On singular behaviour in a plane linear elastostatics problem

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Dedicated to Professor Marcelo Epstein on the 80th anniversary of his birthday.

Abstract

A vector field similar to those separately introduced by Artstein and Dafermos is constructed from the tangent to a monotone increasing one-parameter family of non-concentric circles that touch at the common point of intersection taken as the origin. The circles define and space-fill a lens shaped region Ω whose outer and inner boundaries are the greatest and least circles. The double cusp at the origin creates a geometric singularity at which the vector field is indeterminate and has non-unique limiting behaviour. A semi-inverse method that involves the Airy stress function then shows that the vector field corresponds to the displacement vector field for a linear plane compressible non-homogeneous isotropic elastostatic equilibrium problem in Ω whose boundaries are rigidly rotated relative to each other, possibly causing rupture or tearing at the origin. A sequence of solutions is found for which not only are the Lamé parameters strongly-elliptic, but the non-unique limiting behaviour of the displacement is preserved. Other properties of the vector field are also established.

Keywords: Singular behaviour. Compressible nonhomogeneous isotropic elastostatics. Semi-inverse method. Airy stress function. Lamé parameters.

1 Introduction

This paper continues the investigation relevant to continuum mechanics of a vector field $u(x,t)$ introduced separately by Z. Artstein [1] and C. M. Dafermos [2]. The vector field, defined on some bounded region $\Omega \subset \mathbb{R}^2$, has components

$$u_1(x_1, x_2, t) = x_2, \quad (1.1)$$

$$u_2(x_1, x_2, t) = \frac{(x_2^2 - x_1^2)}{2x_1}, \quad (1.2)$$

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with respect to Cartesian coordinates $(x_1,\, x_2)$ and time variable t.

Artstein considers a variant of (1.1) and (1.2) appropriate to a control system for which a smooth stabilizing control does not exist. Independently, Dafermos used (1.1) and (1.2) with $u_1 = \dot{x}_1$, $u_2 = \dot{x}_2$ as an example of an initial value problem for an evolutionary ordinary differential equation that possesses a non-unique non-smooth solution. Uniqueness is recovered on application of an "entropy rate" admissibility criterion. The present authors [3] extend the study of (1.1) and (1.2) by proving that the vector field $u(x,t)$ forms the Lagrangian trajectories of fluid particles in steady compressible flow. The trajectories can be shown non-unique. The entropy rate criterion is again employed to select a unique trajectory, demonstrating the effectiveness of the criterion,

In this paper, the vector field $u(x)$, supposed time independent, is derived in a manner different to those of either Artstein or Dafermos. As such, it is capable of obvious generalization. The vector field is taken parallel to the tangent at each point of a circle belonging to a monotone increasing one-parameter space-filling family whose members are contiguous at the coordinate origin. This leads immediately not only to the expressions (1.1), (1.2) for the components u_1 and u_2 of u, but also as shown by (1.2) to the indeterminacy of u_2 at the origin. Moreover, the region of definition Ω is that enclosed by circles of the family of least and greatest radii which necessarily creates a double cusp at the origin. An important aim, therefore, of the paper is to explain this singular behaviour by seeking an interpretation in the specific context of a boundary value problem of plane linear elastostatics.

Before doing so, however, it is useful to establish further properties of the vector field u. Thus it is shown that the vector field has constant magnitude c at each point of a member circle of radius c, that the limit of the component u_2 as the origin is approached along circles is ambiguous, and that the inner and outer boundary circles of Ω are rigidly rotated with respect to each other.

The last result provides the boundary conditions for the elastic problem, in which the vector field is assumed to be the prescribed displacement field. However, the elastic problem has still to be defined for which (1.1) and (1.2) are the components of a displacement that corresponds to a stress distribution in equilibrium under zero body force. It is easily checked that the elastic body connot be homogeneous. When further restricted to be linear, isotropic and compressible, both the stress and material parameters are obtained by a semi-inverse method that involves the Airy stress function. The twisting of the inner and outer boundaries, and the singular behaviour at the origin, may then be interpreted as due to tearing, rupture, or brittle fracture. The conjecture is supported by calculating the force and couple acting on Ω together with the strain energy.

Section 2 introduces notation and reviews selected aspects of plane elastostatics. Section 3 derives the components of the vector fields tangential to non-concentric circles belonging to a monotone increasing one-parameter family whose members all touch at the common point of intersection taken to be the origin. Section 4 describes the region of definition Ω and defines the vector field $u(x)$ of current interest as that parallel to the tangent vector field introduced in Section 3. Section 4 also explores further properties of $u(x)$. In particular, at the origin, where the circles intersect and have a common tangent, the second component (1.2) of the vector, besides being indeterminate, exhibits non-unique limiting behaviour. Section 5 employs a semi-inverse method and Airy's stress function to prove that the vector field is the vector displacement field appropriate to displacement boundary value problems on Ω for a nonhomogeneous compressible isotropic plane elastic body in equilibrium subject to zero body force and rigidly rotated boundaries. Conditions are derived for the associated Lamé parameters to be strongly-elliptic. Increased understanding of the singular behaviour at the origin of the displacement is sought in Sections 6 and 7. Section 6 calculates that the total traction acting on Ω is zero whereas the total couple is proportional to the difference between the squares of the outer and inner circles. Section 7 is devoted to calculating the corresponding total strain energy which under certain conditions does not have a singularity while in others the singularity varies inversely as the distance from the origin. Section 8 presents certain directions in which the analysis might be extended and lists some open problems. An Appendix derives the Airy stress function used in Section 5 and proposes a generalisation.

Vector and scalar quantities are not typograhically distinguished although subscripts denote the components of a vector. The subscript comma and summation conventions are adopted throughout with Latin suffixes ranging over $1, 2, 3$ while Greek suffixes have the range $1, 2$. Since the vector field u is prescribed it immediately follows that the solution to the elastic boundary value problem always exists.

2 Notation, basic theory and other preliminaries

Several properties of the vector field leading to (1.1) and (1.2) do not depend upon the relation to elasticity or even continuum mechanics. Later sections, however, require the vector field to be interpreted as the displacement vector occurring in linear plane elasticity. Consequently, this Section, apart from notation, introduces notions from elasticity theory in preparation for the subsequent discussion. The reader is referred to e.g., [5, 8, 9] and other standard texts for complete accounts.

Let $\Omega \subset \mathbb{R}^2$ denote a bounded simply or multiply connected plane region and let (x_1, x_2) be the Cartesian coordinates of a point $x \in \Omega$. Suppose Ω is occupied by material that deforms to the region $\widehat{\Omega}$ and that the particle at x is displaced to the position $y \in \widehat{\Omega}$. The map $\Omega \to \widehat{\Omega}$, represented by $y = y(x)$, is supposed time independent and sufficiently differentiable. The displacement vector field $u(x)$ (c.p., [5, 8]) is defined as

$$u(x) = y(x) - x,$$

so that $u(x)$ as a function of x completely determines $y(x)$. Recall that

$$du_i = u_{i,j} dx_j.$$

Linear theories assume that the displacement gradient is small and that to first order the Cartesian components of the strain are expressed by

$$e_{ij} = (1/2)\left(u_{i,j} + u_{j,i}\right). \tag{2.1}$$

While the vector field $u(x)$ is not necessarily correspondingly small, the change δV in the infinitesimal volume element at a given point may be shown to be

$$\delta V = e_{ii},$$

which may be either positive (extension), negative (compression) or zero (incompressible).

For a compressible linear isotropic plane elastic body, the symmetric stress components satisfy the constitutive relations

$$\sigma_{\alpha\beta} = \lambda e_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}, \tag{2.2}$$

where α, $\beta = 1,\, 2$, the Kronecker delta function is denoted by $\delta_{\alpha\beta}$, and $\lambda(x)$, $\mu(x)$ are the Lamé parameters related to Poisson's ratio $\nu(x)$ by

$$\lambda = \frac{2\mu\nu}{(1-2\nu)}, \qquad \nu \neq 1/2, \tag{2.3}$$

or

$$2\nu = \frac{\lambda}{(\lambda+\mu)}, \qquad \nu \neq 1/2. \tag{2.4}$$

The following relations are easily deduced from (2.2):

$$\begin{aligned} 2\mu &= \frac{\sigma_{12}}{e_{12}} = \frac{(\sigma_{11}-\sigma_{22})}{(e_{11}-e_{22})}, && (2.5)\\ 2(\lambda+\mu) &= \frac{(\sigma_{11}+\sigma_{22})}{(e_{11}+e_{22})}, && (2.6)\\ (\lambda+2\mu) &= \frac{(\sigma_{11}e_{11}-\sigma_{22}e_{22})}{(e_{11}^2-e_{22}^2)}, && (2.7)\\ \lambda &= \frac{(\sigma_{22}e_{11}-\sigma_{11}e_{22})}{(e_{11}^2-e_{22}^2)}. && (2.8) \end{aligned}$$

From relations (2.5), we conclude that the parameter Λ, defined by

$$\Lambda(x_1,\, x_2) := \frac{(e_{11}-e_{22})}{e_{12}} = \frac{(\sigma_{11}-\sigma_{22})}{\sigma_{12}}, \tag{2.9}$$

is explicitly independent of the Lamé parameters in both nonhomogeneous and homogeneous elastic bodies.

The stress tensor in equilibrium under zero body force has zero divergence, and therefore satisfies

$$\sigma_{\alpha\beta,\beta} = 0. \tag{2.10}$$

A general solution to these equilibrium equations, irrespective of constitutive relations, may be expressed in terms of the Airy stress function $\chi(x_1,\, x_2)$ as

$$\sigma_{11} = -\chi_{,22}, \qquad \sigma_{12} = \chi_{,12}, \qquad \sigma_{22} = -\chi_{,11}. \tag{2.11}$$

For compressible non-homogeneous linear elasticity, substitution of (2.2) in the equilibrium equations (2.10) establishes that the vector field $u(x)$ at place x satisfies the Navier equations

$$\mu u_{\alpha,\beta\beta} + (\lambda+\mu)u_{\beta,\beta\alpha} + \lambda_{,\alpha}u_{\beta,\beta} + \mu_{,\beta}(u_{\alpha,\beta}+u_{\beta,\alpha}) = 0, \qquad x \in \Omega, \tag{2.12}$$

which for homogeneous elasticity becomes

$$\mu u_{\alpha,\beta\beta} + (\lambda + \mu) u_{\beta,\beta\alpha} = 0, \qquad x \in \Omega. \tag{2.13}$$

For a bounded domain Ω with piecewise smooth boundary $\partial\Omega$ and subject to standard Dirichlet boundary conditions, (2.12) admits a unique classical or weak solution if the strong-ellipticity condition holds (cf., [6, pp. 19 and 62]);

$$\mu(\lambda + 2\mu) > 0. \tag{2.14}$$

This may be equivalently written as

$$-\infty < \nu < 1/2, \qquad 1 < \nu < +\infty, \qquad \mu \neq 0. \tag{2.15}$$

3 Unit tangent vector field

The vector field $u(x)$ whose components are (1.1) and (1.2) is obtained by assuming it to be parallel to a certain unit tangent field. The argument, motivated by [3, App. D], is different from that of either Artstein [1] or Dafermos [2].

Introduce a one parameter family of non-concentric circles centred at $(c, 0)$, of radius c, and denoted by

$$\psi_c(x) := c^{-2}\left\{(x_1 - c)^2 + x_2^2\right\} = 1 \tag{3.1}$$

where the constant c satisfies $1 \leq c \leq R$ for constant R. The circles touch at the origin $(0, 0)$ which is their common single point of intersection.

The unit tangent vector at the point x on the circle (3.1) has Cartesian components

$$t_1(x_1,\, x_2) = \frac{c}{2}\frac{\partial \psi_c}{\partial x_2} = x_2/c, \tag{3.2}$$

$$t_2(x_1,\, x_2) = -\frac{c}{2}\frac{\partial \psi_c}{\partial x_1} = (c - x_1)/c, \tag{3.3}$$

so that at the origin all curves have the common unit tangent whose components are given by $(t_1(0,0), t_2(0,0) = (0,1)$. Note that for each point on the circle $\psi_c = 1$ *except* the origin, appeal to (3.1) enables (3.3) to be alternatively written as

$$\begin{aligned} t_2(x_1,\, x_2) &= \frac{2x_1(c - x_1)}{2cx_1}, \qquad x_1 \neq 0, \\ &= \frac{(x_2^2 - x_1^2)}{2cx_1}. \end{aligned} \tag{3.4}$$

In the next section, the vector field u is derived from the tangent components (3.2) and (3.3). Certain properties are established irrespective of whether $u(x)$ is interpreted as a displacement vector field.

4 Properties of the vector field

We now specify the shape of the bounded region Ω. Let the outer and inner boundaries $\partial\Omega_\alpha$, $\alpha = 1, 2$ of Ω be determined by $\psi_R(x) = 1$ and $\psi_1(x) = 1$, where $R > 1$. Consequently, the outer boundary is the circle centred at $(R, 0)$ and of radius R while the inner boundary is the circle centred at $(1, 0)$ and of radius 1. The region Ω is lens shaped with symmetrically placed cusps at the origin which therefore create geometric point singularities. Observe that the circles belonging to (3.1) fill the whole space occupied by Ω. Each point $x \in \Omega$ belongs to one and only one member of the family (3.1).

The vector field u is taken parallel to the unit tangent vector t. Set $u = ct$ to obtain at the point $(x_1,\, x_2)$ of the curve (3.1), the expressions (1.1) and (1.2) which as shown in deriving (3.4) may be written, except at the origin, in the form

$$u_1(x_1,\, x_2) = x_2, \tag{4.1}$$

$$u_2(x_1,\, x_2) = \frac{(x_2^2 - x_1^2)}{2x_1} \tag{4.2}$$

$$= (c - x_1). \tag{4.3}$$

Components of the gradient when the vector field has components (4.1) and (4.2) are given by

$$u_{1,1} = 0, \qquad u_{1,2} = 1, \tag{4.4}$$

$$u_{2,1} = -\frac{(x_1^2 + x_2^2)}{2x_1^2}, \tag{4.5}$$

$$u_{2,2} = \frac{x_2}{x_1}, \tag{4.6}$$

while the symmetric components, which by construction are compatible, become

$$e_{11} = 0, \qquad e_{22} = \frac{x_2}{x_1}, \tag{4.7}$$

$$e_{12} = \frac{1}{4x_1^2}\left(x_1^2 - x_2^2\right). \tag{4.8}$$

For given c select the origin of the polar coordinate system (ρ, ϕ), $-\pi \leq \phi \leq \pi$ to be at the centre of the circle $\psi_c = 1$. A second polar coordinate system (r, θ), $-\pi/2 \leq \theta \leq \pi/2$ based on the origin $(0, 0)$ of the Cartesian system is related to the first by

$$x_1 = r\cos\theta = c + \rho\cos\phi, \tag{4.9}$$

$$x_2 = r\sin\theta = \rho\sin\phi. \tag{4.10}$$

Accordingly, the circle (3.1) may be written as

$$r = 2c\cos\theta, \tag{4.11}$$

while the polar coordinates (ρ, ϕ) of points on the circle satisfy

$$\rho = c, \qquad \phi = 2\theta, \tag{4.12}$$

and we have the relations

$$x_1 = r\cos\theta = 2c\cos^2\theta = c\cos 2\theta + c = c(1+\cos\phi), \tag{4.13}$$

$$x_2 = r\sin\theta = 2c\cos\theta\sin\theta = c\sin 2\theta = c\sin\phi. \tag{4.14}$$

In an obvious notation, the radial and tangential components of the vector field whose Cartesian components are (4.1) and (4.3) on using (4.11) and (4.12) in terms of polar coordinates (ρ, ϕ) are given by

$$\begin{aligned} u_\rho(c,\phi) &= u_1(x_1,x_2)\cos\phi + u_2(x_1,x_2)\sin\phi \\ &= [r\sin\theta\cos\phi + (c - r\cos\theta)\sin\phi] \\ &= [-r\sin(\phi-\theta) + c\sin\phi] \\ &= 0, \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} u_\phi(c,\phi) &= -u_1(x_1,x_2)\sin\phi + u_2(x_1,x_2)\cos\phi \\ &= [-r\sin\theta\sin\phi + (c - r\cos\theta)\cos\phi] \\ &= \left[-2c\cos^2\theta + c(2\cos^2\theta - 1)\right] \\ &= -c. \end{aligned} \tag{4.16}$$

We conclude that the vector u is of constant magnitude c and is tangential to the circle in the clockwise direction.

In a mechanical interpretation the boundary circles, for which $c = 1$ and $c = R$, rotate uniformly clockwise by different amounts $1, R$. The singular behaviour in the component (4.2) at the origin, where all members of the family (3.1) touch, perhaps corresponds to rupture or brittle fracture. The infinitesimal volume element in the neighbourhood of the origin is in tension when $x_2 > 0$ and in compression when $x_2 < 0$.

The relative rotation of the boundary circles may be alternatively derived directly from (4.1) and (4.2) on expressing the vector field as

$$u(x_1,x_2) = u_1e_1 + u_2e_2,$$

where $(e_1,\, e_2)$ are the base vectors in the Cartesian coordinate system whose origin is $(0,0)$. Substitution from (4.1) and (4.3) leads to

$$u(x_1,x_2) = ce_2 + (x_2e_1 - x_1e_2), \tag{4.17}$$

which shows that the vector field at each point on a circle belonging to (3.1) may be interpreted as a rigid body displacement consisting of a rigid body translation of amount c in the x_2-direction plus a rigid body unit rotation about the origin in a clockwise direction.

At the origin the vector field with components (4.1) and (4.3) is directed along the common tangent to the family of circles (3.1) i.e., in the direction parallel to $(0,1)$ but the magnitude is given by the parameter c and thus depends upon the particular circle along which the origin is approached. In this sense, the vector field is not uniquely defined at the origin. Indeed, we have proved the following result:

Proposition 4.1 (Limiting behaviour). *For points on a given circle* (3.1)*, the limit of the component* $u_2(x)$ *specified by* (4.3) *as the point* (x_1, x_2) *tends to the origin depends along which member of the family* (3.1) *the origin is approached; that is*

$$\lim_{x_1 \to 0} u_2(x_1, x_2) = c, \tag{4.18}$$

where the constant c *can be arbitrarily chosen in the range* $1 \leq c \leq R$*. Moreover, the component* $u_2(x)$ *when given by* (4.2) *is indeterminate at the origin and is not the limit of* $u_2(x)$ *defined on any sequence of points belonging to a circle of the family* (3.1)*.*

The origin may be regarded as the limit of points belonging, say, to the different circles $\psi_{c_1} = 1$ and $\psi_{c_2} = 1$. We determine the jump in the limit of $u_2(x)$. Consider a point belonging to $\psi_{c_2} = 1$ in the neighbourhood of the origin. Starting from this point and integrating around the separate circle $\psi_{c_1} = 1$ yields

$$\begin{aligned} [u_2] &= \lim_{x_2 \to 0-} u_2(x_1, x_2) - \lim_{x_2 \to 0+} u_2(x_1, x_2) \\ &= c_1 - c_2. \end{aligned} \tag{4.19}$$

We conclude that the jump in $u_2(x)$ in the limit as the origin is approached is ambiguous.

In general, differentiation of the vector field must be performed on the components (4.1) and (4.3) with the condition that the point (x_1, x_2) belongs to a particular member of the family (3.1) applied after differentiation.

Notwithstanding the last remark, observe that the vector field u with components given by (4.1) and either (4.2) or (4.3) is continuously differentiable at each point in Ω except at the origin where, as previously stated, it becomes indeterminate. By construction, at each point $x \in \Omega \backslash \{(0, 0)\}$ the vector field is directed along the tangent to the appropriate circle $\psi_c = 1$ and is of magnitude c. Intuitively, the vector field at all points on a given member of (3.1) is isochoric, when interpreted as a displacement.

The result is proved on converting to the polar coordinate sytem (r, θ) introduced in (4.9) and (4.10) for which the radial and transverse components to (3.1) of the vector field are

$$\begin{aligned} u_r &= u_1 \cos\theta + u_2 \sin\theta \\ &= r \sin\theta \cos\theta + (c - r\cos\theta) \sin\theta \\ &= c \sin\theta, \end{aligned} \tag{4.20}$$

$$\begin{aligned} u_\theta &= -u_1 \sin\theta + u_2 \cos\theta \\ &= -r \sin^2\theta + (c - r\cos\theta)\cos\theta \\ &= -r + c\cos\theta. \end{aligned} \tag{4.21}$$

The corresponding symmetric components of the gradient are given by

$$e_{rr} = u_{r,r}, \qquad e_{\theta\theta} = r^{-1}\left(u_{\theta,\theta} + u_r\right),$$

$$e_{r\theta} = (1/2)\left(u_{\theta,r} - r^{-1}u_\theta + r^{-1}u_{r,\theta}\right).$$

Insertion of (4.20) and (4.21) leads to the infinitesimal dilatation in the direction of a given circle (3.1) becoming

$$
\begin{aligned}
e_{rr} + e_{\theta\theta} &= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \\
&= 0 + \frac{c \sin\theta}{r} + \frac{1}{r}\left(-c\sin\theta\right) \\
&= 0.
\end{aligned}
$$

Therefore, the vector field when its gradient is infinitesimal may be interpreted as isochoric along each circle $\psi_c(x_1, x_2) = 1$ of the space-filling family. The next Section establishes that the vector field $u(x)$ is a displacement field in a linear plane elastostatics problem.

5 Airy stress function

We now prove that the vector field with Cartesian componenets (1.1) and (1.2) is the vector displacement that occurs in a linear non-homogeneous isotropic compressible plane elastic body in equilibrium under zero body-forces. The body must be non-homogeneous otherwise the Navier equations (2.13) are not satisfied without addition of suitable body forces. Nevertheless, subject to our assumptions, the stress has zero divergence and consequently may be expressed in terms of Airy's stress function $\chi(x_1, x_2)$. Substitution of (2.11) in (2.9) yields

$$
\chi_{,11} - \chi_{,22} - \Lambda(x_1, x_2)\chi_{,12} = 0, \qquad \Lambda := \frac{(e_{11} - e_{22})}{e_{12}}, \tag{5.1}
$$

which represents the partial differential equation for χ that is explicitly independent of the Lamé parameters.

Although equation (5.1) is hyperbolic and therefore amenable to the method of characteristics, a solution may also be derived on transforming to the complex variable $z = x_1 + ix_2$ and its conjugate $\bar{z} = x_1 - ix_2$ where $i = \sqrt{-1}$. The respective partial derivatives are defined as

$$
\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \tag{5.2}
$$

$$
\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \tag{5.3}
$$

and the linear strain components (4.7) and (4.8) are

$$
e_{11}(z, \bar{z}) = 0, \tag{5.4}
$$

$$
e_{22}(z, \bar{z}) = i\frac{(\bar{z} - z)}{(\bar{z} + z)}, \tag{5.5}
$$

$$
e_{12}(z, \bar{z}) = \frac{1}{2}\frac{(z^2 + \bar{z}^2)}{(z + \bar{z})^2}, \tag{5.6}
$$

while the parameter Λ from (5.1) is

$$
\Lambda(z, \bar{z}) = 2i\frac{(z^2 - \bar{z}^2)}{(z^2 + \bar{z}^2)}. \tag{5.7}
$$

The differential equation (5.1) after insertion of (5.7) and rearrangement reduces to

$$z^2\chi_{,zz} + \bar{z}^2\chi_{,\bar{z}\bar{z}} = 0. \tag{5.8}$$

Recall that a subscript comma denotes partial differentiation.

A solution to (5.8) is obtained in the particular class that satisfies

$$z^2\chi_{,zz}(z,\bar{z}) = J(z,\bar{z}) = -\bar{z}^2\chi_{,\bar{z}\bar{z}}, \tag{5.9}$$

where

$$J(z,\bar{z}) := z\bar{z} + k, \tag{5.10}$$

for given non-negative constant k. Observe that

$$J_{,zz}(z,\bar{z}) = J_{,\bar{z}\bar{z}}(z,\bar{z}) = 0. \tag{5.11}$$

Remark 5.1 (Generalisation)**.** *The class of solutions satisfying* (5.9) *may be extended when the definition of* $J(z,\bar{z})$ *is generalised to*

$$J(z,\bar{z}) = \sum_{j=1}^{m} \frac{(z\bar{z})^j}{j!} + k, \tag{5.12}$$

where m *is a positive integer and* k *is a prescribed non-negative constant.*

Integration of (5.9) with $J(z,\,\bar{z})$ given by (5.10) and (5.12) is presented in the Appendices. For (5.10), the solution is

$$\chi(z,\bar{z}) = (z\bar{z} - k)\log\frac{z}{\bar{z}}, \qquad z \neq 0, \tag{5.13}$$

which in the previously introduced polar coordinates (r,θ) becomes

$$\chi(x_1,\,x_2) = (r^2 - k)2i\theta. \tag{5.14}$$

The corresponding stress and strain components and Lamé parameters become

$$\sigma_{11} = -2\left[2\theta + \frac{(r^2+k)}{r^2}\sin 2\theta\right], \tag{5.15}$$

$$\sigma_{22} = 2\left[-2\theta + \frac{(r^2+k)}{r^2}\sin 2\theta\right], \tag{5.16}$$

$$\sigma_{12} = 2\frac{(r^2+k)}{r^2}\cos 2\theta, \tag{5.17}$$

$$e_{11} = 0, \tag{5.18}$$

$$e_{22} = \tan\theta, \tag{5.19}$$

$$e_{12} = \frac{\cos 2\theta}{4\cos^2\theta}, \tag{5.20}$$

$$\mu = 4\frac{(r^2+k)}{r^2}\cos^2\theta, \tag{5.21}$$

$$\lambda = -4\theta\cot\theta - 4\frac{(r^2+k)}{r^2}\cos^2\theta, \tag{5.22}$$

which are singular when $r = 0$, $\theta = \pm\pi/2$.

Formulae (5.21) and (5.22) imply $\mu > 0$ and

$$\lambda + 2\mu = -4\theta \cot\theta + 4\frac{(r^2 + k)}{r^2}\cos^2\theta. \tag{5.23}$$

Choose $k = 0$ to give $\mu > 0$ and

$$\begin{aligned} \lambda + 2\mu &= 4\cos\theta\left(\cos\theta - \frac{\theta}{\sin\theta}\right) \\ &< 0, \end{aligned} \tag{5.24}$$

because $\sin\theta < \theta$ for $-\pi/2 \leq \theta < \pi/2$.

Now select $k > 0$ and consider a point on the curve $\psi_c = 1$ for which we have $r = 2c\cos\theta$. The relation (5.23) may be written

$$\begin{aligned} \lambda + 2\mu &= 4\cos\theta\left(\cos\theta - \frac{\theta}{\sin\theta}\right) + 4\frac{k}{r^2}\cos^2\theta \\ &= 4\cos\theta\left(\cos\theta - \frac{\theta}{\sin\theta}\right) + \frac{k}{c^2} \\ &= 4\theta\cot\theta\left(\frac{\sin 2\theta}{2\theta} - 1\right) + \frac{k}{c^2}. \end{aligned} \tag{5.25}$$

But k is arbitrary and can be chosen sufficiently large to ensure that

$$\lambda + 2\mu > 0, \tag{5.26}$$

which is the strong ellipticity condition.

Remark 5.2. *The choice* (5.10) *together with its generalisations provide several examples in nonhomogeneous isotropic compressible plane elasticity for which Lamé parameters are strongly elliptic but for which the displacement possesses ambiguous limiting behaviour at the origin.*

To further examine the problem under consideration, we calculate the total force and couple acting on Ω together with the strain energy when J is given by (5.10).

6 Total force and couple

The total force and couple are obtained by the customary procedure of first deriving these quantities for the region Ω punctured by the deletion of a neighbourhood of the origin. Contracting to zero the diameter of this neighbourhood yields the desired results. The additional notation required is now introduced.

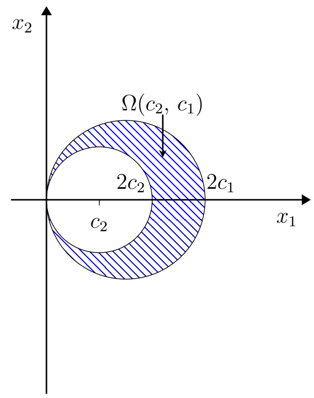

Figure 1: The region $\Omega(c_2,\, c_1)$.

6.1 Geometry

Let $\Omega(c_2,\, c_1)$, $1 \le c_2 < c_1 \le R$, be the region bounded internally by the circle $\psi_{c_2}(x_1, x_2) = 1$ and externally by $\psi_{c_1}(x_1, x_2) = 1$ where ψ_c is given by (3.1). See Figure 1. Let the bounding circles be denoted by $\partial\Omega_{c_\alpha}$, $\alpha = 1, 2$. Let $B(0, a)$, $a \ge 0$ denote the ball of radius a centred at the origin which is at the common point of the intersection of the family of circles $\psi_c(x_1, x_2) = 1$, $1 \le c \le R$. That is

$$B(0, a) := \left\{(x_1,\, x_2) : x_1^2 + x_2^2 \le a^2\right\}. \tag{6.1}$$

Define the punctured region $\Omega_a(c_2,\, c_1)$ by

$$\Omega_a(c_2,\, c_1) := \Omega(c_2,\, c_1) \setminus \{\Omega(c_2,\, c_1) \cap B(0, a)\}, \qquad a \ge 0, \tag{6.2}$$

The boundary of the punctured region consists of the curves

$$\begin{aligned} \partial\Omega_a^{c_\alpha} &:= \partial\Omega_{c_\alpha} \setminus \{\partial\Omega_{c_\alpha} \cap B(0, a)\}, \qquad \alpha = 1, 2, && (6.3)\\ \partial_{AD}B(0, a) &:= \{(x_1, x_2) : (x_1, x_2) \in \partial B(0, a) \cap \Omega(c_2, c_1), x_2 > 0\}, && (6.4)\\ \partial_{CB}B(0, a) &:= \{(x_1, x_2) : (x_1, x_2) \in \partial B(0, a) \cap \Omega(c_2, c_1), x_2 < 0\}, && (6.5) \end{aligned}$$

(Path integrals over $\partial\Omega_a^{c_2}$ and $\partial_{AD}B(0, a)$, $\partial_{CB}B(0, a)$ will be taken anti-clockwise, while that over $\partial\Omega_a^{c_1}$ will be taken clockwise.) The Cartesian coordinates $(x_1,\, x_2)$ of the intersection points A, D, C, B of $\partial B(0, a)$ with $\partial\Omega(c_2, c_1)$ (see Figure 2) are solutions to

$$x_1^2 + x_2^2 = a^2,$$

$$x_1^2 - 2x_1c_\alpha + x_2^2 = 0.$$

Table 1 lists these coordinates and also trigonometric functions of the corresponding polar coordinate θ where, for example, the polar coordinates of point A are $(r_A,\, \theta_A)$.

Obviously, a must satisfy

$$a \le 2 \le 2c_2,$$

whereas

$$\theta_A = -\theta_B, \qquad \theta_D = -\theta_C.$$

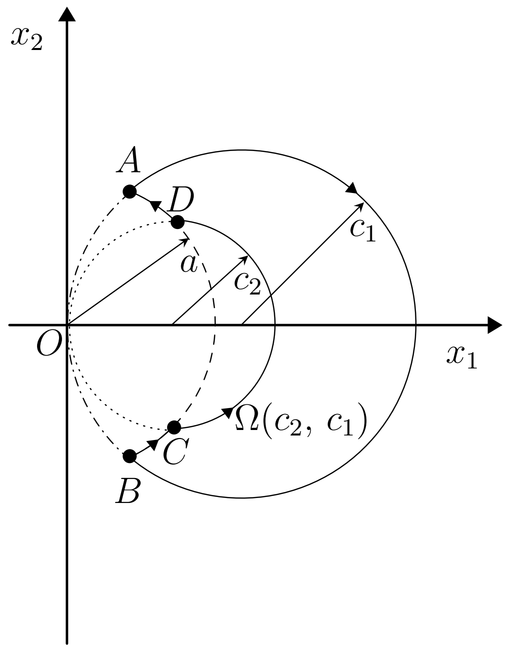

Figure 2: $\partial\Omega(c_2,\, c_1) \cap \partial B(0, a)$.

Point	x_1	x_2	$\tan\theta$	$\cos\theta$	$\sin\theta$
A	$a^2/2c_1$	$(a/2c_1)(4c_1^2 - a^2)^{1/2}$	$a^{-1}(4c_1^2 - a^2)^{1/2}$	$a/2c_1$	$(1/2c_1)(4c_1^2 - a^2)^{1/2}$
D	$a^2/2c_2$	$(a/2c_2)(4c_2^2 - a^2)^{1/2}$	$a^{-1}(4c_2^2 - a^2)^{1/2}$	$a/2c_2$	$(1/2c_2)(4c_2^2 - a^2)^{1/2}$
B	$a^2/2c_1$	$-(a/2c_1)(4c_1^2 - a^2)^{1/2}$	$-a^{-1}(4c_1^2 - a^2)^{1/2}$	$a/2c_1$	$-(1/2c_1)(4c_1^2 - a^2)^{1/2}$
C	$a^2/2c_2$	$-(a/2c_2)(4c_2^2 - a^2)^{1/2}$	$-a^{-1}(4c_2^2 - a^2)^{1/2}$	$a/2c_2$	$-(1/2c_2)(4c_2^2 - a^2)^{1/2}$

Table 1: Points of intersection

Moreover, it is a trivial observation that $a \to 0$ implies

$$x_1 \to 0, \qquad x_2 \to 0,$$

and

$$\theta_A \to \pi/2, \qquad \theta_D \to \pi/2, \qquad \theta_B \to -\pi/2, \qquad \theta_C \to -\pi/2.$$

6.2 Traction and couple

We consider the punctured region $\Omega_a(c_2,\, c_1)$ and compute the resultant traction and couple acting over $\partial_{AD}B(0, a) \cup \partial_{BC}B(0, a)$ or, equivalently, over $\partial\Omega_a^{c_1} \cup \partial\Omega_a^{c_2}$.

The stress components (5.15)-(5.17) combined with the components of the unit outward normal on the circle $\psi_c(x) = 1$ given by

$$n_1 = (x_1 - c)/c, \qquad n_2 = x_2/c, \tag{6.6}$$

provide the traction across the circle. Thus, at the point $(x_1,\, x_2)$ of the punctured curve, we have

$$
\begin{aligned}
F_1(x_1, x_2) &= \sigma_{11} n_1 + \sigma_{12} n_2 \\
&= -2\left[2\theta + \left(1 + \frac{k}{r^2}\right)\sin 2\theta\right]\frac{(x_1 - c)}{c} + 2\frac{x_2}{c}\left(1 + \frac{k}{r^2}\right)\cos 2\theta \\
&= -2\left[2\theta + \left(1 + \frac{k}{r^2}\right)\sin 2\theta\right]\cos 2\theta + 2\left(1 + \frac{k}{r^2}\right)\cos 2\theta \sin 2\theta \\
&= -4\theta\cos 2\theta, \qquad (6.7) \\
F_2(x_1, x_2) &= \sigma_{21} n_1 + \sigma_{22} n_2 \\
&= 2\left(1 + \frac{k}{r^2}\right)\cos 2\theta\frac{(x_1 - c)}{c} + 2\left[-2\theta + \left(1 + \frac{k}{r^2}\right)\sin 2\theta\right]\frac{x_2}{c} \\
&= 2\left(1 + \frac{k}{r^2}\right)\cos^2 2\theta + 2\left[-2\theta + \left(1 + \frac{k}{r^2}\right)\sin 2\theta\right]\sin 2\theta \\
&= -4\theta\sin 2\theta + 2\left(1 + \frac{k}{r^2}\right). \qquad (6.8)
\end{aligned}
$$

These expressions indicate the type of singularity occurring at the origin where $\theta = \pm\pi/2$, $r = 0$. On other parts of the boundary, the traction is non-singular and continuous. Inspection of (6.7) indicates that $F_1(x_1,\, x_2)$ is an odd function of θ in the range $-\pi/2 \leq \theta \leq \pi/2$, so that the total traction acting on Ω in the x_1-direction is zero.

The resultant traction in the x_2 -direction is obtained from expression (6.8) integrated over the boundary curves (6.3) with the unit outward normal on $\psi_{c_1} = 1$ having components

$$
\begin{aligned}
n_1 &= (x_1 - c_1)/c_1 = -u_2/c_1, \qquad (6.9) \\
n_2 &= x_2/c_1 = u_1/c_1, \qquad (6.10)
\end{aligned}
$$

while those on $\psi_{c_2} = 1$ are

$$
\begin{aligned}
n_1 &= -(x_1 - c_2)/c_2 = u_2/c_2, \qquad (6.11) \\
n_2 &= -x_2/c_2 = -u_1/c_2. \qquad (6.12)
\end{aligned}
$$

Let us also recall from (4.11) and (4.12) that in polar coordinates $(r,\, \theta)$ the circle $\psi_{c_\alpha}(x_1,\, x_2) = 1$ is given by

$$r = 2c_\alpha \cos\theta, \qquad (6.13)$$

and the line element by

$$ds = c_\alpha d(2\theta). \qquad (6.14)$$

It follows that the integral of $F_2(.)$ over $\partial\Omega_a^{c_1}$ can be written

$$\begin{aligned}
\int_{\partial\Omega_a^{c_1}} F_2(.)\, ds &= \int_{\theta_A}^{\theta_B} \left[-4\theta\sin 2\theta + 2\left(1+\frac{k}{r^2}\right)\right] c_1\, d(2\theta) &(6.15)\\
&= 2c_1\int_{\theta_A}^{\theta_B} 2\theta\, d\left(\cos 2\theta\right) + 4c_1\int_{\theta_A}^{\theta_B} d\theta + \frac{k}{c_1}\int_{\theta_A}^{\theta_B}\sec^2\theta\, d\theta\\
&= 4c_1\left(\theta_B\cos 2\theta_B - \theta_A\cos 2\theta_A\right) - 2c_1\left(\sin 2\theta_B - \sin 2\theta_A\right) + 4c_1\left(\theta_B-\theta_A\right)\\
&\quad +\frac{k}{c_1}\left(\tan\theta_B - \tan\theta_A\right)\\
&= -8c_1\theta_A\cos 2\theta_A + 4c_1\sin 2\theta_A - 8c_1\theta_A - 2\frac{k}{c_1}\tan\theta_A. &(6.16)
\end{aligned}$$

For the integral over $\partial\Omega_a^{c_2}$, the unit outward normal is given by (6.11) and (6.12). Integration, conducted anti-clockwise, leads to

$$\int_{\partial\Omega_a^{c_2}} F_2(.)\, ds = \int_{\theta_C}^{\theta_D} \left[-4\theta\sin 2\theta + 2\left(1+\frac{k}{r^2}\right)\right] c_2\, d(2\theta), \tag{6.17}$$

which by comparison with (6.15) yields

$$\int_{\partial\Omega_a^{c_2}} F_2(.)\, ds = 8c_2\theta_D\cos 2\theta_D - 4c_2\sin 2\theta_D + 8c_2\theta_D + 2\frac{k}{c_2}\tan\theta_D. \tag{6.18}$$

When $a > 0$, let T_{2a} denote the resultant force in the x_2-direction acting over the surface $\partial_{DA}B(0,a)\cup\partial_{BC}B(0,a)$. The punctured region is in equilibrium under zero body force so that

$$T_{2a} + \int_{\partial\Omega_a^{c_1}} F_2(.)\, ds + \int_{\partial\Omega_a^{c_2}} F_2(.)\, ds = 0,$$

and therefore from (6.15) and (6.18) we obtain

$$\begin{aligned}
T_{2a} &= -8\left(c_2\theta_D\cos 2\theta_D - c_1\theta_A\cos 2\theta_A\right) - 4\left(c_1\sin 2\theta_A - c_2\sin 2\theta_D\right) - 8\left(c_2\theta_D - c_1\theta_A\right)\\
&\quad -2k\left(\frac{\tan\theta_D}{c_2} - \frac{\tan\theta_A}{c_1}\right). &(6.19)
\end{aligned}$$

By reference to Table 1 for sufficiently small a, we have

$$\begin{aligned}
\tan\theta_A &= (2c_1/a)\left(1 - a^2/4c_1^2\right)^{1/2}\\
&= \left(\frac{2c_1}{a}\right)\left[1 - \frac{a^2}{8c_1^2} - \frac{1}{8}\left(\frac{a^2}{4c_1^2}\right)^2 \ldots\right].
\end{aligned}$$

Similarly, we have

$$\tan\theta_D = \left(\frac{2c_2}{a}\right)\left[1 - \frac{a^2}{8c_2^2} - \frac{1}{8}\left(\frac{a^2}{4c_2^2}\right)^2 \ldots\right],$$

which leads to the limits

$$\lim_{a\to 0} |\tan\theta_A - (2c_1/a)| = 0, \tag{6.20}$$
$$\lim_{a\to 0} |\tan\theta_D - (2c_2/a)| = 0. \tag{6.21}$$

Now let $a \to 0$ so that $\theta_A \to \pi/2$, $\theta_D \to \pi/2$. In view of (6.20) and (6.21), the resultant traction T_2 in the x_2-direction becomes

$$T_2 = 0, \qquad a \to 0. \tag{6.22}$$

We next consider the anti-clockwise moment, or couple, of the traction at a point $(x_1,\, x_2)$ of $\psi_c = 1$. On using (6.7) and (6.8), we have

$$\begin{aligned}(x_1F_2 - x_2F_1) &= c(\cos 2\theta + 1)\left[-4\theta\sin 2\theta + 2\left(1 + \frac{k}{r^2}\right)\right] + 4c\theta\sin 2\theta\cos 2\theta \\ &= -4c\theta\sin 2\theta + 4c\cos^2\theta\left(1 + \frac{k}{r^2}\right) \\ &= -4c\theta\sin 2\theta + 2c\,(1+\cos 2\theta) + \frac{k}{c}.\end{aligned} \tag{6.23}$$

Thus, the resultant anti-clockwise moment $\Gamma_a^{c_1}$ acting over the curve $\partial\Omega_a^{c_1}$ from (6.23) is

$$\begin{aligned}\Gamma_a^{c_1} &= -2c_1\int_{\theta_A}^{\theta_B} 2\theta\sin 2\theta\, c_1 d(2\theta) + 2c_1\int_{\theta_A}^{\theta_B} (1+\cos 2\theta)\ c_1 d(2\theta) + 2k(\theta_B - \theta_A) \\ &= 4c_1^2(\theta_B\cos 2\theta_B - \theta_a\cos 2\theta_A) + 2(k + 2c_1^2)(\theta_B - \theta_A) \\ &= -8c_1^2\theta_A\cos 2\theta_A - 4(k+2c_1^2)\theta_A,\end{aligned} \tag{6.24}$$

while the clockwise resultant moment over $\partial\Omega_a^{c_2}$ becomes

$$\Gamma_a^{c_2} = 8c_2^2\theta_D\cos 2\theta_D + 4(k+2c_2^2)\theta_D. \tag{6.25}$$

Because the elastic body occupying $\Omega_a(c_2,\, c_1)$ is in equilibrium, the total moment Γ_a satisfies

$$\Gamma_a + \Gamma_a^{c_1} + \Gamma_a^{c_2} = 0, \qquad a > 0.$$

In consequence from (6.24) and (6.25), we obtain

$$\Gamma_a = 8(c_1^2\theta_A\cos 2\theta_A - c_2^2\theta_D\cos 2\theta_D) + 4(k+2c_1^2)\theta_A - 4(k+2c_2^2)\theta_D. \tag{6.26}$$

Let $a \to 0$ and put $c_2 = 1$, $c_1 = R$, to conclude that the total moment Γ acting on Ω is

$$\Gamma = 4(R^2 - 1)\pi. \tag{6.27}$$

The results of this Section, consistent with the interpretation of rupture or tearing due to twisting of the region Ω, prove the next Proposition:

Proposition 6.1. *The total force acting on the region Ω is zero but the resultant couple is of magnitude $4(R^2-1)\pi$.*

The next Section examines the total strain energy.

7 The strain energy

Define strain energy in the region $\Omega_a(c_2, c_1)$ to be

$$2E_a(c_2, c_1) := \int_{\Omega_a(c_2,c_1)} \sigma_{\alpha\beta} e_{\alpha\beta}\, dx_1 dx_2, \tag{7.1}$$

where the strain components $e_{\alpha\beta}$ are those in (5.18)-(5.20), and the stress components, obtained from the Airy stress function (5.14), are given in (5.15)-(5.17). On recalling the stress equilibrium equations (2.10) and the symmetry of the stress tensor, we apply the divergence theorem to obtain

$$\begin{aligned}
2E_a(c_2, c_1) &= (1/2) \int_{\Omega_a(c_2,c_1)} (\sigma_{\alpha\beta} u_{\alpha,\beta} + \sigma_{\alpha\beta} u_{\beta,\alpha})\, dx_1 dx_2 \\
&= \int_{\Omega_a(c_2,c_1)} \sigma_{\alpha\beta} u_{\alpha,\beta}\, dx_1 dx_2 \\
&= \oint_{\partial\Omega_a(c_2,c_1)} \sigma_{\alpha\beta} u_\alpha n_\beta\, ds \\
&= \oint_{\partial\Omega_a(c_2,c_1)} [(u_1\sigma_{11} + u_2\sigma_{21})n_1 + (u_1\sigma_{12} + u_2\sigma_{22})n_2]\ ds,
\end{aligned} \tag{7.2}$$

where the curvilinear integral is taken clockwise, respectively anticlockwise, as appropriate and n_α are components of the unit outward normal on $\partial\Omega_a(c_2, c_1)$.

In general, the Lamé parameters do not explicity appear in the expression (7.2).

The aim is to evaluate the integral (7.2), compute the limit

$$\lim_{a\to 0} E_a(c_2, c_1),$$

and then set $c_2 = 1$, $c_1 = R$.

We commence by separately computing (7.2) on the respective parts of the boundary. This requires decomposition into four components:

$$\oint_{\partial\Omega_a(c_2,c_1)} (.)\, ds = \int_{\partial\Omega_a^{c1}} (.)\, ds + \int_{\partial_{BC} B(0,a)} (.)\, ds + \int_{\partial\Omega_a^{c2}} (.)\, ds + \int_{\partial_{DA} B(0,a)} (.)\, ds \tag{7.3}$$

which for convenience is written as

$$\oint_{\partial\Omega_a(c_2,c_1)} (.)\, ds = V_1 + W_1 + V_2 + W_2. \tag{7.4}$$

On conversion to polar coordinates $(r,\, \theta)$, the Cartesian components of the displacement at the point $(x_1,\, x_2)$ on the curve $\psi_{c_\alpha}(x_1,\, x_2) = 1$ are

$$\begin{aligned}
u_1 &= x_2 = r\sin\theta = c_\alpha \sin 2\theta, && (7.5)\\
u_2 &= (x_2^2 - x_1^2)/2x_1 = (c_\alpha - x_1) = -c_\alpha \cos 2\theta, && (7.6)
\end{aligned}$$

and on $\partial B(0,a)$ are

$$
\begin{aligned}
u_1 &= a\sin\theta, && (7.7)\\
u_2 &= -a\cos 2\theta/(2\cos\theta), && (7.8)
\end{aligned}
$$

where we note that components of the unit outward normal are

$$
n_1 = -x_1/a = -\cos\theta, \qquad n_2 = -x_2/a = -\sin\theta. \tag{7.9}
$$

On using (6.9), (6.10), (5.15)-(5.17), and integrating in the clockwise sense, we obtain

$$
\begin{aligned}
V_1 &= \int_{\partial\Omega_a^{c_1}} u_\alpha\sigma_{\alpha\beta}n_\beta\, ds\\
&= \int_{\theta_A}^{\theta_B} [(u_\alpha\sigma_{\alpha 1})(-u_2/c_1) + (u_\alpha\sigma_{\alpha 2})(u_1/c_1)]\, c_1\, d(2\theta)\\
&= \int_{\theta_A}^{\theta_B} \left[u_1u_2(\sigma_{22}-\sigma_{11}) + \sigma_{12}(u_1^2-u_2^2)\right]\, d(2\theta)\\
&= 2\int_{\theta_A}^{\theta_B} \left(1+k/r^2\right)\left[2u_1u_2\sin 2\theta + (u_1^2-u_2^2)\cos 2\theta\right]\, d(2\theta),
\end{aligned}
$$

which on substitution from (7.5) and (7.6) yields

$$
\begin{aligned}
V_1 &= 2c_1^2\int_{\theta_A}^{\theta_B} \left(1+k/r^2\right)\left[-2\sin^2 2\theta\cos 2\theta + (\sin^2 2\theta - \cos^2 2\theta)\cos 2\theta\right]\, d(2\theta)\\
&= 2c_1^2(\sin 2\theta_A - \sin 2\theta_B) + 2k(\theta_B-\theta_A) + k(\tan\theta_A - \tan\theta_B)\\
&= 4c_1^2\sin 2\theta_A - 4k\theta_A + 2k\tan\theta_A. && (7.10)
\end{aligned}
$$

The evaluation of V_2 is similar. In view of (6.11) and (6.12), and on taking the integral anti-clockwise, we have

$$
\begin{aligned}
V_2 &= \int_{\partial\Omega_a^{c_2}} u_\gamma\sigma_{\gamma\delta}n_\delta\, ds\\
&= 2c_2^2\int_{\theta_C}^{\theta_D} (1+k/r^2)\cos 2\theta\, d(-2\theta) && (7.11)\\
&= -4c_2^2\sin 2\theta_D + 4k\theta_D - 2k\tan\theta_D. && (7.12)
\end{aligned}
$$

Addition of (7.10) and (7.12) gives in the notation of (7.4)

$$
V_1+V_2 = 4c_1^2\sin 2\theta_A - 4c_2^2\sin 2\theta_D - 4k(\theta_A-\theta_D) + 2k\left(\tan\theta_A - \tan\theta_D\right). \tag{7.13}
$$

Before treating the integrals W_1, W_2, we examine the limiting behaviour of V_1+V_2 as $a\to 0$. We have

$$
\lim_{a\to 0}(V_1+V_2) = 0 - 0 + 2k\lim_{a\to 0}(\tan\theta_A - \tan\theta_D).
$$

which on recalling (6.20) and (6.21), we may write

$$
\lim_{a\to 0}|\,(V_1+V_2)\,| - (4k/a)|\,(c_1-c_2)\,| = 0. \tag{7.14}
$$

The corresponding integrands of W_1, W_2 over appropriate parts of $\partial B(0,a)$ are evaluated as follows:

$$\begin{aligned}
u_1\sigma_{11}+u_2\sigma_{12} &= -2a\sin\theta\left[2\theta+\left(1+k/a^2\right)\sin 2\theta\right]-a\cos^2 2\theta\left(1+k/a^2\right)\sec\theta \\
&= -4a\theta\sin\theta-a\left(1+k/a^2\right)\left[2\sin\theta\sin 2\theta+\cos^2 2\theta\sec\theta\right] \\
&= -4a\theta\sin\theta-a\left(1+k/a^2\right)\sec\theta. \qquad (7.15)
\end{aligned}$$

$$\begin{aligned}
u_1\sigma_{12}+u_2\sigma_{22} &= 2a\left(1+k/a^2\right)\sin\theta\cos 2\theta-a\sec\theta\cos 2\theta\left[-2\theta+\left(1+k/a^2\right)\sin 2\theta\right] \\
&= 2a\theta\sec\theta\cos 2\theta+a\left(1+k/a^2\right)\left[2\sin\theta\cos 2\theta-\cos 2\theta\sin 2\theta\sec\theta\right] \qquad (7.16) \\
&= 2a\theta\sec\theta\cos 2\theta. \qquad (7.17)
\end{aligned}$$

Consequently, (7.9) enables us to obtain for $(x_1,\, x_2)\in\partial B(0,a)$ the expressions

$$\begin{aligned}
(u_1\sigma_{11}+u_2\sigma_{12})n_1+(u_1\sigma_{12}+u_2\sigma_{22})n_2 &= a\cos\theta\left[4\theta\sin\theta+\left(1+k/a^2\right)\sec\theta\right] \\
&\quad -2a\theta\sin\theta\sec\theta\cos 2\theta \\
&= a\left(1+k/a^2\right)+2a\theta\sec\theta\left[\sin 2\theta\cos\theta-\cos 2\theta\sin\theta\right] \\
&= 2a\theta\tan\theta+a\left(1+k/a^2\right). \qquad (7.18)
\end{aligned}$$

Let $\overline{\partial B}(0,a) := \Omega(c_2,\, c_1)\cap\partial B(0,a)$, and take integration over $\overline{\partial B}(0,a)$ in the anti-clockwise sense. Then

$$\begin{aligned}
\int_{\overline{\partial B}(0,a)}\sigma_{\alpha\beta}u_\alpha n_\beta\, ds &= \int_{\overline{\partial B}(0,a)}\left\{2a\theta\tan\theta+a\left(1+k/a^2\right)\right\}\, ds \\
&= \int_{\overline{\partial B}(0,a)}\left\{2a\theta\tan\theta+a\left(1+k/a^2\right)\right\}\, ad(\theta) \\
&= 2a^2\int_{\overline{\partial B}(0,a)}\theta\tan\theta\, d\theta+a^2\left(1+k/a^2\right)(\theta_A-\theta_D+\theta_C-\theta_B) \\
&= 2a^2\int_{\theta_C}^{\theta_B}\log\cos\theta\, d\theta+2a^2\int_{\theta_A}^{\theta_D}\log\cos\theta\, d\theta \qquad (7.19) \\
&\quad -2a^2\left(\theta\log\cos\theta\right)|_{\theta_D}^{\theta_A}-a^2\left(\theta\log\cos\theta\right)|_{\theta_B}^{\theta_C} \qquad (7.20) \\
&\quad +a^2\left(1+k/a^2\right)(\theta_A-\theta_D+\theta_C-\theta_B), \qquad (7.21)
\end{aligned}$$

where we have used an integration by parts, the relation

$$\tan\theta=-\frac{d}{d\theta}\left(\log\cos\theta\right),$$

and where

$$\int_{\overline{\partial B}(0,a)}\log\cos\theta\, d\theta=\int_{\theta_D}^{\theta_A}\log\cos\theta\, d\theta+\int_{\theta_B}^{\theta_C}\log\cos\theta\, d\theta. \qquad (7.22)$$

Consequently (7.21) reduces to

$$\begin{aligned}
\int_{\overline{\partial B}(0,a)}\sigma_{\alpha\beta}u_\alpha n_\beta\, ds &= -2a^2\int_{\overline{\partial B}(0,a)}\log\cos\theta\, d\theta \\
&\quad +4a^2\left[\theta_A\log\cos\theta_A-\theta_D\log\cos\theta_D\right] \\
&\quad +2a^2\left(1+k/a^2\right)(\theta_D-\theta_A). \qquad (7.23)
\end{aligned}$$

But (7.22), the mean value theorem, and the condition $c_1 > c_2$ imply that

$$\lim_{a\to 0}\int_{\overline{\partial B}(0,a)} \log\cos\theta\, d\theta = 0,$$

and thus we conclude, by reference to Table 1, that

$$\begin{aligned}
\lim_{a\to 0}\int_{\overline{\partial B}(0,a)} \sigma_{\alpha\beta}u_\alpha n_\beta\, ds &= 2k\lim_{a\to 0}(\theta_D-\theta_A) + 4\lim_{a\to 0} a^2\left[\theta_A\log\cos\theta_A - \theta_D\log\cos\theta_D\right] \\
&= 4\lim_{a\to 0} a^2\left[\theta_A\log\cos\theta_A - \theta_D\log\cos\theta_D\right] \\
&= 4\lim_{a\to 0} a^2\left[\theta_A\log(a/2c_1) - \theta_D\log(a/2c_2)\right] \qquad (7.24)\\
&= 4\lim_{a\to 0} a^2\left[(\theta_A-\theta_D)\log(a/2c_1) + \theta_D\log(c_2/c_1)\right] \\
&= 0, \qquad (7.25)
\end{aligned}$$

since for $\gamma > 1$ we have $x^\gamma \log x \to 0$ as $x \to 0$.

Hence, we have

$$\lim_{a\to 0}(W_1+W_2) = 0,$$

and on appeal to (7.4) and (7.14), we are led to

$$\lim_{a\to 0}|E_a(c_2,\, c_1) - (2k/a)\,(c_1-c_2)\,| = 0, \qquad (7.26)$$

or

$$E_a(c_2,\, c_1) = O(a^{-1}). \qquad (7.27)$$

The total strain energy of the whole domain Ω is obtained either by setting $c_1 = R$, $c_2 = 1$, or additively, so that (7.26) yields

$$\lim_{a\to 0}|E_a(1,R) - (2k/a)\,(R-1)| = 0. \qquad (7.28)$$

When $k > 0$ and is sufficiently large, (5.26) indicates that the Lamé parameters are strongly-elliptic and the strain energy at the origin behaves like r^{-1}. When $k = 0$ the Lamé parameters by (5.24) cease to be strongly-elliptic but no singularities now occur in the strain energy which indeed vanishes by (7.26).

8 Further remarks

We comment on some generalisations and open problems.

1. A deeper study of the solution presented in this article may clarify the nature of the singular behaviour. We illustrate this in the simpler problem of the Laplace equation $\Delta v = 0$ in an unbounded plane sector $D = \{(r\cos\theta, r\sin\theta) : \theta \in (0,\alpha), r > 0)\}$ of opening angle $\alpha \in (0, 2\pi)$, cp. [4, 7]. Dirichlet boundary conditions $v(r,0) = v_0$, $v(r\cos\alpha, r\sin\alpha) = v_1$, for $r > 0$, are imposed at ∂D, with given constants v_0, v_1. This problem admits a unique solution of finite Dirichlet energy, given by $v = v_0 + (v_1 - v_0)\frac{\theta}{\alpha}$.

However, further solutions of infinite Dirichlet energy are obtained as $v + V$, V being a linear combination of $\log(r)$ and $r^{\frac{\pi k}{\alpha}} \sin\left(\frac{\pi k\theta}{\alpha}\right)$, $k \in \mathbb{Z} \setminus \{0\}$. Uniqueness of such infinite-energy solutions may be recovered by imposing appropriate growth conditions at $r = 0$, respectively $r = \infty$.

2. Let the plane (hollow) domain Ω be formed from the family of space filling simple closed curves

$$\psi_{c,d}(x_1,\, x_2) = 1, \tag{8.1}$$

where c, d are prescribed positive constants. For instance, let

$$\psi_{c,d} := \frac{(x_1 - c - d)^2 + x_2^2}{c^2}, \tag{8.2}$$

which represent a family of circles centred at $(c + d,\, 0)$ and of radius c. The domain Ω is bounded externally by the circle $\psi_{c_1,0}$ and internally by the circle $\psi_{c_2,\,d_2}$ where $0 < c_2 \le c < c_1$, $0 \le d \le d_2$. When $d_2 > 0$, each member of the family intersects the x_1-axis at distinct points $x_1 = d,\, d + 2c$, whereas when $d = d_2 = 0$ the family of circles (8.2) reduces to that previously considered in Section 3 and therefore each member intersects the x_1-axis at points $(0, 0)$ and $(2c, 0)$. The unit tangent vector t to a circle belonging to the family (8.2) is given by

$$t_1 = \frac{x_2}{c}, \tag{8.3}$$

$$t_2 = \frac{(c + d - x_1)}{c}. \tag{8.4}$$

At the two points of intersection of the circle with the x_1-axis the unit tangent has components $(0,\, \pm 1)$. The values are maintained as $d \to 0$. Consequently, the vector field $(ct_1,\, ct_2)$ is uniquely defined at all points of the x_1-axis provided $d_2 > 0$. It is only when $d_2 = 0$ and Ω becomes lens shaped with symmetric cusps at the origin that the vector field fails to be unambigously defined at the origin.

3. As proposed by Dafermos [2], the vector field $u(x_1,\, x_2)$ may be continued into the region $\Omega(0, 1)$ upon defining the respective components to be

$$\begin{aligned} u_1(x_1,\, x_2) &= x_2, \qquad (x_1,\, x_2) \in \Omega(0, 1), \\ u_2(x_1,\, x_2) &= (1 - x_1), \qquad (x_1,\, x_2) \in \Omega(0, 1). \end{aligned}$$

The continuation corresponds to a rigid body rotation of $\Omega(0, 1)$.

Although the vector field similarly may be continued into the external region $\Omega(R, \infty)$, the implications await investigation.

4. When $c_2 = 1$ for the inner boundary $\partial\Omega_{c_2}$ is replaced by $c_2 = \epsilon$ for small but positive ϵ, the lens shaped region $\Omega(\epsilon, R)$ becomes a region pierced by a small hole whose boundary is rigidly rotated. We likewise may take $R \to \infty$ to obtain apparently analogous problems in the positive half-plane $x_1 \ge 0$ with singularities distributed along the $x_1 = 0$ axis.

5. The semi-inverse method developed in Section 5 may be applied to any prescribed conservative plane vector field $u(x_1,\, x_2)$ and the region Ω defined appropriately. The Airy stress function continues to satisfy (5.1), which may be of mixed rather than hyperbolic type, and a solution yields an equilibrium stress distribution in the absence of body force.

6. Other solutions exist to the hyperbolic equation (5.1) which lead to different sets of nonhomogeneous Lamé parameters and corresponding stresses. Nevertheless, in each case the prescribed displacement has the same behaviour at the origin.

7. We comment briefly on another extension of the analysis. Consider the vector field whose Cartesian components are tangential to a general family of simple closed curves $\widehat{\psi}_c = 1$ for variable constant c, and instead of (3.2) and (3.3) for some constant ϖ put

$$u_1 = \varpi \frac{\partial \widehat{\psi}_c}{\partial x_2}, \tag{8.5}$$

$$u_2 = -\varpi \frac{\partial \widehat{\psi}_c}{\partial x_1}. \tag{8.6}$$

Observe that the vector with components (8.5) and (8.6) for smooth $\widehat{\psi}_c$ is isochoric when the point $(x_1,\, x_2)$ is confined to a given curve of the family. The representation is a special case of the plane Helmholtz decomposition

$$u = \nabla \Psi + \nabla \times \psi,$$

for scalar potential function Ψ and vector potential function ψ. The component $\nabla\Psi$ is parallel to the normal to level surfaces of Ψ. The expressions (8.5) and (8.6) are recovered on setting $\Psi = 0$ and $\psi = (0,\, 0,\, \varpi\widehat{\psi}_c)$.

The vector whose components are (8.5) and (8.6) for suitable choices of ϖ and $\widehat{\psi}_c$ correspond to the displacement vector for point defects in plane elastostatic equilibrium problems. For example, when $\varpi = 2$ and $\widehat{\psi}_1(x) = \log\,(x_1^2 + x_2^2)$. Possible relationships are then with Volterra and other dislocations in homogeneous linear isotropic elasticity. See [9].

8. A different generalisation is derived on setting $u(x_1\, x_2) = cw(x_1,\, x_2)t(x_1,\, x_2)$ (see [1]), where $w(x)$ is an arbitrary scalar function. Thus,

$$u_1(x_1,\, x_2) = w(x)x_2, \tag{8.7}$$

$$u_2(x_1, x_2) = w(x)(c - x_1). \tag{8.8}$$

Nevertheless, for simplicity we have set $w = 1$ in the preceding discussion.

9. It is of interest to extend the procedure adopted here to three-dimensions using the Maxwell-Morera functions to generalise the Airy stress function.

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A Airy function. Derivation

This appendix provides the detailed elementary derivation of the solution (5.13) to equation (5.9), subject to (5.10) or more generally to (5.11), which on integration shows that

$$J(z,\bar{z}) = zg(\bar{z}) + f(\bar{z}), \tag{A.1}$$

where the arbitrary functions $g(.)$, $f(.)$ are chosen below. Other choices are presented in Appendix B.

Computations are given only for the left side of (5.9). The right side may be treated similarly.

Consequently, consider

$$z^2\chi_{,zz} = J(z,\bar{z}),$$

which leads to

$$\begin{aligned}\chi_{,zz}(z,\bar{z}) &= \frac{J(z,\bar{z})}{z^2}, \qquad z \neq 0\\ &= -J\frac{d}{dz}\left(\frac{1}{z}\right)\\ &= -\frac{d}{dz}\left(\frac{J}{z}\right)+\frac{J_{,z}}{z}\\ &= -\frac{d}{dz}\left(\frac{J}{z}\right)+J_{,z}\frac{d}{dz}(\log z)\\ &= -\frac{d}{dz}\left(\frac{J}{z}\right)+\frac{d}{dz}\left(J_{,z}\log z\right),\end{aligned} \tag{A.2}$$

on recalling (5.11). Integration of the last expression yields

$$\chi_{,z} = -\frac{J}{z} + J_{,z}\log z + G(\bar{z}),$$

where $G(\bar{z})$ is an arbitrary function of $\bar{z}$. Write the last equation as

$$\begin{aligned}\chi_{,z} &= -J\frac{d}{dz}(\log z) + J_{,z}\log z + G(\bar{z})\\ &= -\frac{d}{dz}(J\log z) + 2J_{,z}\frac{d}{dz}(z\log z - z) + G(\bar{z})\\ &= -\frac{d}{dz}(J\log z) + 2\frac{d}{dz}\left(J_{,z}(z\log z - z)\right) + G(\bar{z}),\end{aligned}$$

which on integration gives

$$\chi(z,\bar{z}) = -J\log z + 2J_{,z}(z\log z - z) + zG(\bar{z}) + F(\bar{z}) \tag{A.3}$$

where $F(\bar{z})$ is an arbitrary function of $\bar{z}$.

Substitution for J from (A.1) in (A.3) after simplification gives

$$\chi(z,\bar{z}) = g(\bar{z})z\log z - f(\bar{z})\log z - 2zg(\bar{z}) + zG(\bar{z}) + F(\bar{z}).$$

Set

$$g(\bar{z}) = k_1\bar{z}, \qquad f(\bar{z}) = k_2, \qquad G(\bar{z}) = 2g(\bar{z}) - k_1\bar{z}\log\bar{z}, \qquad F(\bar{z}) = k_2\log\bar{z}, \tag{A.4}$$

which on putting $k_1 = 1$, $k_2 = k$ leads to (5.13) as required.

B Generalised $J(z,\bar{z})$

Let J be given by

$$J(z,\bar{z}) = \sum_{j=1}^{m}\frac{(z\bar{z})^j}{j!} + k, \qquad m = 1,2,3,\ldots \tag{B.1}$$

from which follows

$$\frac{d^{(m+1)}J(z,\bar{z})}{dz^{(m+1)}} = \frac{d^{(m+1)}J(z,\bar{z})}{d\bar{z}^{(m+1)}} = 0. \tag{B.2}$$

The procedure outlined in Appendix A may now be repeated on noting the formulae

$$\frac{d}{dz}\left(nz^n \log z - z^n\right) = n^2 z^{(n-1)} \log z, \qquad n = 1, 2, 3, \ldots. \tag{B.3}$$

The case $n = 1$ is treated in Appendix A. The general form of the Airy function χ may easily be computed but the result is not recorded.