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On repairable systems with time redundancy and operational constraints

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ABSTRACT

For some repairable systems executing missions/tasks, a functional failure, i.e., a failure of a mission or task can occur not immediately after equipment failure but with some delay. This happens when a failure/defect is not repaired within some specified period of time. Alternatively, a functional failure can also occur when a new failure/defect happens relatively soon after the completion of the previous repair. In this paper, we present a new stochastic model that defines and describes the lifetimes of this kind of repairable systems with operational constraints. A new approach based on Laplace transforms is developed to study the reliability function and the mean time to failure for these systems. Furthermore, we consider the stochastic model when only a finite number of repairs are allowed and obtain relevant reliability indices for this case as well. Detailed numerical examples illustrate our findings.

1. Introduction

In this section, we first discuss motivation of the proposed problem and then we give a brief literature survey.

1.1. Motivation and discussion of the problem

Stochastic modeling of performance for repairable systems is one of the main topics in reliability theory. Through the years, there were numerous ground breaking papers and books devoted to this versatile topic. Along with classical Barlow & Proschan [1,2],

some recent developments relevant to our study include, for example, Ahmadi et al. [3], Asadi [4], Cui & Li [5], Cheng & Zhao [6], de Oliveria Valadares et al. [7], Finkelstein [8], Jubari et al. [9], Levitin et al. [10], Li & Kagaris [11], Li et al. [12], Zhang et al. [13], Zhang et al. [14] to name a few. Although this general field is well developed, practical applications often suggest new settings that demand new approaches to the corresponding modeling. One of these approaches that is applied to a practical problem is developed and justified in our paper.

In many real-life scenarios, systems' performance becomes unacceptable resulting in a functional/mission failure not immediately following a defect or failure of a system as such, but only after some delay . For further convenience, we will use abbreviation "FF" for this type of failure. It can happen, e.g., due to some 'inertia' in the output characteristics and can be considered as 'tolerable' for some limited time after a failure of a system. Thus, if a repair is completed within this interval of time, it does not result in a functional failure (FF). This can be also interpreted in terms of the corresponding *time redundancy* of the model as one can think about some imaginary cold standby system that is switched into operation after the failure of the prime one and has the lifetime described by the distribution of time of repair of the prime system. Some real-world examples that describe this situation are as follows:

(a) For marine navigational systems: if the repair of the failed navigational equipment is completed within a relatively short period of time, then the altitude or longitude for a vessel is not changing much and the initial defect/failure does not become the FF with respect to these navigational parameters;
(b) In certain power generation systems, a defect or failure is classified as catastrophic only if it remains unresolved beyond a specified time frame. For example, in a power station comprising multiple generating units, the continuity of electricity supply can still be deemed acceptable during the repair of one or more generating units, provided the duration of repair does not exceed a predetermined threshold value;
(c) Consider a thermo-stabilization system designed for a highly accurate gyroscopic unit. This system exhibits 'thermo-inertia,' meaning that if it experiences a malfunction, prompt repairs within a defined time frame will not significantly affect the

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Acronyms

FF	functional failure
SF	system failure
iid	independent and identically distributed
LT	Laplace transform
PH	phase-type

Notation

U	underlying non-negative random variable
$f_U(\cdot)$	probability density function of U
$F_U(\cdot)$	cumulative distribution function of U
$\bar{F}_U(\cdot)$	reliability/survival function of U
$E(U)$	mean of U
$\mathscr{L}(g(t))[\cdot]$	LT of the function $g(t)$
X_1	time to the first SF
X_i	time elapsed since the $(i-1)$th repair to the ith SF, $i = 2, 3, 4, \ldots$
R_i	repair time of the system after ith SF, $i = 1, 2, 3, \ldots$
τ	repair time threshold
δ	no-failure time after a repair
M	random variable such that $M = m$ represents the event that the FF occurs due to the mth SF
T_{FF}	time to FF
M_l	number of SFs 'required' for the FF given that l repairs are allowed
T_{FF}^l	time to FF given that l repairs are allowed

gyroscopic unit's performance. However, prolonged delays in repairs could be deemed critical, leading to the FF (the accuracy of obtaining navigational parameters is not within the specified limits).

Other situations (but not limited to) of acceptable delay, where a timely repaired defect or failure does not result in the functional failures (FF) are: pumping systems with intermediate storage, communication and weather satellites, unmanned aerial vehicles, military missions, medical equipment malfunction, etc. Thus, this setting can be considered as rather common in practice.

However, at least, in some of the listed examples, where the initial timely repaired failure prevents from immediate FF; vulnerability to a subsequent failure can be substantially increased if this failure occurs relatively *soon after* the repair is completed. Therefore, this subsequent failure can result in the FF and the corresponding events should form the criterion of a failure for a mission or task. One of the possible explanations in this case is that the relevant 'inertia capability' is not restored to the required extent for 'addressing' the next failure. This can be due to external or internal reasons. Thus, in these cases, technically the failed equipment can be restored to the "as good as new" state but the 'inertia capability' described above needs more time for that. This means, that the repair on a general/functional level is imperfect in the described sense. Therefore, if the next failure of a system occurs relatively soon, this results in the FF . Some of the possible examples are listed below.

For instance, for the pump with intermediate storage, this storage can be significantly depleted after repair, and it starts replenishing but rather slowly, so if the failure of the system occurs soon it means FF (immediately or very soon). Another possible scenario that can result in the FF in this case is when the repair team is unavailable for some time after completing the previous repair (e.g., in some offshore wind farms, when the team leaves the site after repair and should be transported back again to deal with the next repair). Finally, in high precision navigational systems, there external sources for correction/repair are required. For instance, for a submarine on the long patrol mission, the corrections of navigation parameters after replacement of the failed equipment can be performed only in the surface condition. However, due to operational and logistic reasons, this is not usually possible for some period of time after the previous repair and correction. Thus, if the next failure of the navigation system occurs in this period of time, it results in the FF and the mission should be usually terminated or adjusted in some way. The main goal of our study is reliability modeling (i.e., the stochastic description of the time to the FF) of a repairable systems of the described type.

1.2. Literature review

Note that, only a possible acceptable delay for the FF in some specific cases was previously considered in the literature. To the best of our knowledge, the first reference to this topic was Zarudniy [15], where the discussion was inspired by the navigational setting mentioned above. In Vaurio [16], some general approaches to defining the reliability function via the corresponding integral equations and subsequent computational solutions were discussed. Recently, Cha & Finkelstein [17] also have obtained the reliability function for some specific cases (i.e., for a system with exponentially distributed lifetime). Moreover, considerable attention in the latter paper was also devoted to asymptotic 'fast repair' approximations.

For repairable systems, other than reliability analysis, concept of tolerable delay has been considered in various other topics such as availability analysis, optimal spare allocation, etc. Bao and Cui [18] proposed a novel model for evaluating the availability of repairable systems based on Markov processes, wherein failures can either be neglected or delayed based on repair time (see also Qiu and Cui [19]). Park et al. [20,21] provided maintenance models for repairable systems under a renewable warranty policy, where systems are either minimally repaired or replaced if a repair cannot be completed within a certain time. Drefuss and Giat [22] discussed the optimization of spare item allocation in multi-location repair system by considering tolerable delay in service.

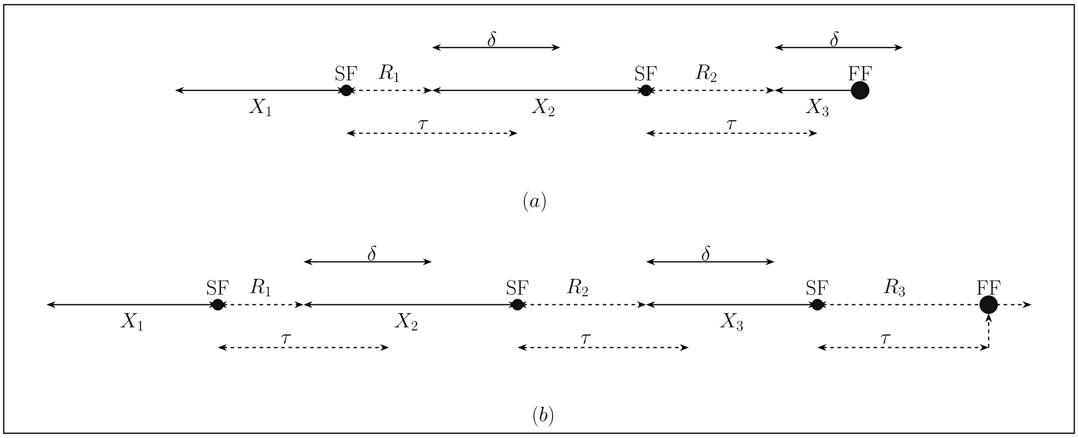


Fig. 1. Realization of events for the defined model.

In the current paper, we develop *a new alternative and more general approach* to obtaining reliability characteristics for the described settings, which allows for explicit general solutions via the Laplace transforms. Moreover, we are the first to consider and analyze stochastically a general and practically sound model combining acceptable delays with the possibility of a mission failure when a system's failure occurs too soon after the previous repair. Note that the latter part, in a way, resembles the delta-shock models (a system fails due to external shocks when the time between two consecutive shocks is less than the pre-fixed threshold delta) extensively studied in the literature (e.g., Li & Zhao [23], Goyal et al. [24,25], Lyu et al. [26], Eryilmaz & Unlu [27], to name a few). We believe that this is the main, innovative contribution of our paper. The main focus is on the case when the acceptable delay time and the no-failure time after the last repair are constants. However, generalization to the case of the corresponding random variables is also provided.

The contribution and novelty of the paper can be summarize as follows:

(i) We present a novel stochastic model for analyzing the reliability of repairable systems operating under some constraints, such as limited repair time and the possibility of FF if a SF occurs too soon after the previous repair.

(ii) A significant contribution is the use of Laplace transforms to derive reliability of the time to first FF and mean time to FF. This approach allows for explicit solutions, making it more general and robust than previous methods that relied on integral equations.

(iii) The model is further extended to consider systems where only a finite number of repairs are permitted, which is crucial for real-world applications with repair limitations. Additionally, this paper expands the model to include cases where delays and no-failure times are random, offering greater flexibility in reliability analysis.

The rest of the paper is organized as follows. In Section 2, we give the model formulation. In Section 3, we derive the reliability function and the mean time to failure for the defined model. In Section 4, we consider systems with finite number of repairs and study different reliability indices for this model. Numerical illustrations are presented in Section 5. In Section 6, we do a simulation study. Conclusions are given in Section 7. Finally, a generalization of the proposed model to the case when the delay time and the no-failure time after the repair are random are discussed in Appendix-B.

2. Model description

Let a new repairable system start operation at time $t = 0$. Assume that its failure is self-announced and can be detected immediately. After each failure, the process of repair starts immediately. If the repair is not completed within time τ, then this event is considered as a functional/mission failure (FF). Note that the FF is different from the system failure (SF). Further, if the next SF occurs too early, say within time δ after the completion of the previous repair, it is also considered as the FF. We are interested in the distribution of the time of the first FF for the repairable system as it terminates a mission or a task. The corresponding criteria of the FF can be defined formally as follows.

Let X_1 be the random variable representing time to the first SF. Further, let X_i be the random time elapsed since the $(i-1)$th repair to the ith SF, $i = 2, 3, 4, \ldots$. Let R_i be the repair time of the system after the ith SF, $i = 1, 2, 3 \ldots$. Then, according to the defined model, the ith SF can be considered as the FF if one of the following cases holds true:

(a) the ith SF is not repaired within time τ, i.e., $R_i > \tau$, $i = 1, 2, 3, \ldots$;

(b) After the $(i-1)$th repaired of SF, the next ith SF occurs within time δ, i.e., $X_i \le \delta$, $i = 2, 3, 4 \ldots$.

Let M be a random variable such that $M = m$ represents the event that the FF occurs due to the mth SF. Let T_{FF} be the random variable representing the time to FF. Then, according to the defined model, the event "$M = 1$" happens if $R_1 > \tau$ and consequently,

$$T_{FF} = X_1 + \tau. \tag{2.1}$$

Further, the event "$M = m$", for $m = 2, 3, 4, \ldots$, can happen in two ways:

(i) $R_1 \le \tau, X_2 > \delta, R_2 \le \tau, \ldots, X_{m-1} > \delta, R_{m-1} \le \tau, X_m \le \delta$,

(ii) $R_1 \le \tau, X_2 > \delta, R_2 \le \tau, \ldots, X_{m-1} > \delta, R_{m-1} \le \tau, X_m > \delta, R_m > \tau$.

For this case,

$$T_{FF} = \sum_{i=1}^{m} X_i + \sum_{i=1}^{m-1} R_i + \tau \mathbf{1}(X_m > \delta), \tag{2.2}$$

where

$$\mathbf{1}(X_m > \delta) = \begin{cases} 1 & if\ X_m > \delta \\ 0 & otherwise, \end{cases}$$

and $m = 2, 3, 4, \ldots$. Next, we illustrate the described model. In Fig. 1(a), the FF occurs due to the 3rd SF, i.e., the failure takes place 'too early' (within time δ) after the second repair. Clearly, from the figure, we have

$$T_{FF} = X_1 + R_1 + X_2 + R_2 + X_3.$$

On the other hand, in Fig. 1(b), the FF occurs because the repair time of the 3rd SF exceeds the threshold τ. Hence, in this case,

$$T_{FF} = X_1 + R_1 + X_2 + R_2 + X_3 + \tau.$$

In what follows, we consider the following assumptions.

(i) The random variables $X_1, X_2, X_3, \dots$ are independent and identically distributed (iid) with a common distribution $F_{X_1}(\cdot)$.
(ii) The random variables $R_1, R_2, R_3, \dots$ are iid with a common distribution $F_{R_1}(\cdot)$.
(iii) The random sequences $\{X_1, X_2, X_3, \dots\}$ and $\{R_1, R_2, R_3, \dots\}$ are independent with each other.
(iv) τ and δ are constants. However, these variables may be random; the results for this case are deferred to Appendix-B.
(v) After each failure, the system as such undergoes a perfect repair (i.e., as good as new).

3. Reliability indices

In this section, we obtain the reliability function and the mean time to failure for the defined model. From (2.1) and (2.2), we see that T_{FF} is a convolution of X_i's and R_i's. Consequently, obtaining the reliability function directly from these equations may be difficult because convolutions are always challenging to deal with analytically and numerically. Therefore, we derive the Laplace transform (LT) of T_{FF}, which fully describes this random variable.

For convenience and notation-sake let us first define the LT for a non-negative function and a random variable. Let $g(t)$ be a real-valued function defined on the interval $(0,\infty)$. Then, the LT of $g(t)$, denoted by $\mathscr{L}(g(t))[\cdot]$, is defined as

$$\mathscr{L}(g(t))[s] = \int_0^\infty e^{-st} g(t)dt,$$

where s is a real number. Further, the LT of a random variable U, with the probability density function $f_U(u)$, is defined by

$$\mathscr{L}(f_U(t))[s] = E\left(e^{-sU}\right) = \int_0^\infty e^{-su} f_U(u)du,$$

where s is a real number.

In the following proposition, we derive the Laplace transform of the lifetime of a system for the defined model.

Proposition 3.1. *Let $p_1 = P(R_1 \le \tau)$ and $p_2 = P(X_1 > \delta)$, and let*

$$L_1(s) = E\left(e^{-sX_1}\right), \tag{3.1}$$

$$L_2(s) = E\left(e^{-sX_1} | X_1 > \delta\right) \tag{3.2}$$

$$L_3(s) = E\left(e^{-sR_1} | R_1 \le \tau\right). \tag{3.3}$$

Then, the LT of T_{FF} is given by

$$E\left(e^{-sT_{FF}}\right) = (1-p_1)e^{-s\tau}L_1(s) + \left(\frac{p_1 L_1(s)L_3(s)(L_1(s) - p_2 L_2(s) + p_2(1-p_1)e^{-s\tau}L_2(s))}{1 - p_1 p_2 L_2(s) L_3(s)}\right).$$

Proof. Let

$$A_m = \text{“}R_1 \le \tau, X_2 > \delta, R_2 \le \tau, \dots, X_{m-1} > \delta, R_{m-1} \le \tau, X_m \le \delta\text{"}, \tag{3.4}$$

and

$$B_m = \text{“}R_1 \le \tau, X_2 > \delta, R_2 \le \tau, \dots, X_{m-1} > \delta, R_{m-1} \le \tau, X_m > \delta, R_m > \tau\text{"}, \tag{3.5}$$

for $m = 2, 3, 4, \dots$. Now, consider

$$\begin{aligned} E\left(e^{-sT_{FF}}\right) &= \sum_{m=1}^{\infty} E\left(e^{-sT_{FF}} | M = m\right) P(M = m) \\ &= E\left(e^{-sT_{FF}} | M = 1\right) P(M = 1) \\ &\quad + \sum_{m=2}^{\infty} E\left(e^{-sT_{FF}} | M = m\right) P(M = m). \end{aligned}$$

Note that, the event “$M = m$” is a union of disjoint events A_m and B_m, $m = 2, 3, \dots$. Thus, from the above equation, we can write

$$\begin{aligned} E\left(e^{-sT_{FF}}\right) &= E\left(e^{-s(X_1+\tau)} | R_1 > \tau\right) P(R_1 > \tau) \\ &\quad + \sum_{m=2}^{\infty} E\left(e^{-s\left(\sum_{i=1}^m X_i + \sum_{i=1}^{m-1} R_i\right)} | A_m\right) P(A_m) \\ &\quad + \sum_{m=2}^{\infty} E\left(e^{-s\left(\sum_{i=1}^m X_i + \sum_{i=1}^{m-1} R_i + \tau\right)} | B_m\right) P(B_m) \\ &= (1-p_1)e^{-s\tau}L_1(s) \\ &\quad + \sum_{m=2}^{\infty} L_1(s)(L_2(s))^{m-2}(L_3(s))^{m-1} \\ &\quad \times E\left(e^{-sX_1} | X_1 \le \delta\right) p_1^{m-1} p_2^{m-2}(1-p_2) \\ &\quad + \sum_{m=2}^{\infty} e^{-s\tau} L_1(s)(L_2(s))^{m-1}(L_3(s))^{m-1} p_1^{m-1}(1-p_1)p_2^{m-1} \\ &= (1-p_1)e^{-s\tau}L_1(s) \\ &\quad + p_1(1-p_2)L_1(s)L_3(s)E\left(e^{-sX_1} | X_1 \le \delta\right) \\ &\quad \times \sum_{m=2}^{\infty} (p_1 p_2)^{m-2}(L_3(s)L_2(s))^{m-2} \\ &\quad + e^{-s\tau}(1-p_1)L_1(s)\sum_{m=2}^{\infty}(L_2(s)L_3(s))^{m-1}(p_1 p_2)^{m-1} \\ &= (1-p_1)e^{-s\tau}L_1(s) \\ &\quad + \left(\frac{p_1(1-p_2)L_1(s)L_3(s)E(\exp\{-sX_1 | X_1 \le \delta)}{1 - p_1 p_2 L_2(s) L_3(s)}\right) \\ &\quad + e^{-s\tau}L_1(s)(1-p_1)\left(\frac{(p_1 p_2 L_2(s) L_3(s))}{1 - p_1 p_2 L_2(s) L_3(s)}\right) \\ &= (1-p_1)e^{-s\tau}L_1(s) \\ &\quad + \left(\frac{p_1 L_1(s)L_3(s)(L_1(s) - p_2 L_2(s) + p_2(1-p_1)e^{-s\tau}L_2(s))}{1 - p_1 p_2 L_2(s) L_3(s)}\right), \end{aligned}$$

where the last equality follows from the fact that

$$(1-p_2)E(\exp\{-sX_1\} | X_1 \le \delta) + p_2 E(\exp\{-sX_1\} | X_1 > \delta) = E(\exp\{-sX_1\}).$$

Hence, the result is proved. □

The following result is an immediate consequence of Proposition 3.1: If $\delta = 0$, then the LT of T_{FF} is given by

$$E\left(e^{-sT_{FF}}\right) = \frac{(1-p_1)e^{-s\tau}L_1(s)}{1 - p_1 L_1(s) L_3(s)}. \quad \square$$

Note that, the reliability function of a random variable U can be derived from its LT. Let $f_U(\cdot)$ and $\bar{F}_U(\cdot)$ be the probability density function and the reliability function of U, respectively. Then, $f_U(\cdot)$ of U can be obtained by applying the inverse LT on $E\left(e^{-sU}\right)$. Further, $\bar{F}_U(t)$ can be obtained by taking inverse LT on $\left[1 - E\left(e^{-sU}\right)\right]/s$ because

$$\mathscr{L}(\bar{F}_U(t))[s] = \left[1 - E\left(e^{-sU}\right)\right]/s, \quad s \ne 0, \tag{3.6}$$

holds.

In the following proposition, we derive the mean time to FF, i.e., $E(T_{FF})$, which is an important reliability characteristic in practice. Formally, it can be obtained also as the derivative of the LT of T_{FF} at the origin. However, this is more cumbersome than the direct, speaking for itself probabilistic reasoning below.

Proposition 3.2. *Let $p_1 = P(R_1 \le \tau)$ and $p_2 = P(X_1 > \delta)$. Then, the mean time to FF of the system is given by*

$$E(T_{FF}) = E(X_1) + \left(E(X_1) + E(R_1 | R_1 \le \tau)\right)\left(\frac{p_1}{1 - p_1 p_2}\right) + \tau\left(\frac{1-p_1}{1 - p_1 p_2}\right).$$

Proof. Let the events A_m and B_m are the same as in (3.4) and (3.5), respectively. Now, consider

$$\begin{aligned} E(T_{FF}) &= \sum_{m=1}^{\infty} E(T_{FF} | M = m) P(M = m) \\ &= E(T_{FF} | M = 1)P(M = 1) + \sum_{m=2}^{\infty} E(T_{FF} | M = m) P(M = m) \end{aligned}$$

$$
\begin{aligned}
&= E(X_1+\tau|R_1>\tau)P(R_1>\tau) \\
&\quad + \sum_{m=2}^{\infty} E\left(\sum_{i=1}^{m} X_i + \sum_{i=1}^{m-1} R_i | A_m\right) P(A_m) \\
&\quad + \sum_{m=3}^{\infty} E\left(\sum_{i=1}^{m} X_i + \sum_{i=1}^{m-1} R_i + \tau | B_m\right) P(B_m) \\
&= (E(X_1)+\tau)(1-p_1) \\
&\quad + \Bigg[\sum_{m=2}^{\infty}\big(E(X_1)+(m-2)E(X_1|X_1>\delta)+E(X_1|X_1\le\delta) \\
&\quad + (m-1)E(R_1|R_1\le\tau)\big)(1-p_2)p_2^{m-2}p_1^{m-1}\Bigg] \\
&\quad + \Bigg[\sum_{m=2}^{\infty}\big(E(X_1)+(m-1)E(X_1|X_1>\delta) \\
&\quad + (m-1)E(R_1|R_1\le\tau)+\tau\big)(1-p_1)p_1^{m-1}p_2^{m-1}\Bigg].
\end{aligned}
$$

After simplifying the above expression, we get

$$
\begin{aligned}
E(T_{FF}) &= E(X_1) + \tau\left(\frac{1-p_1}{1-p_1p_2}\right) + E(R_1|R_1\le\tau)\left(\frac{p_1}{1-p_1p_2}\right) \\
&\quad + E(X_1|X_1\le\delta)\left(\frac{p_1(1-p_2)}{1-p_1p_2}\right) \\
&\quad + E(X_1|X_1>\delta)\left(\frac{p_1p_2}{1-p_1p_2}\right) \\
&= E(X_1) + \big(E(X_1)+E(R_1|R_1\le\tau)\big)\left(\frac{p_1}{1-p_1p_2}\right) + \tau\left(\frac{1-p_1}{1-p_1p_2}\right),
\end{aligned}
$$

where the last equality follows from the fact that

$$E(X_1) = p_2E(X_1|X_1>\delta) + (1-p_2)E(X_1|X_1\le\delta).$$

Hence, the result is proved. □

The following result follows from the above proposition:

If $\delta = 0$, then $E(T_{FF})$ can be expressed as

$$E(T_{FF}) = \left(\frac{p_1}{1-p_1}\right)(E(X_1)+E(R_1|R_1\le\tau)) + E(X_1) + \tau.$$

4. Finite number of repairs

In this section, we explore a potentially significant extension of the proposed model, wherein only a finite number of repairs (or equivalently, SFs), say l, are permitted. Systems subject to a restricted number of repairs are considered in various applications of different engineering and service domains. For instance, consider aircraft engine turbine blades, which endure high temperatures and centrifugal stresses, resulting in elongated microstructure particles. This elongation diminishes fatigue strength, leading to the formation of voids and cracks at the blade tips. These blades can undergo rework only a limited number of times before necessitating replacement to ensure flight safety. Another example involves strategically managing a limited number of available repairs for systems purchased with a constrained warranty. While many warranties operate within fixed time frames, certain warranties impose restrictions on the number of permissible repair actions before becoming void (see Kurt & Kharoufeh [28]).

Note that, in this case, the FF criteria will be the same (i.e., with no restriction on repairs) as in Section 2 until lth SF. However, the system will surely fail due to $(l+1)$th SF because the repair of the system can be done at most l times. This failure may occur due to either early occurrence (within time δ) of $(l+1)$th SF after lth repair or non-repairing the $(l+1)$th SF within time τ (because $(l+1)$th repair is not allowed).

Let M_l denote the number of SFs 'required' for the FF given that l repairs are allowed. According to our model, $P(M_l = m)$ is the same as $P(M = m)$, for all $m = 1, 2, 3, \ldots, l$, where the random variable M is the same as in Section 2. Further,

$$P(M_l = l+1) = P(R_1\le\tau, X_2>\delta, R_2\le\tau, \ldots, X_l>\delta, R_l\le\tau).$$

Let T_{FF}^l denote the time to FF given that l repairs are allowed. Then, $T_{FF}^l = X_1 + \tau$, if $M_l = 1$, and

$$T_{FF}^l = \sum_{i=1}^{M_l} X_i + \sum_{i=1}^{M_l-1} R_i + \tau\mathbf{1}(X_{M_l} > \delta),$$

if $M_l = m$, for $m = 2, 3, 4, \ldots, l+1$.

In the following proposition, we derive the LT of T_{FF}^l. The proof is deferred to Appendix-A.

Proposition 4.1. *Let $p_1 = P(R_1\le\tau)$ and $p_2 = P(X_1>\delta)$. Then, the LT of T_{FF}^l is given by in Box I where $L_1(s)$, $L_2(s)$ and $L_3(s)$ are the same as in (3.1), (3.2), and (3.3), respectively.* □

In the following proposition, we derive the mean time to FF for the defined model. The proof of this proposition is deferred to Appendix-A.

Proposition 4.2. *Let $p_1 = P(R_1\le\tau)$ and $p_2 = P(X_1>\delta)$. Then, the mean time to FF of the system is given by*

$$
\begin{aligned}
E\left(T_{FF}^l\right) &= E(X_1) + (E(X_1)+E(R_1|R_1\le\tau))\left(\frac{p_1}{1-p_1p_2}\right)(1-(p_1p_2)^l) \\
&\quad + \tau\left(\frac{1-p_1}{1-p_1p_2}\right)(1-(p_1p_2)^{l+1}).
\end{aligned}
$$

Remark 4.1. Obviously, when $l\to\infty$, the results of this section coincide with those given in the previous section.

5. Numerical examples

In this section, we present several illustrative examples by assuming that X_1 and R_1 follow the phase-type (PH) distribution.

PH distributions are versatile and powerful family of distributions that arise from the time until absorption in a Markov process with one absorbing state. A non-negative random variable U is said to follow a PH distribution with the parameter set $\{\boldsymbol{\alpha}, \boldsymbol{A}\}$, denoted by $PH(\boldsymbol{\alpha}, \boldsymbol{A})$, if its cumulative distribution function is given by:

$$F_U(u) = 1 - \boldsymbol{\alpha} e^{\boldsymbol{A}u}\boldsymbol{e}, \quad u\ge 0,$$

where $e^{\boldsymbol{A}u} = \sum_{n=0}^{\infty}\boldsymbol{A}^n\frac{u^n}{n!}$, $\boldsymbol{e}$ is a column vector with all elements equal to one, $\boldsymbol{\alpha}$ is a row vector with non-negative elements such that $\boldsymbol{\alpha}\boldsymbol{e} = 1$, and $\boldsymbol{A}$ is a non-singular matrix with negative diagonal elements, non-negative off-diagonal elements, and non-positive row sums.

The PH distribution possesses several valuable properties. Notably, it is dense within the family of distributions with non-negative support, allowing it to approximate any distribution on the interval $(0,\infty)$ with high accuracy. This characteristic makes the PH distribution remarkably versatile. Its use enhances the flexibility of analytical properties while also ensure the practical applicability of models, particularly in engineering contexts. For instance, the lifetimes of systems and maintenance times, which often follow different PH distributions under various conditions, can be effectively modeled using this distribution. As a result, PH distributions are widely employed in maintenance applications (see Li et al. [12], Perez-Ocon & Montoro-Cazorla [29], Wang et al. [30], Juybari et al. [31], Sun & Vatn [32], to name a few).

Various algorithms have been proposed for fitting a phase-type distribution to a sample of non-negative data. Particularly, Thummler et al. [33], presented an EM algorithm that fits a restricted class of PH distributions, namely, mixture of Erlang distributions, to true data.

Below we give three examples that illustrate the results presented in previous sections with the corresponding sensitivity analysis. The latter is fairly obvious, as the parameters of the models has a clear 'physical' meaning. In Example 5.1 (resp. Example 5.2), we use Algorithms 1 and 2 to calculate $\bar{F}_{T_{FF}}(t)$ (resp., $\bar{F}_{T_{FF}^l}(t)$) and $E(T_{FF})$ (resp. $E(T_{FF}^l)$), respectively.

$$E\left(e^{-sT^{l}_{FF}}\right) = (1-p_1)e^{-s\tau}L_1(s) + \left(\frac{p_1L_1(s)L_3(s)(L_1(s)-p_2L_2(s))(1-(p_1p_2L_2(s)L_3(s))^l)}{1-p_1p_2L_2(s)L_3(s)}\right) + \frac{p_1p_2e^{-s\tau}L_1(s)L_2(s)L_3(s)(1-p_1)+p_1(p_1p_2L_2(s)L_3(s))^{l-1}-(p_1p_2L_2(s)L_3(s))^l}{1-p_1p_2L_2(s)L_3(s)},$$

Box I.

In the following two examples we assume that X_1 and R_1 follow the Erlang distribution (a special case of PH distribution). A random variable Z is said to have the Erlang distribution with parameters r and λ, denoted by $Z \sim Erlang(r,\lambda)$, if its cumulative distribution function is given by

$$P(Z \le z) = 1 - \boldsymbol{\beta}_r e^{z\boldsymbol{B}_{r,\lambda}}\boldsymbol{e}_r, \quad z \ge 0,$$

where $\boldsymbol{\beta}_r = (1,0,\dots,0)_{1\times r}$, $\boldsymbol{e}_r = (1,\dots,1)_{r\times 1}$,

$$\boldsymbol{B}_{r,\lambda} = \begin{pmatrix} -\lambda & \lambda & 0 & \dots & 0 \\ 0 & -\lambda & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & -\lambda & \lambda \\ 0 & 0 & \dots & 0 & -\lambda \end{pmatrix}_{r\times r},$$

λ is a positive real number and r is a positive integer.

Example 5.1. Let $R_1 \sim Erlang(r_1,\lambda_1)$ and $X_1 \sim Erlang(r_2,\lambda_2)$. Further, let $r_1 = 2$, $\lambda_1 = 2$, $r_2 = 2$, and $\lambda_2 = 0.1$. Then,

$$\begin{aligned} p_1 &= 1 - \boldsymbol{\beta}_{r_1} e^{\tau\boldsymbol{B}_{r_1,\lambda_1}}\boldsymbol{e}_{r_1}, \\ p_2 &= \boldsymbol{\beta}_{r_2} e^{\delta\boldsymbol{B}_{r_2,\lambda_2}}\boldsymbol{e}_{r_2}, \\ L_1(s) &= \boldsymbol{\beta}_{r_2}(s\boldsymbol{I}_{r_2} - \boldsymbol{B}_{r_2,\lambda_2})^{-1}\boldsymbol{B}^0_{r_2,\lambda_2}, \\ p_2L_2(s) &= \boldsymbol{\beta}_{r_2}(s\boldsymbol{I}_{r_2} - \boldsymbol{B}_{r_2,\lambda_2})^{-1}e^{-\delta(s\boldsymbol{I}_{r_2}-\boldsymbol{B}_{r_2,\lambda_2})}\boldsymbol{B}^0_{r_2,\lambda_2}, \\ p_1L_3(s) &= \boldsymbol{\beta}_{r_1}(s\boldsymbol{I}_{r_1} - \boldsymbol{B}_{r_1,\lambda_1})^{-1}\left(\boldsymbol{I}_{r_1} - e^{-\tau(s\boldsymbol{I}_{r_1}-\boldsymbol{B}_{r_1,\lambda_1})}\right)\boldsymbol{B}^0_{r_1,\lambda_1}, \\ E(X_1) &= -\boldsymbol{\beta}_{r_2}\boldsymbol{B}^{-1}_{r_2,\lambda_2}\boldsymbol{e}_{r_2}, \\ \text{and } p_1E(R_1|R_1\le\tau) &= -\tau\boldsymbol{\beta}_{r1}e^{\tau\boldsymbol{B}_{r1,\lambda_1}}\boldsymbol{e}_{r1} + \boldsymbol{\beta}_{r1}\boldsymbol{B}^{-1}_{r1,\lambda_1}e^{\tau\boldsymbol{B}_{r1,\lambda_1}}\boldsymbol{e}_{r1} - \boldsymbol{\beta}_{r1}\boldsymbol{B}^{-1}_{r1,\lambda_1}\boldsymbol{e}_{r1}, \end{aligned}$$

where $\boldsymbol{B}^0_{r_i,\lambda_i} = -\boldsymbol{B}_{r_i,\lambda_i}\boldsymbol{e}_{r_i}$; I_{r_i} is an identity matrix of size r_i, $i = 1,2$; $p_1 = P(R_1 \le \tau)$, $p_2 = P(X_1 > \delta)$, and $L_1(s)$, $L_2(s)$ and $L_3(s)$ are the same as in (3.1), (3.2), and (3.3), respectively. Now, we plot the survival function and the mean time to FF in Figs. 2 and 3, respectively. In Fig. 2(a), we plot the reliability function for different values of δ, whereas, in Fig. 2(b), we plot the same for different values of τ. From these figures, we can observe that an increment in δ decreases the reliability, whereas an increment in τ increases the reliability. In Fig. 2(a), we compare reliability of the system for the cases $\delta = 0$ (note that considering δ equal to zero reduces our model to the classical time redundancy model) and $\delta > 0$. From this comparison, we can see that consideration of δ may help us to get a better reliability approximation.

Algorithm 1 Find the reliability of T_{FF} (resp. T^l_{FF})

Input: r_i, λ_i, $\boldsymbol{\beta}_{r_i}$, $\boldsymbol{B}_{r_i,\lambda_i}$, $\boldsymbol{e}_{ri}$, τ, δ, t, (l, in case of Example 5.2), for $i = 1,2$

Step 1. Compute p_1, p_2, $L_1(s)$, $L_2(s)$, and $L_3(s)$
Step 2. Compute LT of T_{FF} (resp. T^l_{FF}) from Proposition 3.1 (resp. Proposition 4.1)
Step 3. Compute LT of $\bar{F}_{T_{FF}}(t)$ (resp. $\bar{F}_{T^l_{FF}}(t)$) from Eq. (3.6)
Step 4. Apply *dehoog* inverse LT algorithm to find $\bar{F}_{T_{FF}}(t)$ (resp. $\bar{F}_{T^l_{FF}}(t)$)

Output: $\bar{F}_{T_{FF}}(t)$ (resp. $\bar{F}_{T^l_{FF}}(t)$)

In Fig. 3(a), we plot the mean time to FF with respect to τ for fixed $\delta = 1$, whereas, in Fig. 3(b), we plot the same with respect to δ for fixed $\tau = 1$. These figures show that the mean time to FF is an increasing function with respect to τ, whereas it is decreasing with respect to δ.

In the following example, we illustrate the reliability function and the mean time to FF when l number of repairs are allowed.

Algorithm 2 Find the mean of T_{FF} (resp. T^l_{FF})

Input: r_i, λ_i, $\boldsymbol{\beta}_{r_i}$, $\boldsymbol{B}_{r_i,\lambda_i}$, $\boldsymbol{e}_{ri}$, τ, δ, (l, in case of Example 5.2), for $i = 1,2$

Step 1. Compute p_1, p_2, $E(X_1)$, and $E(R_1|R_1 \le \tau)$
Step 2. Compute $E(T_{FF})$ (resp. $E(T^l_{FF})$) from Proposition 3.2 (resp. Proposition 4.2)

Output: $E(T_{FF})$ (resp. $E(T^l_{FF})$)

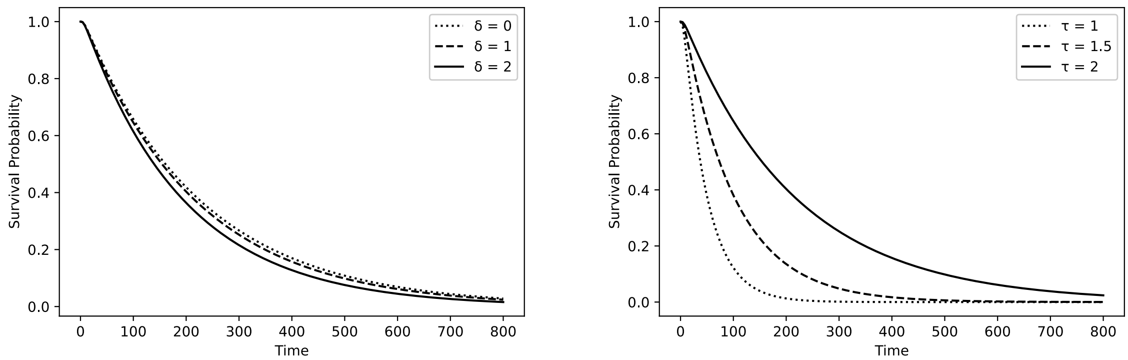
Table 1
Mean time to FF when l repairs are allowed.

l	τ	δ	$E(T^l_{FF})$
2	1	1	40.2058
		2	40.1060
		3	39.9552
	2	1	56.5450
		2	56.3182
		3	55.9754
3	1	1	44.5552
		2	44.3428
		3	44.0246
	2	1	72.1565
		2	71.5276
		3	70.5872
4	1	1	47.1266
		2	46.8153
		3	46.3526
	2	1	86.2719
		2	85.1028
		3	83.3786

Example 5.2. Let $R_1 \sim Erlang(r_1,\lambda_1)$ and $X_1 \sim Erlang(r_2,\lambda_2)$ with $r_1 = 2$, $\lambda_1 = 2$, $r_2 = 2$, and $\lambda_2 = 0.1$. We assume the same set of parameters as in Example 5.1. In Fig. 4, we plot the reliability function for different values of l for fixed $\tau = 2$ and $\delta = 1$. This figure shows that the reliability function increases with the increment in the value of l. Further, in Table 1, we evaluate the mean time to FF for different values of l, τ, and δ. From this table, we observe that, if the value of l or τ increases, considering the other two parameters fixed, the mean time to FF increases. On the other hand, if δ increases, then the mean time to FF decreases. □

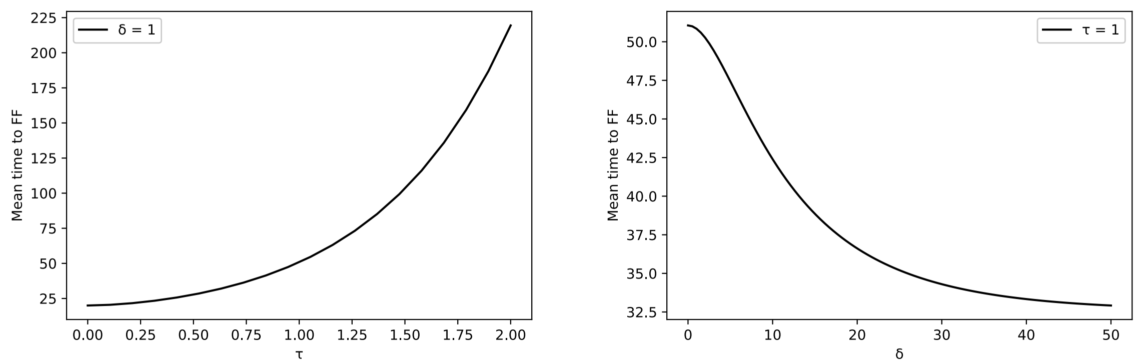
Note that, an increment in τ increases both reliability and mean of T_{FF}. This happens because larger τ means that more times are needed for repairs of SFs. Consequently, the probability of repairing a SF within time τ increases. Further, an increment in δ decreases both reliability and mean of T_{FF} because it increases the chance of occurrence of a SF within time δ (and consequently, more chances of FF). The same conclusions also valid for the case when finite number of repairs are allowed.

In the next example, we conduct a sensitivity analysis similar to that in Examples 5.1 and 5.2, but with different PH distributions for X_1 and R_1. This demonstrates that the effects of the parameters τ, δ, and l on reliability measures remain consistent regardless of the distribution of X_1 and R_1. Additionally, we perform a comparative analysis between the earlier examples and this one to illustrate the impact of the distributional shapes of X_1 and R_1 on system reliability



(a) For fixed $\tau = 2$ (b) For fixed $\delta = 1$

Fig. 2. Plot of reliability function.



(a) For fixed $\delta = 1$ (b) For fixed $\tau = 1$

Fig. 3. Mean time to FF.

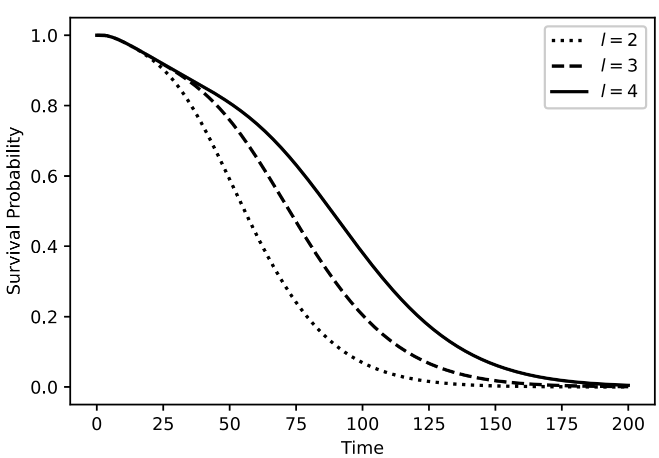


Fig. 4. Reliability of the system for l number of repairs.

measures. Like above examples, we used Algorithm 1 and Algorithm 2 for plotting figures in this example.

Example 5.3. Let $R_1 \sim PH(\boldsymbol{\alpha}_1, \boldsymbol{A}_1)$ and $X_1 \sim PH(\boldsymbol{\alpha}_2, \boldsymbol{A}_2)$, where

$$\boldsymbol{\alpha}_1 = (1, 0), \quad \boldsymbol{A}_1 = \begin{pmatrix} -1 & 0 \\ 1 & -5 \end{pmatrix}$$

and

$$\boldsymbol{\alpha}_2 = (1, 0), \quad \boldsymbol{A}_2 = \begin{pmatrix} -0.2 & 0.1 \\ 0.1 & -0.1 \end{pmatrix}.$$

In this case,

$$\begin{aligned} p_1 &= 1 - \boldsymbol{\alpha}_1 e^{\tau \boldsymbol{A}_1} \boldsymbol{e}_1, \\ p_2 &= \boldsymbol{\alpha}_2 e^{\delta \boldsymbol{A}_2} \boldsymbol{e}_2, \\ L_1(s) &= \boldsymbol{\alpha}_2 (s\boldsymbol{I} - \boldsymbol{A}_2)^{-1} \boldsymbol{A}_2^0, \\ p_2 L_2(s) &= \boldsymbol{\alpha}_2 (s\boldsymbol{I} - \boldsymbol{A}_2)^{-1} e^{-\delta(s\boldsymbol{I} - \boldsymbol{A}_2)} \boldsymbol{A}_2^0, \\ p_1 L_3(s) &= \boldsymbol{\alpha}_1 (s\boldsymbol{I} - \boldsymbol{A}_1)^{-1} \left(\boldsymbol{I} - e^{-\tau(t\boldsymbol{I} - \boldsymbol{A}_1)} \right) \boldsymbol{A}_1^0, \\ E(X_1) &= -\boldsymbol{\alpha}_2 \boldsymbol{A}_2^{-1} \boldsymbol{e}, \\ \text{and } p_1 E(R_1 | R_1 \le \tau) &= -\tau \boldsymbol{\alpha}_1 e^{\tau \boldsymbol{A}_1} \boldsymbol{e} + \boldsymbol{\alpha}_1 \boldsymbol{A}_1^{-1} e^{\tau \boldsymbol{A}_1} \boldsymbol{e} - \boldsymbol{\alpha}_1 \boldsymbol{A}_1^{-1} \boldsymbol{e}, \end{aligned}$$

where $\boldsymbol{A}_i^0 = -\boldsymbol{A}_i \boldsymbol{e}$, $i = 1, 2$; I is an identity matrix of size 2.

For the given PH distributions of X_1 and R_1, we plot the survival function of T_{FF} against τ and δ in Fig. 5. This figure shows that the reliability decreases as δ increases, while an increment in τ leads to an increment in reliability. In Fig. 6, we plot the survival function of T_{FF}^l, which shows that increasing l enhances the system's reliability.

Based on the analysis in Example 5.1, Example 5.2, and this example (so far), it can be seen that the influence of the parameters τ, δ, and l remain unchanged with respect to distributions of X_1 and R_1, which one can expect from the definition of these parameters.

To keep the paper concise, we do not include figures for the mean time to FF. However, the parameters τ, δ, and l influence the mean time to FF in the same manner as they do for reliability.

Now, to capture the effects of the distributional shapes of X_1 and R_1 on system reliability measures, we perform a comparison analysis

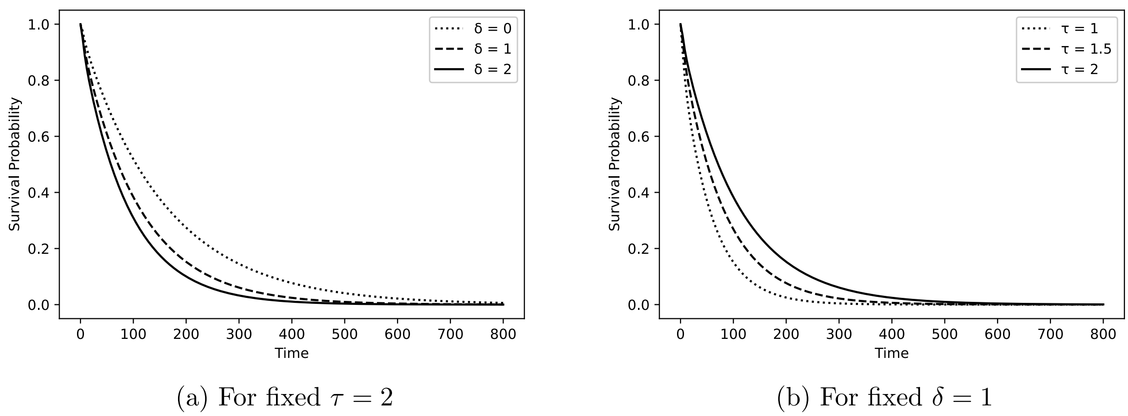


Fig. 5. Plot of reliability function.

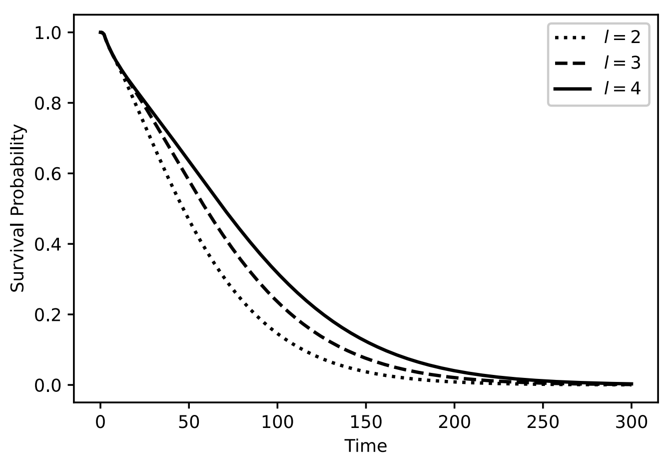


Fig. 6. Reliability of the system for l number of repairs.

between the examples mentioned earlier and this example. Note that for any $U \sim PH(\boldsymbol{\alpha}, \boldsymbol{A})$, $E(U) = -\boldsymbol{\alpha}\boldsymbol{A}^{-1}\boldsymbol{e}$. Therefore, $E(R_1) = E(Erlang(2, 2))$ and $E(X_1) = E(Erlang(2, 0.1))$. In other words, distributions of X_1 and R_1 in Example 5.1 and in this example have the same mean. Under this condition, in Fig. 7, we compare reliabilities of T_{FF} and T^l_{FF} (for Examples 5.1 and 5.3) by considering fixed $\tau = 2$, $\delta = 1$, and $l = 2$. Further, we also compare mean time to FF in Fig. 8 with respect to τ and δ. From these figures, it can be seen that the shapes of the distributions of X_1 and R_1 significantly impact the system's reliability measures such as $\bar{F}_{T_{FF}}(\cdot)$ and $E(T_{FF})$.

In Figs. 7 and 8, the label "Example 5.1" (resp. "Example 5.3") denotes that X_1 and R_1 follow the distribution given in Example 5.1 (resp. Example 5.3).

6. Simulated data analysis

In this section, we validate our model using simulated data, focusing on the scenario with no restriction on the number of repairs. The case of a finite number of repairs can be analyzed similarly.

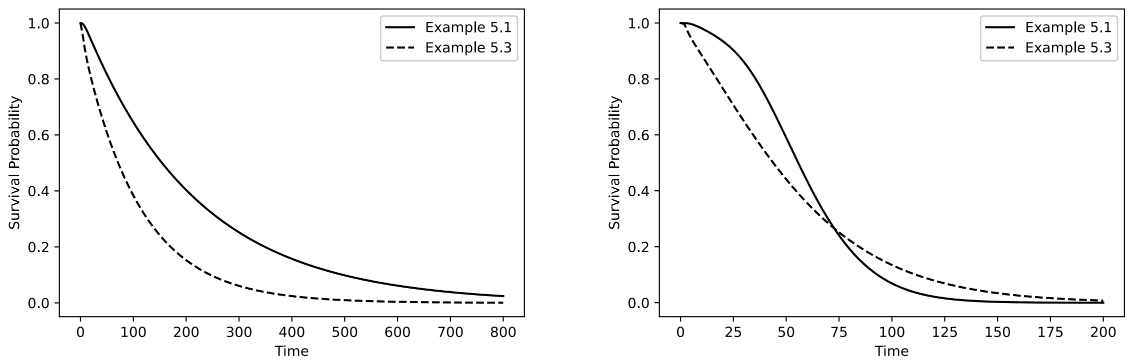
To validate the analytical model developed for assessing system reliability, a Monte Carlo simulation was conducted, generating 10,000 samples for the random variable T_{FF} for each fixed value of t; here we assume that $\tau = 1$, $\delta = 1$, $X_1 \sim Erlang(2, 0.1)$ and $R_1 \sim Erlang(2, 2)$. The empirical cumulative distribution function of T_{FF} was then calculated, and the estimated reliability of T_{FF} was determined for various values of t using the empirical cumulative distribution function (see Algorithm 3 for this simulation procedure).

Fig. 9, presents a graphical comparison between the reliability functions derived from the simulated data (labeled as 'Simulation') and the analytical model (labeled as 'Approximation'). This shows that the proposed theoretical model of the reliability function is closely aligned with the results obtained from the Monte Carlo simulation.

7. Conclusions

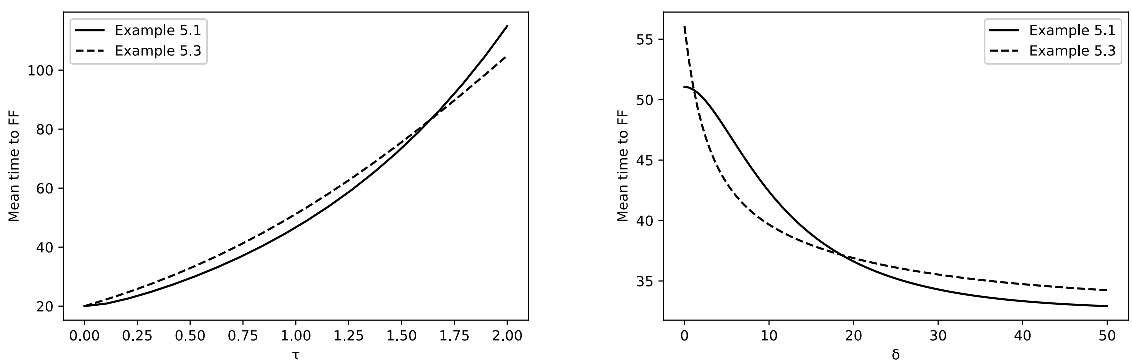
In many real-life scenarios, a functional failure of a repairable system takes place when either a defect/failure of a system as such is not repaired within a grace period of time or the next defect/failure occurs too early after the previous repair.

For these scenarios, we develop the corresponding stochastic model and obtain relevant reliability characteristics (the reliability function and the mean time to failure). These results are illustrated numerically for the case when both the time to failure of a system and the time of repair follow the PH distribution. Furthermore, we consider a practically important case when only the limited number of repairs can be



(a) No restriction on number of repair (b) l number of repair allowed

Fig. 7. Plot of reliability function for fixed $\tau = 2$, $\delta = 1$, and $l = 2$.



(a) For fixed $\delta = 1$ (b) For fixed $\tau = 1$

Fig. 8. Mean time to FF $E(T_{FF})$.

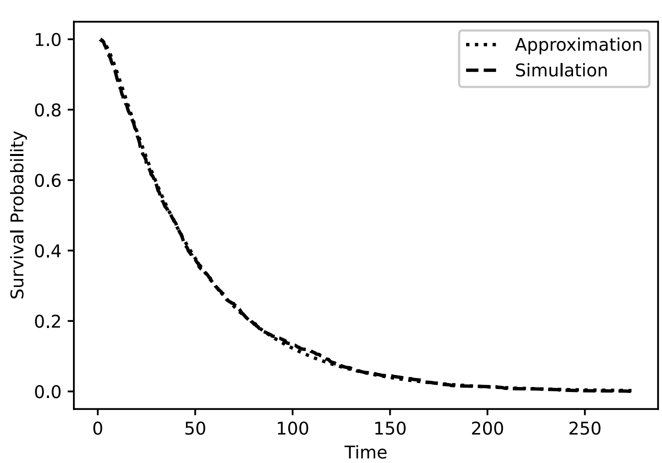


Fig. 9. Comparison of reliability functions obtained from the theoretical model and the simulation.

Algorithm 3 Simulation procedure for T_{FF} reliability estimation

At time t, $n = 1, 2, 3, \dots, J = 10000$

Step 1. Define maximum repair time threshold τ and after a repair no failure time threshold δ

Step 2. Generate a dataset, say $dataset_1$, of size $K = 1000$ from the distribution of R_1

Step 3. Generate a dataset, say $dataset_2$, of size $K = 1000$ from the distribution of X_1

Step 4. Let $dataset_i[k]$ denote the kth entry of $dataset_i$, $i = 1, 2$. Then, find

$$M_1 = min\{k \ge 2 | dataset_1[k] > \tau\},$$
$$M_2 = min\{k \ge 2 | dataset_2[k] \le \delta\}.$$

If $dataset_1[1] > \tau$, define $M = 1$; else, define $M = min(M_1, M_2)$.

Step 5. If $M = 1$, define $T_{FF} = dataset_2[1] + \tau$; else, define

$$T_{FF} = \sum_{k=1}^{M-1} dataset_1[k] + \sum_{i=1}^{M} dataset_2[k] + \tau \mathbf{1}(dataset_2[M] > \delta)$$

Step 6. Find empirical cumulative distribution function by $\hat{F}_{T_{FF}}(t) = \frac{1}{J} \sum_{N=1}^{J} I_N$,
where $I_N = 1$ when $T_{FF}[N] \le t$; otherwise $I_N = 0$.

Step 7. Calculate estimated reliability by $\hat{\bar{F}}_{T_{FF}}(t) = 1 - \hat{F}_{T_{FF}}(t)$.

performed during a mission (defined, e.g., by the number of available spare parts).

Our results are obtained explicitly in the form of Laplace transforms that can be easily inverted numerically. The other option is to derive the corresponding integral equations for the reliability function. However, this approach on our level of generality can encounter substantial analytical and computational problems.

Generalization to the case of random delay time and the time interval after the repair when a failure of a system results in the functional failure is also considered. In the future research, it is reasonable to discuss the case when these random variables are dependent (at least, the second one depends on the first one). The other possible direction of further study could be modeling of shocks impact within the framework of the discussed model (see, e.g., Eryilmaz [34], Goyal et al. [35], Chadjiconstantinidis & Eryilmaz [36], Finkelstein & Cha [37]).

CRediT authorship contribution statement

Dheeraj Goyal: Writing – review & editing, Writing – original draft, Methodology. **Maxim Finkelstein:** Writing – review & editing, Methodology, Conceptualization. **Nil Kamal Hazra:** Writing – review & editing, Methodology, Conceptualization.

Declaration of competing interest

We have nothing to declare in respect with this submission.

Appendix A. Proofs of Propositions 4.1 and 4.2

Proof of Proposition 4.1. Let the events A_m and B_m be the same as in (3.4) and (3.5), respectively (see Box II), where the last equality follows from the fact that

$$(1-p_2)E(\exp\{-sX_1\}|X_1 \le \delta) = E(\exp\{-sX_1\}) - p_2 E(\exp\{-sX_1\}|X_1 > \delta).$$

Hence, the result is proved.

Proof of Proposition 4.2. Let the events A_m and B_m be the same as in (3.4) and (3.5), respectively (see Box III), where the last equality follows from the fact that

$$E(X_1|X_1 \le \delta)(1-p_2) + E(X_1|X_1 > \delta)p_2 = E(X_1).$$

Thus, the result is proved.

Appendix B. Generalized reliability model for random τ and δ

In practice, for various reasons, τ and δ need not to be constant. Here, we consider τ and δ as random variables and derive the reliability and the mean time to FF of the system. The results for this case can be derived in the same line as in Sections 3 and 4 and therefore, the proofs are not included. Here, we assume that δ, $\{X_1, X_2, X_3, \dots\}$ and $\{R_1, R_2, R_3, \dots\}$ are all independent with each other.

We first assume that there is no restriction on the number of repairs. In the first two propositions, we derive the LT and the mean time to FF of the lifetime of a system.

Proposition 7.1. *Let*

$$L_1(s) = E\left(e^{-sX_1}\right),$$
$$L_2(s) = E\left(e^{-sX_1}|X_1 > \delta\right),$$
$$L_3(s) = E\left(e^{-sR_1}|R_1 \le \tau\right),$$
$$L_4(s) = E\left(e^{-s\tau}\right).$$

Then, the LT of T_{FF} is given by

$$E\left(e^{-sT_{FF}}\right) = (1-p_1)L_1(s)L_4(s) + \left(\frac{p_1 L_1(s) L_3(s)(L_1(s) - p_2 L_2(s) + p_2(1-p_1)L_2(s)L_4(s))}{1 - p_1 p_2 L_2(s) L_3(s)}\right),$$

where $p_1 = P(R_1 \le \tau)$ and $p_2 = P(X_1 > \delta)$. □

Proposition 7.2. *The mean time to FF of the system is given by*

$$E(T_{FF}) = E(X_1) + \left(E(X_1) + E(R_1|R_1 \le \tau)\right)\left(\frac{p_1}{1-p_1p_2}\right) + E(\tau)\left(\frac{1-p_1}{1-p_1p_2}\right),$$

where $p_1 = P(R_1 \le \tau)$ and $p_2 = P(X_1 > \delta)$. □

Next, we assume that the system can be repaired only finite number of times. In the following propositions, we derive the reliability indices for this case.

Proposition 7.3. *Let $p_1 = P(R_1 \le \tau)$ and $p_2 = P(X_1 > \delta)$. Then, the LT of T_{FF}^l is given in Box IV where $L_i(t)$ is the same as in Proposition 7.1, for $i = 1, 2, 3, 4$.* □

Proposition 7.4. *The mean time to FF of the system is given by*

$$E(T_{FF}) = E(X_1) + (E(X_1) + E(R_1|R_1 \le \tau))\left(\frac{p_1}{1-p_1p_2}\right)(1-(p_1p_2)^l) + E(\tau)\left(\frac{1-p_1}{1-p_1p_2}\right)(1-(p_1p_2)^{l+1}),$$

where $p_1 = P(R_1 \le \tau)$ and $p_2 = P(X_1 > \delta)$. □

Data availability

No data was used for the research described in the article.

$$
\begin{aligned}
E\left(e^{-sT_{FF}^{l}}\right) &= \sum_{m=1}^{l+1} E\left(e^{-sT_{FF}^{l}}|M_l = m\right) P(M_l = m) \\
&= E\left(e^{-sT_{FF}^{l}}|M_l = 1\right) P(M_l = 1) + \sum_{m=2}^{l} E\left(e^{-sT_{FF}^{l}}|M_l = m\right) P(M_l = m) \\
&\quad + E\left(e^{-sT_{FF}^{l}}|M_l = l+1\right) P(M_l = l+1) \\
&= E\left(e^{-s(X_1+\tau)}|R_1 > \tau\right) P(R_1 > \tau) + \sum_{m=2}^{l} E\left(e^{-s\left(\sum_{i=1}^{m} X_i + \sum_{i=1}^{m-1} R_i\right)}|A_m\right) P(A_m) \\
&\quad + \sum_{m=2}^{l} E\left(e^{-s\left(\sum_{i=1}^{m} X_i + \sum_{i=1}^{m-1} R_i + \tau\right)}|B_m\right) P(B_m) \\
&\quad + \left[E\left(e^{-s\left(\sum_{i=1}^{l+1} X_i + \sum_{i=1}^{l} R_i\right)}|R_1 \le \tau, X_2 > \delta, \dots, X_l > \delta, R_l \le \tau, X_{l+1} \le \delta\right)\right. \\
&\quad \left. \times P(R_1 \le \tau, X_2 > \delta, \dots, X_l > \delta, R_l \le \tau, X_{l+1} \le \delta)\right] \\
&\quad + \left[E\left(e^{-s\left(\sum_{i=1}^{l+1} X_i + \sum_{i=1}^{l} R_i + \tau\right)}|R_1 \le \tau, X_2 > \delta, \dots, X_l > \delta, R_l \le \tau, X_{l+1} > \delta\right)\right. \\
&\quad \left. \times P(R_1 \le \tau, X_2 > \delta, \dots, X_l > \delta, R_l \le \tau, X_{l+1} > \delta)\right] \\
&= (1-p_1)e^{-s\tau} L_1(s) + e^{-s\tau}(1-p_1)L_1(s) \sum_{m=2}^{l} (L_2(s)L_3(s))^{m-1}(p_1p_2)^{m-1} \\
&\quad + p_1(1-p_2)L_1(s)L_3(s)E\left(e^{-sX_1}|X_1 \le \delta\right) \sum_{m=2}^{l} (p_1p_2)^{m-2}(L_3(s)L_2(s))^{m-2} \\
&\quad + L_1(s)(L_2(s))^{l-1}(L_3(s))^{l} p_1^{l} p_2^{l-1}(1-p_2)E\left(e^{-sX_1}|X_1 \le \delta\right) \\
&\quad + e^{-s\tau} L_1(s)(L_3(s))^{l}(L_2(s))^{l} p_1^{l} p_2^{l} \\
&= (1-p_1)e^{-s\tau} L_1(s) + e^{-s\tau}(1-p_1)L_1(s) \sum_{m=2}^{l} (L_2(s)L_3(s))^{m-1}(p_1p_2)^{m-1} \\
&\quad + p_1(1-p_2)L_1(s)L_3(s)E\left(e^{-sX_1}|X_1 \le \delta\right) \sum_{m=2}^{l+1} (p_1p_2)^{m-2}(L_3(s)L_2(s))^{m-2} \\
&\quad + e^{-s\tau} L_1(s)(L_3(s))^{l}(L_2(s))^{l} p_1^{l} p_2^{l} \\
&= (1-p_1)e^{-s\tau} L_1(t) + e^{-s\tau}(1-p_1)p_1p_2L_1(s)L_2(s)L_3(s)\left(\frac{1-(p_1p_2L_2(s)L_3(s))^{l-1}}{1-p_1p_2L_2(s)L_3(s)}\right) \\
&\quad + p_1(1-p_2)L_1(s)L_3(s)E\left(e^{-sX_1}|X_1 \le \delta\right)\left(\frac{1-(p_1p_2L_2(s)L_3(s))^{l}}{1-p_1p_2L_2(s)L_3(s)}\right) \\
&\quad + e^{-s\tau} L_1(s)(L_3(s))^{l}(L_2(s))^{l} p_1^{l} p_2^{l} \\
&= (1-p_1)e^{-s\tau} L_1(s) + \left(\frac{p_1L_1(s)L_3(s)(L_1(s)-p_2L_2(s))(1-(p_1p_2L_2(s)L_3(s))^{l})}{1-p_1p_2L_2(s)L_3(s)}\right) \\
&\quad + \frac{p_1p_2e^{-s\tau} L_1(s)L_2(s)L_3(s)(1-p_1) + p_1(p_1p_2L_2(s)L_3(s))^{l-1} - (p_1p_2L_2(s)L_3(s))^{l}}{1-p_1p_2L_2(s)L_3(s)},
\end{aligned}
$$

Box II.

$$
\begin{aligned}
E(T_{FF}^{l}) &= \sum_{m=1}^{l+1} E(T_{FF}^{l}|M_l = m)P(M_l = m) \\
&= E(T_{FF}^{l}|M_l = 1)P(M_l = 1) + \sum_{m=2}^{l} E(T_{FF}^{l}|M_l = m)P(M_l = m) \\
&\quad + E(T_{FF}^{l}|M_l = l+1)P(M_l = l+1) \\
&= E(X_1 + \tau|R_1 > \tau)P(R_1 > \tau) \\
&\quad + \sum_{m=2}^{l} E\left(\sum_{i=1}^{m} X_i + \sum_{i=1}^{m-1} R_i|A_m\right)P(A_m) + \sum_{m=2}^{l} E\left(\sum_{i=1}^{m} X_i + \sum_{i=1}^{m-1} R_i + \tau|B_m\right)P(B_m) \\
&\quad + \left[E\left(\sum_{i=1}^{l+1} X_i + \sum_{i=1}^{l} R_i|R_1 \le \tau, X_2 > \delta, \ldots, X_l > \delta, R_l \le \tau, X_{l+1} \le \delta\right)\right. \\
&\qquad \left.\times P(R_1 \le \tau, X_2 > \delta, \ldots, X_l > \delta, R_l \le \tau, X_{l+1} \le \delta)\right] \\
&\quad + \left[E\left(\sum_{i=1}^{l+1} X_i + \sum_{i=1}^{l} R_i + \tau|R_1 \le \tau, X_2 > \delta, \ldots, X_l > \delta, R_l \le \tau, X_{l+1} > \delta\right)\right. \\
&\qquad \left.\times P(R_1 \le \tau, X_2 > \delta, \ldots, X_l > \delta, R_l \le \tau, X_{l+1} > \delta)\right] \\
&= (E(X_1) + \tau)(1 - p_1) + \left[\sum_{m=2}^{l}(E(X_1) + E(X_1|X_1 \le \delta) + E(X_1|X_1 > \delta)(m-2) +\right. \\
&\quad \left.(m-1)E(R_1|R_1 \le \tau))p_1^{m-1}p_2^{m-2}(1 - p_2)\right] + \left[\sum_{m=2}^{l}(E(X_1) + E(X_1|X_1 > \delta)(m-1)\right. \\
&\quad \left.+ (m-1)E(R_1|R_1 \le \tau) + \tau)p_1^{m-1}p_2^{m-1}(1 - p_1)\right] + \Big(E(X_1) + (l-1)E(X_1|X_1 > \delta) \\
&\quad + E(X_1|X_1 \le \delta) + lE(R_1|R_1 \le \tau)\Big)p_1^{l}p_2^{l-1}(1 - p_2) + \Big(E(X_1) + lE(X_1|X_1 > \delta) \\
&\quad + lE(R_1|R_1 \le \tau)\Big)p_1^{l}p_2^{l} \\
&= E(X_1)\left[(1 - p_1) + \left(\sum_{m=2}^{l} p_1^{m-1}p_2^{m-2}(1 - p_2)\right) + \left(\sum_{m=2}^{l} p_1^{m-1}p_2^{m-1}(1 - p_1)\right) + p_1^{l}p_2^{l-1}\right] \\
&\quad + \tau\left((1 - p_1) + \sum_{m=2}^{l} p_1^{m-1}p_2^{m-1}(1 - p_1) + p_1^{l}p_2^{l}\right) + E(R_1|R_1 \le \tau)\left[\left(\sum_{m=2}^{l}(m-1)p_1^{m-1}\right.\right. \\
&\quad \left.\left.\times p_2^{m-2}(1 - p_2)\right) + \left(\sum_{m=2}^{l}(m-1)p_1^{m-1}p_2^{m-1}(1 - p_1)\right) + lp_1^{l}p_2^{l-1}\right] + E(X_1|X_1 \le \delta) \\
&\quad \times\left(\sum_{m=2}^{l+1} p_1^{m-1}p_2^{m-2}(1 - p_2)\right) + E(X_1|X_1 > \delta)\left[p_1(1 - p_2)\left(\sum_{m=2}^{l}(m-2)(p_1p_2)^{m-2}\right)\right. \\
&\quad \left.+ (1 - p_1)\left(\sum_{m=3}^{l}(m-1)(p_1p_2)^{m-1}\right) + (l-1)p_1^{l}p_2^{l-1} + p_1^{l}p_2^{l}\right] \\
&= E(X_1) + E(X_1|X_1 \le \delta)\left(p_1(1 - p_2)\left(\frac{1 - (p_1p_2)^{l}}{1 - p_1p_2}\right)\right) \\
&\quad + E(R_1|R_1 \le \tau)\left(\frac{p_1}{1 - p_1p_2}\right)(1 - (p_1p_2)^{l}) + E(X_1|X_1 > \delta)p_1p_2\left(\frac{1 - (p_1p_2)^{l}}{1 - p_1p_2}\right) \\
&\quad + \tau\left(\frac{1 - p_1}{1 - p_1p_2}\right)(1 - (p_1p_2)^{l+1}) \\
&= E(X_1) + (E(X_1) + E(R_1|R_1 \le \tau))\left(\frac{p_1}{1 - p_1p_2}\right)(1 - (p_1p_2)^{l}) + \tau\left(\frac{1 - p_1}{1 - p_1p_2}\right)(1 - (p_1p_2)^{l+1}),
\end{aligned}
$$

Box III.

$$E\left(e^{-sT^l_{FF}}\right) = (1-p_1)L_1(s)L_4(s) + \left(\frac{p_1L_1(s)L_3(s)(L_1(s)-p_2L_2(s))(1-(p_1p_2L_2(s)L_3(s))^l)}{1-p_1p_2L_2(s)L_3(s)}\right) + \frac{p_1p_2L_1(s)L_2(s)L_3(s)L_4(s)((1-p_1)+p_1(p_1p_2L_2(s)L_3(s))^{l-1}-(p_1p_2L_2(s)L_3(s))^l)}{1-p_1p_2L_2(s)L_3(s)},$$

Box IV.

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