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Seven Derivations of the Lorentz Transformations in Special Relativity are Seen to be Invalid

James McKelvie

Department of Mechanical and Aerospace Engineering, University of Strathclyde, 75 Montrose St Glasgow, G1 1XQ, UK

ABSTRACT

Lorentz introduced his transformations as mere proposals for the relations between the E and H fields of light as measured in two reference systems in relative motion [1]. He proffered no argument to justify them. Others have subsequently given their own arguments to derive them, particularly in the context of Special Relativity. Seven derivations of the Lorentz transformations (LT's) are examined in detail. It is seen that all of them are invalid. The Appendix discusses the matter of equations and transformations. It is seen that a transformation is of necessity an inequality.

***Corresponding author**

James McKelvie, Department of Mechanical and Aerospace Engineering, University of Strathclyde, 75 Montrose St Glasgow, G1 1XQ, UK.

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Einstein's first derivation [1905]

In his 1905 paper [2], Einstein derived the Lorentz Transformations, which derivation was fundamental to his discussion that followed. Actually, this was the first derivation. Lorentz introduced them merely as proposals.

Einstein's 1905 paper (translated), is available free to download at http://hermes.ffn.ub.es/luisnavarro/nuevo_maletin/Einstein_1905_relativity.pdf

If there are mistakes in the derivation, what would that imply? (One would presume that it would be serious.) In fact, there two serious mistakes:

First Mistake in Einstein 1905

Einstein considers two Cartesian systems: K(x,y,z,t) which is stationary, and k(ξ,η,ζ,τ) which is moving in the positive x direction at υ with the x- ξ axes parallel. He introduces a parameter $x' = x - \upsilon t$ and [to quote] "it is clear that a point at rest in system k must have a set of values x′,y,z *independent of time*.", [present author's italics because of the significance.]

Using what is a simple thought experiment (a light signal is sent and reflected back) he develops a differential equation: [to quote, - "applying the principle of the constancy of the velocity of light in the stationary system" – which implies "as viewed from within K"]. The equation in question is

$$\frac{\partial \tau}{\partial x'} + \frac{\upsilon}{c^2 - \upsilon^2}\frac{\partial \tau}{\partial t} = 0$$

Since x′ is independent of time, $\frac{\partial x'}{\partial \tau} = 0$ and therefore $\frac{\partial \tau}{\partial x'} = \infty$ and so, since adding anything to ∞ is still infinity, the equation is saying that ∞ = 0.

In other words, the equation has no meaning.

Alternatively, rearranging, $\frac{\partial \tau}{\partial x'} + \frac{\upsilon}{c^2 - \upsilon^2}\frac{\partial \tau}{\partial t} = 0$

By definition $\frac{\partial x'}{\partial \tau} = 0$ The implications then would be that either $\upsilon = c$, or $\frac{\partial t}{\partial \tau} = 0$. The first is not conceivable for the moving clock in the thought experiment, for then the light signal would never reach the target clock (and anyway, according to SR, it is impossible except for light) and the second cannot be true - because it is clear that if τ changes then there must be some kind of appropriate change in t (unless that t-clock is not working).

Thus, we see that $\frac{\partial t}{\partial \tau}$ cannot = 0 and **the differential equation has no validity - and its solution is central in the derivation of the eventual conclusion – the LT′s.**

Second Mistake in Einstein 1905

[This not at all obvious, and requires careful consideration]

At one point the analysis leads to

$$\tau = \varphi(\upsilon)\beta(t - \upsilon x/c^2) \quad (1)$$
$$\xi = \varphi(\upsilon)\beta(x - \upsilon t) \quad (2)$$
$$\eta = \varphi(\upsilon)y \quad (3)$$
$$\zeta = \varphi(\upsilon)z \quad (4)$$
$$\text{where } \beta = [1- \upsilon^2/c^2]^{-1/2} \quad (5)$$

In order to establish φ(υ) he introduces a third system of coordinates K'(x′,y′,z′,t′), with x′ parallel to ξ, which system, relatively to the system k, is in a state of parallel translatory motion parallel to the x′-ξ direction such that the origin of coordinates of system K' moves with velocity -υ on the axis ξ. He continues [to quote] "At the time t = 0 let all three origins coincide, and when t = x = y = z

= 0 let the time t′ of the system K’ be zero”. Now, by a [to quote] “twofold application of the equations of the transformations” i.e. by firstly writing the appropriate analogies of (1),(2),(3),(4) for the K’ and k systems, [and then substituting τ and ξ from (1) & (2)] he arrives at

$$t' = \varphi(-\upsilon)\beta(-\upsilon)(\tau + \upsilon\xi/c^2) \quad \rightarrow \quad = \varphi(\upsilon)\varphi(-\upsilon)t \qquad (6)$$
$$x' = \varphi(-\upsilon)\beta(-\upsilon)\xi + \upsilon t) \quad \rightarrow \quad = \varphi(\upsilon)\varphi(-\upsilon)x \qquad (7)$$
$$y' = \varphi(-\upsilon)\eta \quad \rightarrow \quad = \varphi(\upsilon)\varphi(-\upsilon)y \qquad (8)$$
$$z' = \varphi(-\upsilon)\zeta \quad \rightarrow \quad = \varphi(\upsilon)\varphi(-\upsilon)z \qquad (9)$$

Since k and K’ are clearly identical, then $t' = t$, $x' = x$, etc., so that $\varphi(\upsilon)\varphi(-\upsilon) = 1$, and by examining the behaviour of a rod lying in the y-direction, it is concluded that $\varphi(\upsilon) = \varphi(-\upsilon) = 1$. Thus (1), (2), (3) and (4) are seen to yield the Lorentz transformations for τ, ξ, η, and ζ. [Note that whereas (1), (2), (3) and (4) were for “as viewed from within K”, for (6), (7), (8) and (9) they are, by analogy, *as viewed from within* k. This is central to what follows].

We now consider the question of the meaning of the terms in the LT for, say, τ, - i.e. (1) with $\varphi(\upsilon) = 1$:

It is quite clear then that in (1), we cannot have τ and t both meaning “the time on the clocks as viewed in their respective systems” because a standard clock viewed in its own system will be going at the standard rate - i.e. the transformation would be just $\tau = t$. In fact, the τ actually has a different value from the t. [That is an essence of SR.] The usual interpretation is that τ is the time on the k-clock *as viewed from* within K, and that t is the time on the K-clock *also as viewed from within* K. The same interpretation in (6) gives that the t′ and the τ are both as viewed in k. But, as viewed in k, the τ, for a standard clock, is going at the standard rate, whereas viewed in K it is going at less than standard rate. In short, the τ in (1) has a different value from the τ in (6), since they are views of the same clock *from different systems that are in relative motion*, - K and k. Similarly in (2), ξ is interpreted as a dimension in k (giving the position of a k-clock) but length-contracted *as viewed from* K, whereas in (7) it is also the position of the k-clock, *but as viewed* from k (i.e. not length-contracted). In short, again, the ξ in (7) has a different value from the ξ in (2). That these are *as viewed* interpretations is explicitly confirmed by Einstein himself at various points in the paper.

So, whereas, *prima facie*, the first parts of (6) and (7), as between systems K’ and k, correspond directly to (1) and (2) between k and K, that is in fact *incorrect*.

It is actually the essence of the conclusions [of Einstein 1905] that the times and positions are different as viewed from the different systems, and this makes the substitutions invalid.

So we see that in the 1905 paper the **substitutions used from (1) and (2) into (6) and (7) are invalid**.

Therefore, from Einstein’s derivation, **we cannot, on two counts, have any faith the LT′**s as proper relations as between clock times and distances in different inertial systems.

Møller’s Derivation

Møller [2] gives a derivation of the LT′s based on the constancy of the speed of light. His application of the constancy is to consider that at the instant when the origins of the two moving frames coincide, a flash of light with spherical wave-front is emitted there. Moller’s central assertion is that the constancy of c implies that the sphericity is maintained in both the “stationary” and the “moving” reference frames. He constructs the equation of the sphere for both frames, and after some intermediate work he equates coefficients and arrives at the LT′s. Using the “primed” notation for the moving system, his derived SR time relation of the LT′s for clocks (in his notation) is

$$t' = t[1 - \upsilon^2/c^2]^{1/2} \qquad (10)$$

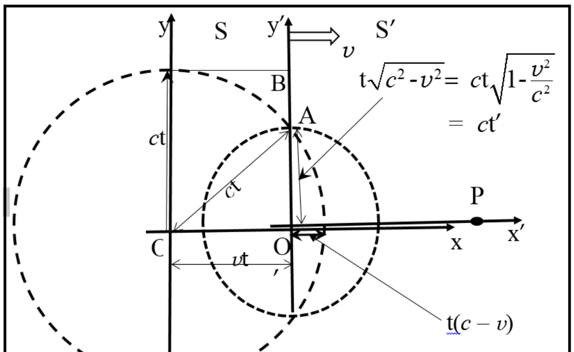


Fig. 1

Figure 1 depicts the x-y plane with the two frames' origins a distance υt apart at a time t after the coincidence of the origins and of the flash. The x- and x′- axes are shown separated for clarity. The large circle represents a section through the spherical wave-front centred on the stationary frame origin O at the time t, with radius ct. That wave-front intercepts the y′ axis at the point A at that instant. It is seen that the intercept AO′ has length $t[c^2 - \upsilon^2]^{1/2}$. That is seen to be ct′, in accordance with (10), and which, because the same geometry would hold in the y′ and z′ axes, would therefore be the radius of a spherical wave-front ($x'^2 + y'^2 + z'^2 = ct'^2$) in the moving frame S′, *precisely* as *Møller specifies*, which sphere is therefore represented by the smaller circle centred on the origin O′ of the moving frame. (The circle is smaller because $t' < t$ according to (10) and c is constant).

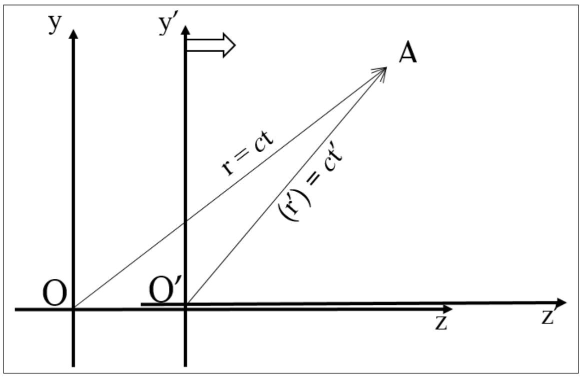
The figure, then, is a depiction of what Møller’s scenario looks like physically. If we consider an observer positioned permanently at the point P in x′, then, because both wave-fronts are proceeding at *c, he will experience a flash* as the “smaller” wave-front reaches him, and he *will see a second flash* as the “larger” one reaches him, that is, *there are two different spherical wave-fronts in the space*.

Looking at the physics: we have one specific point in space where, at one specific point in time, there is a flash, which sends out a spherical wave-front. That is the physics. We now conceptualise two set of Cartesian coordinates, S and S′, one of which S has its origin fixed at the flash-point, while the other S′ moves at υ in the x-direction, and whose origin coincides with the flash-point when it happens. Moller’s thesis is that there are two wave-fronts. *That is not possible from one flash*. There is only one wave-front traversing the space, and an observer can see only one flash as it reaches him. In reality, at time t the wave front will have reached a distance t($c - \upsilon$) beyond O’ on the x′ axis, plus point A on the y′ axis, and a point corresponding to A on the z′ axis, together with similar points on the -y′ axis.

To have another spherical wave would require another flash.

Thus, we see that Møller’s derivation is based on a false premise, and is not valid.

Konopinski's Derivation



Konopinski [3] gives a derivation of the LT′s that is based on the same principle as Møller's, (that the speed of light is the same in both frames). In his case he indicates, at least initially, that there is only one wave-front, and it is observed in both S and S′.

A version of his Figure11.1 is shown here as Figure 2. It is restricted to only one plane. It indicates his distances r and r', from the respective origins, of a point A on the wave-front. [Konopinski's own diagram does not show r' = ct′, but it stated in the text].

Although he does not actually say that there are two spherical waves, he writes $r = (x^2 + y^2 + z^2)^{½}$ and $r' = [(x')^2 + (y')^2 + (z')^2)]^{½}$ which clearly indicates that, since he has taken c to be constant, then *there should be two spherical wave-fronts* centred, respectively at the origins of S and S′ and, in principle, his description is simply a repetition of Møller's and is not valid.

Aharoni's Derivation

Aharoni [4], gives a different derivation of the LT′s. He starts with extremely general expressions for the space and time coordinates lying at completely general angles, all in three dimensions, with system K'(x′,y′,z′,t′) moving at υ w.r.t. K(x,y,z,t) in the direction of the x-axis. These expressions are reduced to the typically utilised geometry of parallel axes in two dimensions. We will use {} for his equation numbers.

After many pages, for viewing from K, he arrives at,

$$x' = ax \;+ a\upsilon t$$
$$y' = by$$
$$z' = bz$$
$$t' = a_{41}x \;+ at \qquad \{2.21\}$$

where the coefficients are to be determined.

He then gives the "reverse equations" – by which he means the *algebraically reversed* equations –developed from his original {2.13}:

$$x = (a - a_{41})^{-1}x' \qquad - (a - a_{41})^{-1}\upsilon t'$$
$$y = b^{-1}y'$$
$$z = b^{-1}z'$$
$$t = -a^{-1}(a - a_{41}\upsilon)^{-1}a_{41}x' \qquad + (a - a_{41}\upsilon)^{-1}t' \qquad \{2.23\}$$

He now says that "the coefficients of {2.23} should be obtainable from those of {2.21} by replacing $a(\upsilon)$ by $a(-\upsilon)$……" What he is saying here is the if we take the standard procedure for converting from the view from the stationary system K to the view from the moving system K' (now being regarded as the stationary one), - which is to interchange the x,y,z,t with x′,y′,z′,t′ and change the sign of υ) - then we will have two independent equations. So, he is saying that simply by writing the equations for the views from K' and from K he can determine the coefficients. That is not correct, for the following reason: in Einstein's derivation he says "let… all the clocks of the two systems, be in all respects alike". Aharoni's version must include the same condition, i.e. viewing from the stationary K, the clock represented by t is taken to be a standard clock viewed from its rest system and therefore is seen to be going at standard rate, and t′, although also a standard clock, is,by the conclusion arrived at, seen to be going at a different, slower, rate. In the case of viewing from K' (as he suggests we consider), it is t′ that is being viewed from its own rest system, and by analogy with the K-view of t, it is now t′ that is seen to be 4going at standard rate. Thus the two t′ in the two views represent clocks going at different rates, as do the two t′s and we cannot solve for the coefficients because there are too many unknowns. [In this, he makes the same error as in the second derivation of Einstein at (Second mistake in Einstein 1905)].

This is a fundamental error, and the derivation that then proceeds cannot be correct.

Thus, we cannot have any faith in Aharoni's derivation of the LT′s

Schwartz′s Derivation

Schwartz [5] starts with algebraic transposing of the general linear relations between the coordinate systems using Einstein's notation. Quoting now, and using {} to denote Schwartz′s equation numbers, with Greek notation representing variables in the moving frame

$$\xi = Ax + Bt, \quad = Cx + Dt \qquad \{3.6\}$$

which yield $x = \frac{1}{\Delta}(D\xi - B\tau)$

where $\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$ {3.7}

$$t = \frac{1}{\Delta}(-C\xi + A\tau)$$

End of quotation

Those equations are correct. He now proceeds to say "From the requirement …..that the velocity of S relative to Σ as measured in Σ should be the negative of the velocity of Σ relative to S as measured in S, it follows that

$$\left.\frac{dx}{dt}\right|_{\xi=0} = \upsilon = -\frac{B}{A} \qquad \left.\frac{d\xi}{d\tau}\right|_{x=0} = -\upsilon = \frac{B}{D} \qquad \{3.8\}$$

Hence $\quad A = D \qquad \{3.9\}$

End of quotation.

His statement about the velocities is correct, as are equations {3.6} to {3.9}. He then proceeds to say "Again…… it can be concluded that the contraction of lengths in Σ as found in S must be the same as the corresponding contraction in S as measured in Σ, with a *similar result holding for dilation of time intervals*" (present author's italics), and his expression for mutual length contraction is

$$\left.\frac{d\xi}{dx}\right|_{t=0} = \left.\frac{dx}{d\xi}\right|_{\tau=0} = \frac{D}{\Delta} \qquad \{3.10\}$$

From either one of these equations and Equation {3.9} it then follows that

$$\Delta = 1$$

Schwartz says that it is a matter of "in Σ as found in S" and "in S as measured in Σ". He is saying *as viewed*. That wording too is a correct statement, consistent with the Principle of Relativity. If we examine the length relation of {3-10} *in the physics*, what does it mean? It means that his statement on velocities in {3.9} above) is saying that just as $\frac{d\xi}{dx} = \frac{dx}{d\xi}$ with appropriate viewing, so, by the same mutuality, his analogous result is $\frac{d\tau}{dt} = \frac{dt'}{d\tau'}$ which we will number as (3.11)

where the primes represent variables "as measured from the Σ system" [*not* "variables in the Σ system"], and un-primed means as measured in the S system. But measured in S, dt is standard rate, and measured in Σ, dτ is standard rate (as explained in connection with Aharoni's derivation in §3). That means that in his *analogous result* the two denominators are numerically equal, and so the numerators are numerically equal. That is, dτ viewed from S is the same as dt viewed from Σ. By mutuality, that means that dτ = dt.

Therefore Schwartz′s argument stalls, because he is at this point saying, dτ = dt. This contradicts his conclusions which say that $d\tau = dt[1-\upsilon^2/c^2]^{½}$. So the analysis is seen to be seriously flawed, and **we can have no faith in Schwatrz′s derivation of the LT′s**.

A Second Derivation by Einstein

In Appendix 1 of his "popular" text, Einstein [6], gives his second derivation of the LT′s. Again the author's own equation numbers are shown in braces {}

He says, in the stationary frame, a light wavefront, for a flash at x = x′ = t = t′ =0, will be positioned at x = *c*t, and in the moving frame, at x′ = *c*t′, i.e.

i.e. $x - ct = 0$ {1}
and $x' - ct' = 0.$ {2}

He restates a consequence of these, more generally, as

$(x' - ct') = \lambda(x - ct)$ {3}

and, for the case of υ being reversed,
$(x' + ct') = \mu(x + ct)$ {4}

After adding and subtracting {3} and {4}, he introduces parameters *a* and *b*

$$a = \frac{\lambda + \mu}{2}$$

and

$$b = \frac{\lambda - \mu}{2}$$

These yield

$$x' = ax - bct$$
$$ct' = act - bx \quad \{5\}$$

He says "For the origin of K' [the moving system] we have permanently x′ = 0", and hence according to the first of equations {5}

$x = \frac{bc}{a}t$ [which he does not number; but we shall call it …….{5a}]

Now, x in {5} is the instantaneous position of a wavefront [see before eqn {1}], whereas in {5a} it is *stated* as the position of the origin of K' in K. Equally in {5}, x′ is the position of a wavefront. He cannot put x′ = 0 into {5} as the position of the origin of x′ and get $x = \frac{bc}{a}t$ as the general position of the wavefront and to be used as such thereafter. It would mean that the wavefront is stuck at x′= 0 thereafter. Continuing with this confusion (between x for both the positions of the origin of K' and the position of the wavefront) is futile: if we compare the equation {5a} with his first statement, {1}, x = *c*t, we have *a* = *b*, which would mean that his μ is zero.

Now he says "if we call υ the velocity with which the origin of K' is moving relative to K we then have $x = \frac{bc}{a}t$

$$\upsilon = \frac{bc}{a} \text{"} \quad \{6\}$$

[end of quotation]

Since *a* = *b*, that means that $\upsilon = c$. Something has clearly gone awry. *His use of his nomenclature and of his equations* {5} *and* {5a} *is confused, and so* {6} *is invalid.*

Using equation {6} is an essential step in his further argument leading to the LT′s, and so **we can have no faith in this second derivation by Einstein**.

A Derivation by Eddington

Eddington [7] gives a derivation of the LT′s. However, at one point he says without any justification "Make the following transformation of coordinates..." So, mathematically, that is not a valid derivation. [He makes that transformation simply from having knowledge of Einstein's result.]

Therefore, we can have no faith in Eddington's derivation.

Conclusion

Seven derivations of the LT′s have been examined and all have been shown to be invalid. However, that does not mean necessarily that Lorentz′s proposals for determining the relations for the E and H fields of light in moving reference frames are invalid. This is discussed in the Appendix. Further, it is obviously possible that some other worker may have presented a derivation which is not flawed, - but that seems to be unlikely, given the now general acceptance of the LT′s, there being no motivation to present yet another derivation.

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Appendix

Reconciling the repudiation of the LT's with Lorentz's original proposal

The derivations of the Lorentz transformations have been shown to be invalid in the context of SR, but that does not mean that they may not be applied for the purpose for which they were originally proposed by Lorentz.

A1. Lorentz's Introduction

In Lorentz [1], a major purpose for the introduction of his transformations was for defining the Maxwell equations for light in a reference system moving at constant speed relative to the source. That is, to determine how the E and H fields change. He did not develop them by mathematical derivation. He simply "proposed" them. For instance, he says at his equation {3} "We shall further *transform* these formulae [present author's italics] by a change of variables

$$\frac{c^2}{c^2 - \upsilon^2} = \beta^2 \tag{1a}$$

[end of quotation].

That was merely a *proposal*, with no preceding argument specifically for it. It is a relation to be used to develop a "transformation". He does proceed thereafter with mathematical derivation using that proposal, but that means that his eventual conclusions are actually not truly totally derived. They are, at their heart, proposals.

A2. Of Equations and Transformations

It is a matter of regret that, in considering transformations, mathematicians have always used the equality sign "=". If we consider a transformation that moves the line y = x by two units in the negative x-direction, it is given by x = x + 2. There is no such equality. It is an equation that that does not have any physical meaning. In mathematics and physics, it represents an impossibility.

But *it is an effective transformation*. What are we to make of that? The answer is that such a transformation represents, not an equality, but a *replacement process*. We *replace* x by x + 2. Strictly speaking we should have a symbol that means "is to be replaced by" – but such a symbol is not available to us.

So, whereas it has been shown above that Einstein's (and others') derivation of the transformations, - using only mathematics and ending up with an "equals" sign – is flawed, that does not mean that they are flawed *as transformations*, as Lorentz intended. As transformations are not equalities, we cannot make the kind of subsequent mathematical deductions that we are used to with equalities. In fact, a transformation is *of necessity* an inequality, for if it be an equality, then we are replacing the variable by something that is actually the same thing in a different guise, and there can be no resulting transformation.

A3. Actual transformations

There is nothing wrong with the following "transformation" which Lorentz might just as readily have written

$$\mathrm{t} = \frac{1}{\sqrt{1-\upsilon^2/c^2}}\left(\mathrm{t} - \frac{\upsilon x}{c^2}\right) \tag{2a}$$

which is equation (1) with φ(υ) = 1 and the τ replaced by t, – just as there is nothing wrong with the transformation x = x + 2. The t's are the same, and the equation looks to be impossible, but it is a perfectly valid transformation. It will give results for the E and H fields. (Of course, using that transformation (2a), nothing of SR would follow.)

Thus, Lorentz's statement that "the variable t' [his nomenclature] may be called 'local time'" is not necessary. He could have dispensed with t', by writing just t, and while having a "nonsense" equation, it would still be a valid transformation.

A4. The transformed Maxwell equations

So, whereas the derivations of the LT's considered here have been shown to be invalid, that does not mean that they are not valid for application to the Maxwell equations, because a transformation (2a), suffices, albeit it makes no sense as an equation.

It is a pity that there does not appear to be any experimental work having been done to investigate whether the proposals according to Lorentz for the E and H fields in a moving reference frame might be verified. But it is entirely possible that they are correct – though the LTs, as derived, expressed as *equations* and utilised in SR, are not valid - as derived in the seven examples quoted. [Of course, it is possible that a derivation that shows them to be valid may have been given and not come to the attention of the present author. But if they were shown to be equalities, then that would make them invalid as transformations, because to achieve a transformation the relation must, of necessity, be an inequality].

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