{1, 2, \dots, n\}$. Denote by S_n the set of permutations of $[n]$. For $\pi \in S_n$, let $\pi^r = \pi_n \pi_{n-1} \cdots \pi_1$ and $\pi^c = (n+1-\pi_1)(n+1-\pi_2) \cdots (n+1-\pi_n)$ denote the *reverse* and *complement* of π , respectively. Then $\pi^{rc} = (n+1-\pi_n)(n+1-\pi_{n-1}) \cdots (n+1-\pi_1)$. A permutation $\pi_1 \pi_2 \cdots \pi_n \in S_n$ avoids a *pattern* $p = p_1 p_2 \cdots p_k \in S_k$ if there is no subsequence $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ such that $\pi_{i_j} < \pi_{i_m}$ if and only if $p_j < p_m$. For example, the permutation 32145 avoids the pattern 132. $S_n(p)$ denotes the number of permutations of length n avoiding a pattern p .

Patterns p_1 and p_2 are *Wilf-equivalent*, denoted $p_1 \sim p_2$, if $|S_n(p_1)| = |S_n(p_2)|$ for all $n \geq 0$. Also, p_1 and p_2 are *equidistributed*, denoted $p_1 \sim_d p_2$, if the number of permutations of length n with k occurrences of p_1 is equal to that with k occurrences of p_2 for any $k, n \geq 0$. Moreover, p_1 and p_2 are *jointly equidistributed*, denoted $p_1 \sim_{jd} p_2$, if the number of permutations of length n with k occurrences of p_1 and ℓ occurrences of p_2 is equal to that with ℓ occurrences of p_1 and k occurrences of p_2 for any $k, \ell, n \geq 0$.

Patterns in permutations have attracted much attention in the literature (see [12] and references therein), and this area of research continues to grow rapidly. A significant problem in the field that has garnered considerable attention since 1973 is finding a bijective proof of the equivalence $123 \sim 132$, specifically that $|S_n(123)| = |S_n(132)| = \frac{1}{n+1} \binom{2n}{n}$, the n -th *Catalan number*, for $n \geq 1$ (for example, see [9, 16, 17, 18, 19, 21, 24]). Claesson and Kitaev [7] discovered that some of the published bijections can be easily derived from others via “trivial” bijections. They provide a comprehensive survey and systematic analysis of these bijections. Subsequent work in this area includes [3, 20]. Note that the patterns 123 and 132 are not equidistributed; for instance, there are 6 (resp., 5) 4-permutations with one occurrence of the pattern 123 (resp., 132).

The notion of a *mesh pattern*, generalizing several classes of patterns, was introduced by Brändén and Claesson [5] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns. A pair (τ, R) , where τ is a permutation of length k and R is a subset of $\llbracket 0, k \rrbracket \times \llbracket 0, k \rrbracket$, where $\llbracket 0, k \rrbracket$ denotes the interval of the integers from 0 to k , is a mesh pattern of length k . Let (i, j) denote the box whose corners have coordinates $(i, j), (i, j+1), (i+1, j+1)$, and $(i+1, j)$. Let the horizontal lines represent the values, and the vertical lines

denote the positions in the pattern. Mesh patterns can be drawn by shading the boxes in R . The picture



represents the mesh pattern with $\tau = 132$ and $R = \{(0, 0), (1, 2), (2, 1), (3, 1)\}$.

A subsequence $\pi' = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is an occurrence of a mesh pattern (τ, R) if (a) π is order-isomorphic to τ , and (b) the shaded squares given by R do not contain any elements of π not appearing in π' . Note that occurrences of patterns and are known as *ascents* and *descents*, respectively.

Many papers are dedicated to the study of mesh patterns and their generalizations; for example, see [1, 2, 4, 10, 13, 22, 23]. However, the first systematic study of avoidance of mesh patterns was not conducted until [11], and the study of their distribution not until [14, 15].

In relation to our paper, Kitaev and Liese [13] have demonstrated Wilf-equivalence of the patterns and . These patterns are not equidistributed; for example, there are 4 (resp., 3) 4-permutations containing exactly one occurrence of the former (resp., latter) pattern. However, as proved in Theorem 3.1, the patterns and are not only Wilf-equivalent but also equidistributed. On another note, Claesson [6] proved that and are Wilf-equivalent and are both counted by the *Bell numbers* 1, 1, 2, 5, 15, 52, 203, 877, 4140, \dots . These patterns are not equidistributed; for example, there are 7 (resp., 6) 4-permutations containing exactly one occurrence of the former (resp., latter) pattern. For another relevant fact, as discussed by Avgustinovich at el. [1], the patterns and are not Wilf-equivalent. However, as proved in Theorem 4.6, the patterns and are equidistributed. Interestingly, in some cases, four patterns can be equidistributed. Three (resp., two and two) examples of such situations are presented in Table 1 (resp., Tables 3 and 5).

Note that the patterns 123 and 132 remain invariant under the usual group-theoretic inverse, reflecting the elements of a pattern across the diagonal. This motivated us to restrict our attention in systematic studies to shadings symmetric with respect to the diagonal. Computer experiments, for $n \leq 9$, suggest that the mesh patterns 123 and 132 with symmetric shadings

may be potentially equidistributed in a maximum of 93 cases. Interestingly, the patterns and (resp., and) are equidistributed for $n \leq 7$, but not for $n = 8$.

In this paper, we will establish 75 joint equidistributions (presented in Tables 1–5) out of the potential 93 equidistributions, leaving the justification of the remaining cases as open problems (see Table 6 and five conjectures in Section 5). As a by-product, we also note 36 relevant non-symmetric joint equidistributions (see Table 7). All our proofs are bijective, interchanging occurrences of the patterns in question. In particular, the results in Section 3 (resp., Section 4) rely on the bijection presented in the proof of Theorem 3.1 (resp., Theorem 4.1). In two cases (see Section 2.3) we could have used known equidistribution results for mesh patterns of length 2 to prove the equidistribution. However, we needed to derive alternative proofs to establish joint equidistribution. Our findings continue the systematic exploration of distributions of short-length mesh patterns initiated by Kitaev and Zhang in 2019.

For a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$, π_i is a *left-to-right minimum* (resp., *right-to-left maximum*) if $\pi_i < \pi_j$ (resp., $\pi_i > \pi_j$) for all $j < i$ (resp., $j > i$). In particular, π_1 is a left-to-right minimum. For example, in the permutation 426153 the elements 4, 2, and 1 are left-to-right minima. Throughout the paper, p_1 and p_2 refer to the mesh patterns 123 and 132, respectively, with identical shading in question.

Our paper is organized as following. In Section 2 (resp., 3, 4) we explain 22 (resp., 23, 30) joint equidistributions. The cases unsolved by us are presented, along with concluding remarks, in Section 5. Note that, while we provide thorough proofs and examples for our key results, as well as counterexamples for cases where our methods do not apply, many of the other justifications are relatively sketchy. This is because it is not feasible to provide detailed proofs for all 111 equidistributions discussed.

2 The first few cases

In this section, we justify more straightforward joint equidistributions. From now on, for brevity, we usually omit “and” in “ p_1 and p_2 ” when referring to a pair of mesh patterns $p_1 = 123$ and $p_2 = 132$ with the same shading.

nr.	patterns	nr.	patterns	nr.	patterns	nr.	patterns
1		2		3		4	
5		6		7		8	
9		10		11		12	

Table 1: Easy explainable joint equidistributions; pairs 2 and 3, 4 and 5, and 7 and 8 have the same joint distributions by applying the complement and reverse operations to the monotone pattern.

2.1 Cases justifiable by simple direct arguments

Occurrences of the patterns in this subsection can be easily controlled in permutations. Therefore, we provide just brief justifications for the joint equidistributions of the patterns listed in Table 1 as bullet points. Additionally, a map that establishes the joint equidistribution of the patterns in question is denoted as f .

- For pair 1, only permutation 123 (resp., 132) contains one occurrence of p_1 (resp., p_2); other permutations avoid the patterns. Hence, $p_1 \sim_{jd} p_2$.
- For pair 2, only n -permutations $123\pi_4 \dots \pi_n$ (resp., $132\pi_4 \dots \pi_n$), for $n \geq 3$, contain one occurrence of p_1 (resp., p_2); other permutations avoid the patterns. Letting $f(123\pi_4 \dots \pi_n) = 132\pi_4 \dots \pi_n$ and $f(132\pi_4 \dots \pi_n) = 123\pi_4 \dots \pi_n$ and $f(\pi) = \pi$ for any other permutation π , we obtain a bijective map proving $p_1 \sim_{jd} p_2$.
- Pair 3 is similar to pair 2 with $123\pi_4 \dots \pi_n$ (resp., $132\pi_4 \dots \pi_n$) replaced by $\pi_1 \dots \pi_{n-3}(n-2)(n-1)n$ (resp., $\pi_1 \dots \pi_{n-3}(n-2)n(n-1)$). We omit the details.
- For pair 4, only n -permutations $12\pi_3 \dots \pi_{n-1}n$ (resp., $1n\pi_3 \dots \pi_{n-1}2$), for $n \geq 3$, contain one occurrence of p_1 (resp., p_2); other permutations avoid the patterns. Letting $f(12\pi_3 \dots \pi_{n-1}n) = 1n\pi_3 \dots \pi_{n-1}2$ and $f(1n\pi_3 \dots \pi_{n-1}2) = 12\pi_3 \dots \pi_{n-1}n$ and $f(\pi) = \pi$ for any other permutation π , we obtain a bijective map proving $p_1 \sim_{jd} p_2$.
- Pair 5 is similar to pair 4 with $12\pi_3 \dots \pi_{n-1}n$ (resp., $1n\pi_3 \dots \pi_{n-1}2$) replaced by $1\pi_2 \dots \pi_{n-2}(n-1)n$ (resp., $1\pi_2 \dots \pi_{n-2}n(n-1)$). We omit the details.

- For pairs 6, 7 and 8, each occurrence of p_1 (resp., p_2) in a permutation π is as consecutive elements $\pi_i\pi_{i+1}\pi_{i+2}$ equal to $a(a+1)(a+2)$ (resp., $a(a+2)(a+1)$) for some a , where π_i is a left-to-right minimum for pairs 6 and 7 and for pairs 6 and 8 and p_1 , π_{i+2} is a right-to-left maximum. Note that two different occurrences of these patterns cannot overlap (otherwise, some elements in an occurrence would be in a shaded box). But then, a bijective map f is described by traversing from left to right in π and replacing occurrences $\pi_i\pi_{i+1}\pi_{i+2}$ of p_1 or p_2 by $\pi_i\pi_{i+2}\pi_{i+1}$. This proves that $p_1 \sim_{jd} p_2$.

- For pairs 9, 10 and 11, all occurrences of p_1 (resp., p_2) are $\pi_i\pi_{n-1}\pi_n = \pi_i(n-1)n$ (resp., $\pi_i\pi_{n-1}\pi_n = \pi_i n(n-1)$) for some i , where for pair 10 (resp., 11), π_i is a left-to-right minimum (resp., a right-to-left maximum in the permutation obtained by removing $n-1$ and n). Letting $f(\pi_1 \dots \pi_{n-2}(n-1)n) = \pi_1 \dots \pi_{n-2}n(n-1)$ and $f(\pi_1 \dots \pi_{n-2}n(n-1)) = \pi_1 \dots \pi_{n-2}(n-1)n$ and $f(\pi) = \pi$ for any other permutation π , we obtain a bijective map proving $p_1 \sim_{jd} p_2$.

We conclude this subsection by proving a joint equidistribution that is not straightforward but still relatively easy to justify.

Theorem 2.1 (Pair 12). *We have $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$.*

Proof. Note that any occurrence of the patterns begins with a left-to-right minimum, and any left-to-right minimum can be involved in at most one occurrence of the patterns. Let $\pi = \pi_1 \dots \pi_n \in S_n$. If $\pi_i\pi_j\pi_k$ is an occurrence of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ (resp., $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$), then $j = i + 1$, $\pi_j = \pi_i + 1$ (resp., $\pi_k = \pi_i + 1$), and π_j and π_k are the leftmost and the smallest elements in the North-East area relative to π_i . Suppose that $x_1 > \dots > x_k$ is the sequence of left-to-right minima in π .

Replace the occurrence $\pi_i\pi_{i+1}\pi_k$ of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ (resp., $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$), if any, starting at x_1 , by $\pi_i\pi_k\pi_{i+1}$, which is an occurrence of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ (resp., $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$). Then do the same for x_2, x_3 , etc. Note that even though an element can be present in more than one occurrence of the patterns, after conducting the swap for x_i , no new occurrences of the patterns in π can be introduced and no occurrences can be lost (the elements π_{i+1} and π_k are indistinguishable in π for the elements $< x_i$). The procedure described by us is clearly reversible, and it establishes joint equidistribution of the patterns.

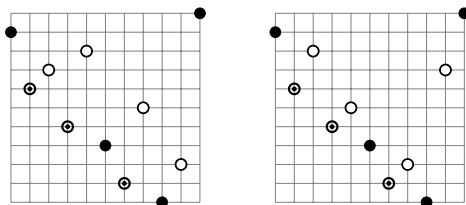


Figure 1: Permutations $\pi = (10)785942613(11)$ (to the left) and $\pi' = (10)795642318(11)$ (to the right) illustrating the proof of Theorem 2.1. The circled dots represent the first elements of occurrences (certain left-to-right minima in π), and the white dots represent the second and third elements of occurrences of p_1 or p_2 .

For example, for the permutation $\pi = (10)785942613(11)$ in Figure 1, the order of occurrences of the two patterns from left to right are $\pi_2\pi_3\pi_5 = 789, \pi_4\pi_5\pi_8 = 596, \pi_7\pi_8\pi_{10} = 263$, we do the procedure as follows: $\pi = (10)785942613(11) \xrightarrow{\text{swap } 8 \ \& \ 9} (10)795842613(11) \xrightarrow{\text{swap } 8 \ \& \ 6} (10)795642813(11) \xrightarrow{\text{swap } 8 \ \& \ 3} (10)795642318(11) = \pi'$. Note that the occurrence 789 of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ in π is replaced by the occurrence 798 of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ in π' , while the occurrences 596 and 263 of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ in π are replaced by the occurrences 568 and 238 of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ in π' . \square

2.2 Cases justifiable by the complement operation

Occurrences of any pattern in Table 2 in a permutation π begin with $\pi_1 = 1$, and removing 1 from the pattern results in a symmetrically shaded pattern with respect to a horizontal line, allowing us to apply the complement operation. Specifically, letting $f(\pi) = \pi$ for any π with $\pi_1 \neq 1$, and $f(1\pi_2 \dots \pi_n) = 1(\pi_2 \dots \pi_n)^c = 1\pi_2^c \dots \pi_n^c$, where the complement is taken on the set $\{2, \dots, n\}$ (for example, $(2435)^c = 5342$) gives a bijective map proving joint equidistribution of any pair in Table 2, because $1\pi_i\pi_j$ is an occurrence of p_1 if and only if $1\pi_i^c\pi_j^c$ is an occurrence of p_2 .

nr.	patterns	nr.	patterns	nr.	patterns
13		14		15	
16		17		18	

Table 2: Joint equidistributions explainable by the complement operation.

nr.	patterns	nr.	patterns	nr.	patterns	nr.	patterns
19		20		21		22	

Table 3: Joint equidistributions explainable by reduction to known results; pairs 19 and 20 (resp., 21 and 22) have the same joint distributions.

2.3 Reduction to joint equidistribution of patterns of length 2

In this subsection, we explain joint equidistributions in Table 3, which cannot be justified using the complement operation, as done in Section 2.2. The following theorem pertains to patterns nr. 64 and 65 in [14].

Theorem 2.2 ([14]). *We have $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix} \sim_d \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$.*

The following corollary follows from Theorem 2.2 by noting that the complement of $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ is $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$. However, for our purposes, we need a different proof that establishes the joint equidistribution of the patterns in question. The procedure of moving elements described by us is equivalent, when applying reverse and complement, to the procedure described in the proof of Theorem 4.1 below (in particular, instead of replacing elements while going from left to right, in the next proof we go from right to left). The justifications that everything works are essentially the same as in the proof of Theorem 4.1 and are therefore omitted.

Theorem 2.3. *We have $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix} \sim_{jd} \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$.*

Proof. Note that any occurrence of the patterns begins with a left-to-right minimum, and occurrences of $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ can only be formed by (not necessarily all) consecutive left-to-right minima. Also, any element can be the second

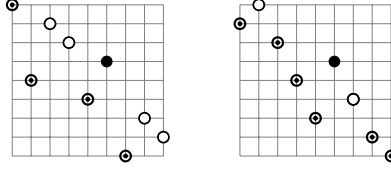


Figure 2: Permutations $\pi = 958746132$ (to the left) and $\pi' = 897536421$ (to the right) illustrating the proof of Theorem 2.3. The circled dots represent the left-to-right minima in π , and the white dots represent the second elements of occurrences of either pattern.

element in at most one occurrence of a pattern. Finally, the second element in an occurrence of $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ must be a right-to-left maximum.

Let $\pi_{i_1}\pi_{i_2}\dots\pi_{i_s}$ be the subsequence of a permutation $\pi = \pi_1\dots\pi_n$ that includes all elements in all occurrences of the patterns. In the procedure we are about to describe, we will permute the elements of $\pi_{i_1}\pi_{i_2}\dots\pi_{i_s}$ in a certain way while keeping the positions in π occupied by these elements unchanged. The permutation π' obtained from π by permuting $\pi_{i_1}\pi_{i_2}\dots\pi_{i_s}$ will have the desired properties.

Consider the unique occurrence $\pi_{i_x}\pi_{i_s}$ of a pattern that ends with π_{i_s} and swap π_{i_x} and π_{i_s} in π . Such a swap will change exactly one occurrence of a pattern to an occurrence of the other pattern, without introducing or removing any other occurrences of the patterns. Next, consider $\pi^* = \pi_1^*\dots\pi_n^*$ obtained from π by replacing π_{i_x} with π_{i_s} , and suppose $i_j < i_s$ is largest such that $\pi_{i_j}^*$ is the second element in an occurrence $\pi_{i_\ell}^*\pi_{i_j}^*$ of a pattern. Swap $\pi_{i_\ell}^*$ and $\pi_{i_j}^*$. Now, consider $\pi^{**} = \pi_1^{**}\dots\pi_n^{**}$ obtained from π^* by replacing $\pi_{i_\ell}^*$ and $\pi_{i_j}^*$, and suppose $i_k < i_j$ is largest such that $\pi_{i_k}^{**}$ is the second element in an occurrence $\pi_{i_\ell}^{**}\pi_{i_k}^{**}$ of a pattern. Swap $\pi_{i_\ell}^{**}$ and $\pi_{i_k}^{**}$. And so on. Eventually, all occurrences of the patterns in π will be swapped, which proves their joint equidistribution.

For example, for the permutation π in Figure 2, we do the procedure as follows: $\pi = 958746132 \xrightarrow{\text{swap 1 \& 2}} 958746231 \xrightarrow{\text{swap 2 \& 3}} 958746321 \xrightarrow{\text{swap 3 \& 4}} 958736421 \xrightarrow{\text{swap 5 \& 7}} 978536421 \xrightarrow{\text{swap 7 \& 8}} 987536421 \xrightarrow{\text{swap 8 \& 9}} 897536421$, so that $\pi' = 897536421$.

Note that the occurrences 58, 57, 13 and 12 of $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ in π are replaced, respectively, by the occurrence 87, 75, 32 and 21 of $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ in π' , while the occurrences 95 and 41 of $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ in π are replaced by the occurrences 89 and 34 of $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ in π' . \square

Theorem 2.4. We have $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$.

Proof. Copying and pasting the proof of Theorem 2.3 yields the desired result. \square

Let $T_{n,k}$ be the number of n -permutations with k occurrences of $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$. Theorem 4.1 in [14] states that

$$T_{n,k} = T_{n-1,k-1} + (n-1)T_{n-1,k} \quad (1)$$

with the initial conditions $T_{n,0} = (n-1)!$ for $n \geq 1$ and $T_{0,0} = 1$, which shows that $T_{n,k} = C(n, k+1)$, the *unsigned Stirling number of the first kind*. The next theorem (demonstrating only equidistribution, not joint equidistribution) is a weaker version of Theorem 2.4. It is proved by showing that the distribution of the pattern $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$ is given by $T_{n,k}$. The proof of Theorem 2.5 follows the style of the corresponding proof in [14]. We included this result in the paper to demonstrate an alternative approach to tackle the problem.

Theorem 2.5. We have $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} \sim_d \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$.

Proof. Let $T'_{n,k}$ be the number of n -permutations with k occurrences of $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$. A permutation counted by $T'_{n,k}$ can either be obtained by inserting the new largest element n

- to the left of a permutation counted by $T'_{n-1,k-1}$ (which clearly results in one extra occurrence of $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$), or
- any other but the leftmost position in a permutation π' counted by $T'_{n-1,k}$, which cannot result in a new occurrence of $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$ because either n is in the rightmost possible position, or the leftmost element in π' is in the shaded area.

Hence, $T'_{n,k}$ satisfies the same recursion as (1). A similar argument shows that $T'_{n,0} = (n-1)T'_{n-1,0}$ because inserting n in any except the leftmost position in an $(n-1)$ -permutation avoiding $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$ results in an n -permutation also avoiding $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$. Since, $T'_{0,0} = 1$, we have that $T'_{n,k} = T_{n,k}$ for all $n, k \geq 0$, proving that $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} \sim_d \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$. \square

Occurrences of any pattern in Table 3 in a permutation π begin with $\pi_1 = 1$. Removing 1 from such a pattern results in a mesh pattern of length 2.

nr.	patterns	nr.	patterns	nr.	patterns	nr.	patterns
23		24		25		26	
27		28		29		30	
31		32		33		34	
35		36		37		38	
39		40		41		42	
43		44		45			

Table 4: Joint equidistributions explainable based on the proof of Theorem 3.1. Pairs 23–29, 32–38, 40–43 and 45 (and pair 6 in Table 1) have the property that shading in the North-East area, relative to the minimum element, is symmetric with respect to a horizontal line. Pairs 30 and 46, and 31 and 47 have the same joint distributions by Theorem 4.5.

But then, joint equidistribution of (resp.,) follows from joint equidistribution of (resp.,) by Theorem 2.4 (resp., Theorem 2.3).




The joint equidistribution of pairs 19 and 20 (resp., 21 and 22) follows from the fact that under reverse and complement the pair (resp.,) goes to (resp., .



3 Cases based on the proof of Theorem 3.1





In this section, we explain joint equidistributions of pairs of patterns in Table 4.




Theorem 3.1 (Pair 23). *We have $\begin{smallmatrix} \blacksquare & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{smallmatrix} \sim_{jd} \begin{smallmatrix} \bullet & \bullet & \bullet \\ \bullet & \blacksquare & \bullet \end{smallmatrix}$.*

Proof. Let $\pi = \pi_1 \dots \pi_n \in S_n$ and $x_1 > x_2 > \dots > x_k$ be the sequence of left-to-right minima in π . If π has no occurrences of the patterns, π will be mapped to itself. Otherwise, by definitions, any occurrence of or


 in π must begin with an x_i , $1 \leq i \leq k$. Moreover, referring to a schematic representation of π in Figure 3, occurrences of the patterns must be entirely inside $\{x_i\} \cup A_i$ for some i , $1 \leq i \leq k$, where any $y \in A_i$ satisfies $x_i < y < x_{i-1}$ (some A_i 's can be empty, and A_i 's with just one element do not give any occurrences of the patterns). Indeed, the element x_{i-1} (resp., x_{i+1}) ensures that elements in occurrences of the patterns cannot be above (resp., to the right of) A_i . Also, note that if A_i is of length at least 2, then *each* pair of elements in it is involved in an occurrence of either  or . This is because the rightmost and leftmost elements in A_i , as well as the maximum and minimum elements, are involved in occurrences of these patterns (so there are no elements below A_i and to the left of it).

Let $X_i = \{x_i + 1, x_i + 2, \dots, x_{i-1} - 1\}$, where $1 \leq i \leq k$ and $x_0 := n + 1$. We define the operation of complementation of X_i 's, $1 \leq i \leq k$. Suppose X_i is formed by the elements $\pi_{i_1} \pi_{i_2} \dots \pi_{i_m}$ with $i_1 < i_2 < \dots < i_m$. Then, for $j = 1, 2, \dots, m$, if π_{i_j} is the ℓ -th smallest element in X_i then replace it by the $(m + 1 - \ell)$ -th smallest element in X_i . The complementation of X_i 's induces complementation on A_i and exchanges occurrences of the patterns 12 and 21 in each A_i , hence it replaces each occurrence of  by an occurrence of , and vice versa, and no new occurrences of these patterns in the obtained permutation π' can appear (any new occurrence must involve an element outside of an A_i , which is impossible because if an element was in a shaded area with respect to two elements a and b making ab non-occurrence of a pattern, it will continue to be in the shaded area for ab).

To complete the proof of the theorem, apply the operation of complementation to all X_i 's in every permutation in S_n , which will turn, in a bijective manner, permutations with x occurrences of  and y occurrence of  into permutations with y occurrences of  and x occurrence of . \square

To illustrate our proof of Theorem 3.1, consider the permutation $\pi = 9(11)4(12)8(10)5713(13)62$ with $x_1 = 9$, $x_2 = 4$, $x_3 = 1$, $A_1 = (11)$, $A_2 = 857$, $A_3 = 32$, $X_1 = (11)(12)(10)(13)$, $X_2 = 8576$, and $X_3 = 32$. The complementation process will result in the permutation $\pi' = 9(12)4(11)5(13)8612(10)73$. See Figure 4 for the permutation diagrams for π and π' . Note that the occurrence 457 of  in π is replaced by the occurrence 486 of  in π' , while the occurrences 485, 487, 132 of  in π are replaced, respectively,

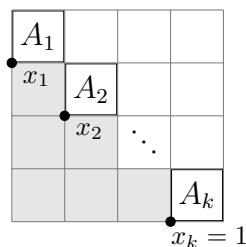


Figure 3: The structure of permutations in the proof of Theorem 3.1. Each A_i is of length ≥ 0 .

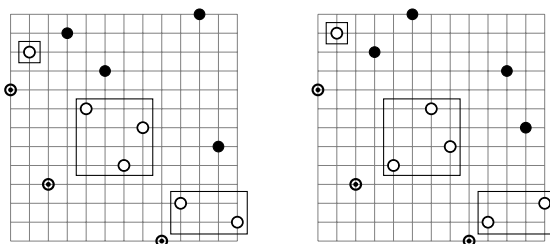


Figure 4: Permutations π (to the left) and π' (to the right) illustrating the proof of Theorem 3.1. The circled dots represent the left-to-right minima, and the white dots represent the elements in A_i 's, $i = 1, 2, 3$.

by the occurrences 458, 456, 123 of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ in π' ; no new occurrences of the patterns are introduced in π' .

Essentially copying and pasting the proof of Theorem 3.1, we can justify several other joint equidistributions, which we proceed to do next. Instead of providing full proofs, we will simply explain the differences with, and necessary adjustments to the proof of Theorem 3.1. The common feature of the mesh patterns in the next theorem is that the shading in the North-East area relative to the minimum element is symmetric with respect to a horizontal line (similarly to the patterns considered in Section 2.2). Note that the joint equidistribution $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ has already been explained in Section 2.1 (this is pair 6 in Table 1), but we still include this pair in Theorem 3.2 for completeness.

Theorem 3.2 (Pairs 24–29, 6). *We have* $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array},$
 $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}.$

Proof. The only difference from the proof of Theorem 3.1 is that now not any pair of elements in A_i is involved in an occurrence of a pattern in question, but only pairs forming an occurrence of respective patterns of length 2 in the North-East area relative to x_i . This does not affect our arguments in the proof of Theorem 3.1 since the only changes in π occur inside X_i 's and under complementing X_i , occurrences of $\begin{smallmatrix} \blacksquare & \blacksquare \\ \bullet & \bullet \end{smallmatrix}$ and $\begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \blacksquare \end{smallmatrix}$ (resp., $\begin{smallmatrix} \blacksquare & \bullet \\ \bullet & \blacksquare \end{smallmatrix}$ and $\begin{smallmatrix} \bullet & \blacksquare \\ \blacksquare & \bullet \end{smallmatrix}$, $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ and $\begin{smallmatrix} \blacksquare & \bullet \\ \blacksquare & \bullet \end{smallmatrix}$, $\begin{smallmatrix} \bullet & \blacksquare \\ \bullet & \blacksquare \end{smallmatrix}$ and $\begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \blacksquare \end{smallmatrix}$, $\begin{smallmatrix} \blacksquare & \blacksquare \\ \bullet & \bullet \end{smallmatrix}$ and $\begin{smallmatrix} \blacksquare & \bullet \\ \bullet & \bullet \end{smallmatrix}$) go to each other. \square

Theorem 3.3 (Pair 30). *We have $\begin{smallmatrix} \blacksquare & \bullet \\ \bullet & \bullet \end{smallmatrix} \sim_{jd} \begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \bullet \end{smallmatrix}$.*

Proof. The desired result cannot be obtained through the complementation operation on X_i 's. Nevertheless, we proceed following the same steps as in the proof of Theorem 3.2. The joint equidistribution $\begin{smallmatrix} \blacksquare & \bullet \\ \bullet & \bullet \end{smallmatrix} \sim_{jd} \begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \bullet \end{smallmatrix}$ follows from the joint equidistribution of $\begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \bullet \end{smallmatrix} \sim_{jd} \begin{smallmatrix} \bullet & \bullet \\ \bullet & \blacksquare \end{smallmatrix}$, by Theorem 2.4, in each A_i (we can use the respective bijection for the patterns of length 2 to permute the elements of A_i in the same positions accordingly). \square

Theorem 3.4 (Pair 31). *We have $\begin{smallmatrix} \blacksquare & \bullet \\ \bullet & \bullet \end{smallmatrix} \sim_{jd} \begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \bullet \end{smallmatrix}$.*

Proof. Again, the desired result cannot be obtained through the complementation operation on X_i 's. However, we can proceed with the same steps as in the proof of Theorem 3.3. The joint equidistribution $\begin{smallmatrix} \blacksquare & \bullet \\ \bullet & \bullet \end{smallmatrix} \sim_{jd} \begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \bullet \end{smallmatrix}$ follows from our proof of Theorem 2.3 of the joint equidistribution of $\begin{smallmatrix} \bullet & \bullet \\ \bullet & \blacksquare \end{smallmatrix} \sim_{jd} \begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \bullet \end{smallmatrix}$ applied to each A_i (the permuted elements in A_i collectively occupy the same positions and have the same values, so conflict like creating or removing an extra occurrence can occur with the shaded box (2,2) giving restrictions outside of A_i). \square

To illustrate the proof of Theorem 3.4, see the example in Figure 5 giving the permutation $\pi = (10)47986152$ with $x_1 = 4, x_2 = 1$. We can see that $A_1 = 7986$ and $A_2 = 23$ in π are replaced, respectively, by $A'_1 = 9867$ and $A'_2 = 32$ in π' , by doing the same procedure as Theorem 2.3. Note that the occurrences 479, 478 and 123 of $\begin{smallmatrix} \blacksquare & \bullet \\ \bullet & \bullet \end{smallmatrix}$ in π are replaced, respectively, by the occurrences 498, 486 and 132 of $\begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \bullet \end{smallmatrix}$ in π' , while the occurrence 476 of $\begin{smallmatrix} \bullet & \bullet \\ \bullet & \blacksquare \end{smallmatrix}$ in π is replaced by the occurrence 467 of $\begin{smallmatrix} \bullet & \bullet \\ \blacksquare & \bullet \end{smallmatrix}$ in π' .

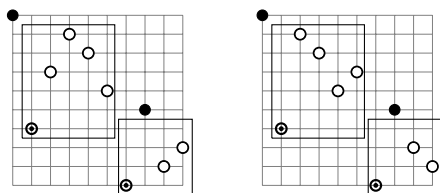


Figure 5: Permutations $\pi = (10)479861523$ (to the left) and $\pi' = (10)498671532$ (to the right) illustrating the proof of Theorem 3.4. The circled dots represent the first elements of occurrences of the patterns (certain left-to-right minima in π), and the white dots represent the second and third elements of occurrences of one of the two patterns.

Theorem 3.5 (Pair 32). *We have $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$.*

Proof. The differences from the proof of Theorem 3.1 are that now

- x_i does not have to be the only element in π in the South-West direction relative to A_i , allowing multiple left-to-right minima to potentially be the starting points of occurrences of the patterns for elements within the same A_i ;
- $A_i = X_i$; specifically, A_i 's can overlap vertically, and not all elements within an A_i need to be involved in occurrences of the patterns (some A_i 's, even those of length greater than 2, might not contribute any occurrences of the patterns). However, the second and third elements in an occurrence of a pattern cannot be in different A_i 's (as is the case in the proof of Theorem 3.1).

Refer to Figure 6 for a schematic structure of a permutation π . The observations above do not affect our arguments in the proof of Theorem 3.1, since complementation of A_i 's swaps occurrences of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ and $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$, and does not introduce any new occurrences of these patterns. \square

In the following theorem, we once more utilize the fact that the shading in the North-East area, relative to the minimum element in a pattern, exhibits symmetry with respect to a horizontal line.

Theorem 3.6 (Pairs 33–38, 40). *We have $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$, $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$, $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$, $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$, $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$.*

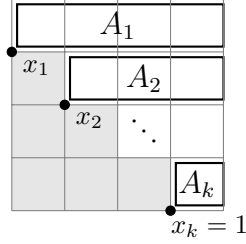


Figure 6: The structure of permutations in the proof of Theorem 3.5.

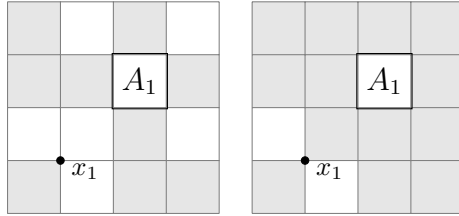


Figure 7: Permutations in the proof of Theorem 3.7 corresponding to the respective patterns. A_1 is of length at least 2.

Proof. Our arguments here are analogous to those in Theorem 3.2, but the structure of permutations corresponds to the one considered in Theorem 3.5 and depicted in Figure 6. \square

Shading additional boxes $(0, 3)$ and $(3, 0)$ in the patterns from Theorem 3.6, we observe through computer experiments that $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \not\sim_{jd} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$, $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \not\sim_{jd} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$, and $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \not\sim_{jd} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$. Additionally, $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \not\sim_{jd} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$. As mentioned above, $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$ and $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$ (resp., $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$ and $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$) are equidistributed for $n \leq 7$, but not for $n = 8$. However, other patterns in Theorem 3.6 do have joint equidistribution, as stated in the next theorem. Note that a similar argument as that for pair 10 can be applied to prove $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$. Finally, we refer to Remark 3.8 explaining why the complement does not work for all patterns in Theorem 3.6 when boxes $(0,3)$ and $(3,0)$ are shaded.

Theorem 3.7 (Pairs 41–43, 45). *We have $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$, $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$, $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$, and $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$.*

Proof. Our arguments here are analogous to those in Theorem 3.6, but in this case we have at most one A_i , as depicted in Figure 7: the picture to

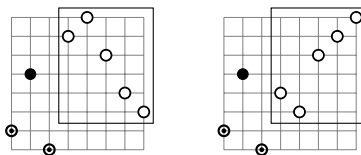


Figure 8: Permutations π (to the left) and π' (to the right) supporting Remark 3.8. The circled dots represent the left-to-right minima, and the white dots represent the elements in A_1 .

the left (resp., to the right) corresponds to pairs 41 and 42 (resp., 43 and 45). In both pictures, we assume x_1 to be the largest left-to-right minima starting occurrences of the patterns. Note that we need to unshade the box adjacent to x_1 in its bottom-right corner in the leftmost picture because the box (3,0) in the patterns is shaded. Thus, the unshaded area can contain left-to-right minima not involved in occurrences of the patterns. Recall that if A_1 has length at most 1, such a permutation will have no occurrences of the patterns and is mapped to itself. Otherwise, by complementing A_1 (not X_1 as is done previously!), we obtain the desired result. \square

Remark 3.8. *It is not entirely clear why the complement does not work for all patterns in Theorem 3.6 when boxes (0,3) and (3,0) are shaded. The reason for that is rather subtle and lies in the fact that, in the structures in Figure 7, A_1 must be above (and to the right) of the only unshaded square in the first column for the complement to work. Otherwise, occurrences of the patterns may become non-occurrences. For example, for the pair $\begin{array}{cc} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \end{array}$, a counterexample of length $n = 8$ demonstrating this phenomenon is the permutation $\pi = 25178643$ mapped to the permutation $\pi' = 25143678$; see Figure 8 for the respective permutation diagrams. Here, $A_1 = 78643$ and the total number of occurrences of the patterns beginning with $x_1 = 2$ (and defining A_1) is preserved. However, π has three occurrences of the patterns beginning with $x_2 = 1$, namely, 178, 176 and 186, while π' has two occurrences of the patterns beginning with $x_2 = 1$, namely, 167 and 178.*

Theorem 3.9 (Pairs 39, 44). *We have $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ and $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$.*

Proof. Considering the structure of permutations in Theorem 3.5, we observe that for the given patterns, if ab is the second and third elements in an occurrence of a pattern in a permutation $\pi = \pi_1\pi_2 \dots \pi_n$, then $ab = \pi_i\pi_{i+1} =$

nr.	patterns	nr.	patterns	nr.	patterns	nr.	patterns
46		47		48		49	
50		51		8		6	
52		53		54		55	
56		57		58		59	
60		61		62		63	
64		65		66		67	
68		69		70		20	
71		72		73		22	
74		75					

Table 5: Joint equidistributions explainable based on the proof of Theorem 4.1. Pairs 6, 8, 20 and 22 have been added to the table for completeness. Pairs 30 (resp., 31) and 46 (resp., 47) have the same joint distributions by Theorem 4.5.

$j(j+1)$ for some i and j , and j and $j+1$ are two smallest elements in their A_i (in particular, there is only A_1 for the second pair of patterns), so that each A_i has at most one occurrence of a pattern, and neither the second nor the third elements in an occurrence can be part of another occurrence. The involution that replaces occurrences of the patterns is then given by swapping ab by ba for every occurrence. \square

4 Cases based on the proof of Theorem 4.1

In this section, we explain joint equidistributions of pairs of patterns in Table 5.

Theorem 4.1 (Pair 46). *We have $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix} \sim_{jd} \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$.*

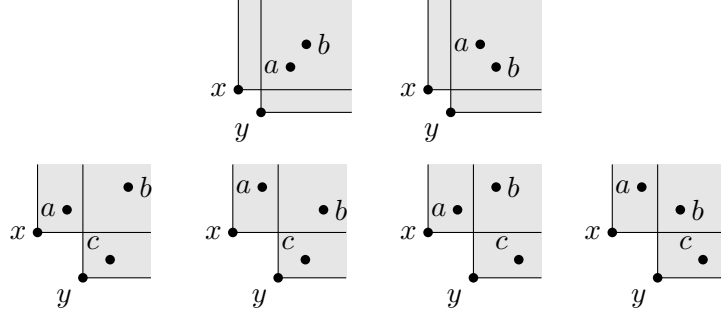


Figure 9: Observations (ii) and (iii) in the proof of Theorem 4.1.

Proof. Let $\pi = \pi_1 \dots \pi_n \in S_n$, $p_1 = \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$ and $p_2 = \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$. Suppose xab is an occurrence of p_1 or p_2 in π . We make the following observations.

- (i) No other occurrence of p_1 or p_2 can begin with x (otherwise, xab cannot be an occurrence of p_1 or p_2 because of some elements in π in a shaded area).
- (ii) If $x \neq y$ and yac is another occurrence of p_1 or p_2 then $b = c$ (which is illustrated schematically on the first line in Figure 9). Indeed, suppose $b \neq c$ and y is to the right of x (the case when y is to the left of x can be considered similarly). If $y > x$ then xab cannot be an occurrence of p_1 or p_2 since in this case, the element y is in a shaded area. Hence $y < x$, but then yac is not an occurrence of p_1 or p_2 because the element b is in a shaded area. This contradiction shows that $b = c$.
- (iii) If $x > y$, $a \neq c$, and ycb (resp., ybc) is an occurrence of p_1 or p_2 then the only two possibilities are the leftmost (resp., rightmost) two drawings on the second line in Figure 9. Indeed, element a must be between elements x and y or else ycb is not an occurrence because of a . Also, we must have $x > c$ or else xab is not an occurrence because of c .

Assume that π has x occurrences of p_1 and y occurrences of p_2 . Consider all occurrences of p_1 and p_2 in π and let $x_1 > x_2 > \dots > x_k$ be the first elements in these occurrences (note that $x + y = k$). By observations (i)–(iii), to the right of each x_i there are exactly two elements in π larger than x_i . We subdivide x_i 's into blocks as follows: x_i and x_{i+1} belong to the same block if occurrences of p_1 or p_2 starting at x_i and x_{i+1} share at least one element.

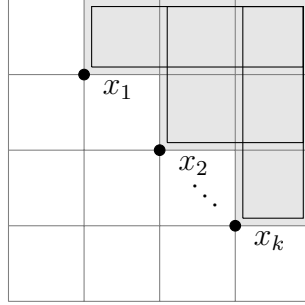


Figure 10: The structure of permutations in the proof of Theorem 4.1.

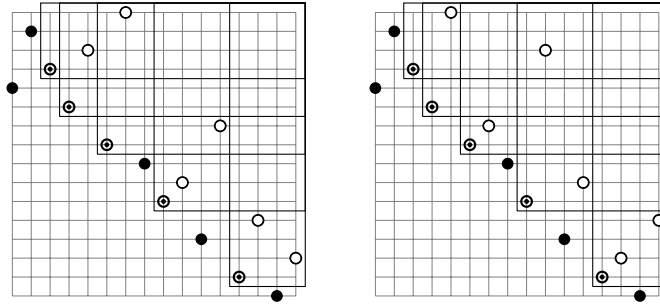


Figure 11: Permutations $\pi = (12)(15)(13)(11)(14)9(16)8674(10)2513$ (to the left) and $\pi' = (12)(15)(13)(11)(16)9(10)86(14)4(7)2315$ (to the right) illustrating the proof of Theorem 4.1. The circled dots represent the first elements of occurrences of p_1 or p_2 (the elements in X_1 and X_2), and the white dots represent $\pi_{i_1} \dots \pi_{i_s}$ for X_1 and X_2 .

Let $X_i = \{x_{m_{i-1}+1}, x_{m_{i-1}+2}, \dots, x_{m_i}\}$ be the i -th block, where $1 \leq i \leq t$, $m_0 = 0$, $t \geq 1$, and $m_1 < m_2 < \dots < m_t$. For example, for the permutation π in Figure 11, $t = 2$, $m_1 = 6$, $m_2 = 2$, $X_1 = \{6, 9, 11, 13\}$, and $X_2 = \{2\}$.

We will next describe a procedure of permuting the second and third elements, forming a subsequence $\pi_{i_1} \dots \pi_{i_s}$ of π , in occurrences of p_1 and p_2 that begin with the elements in block X_i , $1 \leq i \leq t$ (the subsequence $\pi_{i_1} \dots \pi_{i_s}$ of π is between $x_{m_{i-1}+1}$ and x_{m_i+1} in π ; $x_t + 1$ is defined as the right end of π). All of the π_i 's are in the shaded area in the schematic representation of π in Figure 10. The resulting permutation will be placed in π in positions i_1, \dots, i_s . We say that the procedure has been applied to block X_i . For example, for the permutation π in Figure 11, $\pi_{i_1} \dots \pi_{i_s}$ for X_1 is $\pi_5 \pi_7 \pi_{10} \pi_{12} = (14)(16)7(10)$, and for X_2 , it is $\pi_{14} \pi_{16} = 53$. Applying the procedure to each block will result, in a bijective manner, in a permutation

π' that has y occurrences of p_1 and x occurrences of p_2 .

Let $\pi_{i_1}^{(1)} \dots \pi_{i_s}^{(1)} = \pi_{i_1} \dots \pi_{i_s}$. First, suppose $\pi_{i_a}^{(1)}$ is the largest element among $\pi_{i_2}^{(1)} \dots \pi_{i_s}^{(1)}$. Note that $x_i \pi_{i_1}^{(1)} \pi_{i_a}^{(1)}$ is an occurrence of p_1 or p_2 for $m_{i-1} + 1 \leq i \leq j_1 \leq m_i$ for some j_1 . Let $\pi_{i_1}^{(2)} \dots \pi_{i_s}^{(2)}$ be the permutation obtained from $\pi_{i_1}^{(1)} \dots \pi_{i_s}^{(1)}$ by swapping $\pi_{i_1}^{(1)}$ and $\pi_{i_a}^{(1)}$. Note that this move swaps occurrences of p_1 and p_2 in π starting at x_i for $m_{i-1} + 1 \leq i \leq j_1 \leq m_i$ and does not affect any other occurrences of p_1 and p_2 (i.e., no new occurrences are introduced, and no other occurrences are changed). We let $\pi^{(2)}$ be the permutation obtained from π by the swap.

Next, suppose $\pi_{i_b}^{(2)}$ is the largest element among $\pi_{i_3}^{(2)} \dots \pi_{i_s}^{(2)}$. Note that $x_i \pi_{i_2}^{(2)} \pi_{i_b}^{(2)}$ is an occurrence of p_1 or p_2 for $j_1 + 1 \leq i \leq j_2 \leq m_i$ for some j_2 . Let $\pi_{i_1}^{(3)} \dots \pi_{i_s}^{(3)}$ be the permutation obtained from $\pi_{i_1}^{(2)} \dots \pi_{i_s}^{(2)}$ by swapping $\pi_{i_2}^{(2)}$ and $\pi_{i_b}^{(2)}$. Note that this move swaps occurrences of p_1 and p_2 in $\pi^{(2)}$ starting at x_i for $j_1 + 1 \leq i \leq j_2$ and does not affect any other occurrences of p_1 and p_2 . We let $\pi^{(3)}$ be the permutation obtained from $\pi^{(2)}$ by the swap.

And so on. Continuing in this way, eventually we arrive to swapping the elements $\pi_{i_{s-1}}^{(s-1)}$ and $\pi_{i_s}^{(s-1)}$ and obtaining $\pi_{i_1}^{(s)} \dots \pi_{i_s}^{(s)}$, which completes swapping all occurrences of p_1 and p_2 in π starting at X_i . Recall that π' is obtained by applying the procedure to each block X_i , $1 \leq i \leq t$.

For example, for the permutation π in Figure 11, for X_2 we have a single swap of the elements 3 and 5, while for X_1 ,

$$\begin{aligned} \pi_{i_1}^{(1)} \pi_{i_2}^{(1)} \pi_{i_3}^{(1)} \pi_{i_4}^{(1)} &= \pi_5^{(1)} \pi_7^{(1)} \pi_{10}^{(1)} \pi_{12}^{(1)} = (14)(16)7(10) \xrightarrow{\text{swap 14 \& 16}} \\ \pi_{i_1}^{(2)} \pi_{i_2}^{(2)} \pi_{i_3}^{(2)} \pi_{i_4}^{(2)} &= \pi_5^{(2)} \pi_7^{(2)} \pi_{10}^{(2)} \pi_{12}^{(2)} = (16)(14)7(10) \xrightarrow{\text{swap 10 \& 14}} \\ \pi_{i_1}^{(3)} \pi_{i_2}^{(3)} \pi_{i_3}^{(3)} \pi_{i_4}^{(3)} &= \pi_5^{(3)} \pi_7^{(3)} \pi_{10}^{(3)} \pi_{12}^{(3)} = (16)(10)7(14) \xrightarrow{\text{swap 7 \& 14}} \\ \pi_{i_1}^{(4)} \pi_{i_2}^{(4)} \pi_{i_3}^{(4)} \pi_{i_4}^{(4)} &= \pi_5^{(4)} \pi_7^{(4)} \pi_{10}^{(4)} \pi_{12}^{(4)} = (16)(10)(14)7. \end{aligned}$$

Note that the occurrence (13)(14)(16), (11)(14)(16) and 67(10) of p_1 in π are replaced, respectively, by the occurrence (13)(16)(14), (11)(16)(14), 6(14)(7) of p_2 in π' , while the occurrences 9(16)(10) and 253 of p_2 in π are replaced, respectively, by the occurrences 9(10)(14), 235 of p_1 in π' ; no new occurrences of the patterns are introduced in π' .

Remark 4.2. *Note that no element in $\pi_{i_1} \dots \pi_{i_s}$ in π will be in the same position in π' . Indeed, at each step in our procedure, an element π_{i_j} either moves to the left and then moves nowhere else, or it moves to the right, and if it moves again, it moves to a position $> i_j$.*

It remains to explain why the procedure described above is a bijection. Clearly, the steps are reversible (instead of conducting swaps in $\pi_{i_1} \dots \pi_{i_s}$ from left to right, we do it from right to left). Now, for injectivity, suppose $\pi = \pi_1 \dots \pi_n$ and $\sigma = \sigma_1 \dots \sigma_n$ are two different permutations, so that $\pi_i \neq \sigma_i$ for some i , and we assume i is the smallest such index, but $\pi' = \sigma'$, so that $\pi'_i = \sigma'_i$. The procedure only changes the places of elements in $\pi_{i_1} \dots \pi_{i_s}$ and $\sigma_{k_1} \dots \sigma_{k_\ell}$. Hence, if π_i is not in $\pi_{i_1} \dots \pi_{i_s}$ or σ_i is not in $\sigma_{k_1} \dots \sigma_{k_\ell}$ then $\pi'_i = \sigma'_i \in \{\pi_i, \sigma_i\}$. It is impossible because either π_i and σ_i are not changed in their original position, or one of them would be involved as a second or third element in an occurrence of p_1 or p_2 in $\pi' = \sigma'$, while the other one would not.

Therefore, the remaining case to consider is when π_i is in $\pi_{i_1} \dots \pi_{i_s}$ and σ_i is in $\sigma_{k_1} \dots \sigma_{k_\ell}$. Because $\pi' = \sigma'$, $\{i_1, \dots, i_s\} = \{k_1, \dots, k_\ell\}$, in particular, $s = \ell$, and $\pi_{i_1} \dots \pi_{i_s}$ and $\sigma_{i_1} \dots \sigma_{i_s}$ correspond to the same block X_i . Let $x = \pi_i$, $y = \sigma_i$, $z = \pi'_i = \sigma'_i$, and assume, w.l.o.g., that $y > x$. By Remark 4.2, $z \notin \{x, y\}$. Because we assumed that $\pi_j = \sigma_j$ for $j < i$, we have that y is to the right of x in π , and x is to the right of y in σ . One can see that z must be in position $< i$ because otherwise it cannot be in position i in $\pi' = \sigma'$. Clearly, $z > x$, and we consider two subcases.

- Suppose $x < z < y$. When implementing the process in σ , z and y will be swapped, but z will then move to the right again because x is still to the right of z (and maybe some other elements in $\sigma_{i_1} \dots \sigma_{i_s}$). This contradicts to $z = \sigma'_i$.
- Suppose $z > y$. Then $z = \pi_{i_1} = \sigma_{i_1}$ since otherwise z must be swapped with an element to its left. And because $y > x$ in π , xy must be the second and third elements in an occurrence of p_1 in π , which implies that yx must be the second and third elements in an occurrence of p_2 in σ because all elements to the left of y are equal. Therefore, y must be the largest element in $\sigma_{i_1} \dots \sigma_{i_s}$ because of the shaded area in the definition of an occurrence of the mesh pattern p_2 . Consequently, because $\sigma_i = z$, z will then move to the right again because x is still to the right of z (and maybe some other elements in $\sigma_{i_1} \dots \sigma_{i_s}$). This contradicts to $z = \sigma'_i$.

We showed that the procedure is bijective, which completes our proof of the theorem. \square

The following theorem is also true for any shading of the boxes (0,0), (0,1), (0,2), (1,0), and (2,0), whether symmetric or not, provided that the boxes (3,0) and (0,3) are unshaded and the other boxes are shaded. Note that this theorem offers alternative proofs for pairs 6 and 8. If the boxes (3,0) and (0,3) are shaded, along with the boxes shaded in Theorem 4.1, then only half of the patterns with additional symmetric shading of boxes are jointly equidistributed, and they are pairs 1, 3, 11, and 45.

Theorem 4.3 (Pairs 47–51, 8, 6). *We have* $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array},$
 $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}.$

Proof. Our arguments in the proof of Theorem 4.1 are independent from any (symmetric or not) shading of (0,0), (0,1), (0,2), (1,0), and (2,0). \square

Remark 4.4. *Note that the arguments in the proof of Theorem 4.3 do not hold if we shade the boxes (0,3) and (3,0). For example, consider the pair $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$, for which our experiments show that $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \not\sim_d \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$. Then, for instance, consider the permutation 34125 with two occurrence of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ and no occurrences of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$. Moving the elements results in the permutation 35142 with no occurrences of either of the patterns. In fact, even the first step of moving elements (resulting in the permutation 35124) already fails, as two occurrences of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ are replaced by a single occurrence of $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$.*

Theorem 4.5. *The patterns in the sets $\left\{ \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \right\}$ and $\left\{ \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \right\}$ have the same joint distribution.*

Proof. Applying the complement and reverse operations, we have:

$$\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \xrightarrow{r} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \xrightarrow{c} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \xrightarrow{r} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \xrightarrow{c} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$$

Hence, $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$ and $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$. The result now follows from Theorems 3.3, 3.4, 4.1 and 4.3. \square

Theorem 4.6 (Pair 52). *We have* $\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}.$

Proof. The arguments in the proof of Theorem 4.1 can be copied and pasted. Indeed, the first element x_i in an occurrence of a pattern cannot be the second or third element in another occurrence of a pattern. The only difference now is that each x_i can have more than two elements to its right that are larger than x_i , and also we can have occurrences of patterns x_iab and x_jab such that $x_i < x_j$ and $i < j$. This does not change the procedure for permuting the second and third elements in occurrences of the patterns, although at each step we might be swapping multiple occurrences of the patterns, not necessarily just one. \square

The following theorem is also true for any shading of the boxes (0,0), (0,1), (1,0) and (1,1), whether symmetric or not, provided the boxes (2,2), (2,3), (3,2), and (3,3) are shaded.

Theorem 4.7 (Pairs 53–59). *We have* $\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$
 $\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}, \text{ and } \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}.$

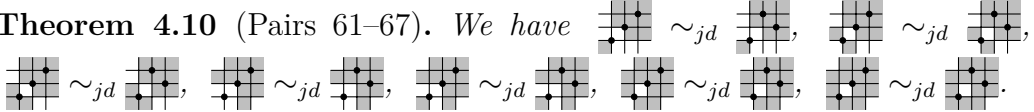
Proof. Our arguments in the proof of Theorem 4.6 are independent of (symmetric) shadings of the boxes (0,0), (0,1), (1,0), and (1,1). This is because the first element x_i in an occurrence of a pattern cannot be the second or third element in another occurrence, and the shaded areas do not affect the procedures described in Theorems 4.1 and 4.6. Hence, the result follows. \square

Theorem 4.8 (Pair 60). *We have* $\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \sim_{jd} \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}.$

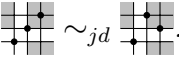
Proof. The arguments in the proof of Theorem 4.6 can be copied and pasted. The only difference is that if x_iab is an occurrence of a pattern in question then there are no elements greater than $\min\{a, b\}$ in positions between x_i and a , and there are no elements to the right of a , different from b , that are larger than x_i . This does not affect the procedure of moving the elements. \square

Remark 4.9. *Note that the arguments in the proof of Theorem 4.8 do not work for the pair* $\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$, *for which our experiments show that* $\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \not\sim_a \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$. *For example, consider the permutation 1324 with two occurrence of* $\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$ *and no occurrences of* $\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$. *Moving the elements results in the permutation 1432 with no occurrences of either of the patterns. In fact, even the first step of moving elements (resulting in the permutation 1423) already fails, as two occurrences of* $\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$ *are replaced by a single occurrence of* $\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$.

The following theorem is also true for any shading of the boxes (0,0), (0,1), (0,2), (1,0) and (2,0), whether symmetric or not, provided the shading in Theorem 4.8.

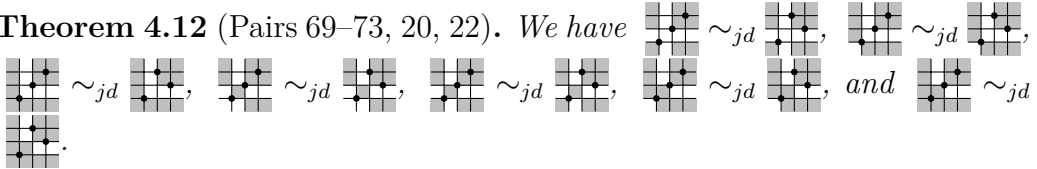
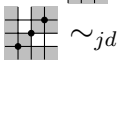
Theorem 4.10 (Pairs 61–67). *We have* .

Proof. Our arguments in the proof of Theorem 4.8 are independent of (symmetric) shadings of the boxes (0,0), (0,1), (0,2), (1,0) and (2,0). Hence, the result follows. \square

Theorem 4.11 (Pair 68). *We have* .

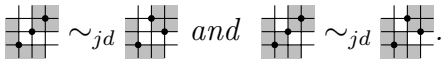
Proof. Our procedure for moving elements in the proof of Theorem 4.6 is not affected by shading the boxes (0,2), (0,3), (2,0), and (3,0). \square

The following theorem is also true for any shading of the boxes (0,0), (0,1), (1,0) and (1,1), whether symmetric or not, provided the shading in Theorem 4.11.

Theorem 4.12 (Pairs 69–73, 20, 22). *We have*  and .

Proof. Our arguments in the proof of Theorem 4.11 are independent of (symmetric) shadings of the boxes (0,0), (0,1), (1,0), and (1,1). We are done. \square

We conclude this section by proving joint equidistribution for two additional pairs of patterns, this time using both Theorems 3.1 and 4.1.

Theorem 4.13 (Pairs 74, 75). *We have* .

Proof. Referring to the notation in the proof of Theorem 3.1, each occurrence of any of the four patterns must contain the second and third elements within an A_i . Therefore, the procedure of permuting some elements, as explained in the proof of Theorem 4.1, will be applied independently within each A_i . All arguments will hold regardless of whether the box (1,1) is shaded or not. \square

nr.	patterns	nr.	patterns	nr.	patterns	nr.	patterns
76		77		78		79	
80		81		82		83	
84		85		86		87	
88		89		90		91	
92		93					

Table 6: Conjectured equidistributions.

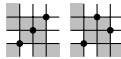
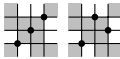
5 Concluding remarks

In this paper, we proved 75 joint equidistributions for mesh patterns 123 and 132 out of a maximum of 93 potential equidistributions, where shading is symmetric relative to the diagonal. The cases we did not solve are presented in Table 6. A few comments on these cases are in order.

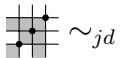

The pairs of patterns and appear to have the same joint distributions. Proving the equidistribution for the second pair is equivalent to proving Kitaev and Zhang's conjecture [14] on equidistribution of patterns nr. 57 and 58, respectively, and , which was reposted in [10]. Indeed, the equivalence of the conjectures follows from reducing to the patterns of length 2 and applying the complement operation.

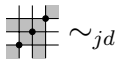
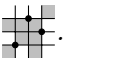
Also, we encountered issues while trying to explain the equidistributions for the pairs and using our approaches. For the former pair, consider the permutation $\pi = 265143$ for which A_1 is given by 6543, in the reduced form 4321. Under the bijection in Theorem 2.4, 4321 (the only permutation of length 4 with the maximum number of occurrences of) must go to 1432 (the only permutation of length 4 with the maximum number of occurrences of) and hence, π must go to $\pi' = 236154$ if we want to apply the reduction to the smaller case for A_1 . But π has three occurrences of (265, 243, 143), while π' has just one occurrence of (236), so our approach does not work in its present form.

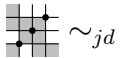

Note that the arguments used in the proof of Theorem 3.5 are not ap-

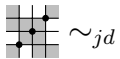

plicable for proving equidistribution for the pairs  and  as Theorem 2.3 and Theorem 2.4 are not particularly useful in this context. Indeed, for the permutation $\pi = 36178254$, we have $A = 7854 \rightarrow 7845 \rightarrow 4875 \rightarrow 8475$, resulting in $\pi' = 36184275$. While π contains occurrences 178 and 354 of the patterns with either of the two shadings, π' does not contain any occurrences of these patterns.

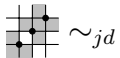
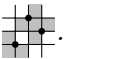
Finally, computer experiments suggest that in Table 6, pairs 83, 84, 86, 87, and 92 appear to have joint equidistributions, while the remaining pairs do not. Thus, we propose the following conjectures, which strengthen the corresponding conjectures in Table 6.

Conjecture 5.1 (Pair 83). *We have  \sim_{jd} .*

Conjecture 5.2 (Pair 84). *We have  \sim_{jd} .*

Conjecture 5.3 (Pair 86). *We have  \sim_{jd} .*

Conjecture 5.4 (Pair 87). *We have  \sim_{jd} .*

Conjecture 5.5 (Pair 92). *We have  \sim_{jd} .*

Even though our focus in this paper has been on symmetric shadings, it follows from the proofs of Theorems 4.1 and 4.7, where the symmetry of additional shadings is irrelevant, that the pairs of mesh patterns listed in Table 7 are jointly equidistributed. Interestingly, the patterns in the set $\left\{ \begin{array}{c} \text{---} \\ | \text{---} \\ | \text{---} \\ \text{---} \end{array} \right\}$ appear to have the same joint distribution. Why is this the case?

Acknowledgement. The authors are grateful to Philip B. Zhang for sharing with us the results of computer experiments related to the (joint) equidistributions of mesh patterns in question.

References

- [1] S. Avgustinovich, S. Kitaev, and A. Valyuzhenich. Avoidance of boxed mesh patterns on permutations, *Discrete Appl. Math.*, **161** (2013), 43–41.

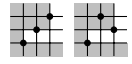
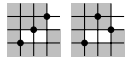
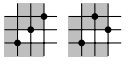
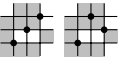
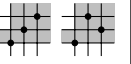
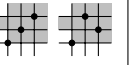
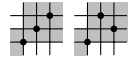
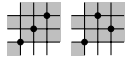
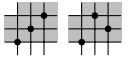
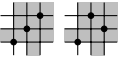
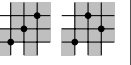
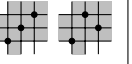
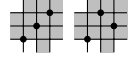
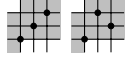
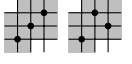
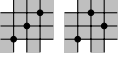
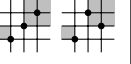

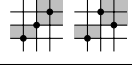
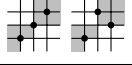
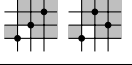
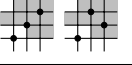
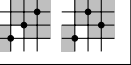
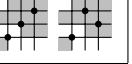
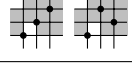
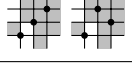
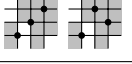
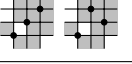
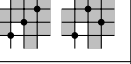
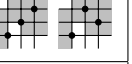
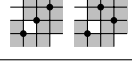
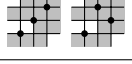
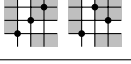
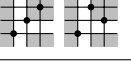
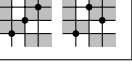
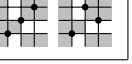
					
					
					
					
					
					

Table 7: Non-symmetric joint equidistributions, up to reflection with respect to the diagonal, obtained as a by-product in this paper. The first four joint equidistributions in the top row are obtained by applying the complement to the patterns in Theorem 2.3 and Theorem 2.4. The other joint equidistributions are obtained from the proofs of Theorems 4.1, 4.6, 4.8 and 4.11.

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