A bathtub pattern for repairable systems: from better-than-minimal to minimal and worse-than-minimal repairs

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Abstract

We introduce a new combined repair process to describe repairs that are initially better-than-minimal, then become minimal, before finally becoming worse-than-minimal. The extended generalized Polya process (EGPP), non-homogeneous Poisson process (NHPP), and generalized Polya process (GPP) are used to describe this repair pattern, respectively. Several useful properties are derived for the combined process under two settings: change in repair type after a specified time and change in repair type after a specified number of failures/repairs. As an application, the optimal age replacement problem is defined and its optimal solution is analyzed. Detailed numerical examples support our findings.

Keywords: Better-than-minimal repair; minimal repair; worse-than-minimal repair; minimal repair; optimal age replacement.

1 Introduction

In reliability literature, various repair models have been proposed to describe processes of failures and their subsequent repairs. These models can be vital tools for optimization, defining maintenance policies, and analyzing the performance of various repairable systems. As the repair time of systems in practice is usually much smaller than the corresponding lifetime, it is often assumed that each failure of the repairable system is instantaneously repaired such that the failure and repair processes coincide. This has allowed for the use of a number of well-defined point stochastic processes for modelling and as such, will be the assumption throughout this paper.

Two basic repair types that are commonly used in reliability applications are perfect and minimal repairs. Perfect repairs restore a system to the 'as good as new' state and can be thought of as an equivalent

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to instantaneously replacing the system with a new one. As such, the corresponding process of repairs can be modeled by the classical renewal process (see, e.g., Barlow and Proschan [1]). On the other hand, minimal repairs restore a system to the 'as bad as old' state. A commonly used example of this is in a multi-component system, where one of the minor components fails, and only this component is repaired or replaced. This imperfect repair model, proposed by Barlow and Hunter [2], is widely used in numerous applications. Minimal repair provides a more realistic and mathematically tractable model to describe repairs in practice. It is well known that the process of instantaneous minimal repairs can be described by the non-homogeneous Poisson process (NHPP) with rate corresponding to the failure rate of the underlying lifetime distribution. As the NHPP allows for closed-form analytical results, a number of minimal repair-based models have been considered in the literature (see, for example, Tadj et al. [3] and references therein). It is important to note that, in practice, a repair is neither perfect nor minimal, and thus numerous other (intermediate) imperfect repair models (see, for example, Luo et al. [4] for a recent overview, as well as Tanwar et al. [5] and Pham and Wang [6]) have been extensively studied as some kind of natural generalizations to the concept of minimal repair.

In slightly more recent literature, there has been a move to describe other types of repairs. This includes repairs that are worse-than-minimal, which restore a system to the 'worse than old' state, and, as already mentioned above, repairs that are better-than-minimal, which implies that the state of the system after the repair is better than that just before the failure. Worse-than-minimal repairs take into account the impact that previous failures/repairs have had on the system and, in certain situations, can be a more realistic assumption than that of minimal repair. One of the simplest and most commonly used examples to illustrate this type of repair is that of the failure of a single component in a multi-component system. This single-component failure can increase the load on or cause additional damage to the non-failed components, increasing the overall failure rate. A number of more specific real-world applications can be found in Lee and Cha [7]. Further, Lee and Cha [7] have shown that worse-than-minimal repairs can be effectively described using the generalized Polya process (GPP), introduced and extensively studied in Cha [8]. Some other notable contributions to the applications of the GPP to repair modeling can be seen in, for example, Badía et al. [9] and Cha and Finkelstein [10].

On the other hand, and somewhat dual to the idea of worse-than-minimal repair (see the next section for a formal justification of this claim), is that of better-than-minimal repair of the considered *specific type*. This type of repair can occur when, for example, under some reasonable assumptions, a defect is eliminated from the failed system during the repair operation. This would result in a smaller failure rate than that before the failure/repair. These defects could be as a result of manufacturing discrepancies, damaged parts that may have initially gone unnoticed, some design faults, bugs in software, etc. Therefore, this type of repair can be considered in the framework of reliability growth modeling (see, for example, Finkelstein [11] and Singpurwalla and Wilson [12] for several models that were developed subsequent to the introduction of the basic Jelenski-Moranda [13] model for software reliability growth, as well as Al Turk and Alsolami [14] and references therein). In order to model the process of repairs that are better-than-minimal of the described type, a new point process called the extended generalized Polya process (EGPP) can be used. This process was recently suggested and studied in Cha [15].

There have also been a number of recent advances in combining various point processes to describe possible changes in the pattern of repairs. Cha et al. [16], for example, consider a delayed worse-than-minimal repair model, which combines the NHPP with the GPP to model repairs that are minimal up to a certain event or time, and then become worse-than-minimal. Similarly, Langston et al. [17] define a point process with finite memory, which combines the GPP with the NHPP to model repairs that are worse-than-minimal up to a certain event or time, and then become minimal. On the other hand, Cha and Finkelstein [18] and Finkelstein and Cha [19] also define a point process combining the NHPP with the GPP/EGPP in a different way to model repairs that are minimal with some given probability and are worse-than-minimal/better-than-minimal with the complimentary probability. These are meaningful generalizations to the well-known Brown-Proschan model [20], which accounts for the randomness in combining minimal and perfect repair.

In line with these recent advances, our paper goes further and defines a new combined (three-stage) repair process to describe repair operations that begin as better-than-minimal, which then become minimal, and then finally become worse-than-minimal. To describe this process, we introduce the EGPP+NHPP+GPP model, discuss some of its relevant stochastic properties, and, finally, develop the methodology for application to the optimal replacement problem. Our approach is innovative as we are the first to propose and describe the natural *long-term repair pattern* that has a loose analogy with the well-known bathtub curve for the failure rate of an item during its entire life cycle. Indeed, the EGPP, as discussed above, is decreasing the failure rate with each repair (similar to modeling the 'infant mortality' phase), minimal repairs do not change the failure rate (similar to modeling the 'normal' life phase), whereas the GPP increases the failure rate with each repair (similar to modeling the degradation phase).

Notably, all stages in this new process have a uniform stochastic description via the Poisson-induced point processes, which is not straightforward. Therefore, the main contribution of this paper is in proposing this three-stage process and developing a theoretical framework for its description. In addition to this, and to illustrate an application of the proposed model, we consider the corresponding optimal age replacement problem.

The paper is organized as follows. In Section 2, we provide some definitions and a brief preliminary discussion, as well as define the combined repair processes for the two settings. Section 3 is devoted to deriving the long-run expected cost rates, and an application of the combined repair process to the optimal age replacement problem. Numerical illustrations are presented in Section 4. Section 5 provides some concluding remarks.

2 Model Descriptions

2.1 Preliminaries

Point processes can be characterized in a number of different ways, and are most commonly characterized via the joint distribution of the times between successive events, or the joint distribution of the number of events in all finite sets of disjoint intervals. Throughout this paper, however, we will characterize our point process of interest through the notion of stochastic intensity. This is in line with the characterizations of the EGPP and the GPP that underlie our process. The stochastic intensity (or the intensity process), λ_t, $t \geq 0$ for some orderly (without multiple occurrences) point process $\{N(t), t \geq 0\}$ is defined as the following limit:

$$\lambda_t = \lim_{\Delta t \to 0} \frac{P\left(N\left(t, t+\Delta t\right) = 1 \mid H_{t-}\right)}{\Delta t} = \lim_{\Delta t \to 0} \frac{E\left[N\left(t, t+\Delta t\right) \mid H_{t-}\right]}{\Delta t},$$

where $N\left(t, t+\Delta t\right)$ represents the number of events in $[t, t+\Delta t)$ and $H_{t-} \equiv \{N\left(u\right), 0 \le u < t\}$ is the history (internal filtration) of the process in $[0, t)$, which can be equivalently defined in terms of $N\left(t-\right)$ and the sequential arrival times of the events $0 \le T_1 \le T_2 \le \cdots \le T_{N(t-)} < t$, where T_i is the time from 0 until the arrival of the ith event in $[0, t)$ (see, for example, Finkelstein [11] and Cha and Finkelstein [10]). For a more detailed discussion of the notion of stochastic intensity, we refer the reader to Aven and Jensen [21, 22].

It is well known that for the NHPP with rate $\lambda\left(t\right)$, the stochastic intensity is deterministic and reduces to the rate of the process. That is, for the NHPP, $\lambda_t = \lambda\left(t\right), t \ge 0$. The GPP, discussed and extensively studied in Cha [8], is defined as follows.

Definition 1. Generalized Polya Process

A counting process $\{N\left(t\right), t \ge 0\}$ is said to be a generalized Polya process (GPP) with the set of parameters $\left(\lambda\left(t\right), \alpha, \beta\right)$, $\alpha \ge 0$, $\beta > 0$, if

i. $N\left(0\right) = 0$;

ii. $\lambda_t = \left(\alpha N\left(t-\right) + \beta\right) \lambda\left(t\right)$.

From Cha [8], the GPP can be understood as a generalization of the NHPP, as the GPP with parameter set $\left(\lambda\left(t\right), \alpha = 0, \beta = 1\right)$ reduces to the NHPP with rate $\lambda\left(t\right)$. It is also clear from this definition that $\lambda\left(t\right)$ is some baseline rate function for the process, whereas α models the extent of positive dependence in the corresponding increments. That is, for $\alpha > 0$, the probability of occurrence of an event in the next infinitesimal time interval increases with the number of events that have occurred in the previous time interval. Lee and Cha [7] have shown that the general three parameter definition of the GPP given in Definition 1 can be re-parameterized, without loss of generality, using two parameters by considering $\beta = 1$. It obviously follows that the stochastic intensity corresponding to this re-parameterization is given by

$$\lambda_t = \left(\alpha N\left(t-\right) + 1\right) \lambda\left(t\right).$$

Throughout this paper, we will use the re-parameterization of this process as it sufficiently describes repairs that are worse-than-minimal [7]. For ease of notation, we will refer to the GPP with parameter set $\left(\lambda\left(t\right), \alpha\right)$. In order to define our point process of interest probabilistically, we will require the following supplementary results for the GPP:

Lemma 1. *For the GPP with set of parameters $\left(\lambda\left(t\right), \alpha\right), \alpha > 0$, the following properties hold:*

i. The distribution of $N\left(t\right)$ is given by

$$P\left(N\left(t\right) = n\right) = \frac{\Gamma\left(\frac{1}{\alpha} + n\right)}{\Gamma\left(\frac{1}{\alpha}\right) n!} \left(1 - \exp\left\{-\alpha \Lambda\left(t\right)\right\}\right)^n \exp\left\{-\Lambda\left(t\right)\right\}, \quad n = 0, 1, 2, \ldots \tag{1}$$

where $\Lambda\left(t\right) \equiv \int_0^t \lambda\left(u\right) du$.

ii. $E[N(t)] = \frac{1}{\alpha}(\exp\{\alpha\Lambda(t)\} - 1)$.

The proofs for these properties, as well as an extensive discussion of other properties for the GPP with parameter set $(\lambda(t), \alpha, \beta)$, $\alpha \geq 0$, $\beta > 0$, can be found in Cha [8].

The EGPP, recently introduced and stochastically described by Cha [15], is defined as follows.

Definition 2. Extended Generalized Polya Process

A counting process $\{N(t), t \geq 0\}$ is said to be an extended generalized Polya process (EGPP) with the set of parameters $(\lambda(t), l_0)$, where l_0 is a positive integer, if

i. $N(0) = 0$;

ii. $\lambda_t = (l_0 - N(t-))\lambda(t)$,

where, as previously, $\lambda(t)$ is some baseline rate function for the process, but now there is a kind of negative dependence in the corresponding increments. That is, for $N(t-) < l_0$, the probability of occurrence of an event in the next infinitesimal time interval decreases with the number of events that have occurred in the previous time interval. On the other hand, once $N(t-) = l_0$, no additional events can occur as the process *terminates* in this case. This process, therefore, can describe repairs that are better-than-minimal (see, for example, Finkelstein and Cha [19] and our discussion in the Introduction above). In order to define our point process of interest probabilistically, we will require the following supplementary results for the EGPP:

Lemma 2. *For the EGPP with set of parameters* $(\lambda(t), l_0)$*,where* l_0 *is a positive integer, the following properties hold:*

i. The distribution of $N(t)$ *is given by*

$$P(N(t) = n) = \binom{l_0}{n}(1 - \exp\{-\Lambda(t)\})^n (\exp\{-\Lambda(t)\})^{l_0 - n}, \quad n = 0, 1, 2, \ldots, l_0 \tag{2}$$

where $\Lambda(t) \equiv \int_0^t \lambda(u)\,du$.

ii. $E[N(t)] = l_0(1 - \exp\{-\Lambda(t)\})$.

The proofs for these properties, as well as an extensive discussion of other properties for the EGPP, can be found in Cha [15].

In addition to the notation and definitions above, note that throughout this paper, we will define the cumulative baseline rate function in the interval $(a, b]$ as $\Lambda(b, a) = \int_a^b \lambda(u)\,\mathrm{d}u = \int_0^b \lambda(u)\,\mathrm{d}u - \int_0^a \lambda(u)\,\mathrm{d}u$.

In the sections to follow, two models to describe the new combined repair process will be defined.

2.2 Model 1: Repair type changes after some time

The repair is better-than-minimal (EGPP) for $t \leq s$, the repair is minimal (NHPP) for $s < t \leq u$, and the repair is worse-than-minimal (GPP) for $t > u$. The described combined repair process can be defined as follows.

Definition 3. Combined Repair Process I

i. $N(0) = 0$;

ii. $\lambda_t = (l_0 - N(t-))\lambda(t), \quad t \le s$;

iii. $\lambda_t = (l_0 - N(s))\lambda(t), \quad s < t \le u$;

iv. $\lambda_t = (\alpha(N(t-) - N(u)) + 1)(l_0 - N(s))\lambda(t), \quad t > u$.

In some sense, we can think of the initial stage of this process when $t \le s$ as some kind of testing period for the system where the majority of major defects are eliminated. This would usually imply (but not necessary) that s is relatively small in comparison to the remaining operational period of the system. As a practical example of this in application, we can think of some type of software system. Before a software is released to the market, it will undergo some type of initial run or early software testing. During this period, majority of the major bugs in the system will be addressed/eliminated. This testing period will be relatively short in comparison to the period for which the software is expected to operate. Also, in reality, it is fairly uncommon to achieve a completely bug-free software, meaning that the number of major bugs eliminated by the release date is unlikely to reach the threshold l_0. Obviously, there can be other examples for this set up as well.

We will now characterize the process probabilistically. Therefore, let us derive $P(N(t) = n)$.

1. For $t \le s$, the process is the EGPP. Obviously, in this case $P(N(t) = n)$ is given by (2).

2. For $s < t \le u$, the process is the NHPP. Observe that

$$\begin{aligned} P(N(t) = n) &= \sum_{j=0}^{n} P(N(t) = n \mid N(s) = j) P(N(s) = j) \\ &= \sum_{j=0}^{n} P(N(t) - N(s) = n - j \mid N(s) = j) P(N(s) = j), \end{aligned}$$

for $n = 0, 1, 2, \ldots, \infty$. Although it is possible to observe $n > l_0$ failures/repairs in the interval $(s, u]$ note that $P(N(t) = n)$ reduces to

$$P(N(t) = n) = \sum_{j=0}^{l_0} P(N(t) - N(s) = n - j \mid N(s) = j) P(N(s) = j)$$

for all $n > l_0$. This holds as $P(N(s) = j) = 0$ for all $j > l_0$.

Now, in the interval $(s, u]$, the process is the NHPP with intensity function $\lambda_t = (l_0 - N(s))\lambda(t)$. Therefore,

$$P(N(t) - N(s) = n - j \mid N(s) = j) = \frac{(W_j(t, s))^{n-j}}{(n-j)!} \exp\{-W_j(t, s)\},$$

where $W_j(t,s) = W_j(t) - W_j(s)$, where $W_j(t) = \int_0^t (l_0 - j)\lambda(u)\,\mathrm{d}u = (l_0 - j)\Lambda(t)$.
On the other hand, $P(N(s) = j)$ follows from (2). Therefore,

$$\begin{aligned} P(N(t) = n) &= \sum_{j=0}^{n} P(N(t) - N(s) = n - j \mid N(s) = j)\, P(N(s) = j) \\ &= \sum_{j=0}^{n} \frac{(W_j(t,s))^{n-j}}{(n-j)!} \exp\{-W_j(t,s)\} \begin{pmatrix} l_0 \\ j \end{pmatrix} (1 - \exp\{-\Lambda(s)\})^j (\exp\{-\Lambda(s)\})^{l_0 - j}. \end{aligned}$$

3. For $t > u$, the process is the GPP. Observe that now

$$P(N(t) = n) = \sum_{k=0}^{n}\sum_{j=0}^{k} P(N(t) = n \mid N(u) = k, N(s) = j)\, P(N(u) = k \mid N(s) = j)\, P(N(s) = j),$$

for $n = 0, 1, 2, \ldots, \infty$. Once again, although it is possible to observe $n > l_0$ failures/repairs in the interval (u, ∞), note that $P(N(t) = n)$ reduces to

$$P(N(t) = n) = \sum_{k=0}^{n}\sum_{j=0}^{\min\{k, l_0\}} P(N(t) = n \mid N(u) = k, N(s) = j)\, P(N(u) = k \mid N(s) = j)\, P(N(s) = j)$$

for all $n > l_0$. This holds as $P(N(s) = j) = 0$ for all $j > l_0$.

Now, consider,

$$P(N(t) = n \mid N(u) = k, N(s) = j) = P(N(t) - N(u) = n - k \mid N(u) = k, N(s) = j).$$

Now, when $t > u$, it follows from Definition 3 that the process $\{M_u(v), v \geq 0\}$, where $M_u(v) = N(v + u) - N(u)$ is the GPP with parameter set $(w(u + v), \alpha)$, where $w(t) = (l_0 - N(s))\lambda(t)$, and is independent of $N(u)$. Therefore, $N(t) - N(u) = n - k \mid N(u) = k, N(s) = j$ is stochastically equivalent to $M_u(t - u)$, $t > u$, regardless of k. Thus,

$$\begin{aligned} P(N(t) = n \mid N(u) = k, N(s) = j) &= P(N(t) - N(u) = n - k \mid N(u) = k, N(s) = j) \\ &= P(N(t) - N(u) = n - k \mid N(s) = j) \\ &= \frac{\Gamma\left(\frac{1}{\alpha} + n - k\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n-k)!} \exp\{-W_j(t,u)\} (1 - \exp\{-\alpha W_j(t,u)\})^{n-k}, \end{aligned}$$

where $W_j(t,u) = (l_0 - j)\Lambda(t,u)$. While, as from above,

$$\begin{aligned} P(N(u) = k \mid N(s) = j) &= P(N(u) - N(s) = k - j \mid N(s) = j) \\ &= \frac{(W_j(u,s))^{k-j}}{(k-j)!} \exp\{-W_j(u,s)\}, \end{aligned}$$

where $W_j(u,s) = (l_0 - j)\Lambda(u,s)$, and $P(N(s) = j)$ follows from (2). Therefore,

$$\begin{aligned}
&P(N(t) = n) \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k}\left[\frac{\Gamma\left(\frac{1}{\alpha}+n-k\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n-k)!}\exp\{-W_j(t,u)\}\left(1-\exp\{-\alpha W_j(t,u)\}\right)^{n-k}\frac{(W_j(u,s))^{k-j}}{(k-j)!}\exp\{-W_j(u,s)\}\right. \\
&\quad\left.\times\binom{l_0}{j}\left(1-\exp\{-\Lambda(s)\}\right)^{j}\left(\exp\{-\Lambda(s)\}\right)^{l_0-j}\right].
\end{aligned}$$

Given the above, we can define the mean number of repairs in $[0,t)$.

Theorem 1. *The mean number of repairs in $[0,t)$ is given by*

$$E[N(t)] = l_0\left(1-\exp\{-\Lambda(t)\}\right) \tag{3}$$

for $t \le s$,

$$E[N(t)] = l_0\Lambda(t,s)\exp\{-\Lambda(s)\} + l_0\left(1-\exp\{-\Lambda(s)\}\right) \tag{4}$$

for $s < t \le u$, and

$$\begin{aligned}
E[N(t)] =& \sum_{j=0}^{l_0}\frac{1}{\alpha}\left(\exp\{\alpha(l_0-j)\Lambda(t,u)\}-1\right)\binom{l_0}{j}\left(1-\exp\{-\Lambda(s)\}\right)^{j}\left(\exp\{-\Lambda(s)\}\right)^{l_0-j} \\
&+ l_0\Lambda(u,s)\exp\{-\Lambda(s)\} + l_0\left(1-\exp\{-\Lambda(s)\}\right)
\end{aligned} \tag{5}$$

for $t > u$.

Proof. The proof for $t \le s$ is straightforward as it follows from $E[N(t)]$ when $\{N(t), t \ge 0\}$ is the EGPP with parameter set $(\lambda(t), l_0)$, where l_0 is a positive integer.

For $s < t \le u$, observe that

$$E[N(t)] = E[E[N(t) \mid N(s)]].$$

Consider,

$$\begin{aligned}
E[N(t) \mid N(s) = j] &= E[N(t) - N(s) + N(s) \mid N(s) = j] \\
&= E[N(t) - N(s) \mid N(s) = j] + E[N(s) \mid N(s) = j] \\
&= W_j(t,s) + j \\
&= (l_0 - j)\Lambda(t,s) + j.
\end{aligned}$$

Therefore,

$$E[N(t)] = \sum_{j=0}^{l_0}\left[(l_0-j)\Lambda(t,s) + j\right]P(N(s) = j).$$

Finally, for $t > u$, observe that

$$E[N(t)] = E[E[N(t) \mid N(s)]].$$

Consider,

$$\begin{aligned}
E[N(t) \mid N(s)=j] &= E[N(t)-N(u)+N(u)-N(s)+N(s) \mid N(s)=j] \\
&= E[N(t)-N(u) \mid N(s)=j] + E[N(u)-N(s) \mid N(s)=j] + E[N(s) \mid N(s)=j] \\
&= \frac{1}{\alpha}\left(\exp\{\alpha W_j(t,u)\}-1\right) + W_j(u,s) + j \\
&= \frac{1}{\alpha}\left(\exp\{\alpha W_j(t,u)\}-1\right) + (l_0-j)\,\Lambda(u,s) + j.
\end{aligned}$$

Therefore,

$$E[N(t)] = \sum_{j=0}^{l_0}\left[\frac{1}{\alpha}\left(\exp\{\alpha W_j(t,u)\}-1\right) + (l_0-j)\,\Lambda(u,s) + j\right] P(N(s)=j).$$

□

Alternate proofs for $E[N(t)]$ when $s < t \leq u$ and $t > u$ can be found in Appendix I.

2.3 Model 2: Repair type changes after some number of events

From the start of operation until the kth repair, the repairs are better-than-minimal (EGPP). After the kth repair and until the mth repair, where $m \geq k$, the repairs are minimal (NHPP). Just after the mth repair, the repairs start to be worse-than-minimal (GPP). The described combined repair process can be defined as follows.

Definition 4. Combined Repair Process II

i. $N(0)=0$;

ii. $\lambda_t = (l_0 - N(t-))\,\lambda(t), \quad N(t-) = 0, 1, \ldots k$;

iii. $\lambda_t = (l_0 - k)\,\lambda(t), \quad N(t-) = k+1, k+2, \ldots, m$;

iv. $\lambda_t = (\alpha(N(t-)-m)+1)(l_0-k)\,\lambda(t), \quad N(t-) = m+1, m+2, \ldots$.

As a practical example of this model in application, we can think of a mechanical system containing multiple components, where l_0 of the components have some major defects. A technician may only be able to repair the major defects in k of these components due to his technical skills, availability of spare parts, the ability to access the components, etc. Therefore, although the system is operational, there are still $l_0 - k$ components with major defects.

We will now characterize the process probabilistically. Therefore, let us derive $P(N(t)=n)$.

1. First let $n = 0, 1, \ldots, k$.
   Let s_n denote the arrival time of the nth failure/repair. Therefore, when we consider the event $\{N(t)=n\}$ for $n = 1, 2, \ldots, k-1$, we have that $s_n \leq t < s_{n+1} \leq s_k$. That is, the time instant t would be before the change point s_k and the process is the EGPP. Obviously, in this case $P(N(t)=n)$ is

given by (2) for $n = 1, 2, \ldots, k-1$. Now, when $n = k$, for the event $\{N(t) = k\}$ the time instant t satisfies $s_k \leq t < s_{k+1}$. In the interval $[s_k, t)$, the process is already the NHPP. However, from property (iii) in Definition 4, it follows that the stochastic intensity is still given by $\lambda_t = (l_0 - k)\lambda(t)$ in the interval $[s_k, t)$, which also results in

$$P(N(t) = k) = \binom{l_0}{k} (1 - \exp\{-\Lambda(t)\})^k (\exp\{-\Lambda(t)\})^{l_0 - k},$$

as from (2).

2. Now consider $n = k+1, k+2, \ldots, m$.
When we consider the event $\{N(t) = n\}$ for $n = k+1, k+2, \ldots, m-1$, we have that $s_k < s_n \leq t < s_{n+1} \leq s_m$, where once again s_n is the time of the nth failure/repair. Thus the time instant t is after the first change point s_k but before the next change point s_m, and the process between these change points is the NHPP. Observe then that

$$P(N(t) = n) = \int_0^t P(N(t) = n \mid S_k = u) f_{S_k}(u)\,\mathrm{d}u,$$

where $f_{S_k}(u)$ is the pdf of S_k. Consider

$$\begin{aligned} P(N(t) = n \mid S_k = u) &= P(N(t) = n \mid N(s) < k \text{ for } 0 \leq s < u, N(u) = k) \\ &= P(N(t) - N(u) = n - k \mid N(s) < k \text{ for } 0 \leq s < u, N(u) = k) \\ &= P(N(t) - N(u) = n - k). \end{aligned}$$

Note that the last equality holds due to the independent increment property of the NHPP. That is in the interval $[u, t)$, the process is the NHPP with rate $(l_0 - k)\lambda(t)$. It then follows that

$$\begin{aligned} P(N(t) = n \mid S_k = u) &= P(N(t) - N(u) = n - k) \\ &= \frac{(W(t,u))^{n-k}}{(n-k)!} \exp\{-W(t,u)\}, \end{aligned}$$

where $W(t,u) = (l_0 - k)\Lambda(t,u)$. On the other hand,

$$\begin{aligned} f_{S_k}(u)\Delta u &\simeq P(u \leq S_k \leq u + \Delta u) \\ &= P(u \leq S_k \leq u + \Delta u, N(u-) = k-1) \\ &= P(u \leq S_k \leq u + \Delta u \mid N(u-) = k-1) P(N(u-) = k-1) \\ &= (l_0 - k + 1)\lambda(u)\Delta u \binom{l_0}{k-1} (1 - \exp\{-\Lambda(u)\})^{k-1} (\exp\{-\Lambda(u)\})^{l_0 - k + 1}. \end{aligned}$$

Therefore,

$$f_{S_k}(u) = \frac{l_0!}{(k-1)!(l_0 - k)!} \lambda(u) \exp\{-l_0\Lambda(u)\} (\exp\{\Lambda(u)\} - 1)^{k-1}.$$

Note that the pdf of S_k could also be derived using the fact that

$$f_{S_k}(t) = -\sum_{j=0}^{k-1} \frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = j).$$

This alternative proof is deferred to Appendix II.
Finally,

$$\begin{aligned} P(N(t) = n) &= \int_0^t P(N(t) = n \mid S_k = u) f_{S_k}(u) \,\mathrm{d}u \\ &= \int_0^t \frac{(W(t,u))^{n-k}}{(n-k)!} \exp\{-W(t,u)\} \frac{l_0!}{(k-1)!(l_0-k)!} \lambda(u) \exp\{-l_0 \Lambda(u)\} (\exp\{\Lambda(u)\} - 1)^{k-1} \mathrm{d}u, \end{aligned}$$

where $W(t,u) = (l_0 - k)\Lambda(t,u)$.
Now, when $n = m$, for the event $\{N(t) = m\}$ the time instant of interest t satisfies $s_m \leq t < s_{m+1}$. Thus, in the interval $[s_m, t)$, the process is already the GPP. However, due to property (iv) of Definition 4, the stochastic intensity in the interval $[s_m, t)$ is $\lambda_t = (l_0 - k)\lambda(t)$ which also results in

$$P(N(t) = m) = \int_0^t \frac{(W(t,u))^{m-k}}{(m-k)!} \exp\{-W(t,u)\} \frac{l_0!}{(k-1)!(l_0-k)!} \lambda(u) \exp\{-l_0 \Lambda(u)\} (\exp\{\Lambda(u)\} - 1)^{k-1} \mathrm{d}u,$$

where $W(t,u) = (l_0 - k)\Lambda(t,u)$.

3. Finally, consider $n = m+1, m+2, \ldots$
Once again letting s_n denote the arrival time of the nth failure/repair, we have that $s_m < s_n \leq t$ when we consider the event $\{N(t) = n\}$ for $n = m+1, m+2, \ldots$. Observe that,

$$P(N(t) = n) = \int_0^t \int_0^s P(N(t) = n \mid S_m = s, S_k = u) f_{S_m, S_k}(s,u) \,\mathrm{d}u\mathrm{d}s,$$

where $f_{S_m,S_k}(s,u)$ is the joint pdf of (S_m, S_k). Consider

$$\begin{aligned} P(N(t) = n \mid S_m = s, S_k = u) &= P(N(t) = n \mid N(s) = m, k \leq N(c) < m \text{ for } u \leq c < s, S_k = u) \\ &= P(N(t) - N(s) = n - m \mid N(s) = m, k \leq N(c) < m \text{ for } u \leq c < s, S_k = u). \end{aligned}$$

Note that, given $N(s) = m, k \leq N(c) < m$ for $u \leq c < s, S_k = u$, $\{N(c+s) - N(s), c \geq 0\}$ is the GPP with parameter set $(w(s+c), \alpha)$, where $w(t) = (l_0 - k)\lambda(t)$. Therefore, from Cha [8] we have that

$$\begin{aligned} &P(N(t) - N(s) = n - m \mid N(s) = m, k \leq N(c) < m \text{ for } u \leq c < s, S_k = u) \\ =&\frac{\Gamma\left(\frac{1}{\alpha} + n - m\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n-m)!} \exp\{-W(t,s)\} (1 - \exp\{-\alpha W(t,s)\})^{n-m}, \end{aligned}$$

where $W(t,s) = (l_0 - k)\Lambda(t,s)$. Now, consider that

$$f_{S_m,S_k}(s,u) = \lim_{\Delta s \to 0} \lim_{\Delta u \to 0} \frac{1}{\Delta s \Delta u} P(s \leq S_m \leq s + \Delta s, u \leq S_k \leq u + \Delta u),$$

where,

$$
\begin{aligned}
&P\left(s \leq S_m \leq s+\Delta s, u \leq S_k \leq u+\Delta u\right) \\
=&P\left(s \leq S_m \leq s+\Delta s \mid u \leq S_k \leq u+\Delta u\right) P\left(u \leq S_k \leq u+\Delta u\right).
\end{aligned}
$$

As from above,

$$
P\left(u \leq S_k \leq u+\Delta u\right)=\frac{l_0!}{(k-1)!\left(l_0-k\right)!} \lambda(u) \Delta u(1-\exp \{-\Lambda(u)\})^{k-1}(\exp \{-\Lambda(u)\})^{l_0-k+1}.
$$

On the other hand,

$$
\begin{aligned}
&P\left(s \leq S_m \leq s+\Delta s \mid u \leq S_k \leq u+\Delta u\right) \\
=&P\left(s \leq S_m \leq s+\Delta s, N(s-)=m-1 \mid N(u-)=k-1, N(u+\Delta u)=k\right) \\
=&P\left(s \leq S_m \leq s+\Delta s, N(s-)-N(u+\Delta u)=m-k-1 \mid N(u-)=k-1, N(u+\Delta u)=k\right) \\
=&P\left(s \leq S_m \leq s+\Delta s \mid N(s-)-N(u+\Delta u)=m-k-1, N(u-)=k-1, N(u+\Delta u)=k\right) \\
&\times P(N(s-)-N(u+\Delta u)=m-k-1 \mid N(u-)=k-1, N(u+\Delta u)=k) \\
=&\left(l_0-k\right) \lambda(s) \Delta s \frac{(W(s, u+\Delta u))^{m-k-1}}{(m-k-1)!} \exp \{-W(s, u+\Delta u)\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_{S_m, S_k}(s, u)=&\frac{l_0!}{(k-1)!\left(l_0-k-1\right)!(m-k-1)!} \lambda(u) \lambda(s)(W(s, u))^{m-k-1} \exp \{-W(s, u)\} \\
&\times \exp \left\{-l_0 \Lambda(u)\right\}(\exp \{\Lambda(u)\}-1)^{k-1},
\end{aligned}
$$

where $W(s, u)=\left(l_0-k\right) \Lambda(s, u)$.

Note that the joint pdf of $\left(S_m, S_k\right)$ could also be derived using the fact that

$$
f_{S_m, S_k}(t, u)=f_{S_m \mid S_k}(t \mid u) f_{S_k}(u)
$$

where,

$$
f_{S_m \mid S_k}(t \mid u)=-\sum_{j=k}^{m-1} \frac{\mathrm{d}}{\mathrm{d} t} P\left(N(t)=j \mid S_k=u\right).
$$

This alternative proof is deferred to Appendix III.

Finally,

$$
\begin{aligned}
&P\left(N\left(t\right)=n\right)\\
=&\int_0^t\int_0^s P\left(N\left(t\right)=n\mid S_m=s,S_k=u\right)f_{S_m,S_k}\left(s,u\right)\mathrm{d}u\mathrm{d}s\\
=&\int_0^t\int_0^s\left[\frac{\Gamma\left(\frac{1}{\alpha}+n-m\right)}{\Gamma\left(\frac{1}{\alpha}\right)\left(n-m\right)!}\exp\left\{-W\left(t,s\right)\right\}\left(1-\exp\left\{-\alpha W\left(t,s\right)\right\}\right)^{n-m}\frac{l_0!}{\left(k-1\right)!\left(l_0-k-1\right)!\left(m-k-1\right)!}\right.\\
&\left.\times\lambda\left(u\right)\lambda\left(s\right)\left(W\left(s,u\right)\right)^{m-k-1}\exp\left\{-W\left(s,u\right)\right\}\exp\left\{-l_0\Lambda\left(u\right)\right\}\left(\exp\left\{\Lambda\left(u\right)\right\}-1\right)^{k-1}\right]\mathrm{d}u\mathrm{d}s,
\end{aligned}
$$

where $W\left(s,u\right)=\left(l_0-k\right)\Lambda\left(s,u\right)$.

Given the above, we can define the mean number of repairs in $[0,t)$.

Theorem 2. *The mean number of repairs in $[0,t)$ is given by*

$$
\begin{aligned}
E\left[N\left(t\right)\right]=&\sum_{n=0}^{k}n\binom{l_0}{n}\left(1-\exp\left\{-\Lambda\left(t\right)\right\}\right)^n\left(\exp\left\{-\Lambda\left(t\right)\right\}\right)^{l_0-n}\\
&+\sum_{n=k+1}^{m}n\int_0^t\frac{\left(W\left(t,u\right)\right)^{n-k}}{\left(n-k\right)!}\exp\left\{-W\left(t,u\right)\right\}\frac{l_0!}{\left(k-1\right)!\left(l_0-k\right)!}\lambda\left(u\right)\exp\left\{-l_0\Lambda\left(u\right)\right\}\left(\exp\left\{\Lambda\left(u\right)\right\}-1\right)^{k-1}du\\
&+\int_0^t\int_0^s\left[\left(\frac{1}{\alpha}\left(\exp\left\{\alpha W\left(t,s\right)\right\}-1\right)+m\left(1-\exp\left\{W\left(t,s\right)\right\}\right)\right)\frac{l_0!}{\left(k-1\right)!\left(l_0-k-1\right)!\left(m-k-1\right)!}\right.\\
&\left.\times\lambda\left(u\right)\lambda\left(s\right)\left(W\left(s,u\right)\right)^{m-k-1}\exp\left\{-W\left(s,u\right)\right\}\exp\left\{-l_0\Lambda\left(u\right)\right\}\left(\exp\left\{\Lambda\left(u\right)\right\}-1\right)^{k-1}\right]duds.
\end{aligned}
$$

For brevity, the proof of this has been deferred to Appendix IV.

3 Optimal age replacement policies

In what follows we will investigate the application of the new combined repair process to the optimal age replacement problem. Therefore, let $\{N(t), t\geq 0\}$ be the stochastic repair process with baseline function (underlying failure rate of the system) $\lambda(t)$, $\alpha>0$, and l_0 as some positive integer.

Assume that the system is replaced when it reaches age T (periodic replacement policy). That is, after each replacement, a new cycle begins and so on. Between these successive replacements, the repairs/failures occur in accordance with one of the described models above.

3.1 Model 1

Recall that for Model 1, the repair is better-than-minimal (EGPP) for $t\leq s$, the repair is minimal (NHPP) for $s<t\leq u$, and then the repair is worse-than-minimal (GPP) for $t>u$.

Let c_b, c_m, c_w, and c_r be the cost of better-than-minimal repair, minimal repair, worse-than-minimal repair, and replacement, respectively, such that $c_w\leq c_m<c_b<c_r$. The corresponding long-run expected

cost rate, $C(T)$, for this periodic setting is given by

$$C(T) = \frac{c_b E[N(T)] + c_r}{T},$$

for $T \leq s$, where $E[N(T)]$ is given by (3),

$$C(T) = \frac{c_b E[N(s)] + c_m (E[N(T)] - E[N(s)]) + c_r}{T},$$

for $s < T \leq u$, where $E[N(T)]$ is now given by (4), and

$$C(T) = \frac{c_b E[N(s)] + c_m (E[N(u)] - E[N(s)]) + c_w (E[N(T)] - E[N(u)]) + c_r}{T}, \tag{6}$$

for $T > u$, where $E[N(T)]$ is now given by (5).

Assume that a system's lifetime is described by the baseline failure rate $\lambda(t)$ before the first failure/repair. Methodologically, it is important to consider the cases when this baseline failure rate is either constant or is increasing such that $\lim_{t\to\infty} \lambda(t) = \infty$. The latter describes the natural aging (or degradation) of a system with time (see Nakagawa [23]) and is a common assumption in optimal age replacement problems in reliability literature. Under both of these baseline failure rates it can be shown that

$$\lim_{T\to 0} C(T) = \infty, \quad \lim_{T\to\infty} C(T) = \infty,$$

and $C(T)$ is decreasing in the vicinity of $T = 0$. Therefore, there exists an optimal replacement time T^* that satisfies

$$C(T^*) = \min_{0<T<\infty} C(T).$$

An explicit expression to determine the optimal replacement time T^*, found by differentiating (6) and equating to 0, is too cumbersome to provide an analytical solution. Therefore, in the next section, for the given values of parameters, we obtain it numerically by plotting the corresponding expected long-run cost rate function for the considered examples. This is done using R.

3.2 Model 2

Recall that for Model 2, from the start of operation until the kth repair, the repairs are better-than-minimal (EGPP). Just after repair k and until repair m, where $m \geq k$, the repairs are minimal (NHPP). Just after repair m, the repairs are worse-than-minimal (GPP). Below, we only briefly describe the calculations for the long-run expected cost rate. The procedure for analyzing an optimal solution is similar to that for Model 1.

Once again, let c_b, c_m, c_w, and c_r be the cost of better-than-minimal repair, minimal repair, worse-than-minimal repair, and replacement, respectively, such that $c_w \leq c_m < c_b < c_r$. The corresponding long-run expected cost rate, $C(T)$, for this periodic setting is given by

$$C(T) = \frac{1}{T}\left[c_b \sum_{j=0}^{k} jP(N(T)=j) + c_b \sum_{j=k+1}^{\infty} kP(N(T)=j) + c_m \sum_{j=k+1}^{m} (j-k)P(N(T)=j) \right.$$
$$\left. + c_m \sum_{j=m+1}^{\infty} mP(N(T)=j) + c_w \sum_{j=m+1}^{\infty} (j-m)P(N(T)=j) + c_r \right].$$

4 Numerical illustrations and discussion

Here, we will only present numerical examples for the optimal replacement results under Model 1. The results and conclusions for Model 2 follow similarly. For demonstration purposes, we will consider the constant baseline failure rate, $\lambda(t) = \lambda$, and an increasing baseline failure rate, $\lambda(t) = \lambda t + 1$. Other increasing baseline failure rates were considered. However, these numerical illustrations demonstrated similar properties for the considered optimal model and have been omitted for brevity. Further, the latter baseline failure rate is in line with the discussions in Section 3 and the failure rates used in similar reliability literature (see, for example, Badía et al. [9], Cha et al. [16], Langston et al. [17], Cha and Finkelstein [18], to name a few recent considerations).

Let $c_b = 20$, $c_m = 15$, and $c_w = 10$, and assume that the time at which the repair type changes from minimal to worse-than-minimal is fixed at $u = 4$. The corresponding long-run expected cost rate for different values of λ under the two baseline failure rates for $l_0 = 10$, $s = 1$, $\alpha = 1$, and $c_r = 35$ are given in Figure 1. We also conducted numerical experiments for other parameter values, which show a similar general 'picture'. It is clear from this figure, and all subsequent figures, that an optimal replacement time T^* exists. Further, for fixed values of λ the optimal replacement time T^* is smaller in the case of the increasing baseline failure rate compared to the constant baseline failure rate. For example, in Figure 1 when $\lambda = 1$, the optimal replacement time T^* is 4.17 and 2.89 for $\lambda(t) = \lambda$ and $\lambda(t) = \lambda t + 1$, respectively. This follows rather intuitively (see, for example, Finkelstein et al. [24]) as increasing the baseline failure rate would increase the amount of overall aging (or deterioration) of the system over time. This would lead to more frequent system failures, and as such, more frequent repair actions would need to occur. Therefore, to reduce the long-run expected cost rate, replacement of the system would need to be implemented sooner. From Figure 1 specifically, it can be also seen that as λ increases, the optimal replacement time T^* decreases. For example, in the case of the increasing baseline failure rate, T^* is 3.40 and 2.89 for $\lambda = 0.5$ and $\lambda = 1$, respectively. This follows a similar reasoning: increasing λ results in more failures, and as such, replacement should be carried out sooner.

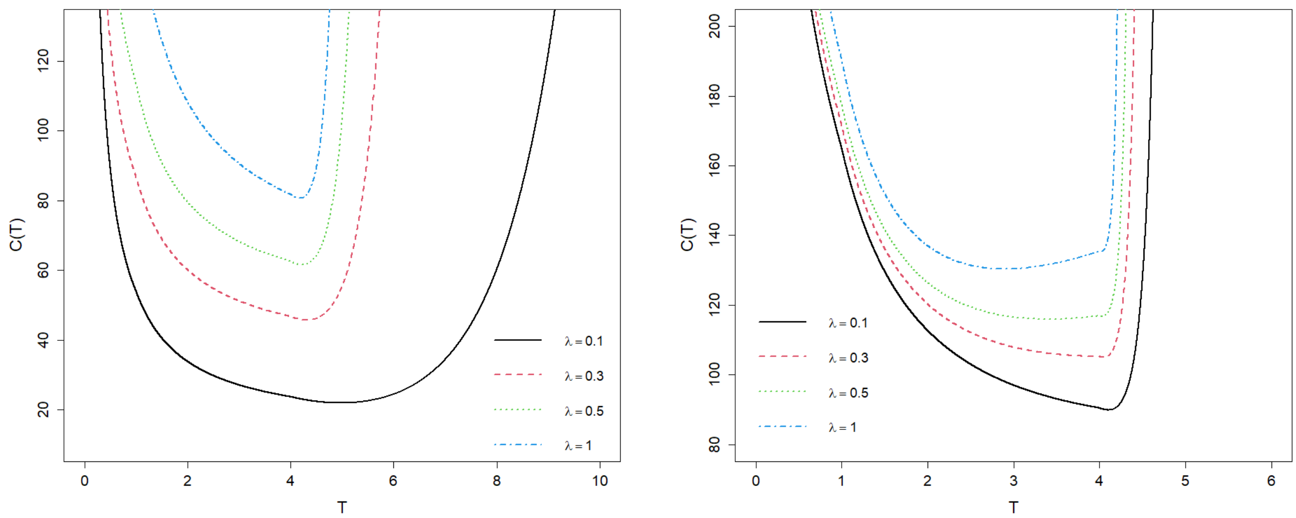

Figure 1: The long-run expected cost rates for $l_0 = 10$, $s = 1$, $u = 4$, $\alpha = 1$, $c_b = 20$, $c_m = 15$, $c_w = 10$, $c_r = 35$, and varying λ for $\lambda(t) = \lambda$ (left) and $\lambda(t) = \lambda t + 1$ (right).

The corresponding long-run expected cost rate for different values of α under the two baseline failure rates for $l_0 = 10$, $s = 1$, $\lambda = 0.1$, and $c_r = 35$ are given in Figure 2. It follows from Definition 3 that changes in α would only impact the worse-than-minimal (GPP) repair stage of the process. This is clearly illustrated in the figure where differences in the long-run expected cost rate only occur after $T = u = 4$. Further, in this stage of the process, we note that for increasing α, the optimal replacement time T^* is decreasing. For example, in the case of a constant baseline failure rate, T^* is 10.67 and 4.96 for $\alpha = 0.1$ and $\alpha = 1$, respectively. This follows from the foregoing discussion and is in line with the reasoning for increasing λ. That is, if previous failures (and their subsequent repairs) have more influence on the system's susceptibility to future failure, this would suggest that replacement of the system should happen sooner.

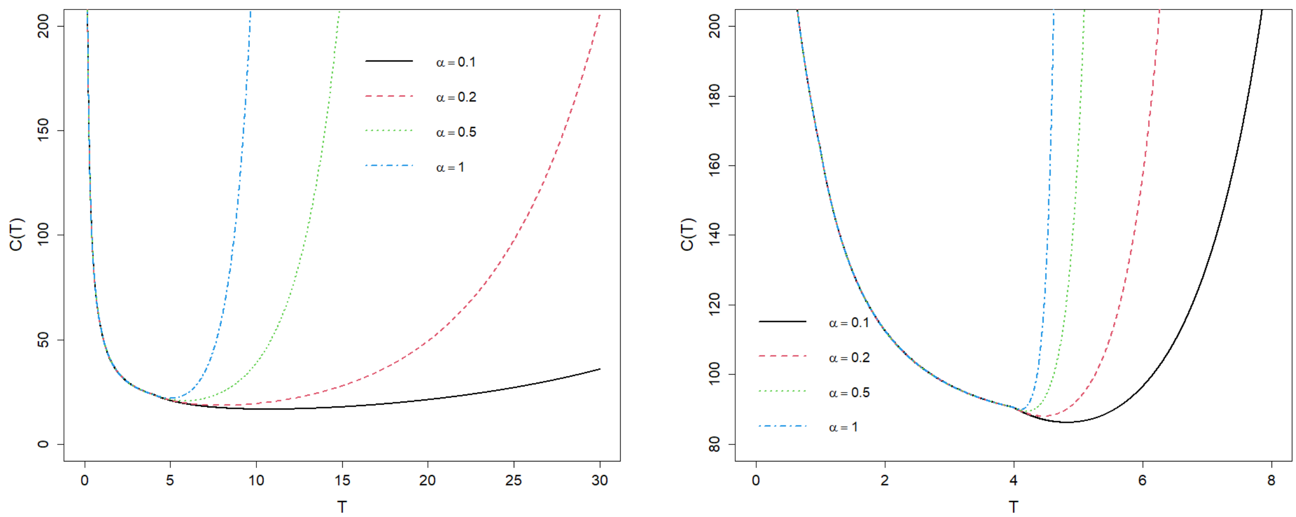

Figure 2: The long-run expected cost rates for $l_0 = 10$, $s = 1$, $u = 4$, $\lambda = 0.1$, $c_b = 20$, $c_m = 15$, $c_w = 10$, $c_r = 35$, and varying α for $\lambda(t) = \lambda$ (left) and $\lambda(t) = \lambda t + 1$ (right).

The corresponding long-run expected cost rate for different values of c_r under the two baseline failure rates for $l_0 = 10$, $s = 1$, $\lambda = 0.1$, and $\alpha = 1$ are given in Figure 3. From this figure it can be noted that as the cost of replacement c_r increases, so does the optimal replacement time T^*. For example, in the case of the increasing baseline failure rate, T^* is 4.10 and 4.12 for $c_r = 35$ and $c_r = 100$, respectively. Although the change is not significant for the given values of other parameters, it illustrates a general, intuitively clear result: the higher replacement cost delays replacement.

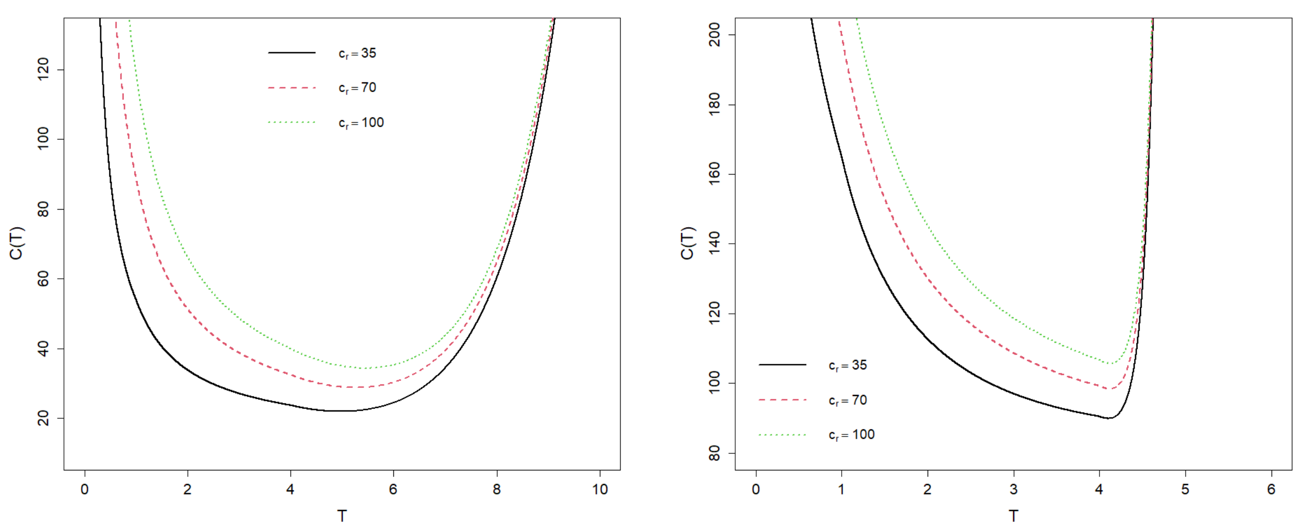

Figure 3: The long-run expected cost rates for $l_0 = 10$, $s = 1$, $u = 4$, $\lambda = 0.1$, $\alpha = 1$, $c_b = 20$, $c_m = 15$, $c_w = 10$, and varying c_r for $\lambda(t) = \lambda$ (left) and $\lambda(t) = \lambda t + 1$ (right).

The corresponding long-run expected cost rate for different values of s under the two baseline failure rates for $l_0 = 10$, $\lambda = 0.1$, $\alpha = 1$, and $c_r = 35$ are given in Figure 4. From this figure, we see that as s is increasing, the optimal replacement time T^* is also increasing. For example, in the case of a constant baseline failure rate, T^* is 5.15 and 5.38 for $s = 2$ and $s = 3$, respectively. Recall here that s is the time at which the type of repair changes from better-than-minimal repair to minimal repair. Therefore, by increasing s, the expected number of better-than-minimal repairs is also increasing (for example, $E[N(2)] = 1.81$ and $E[N(3)] = 2.59$ for the constant baseline failure rate case). As per Definition 2, if a system undergoes a higher number of better-than-minimal repairs (obviously still under the constraint that $N(s) < l_0$), this would decrease the system's susceptibility to future failure and would suggest that replacement of the system should be postponed to a later time.

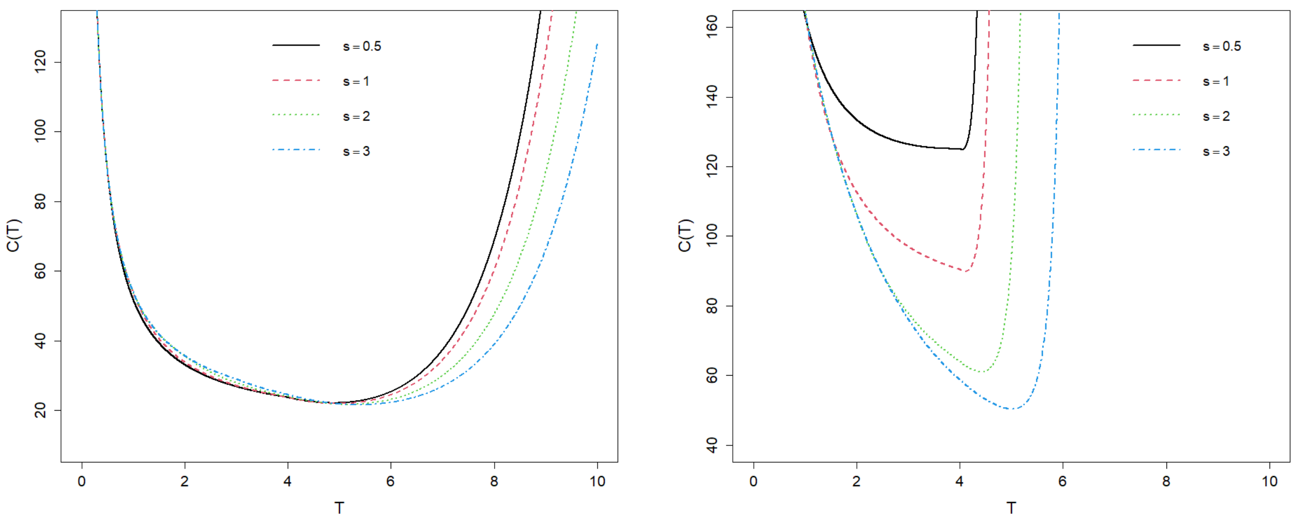

Figure 4: The long-run expected cost rates for $l_0 = 10$, $u = 4$, $\lambda = 0.1$, $\alpha = 1$, $c_b = 20$, $c_m = 15$, $c_w = 10$, $c_r = 35$, and varying s for $\lambda(t) = \lambda$ (left) and $\lambda(t) = \lambda t + 1$ (right).

The corresponding long-run expected cost rate for different values of l_0 under the two baseline failure rates for $s = 1$, $\lambda = 0.1$, $\alpha = 1$, and $c_r = 35$ are given in Figure 5. It is clear from this figure that the optimal replacement time T^* is decreasing for increasing l_0. For example, in the case of the constant baseline failure rate, T^* is 6.31 and 4.40 for $l_0 = 5$ and $l_0 = 20$, respectively. If the number of major defects/bugs in a system is increased, a smaller proportion of these will be repaired by the fixed time s. Therefore, there would be a larger number of remaining major defects in the system. As evident from Definition 3, this would contribute to more frequent failures, and therefore, replacement should be conducted sooner.

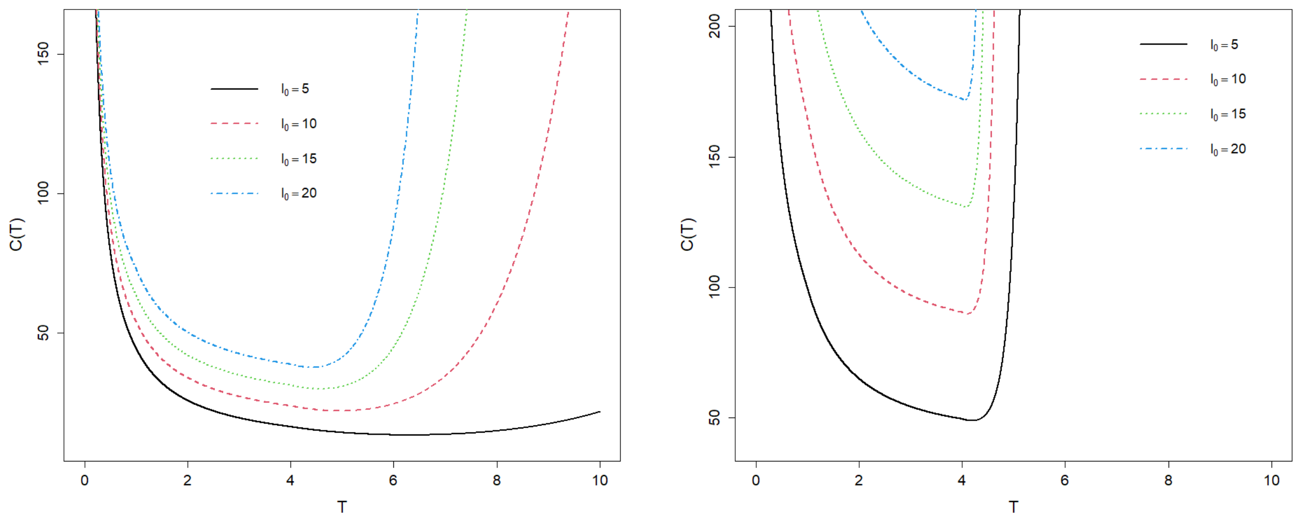

Figure 5: The long-run expected cost rates for $s = 1$, $u = 4$, $\lambda = 0.1$, $\alpha = 1$, $c_b = 20$, $c_m = 15$, $c_w = 10$, $c_r = 35$, and varying l_0 for $\lambda(t) = \lambda$ (left) and $\lambda(t) = \lambda t + 1$ (right).

5 Conclusions

In recent reliability literature, a number of combined point processes have been introduced to define and describe various repair models. In line with this, we have defined a new combined process using the extended generalized Polya process (EGPP), non-homogeneous Poisson process (NHPP), and generalized Polya process (GPP) to describe repairs that begin as better-than-minimal, then become minimal, before finally becoming worse-than-minimal. This, in a way, resembles the bathtub curve for the failure rate of some items when we move from 'infant mortality' to the 'normal' operation stage and, finally, to the stage of degradation.

Several useful properties have been derived for this combined repair process under two settings, namely, repair type changes after a specified time and repair type changes after a specified number of repairs/failures. As such, the corresponding optimal replacement policy was defined under the two settings, and the existence of an optimal solution was discussed. Various numerical illustrations were provided to support our findings, and a sensitivity analysis was conducted for the main parameters of the considered model.

As further research, the change points discussed under the two settings could be considered as random variables. Naturally, appropriate distributions would need to be assumed in this case, and the stochastic properties would need to be carefully defined in the context of this more complex model. Also, the setting described here could be regularized in some way.

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Appendix I

Alternative derivations for $E[N(t)]$ for Model 1:

For $s < t \leq u$,

$$\begin{aligned}
E[N(t)] &= \sum_{n=0}^{\infty} n \sum_{j=0}^{n} \frac{(W_j(t,s))^{n-j}}{(n-j)!} \exp\{-W_j(t,s)\} \binom{l_0}{j} (1-\exp\{-\Lambda(s)\})^j (\exp\{-\Lambda(s)\})^{l_0-j} \\
&= \sum_{j=0}^{l_0} \sum_{n=j}^{\infty} n \frac{(W_j(t,s))^{n-j}}{(n-j)!} \exp\{-W_j(t,s)\} \binom{l_0}{j} (1-\exp\{-\Lambda(s)\})^j (\exp\{-\Lambda(s)\})^{l_0-j} \\
&= \sum_{j=0}^{l_0} \binom{l_0}{j} (1-\exp\{-\Lambda(s)\})^j (\exp\{-\Lambda(s)\})^{l_0-j} \sum_{n=j}^{\infty} n \frac{(W_j(t,s))^{n-j}}{(n-j)!} \exp\{-W_j(t,s)\} \\
&= \sum_{j=0}^{l_0} \binom{l_0}{j} (1-\exp\{-\Lambda(s)\})^j (\exp\{-\Lambda(s)\})^{l_0-j} \sum_{n=0}^{\infty} (n+j) \frac{(W_j(t,s))^{n}}{n!} \exp\{-W_j(t,s)\} \\
&= \sum_{j=0}^{l_0} \binom{l_0}{j} (1-\exp\{-\Lambda(s)\})^j (\exp\{-\Lambda(s)\})^{l_0-j} (W_j(t,s)+j) \\
&= \sum_{j=0}^{l_0} (l_0-j)\Lambda(t,s) \binom{l_0}{j} (1-\exp\{-\Lambda(s)\})^j (\exp\{-\Lambda(s)\})^{l_0-j} + l_0(1-\exp\{-\Lambda(s)\}) \\
&= l_0\Lambda(t,s) - l_0\Lambda(t,s)(1-\exp\{-\Lambda(s)\}) + l_0(1-\exp\{-\Lambda(s)\}) \\
&= l_0\Lambda(t,s)\exp\{-\Lambda(s)\} + l_0(1-\exp\{-\Lambda(s)\}).
\end{aligned}$$

For $t > u$,

$$E\left[N\left(t\right)\right] = \sum_{n=0}^{\infty} n \sum_{k=0}^{n} \sum_{j=0}^{k} \left[\frac{\mathit{\Gamma}\left(\frac{1}{\alpha}+n-k\right)}{\mathit{\Gamma}\left(\frac{1}{\alpha}\right)(n-k)!} \exp\left\{-W_j\left(t,u\right)\right\}\left(1-\exp\left\{-\alpha W_j\left(t,u\right)\right\}\right)^{n-k} \frac{\left(W_j\left(u,s\right)\right)^{k-j}}{(k-j)!} \exp\left\{-W_j\left(u,s\right)\right\} \right.$$
$$\left. \times \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j} \right]$$
$$= \sum_{j=0}^{l_0} \sum_{k=j}^{\infty} \sum_{n=k}^{\infty} \left[n \frac{\mathit{\Gamma}\left(\frac{1}{\alpha}+n-k\right)}{\mathit{\Gamma}\left(\frac{1}{\alpha}\right)(n-k)!} \exp\left\{-W_j\left(t,u\right)\right\}\left(1-\exp\left\{-\alpha W_j\left(t,u\right)\right\}\right)^{n-k} \frac{\left(W_j\left(u,s\right)\right)^{k-j}}{(k-j)!} \exp\left\{-W_j\left(u,s\right)\right\} \right.$$
$$\left. \times \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j} \right]$$
$$= \sum_{j=0}^{l_0} \sum_{k=j}^{\infty} \left[\frac{\left(W_j\left(u,s\right)\right)^{k-j}}{(k-j)!} \exp\left\{-W_j\left(u,s\right)\right\} \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j} \right.$$
$$\left. \times \sum_{n=k}^{\infty} n \frac{\mathit{\Gamma}\left(\frac{1}{\alpha}+n-k\right)}{\mathit{\Gamma}\left(\frac{1}{\alpha}\right)(n-k)!} \exp\left\{-W_j\left(t,u\right)\right\}\left(1-\exp\left\{-\alpha W_j\left(t,u\right)\right\}\right)^{n-k} \right]$$
$$= \sum_{j=0}^{l_0} \sum_{k=j}^{\infty} \left[\frac{\left(W_j\left(u,s\right)\right)^{k-j}}{(k-j)!} \exp\left\{-W_j\left(u,s\right)\right\} \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j} \right.$$
$$\left. \times \sum_{n=0}^{\infty} (n+k) \frac{\mathit{\Gamma}\left(\frac{1}{\alpha}+n\right)}{\mathit{\Gamma}\left(\frac{1}{\alpha}\right) n!} \exp\left\{-W_j\left(t,u\right)\right\}\left(1-\exp\left\{-\alpha W_j\left(t,u\right)\right\}\right)^{n} \right]$$
$$= \sum_{j=0}^{l_0} \sum_{k=j}^{\infty} \frac{\left(W_j\left(u,s\right)\right)^{k-j}}{(k-j)!} \exp\left\{-W_j\left(u,s\right)\right\} \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j} \left(\frac{1}{\alpha}\left(\exp\left\{\alpha W_j\left(t,u\right)\right\}-1\right)+k\right)$$
$$= \sum_{j=0}^{l_0} \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j} \sum_{k=j}^{\infty} \frac{\left(W_j\left(u,s\right)\right)^{k-j}}{(k-j)!} \exp\left\{-W_j\left(u,s\right)\right\} \left(\frac{1}{\alpha}\left(\exp\left\{\alpha W_j\left(t,u\right)\right\}-1\right)+k\right)$$
$$= \sum_{j=0}^{l_0} \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j} \sum_{k=0}^{\infty} \frac{\left(W_j\left(u,s\right)\right)^{k}}{k!} \exp\left\{-W_j\left(u,s\right)\right\} \left(\frac{1}{\alpha}\left(\exp\left\{\alpha W_j\left(t,u\right)\right\}-1\right)+k+j\right)$$
$$= \sum_{j=0}^{l_0} \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j} \left(\frac{1}{\alpha}\left(\exp\left\{\alpha W_j\left(t,u\right)\right\}-1\right)+W_j\left(u,s\right)+j\right)$$
$$= \sum_{j=0}^{l_0} \frac{1}{\alpha}\left(\exp\left\{\alpha\left(l_0-j\right)\Lambda\left(t,u\right)\right\}-1\right) \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j}$$
$$+ \sum_{j=0}^{l_0} \left(l_0-j\right)\Lambda\left(u,s\right) \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j} + l_0\left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)$$
$$= \sum_{j=0}^{l_0} \frac{1}{\alpha}\left(\exp\left\{\alpha\left(l_0-j\right)\Lambda\left(t,u\right)\right\}-1\right) \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j}$$
$$+ l_0\Lambda\left(u,s\right) - l_0\Lambda\left(u,s\right)\left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right) + l_0\left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)$$
$$= \sum_{j=0}^{l_0} \frac{1}{\alpha}\left(\exp\left\{\alpha\left(l_0-j\right)\Lambda\left(t,u\right)\right\}-1\right) \binom{l_0}{j} \left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right)^{j} \left(\exp\left\{-\Lambda\left(s\right)\right\}\right)^{l_0-j}$$
$$+ l_0\Lambda\left(u,s\right)\exp\left\{-\Lambda\left(s\right)\right\} + l_0\left(1-\exp\left\{-\Lambda\left(s\right)\right\}\right).$$

Appendix II

Alternative derivation for the pdf of S_k:

The pdf of S_k can be derived from first principals using the fact that

$$f_{S_k}(t) = -\sum_{j=0}^{k-1} \frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = j),$$

where

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = 0) = -l_0 \lambda(t) \exp\{-l_0 \Lambda(t)\}$,

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = 1) = -l_0 \lambda(t) \exp\{-l_0 \Lambda(t)\} [(l_0 - 1)(\exp\{\Lambda(t)\} - 1) - 1]$,

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = 2) = -l_0 \lambda(t) \exp\{-l_0 \Lambda(t)\} \left[\frac{1}{2!}(l_0 - 1)(l_0 - 2)(\exp\{\Lambda(t)\} - 1)^2 - (l_0 - 1)(\exp\{\Lambda(t)\} - 1)\right]$,

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = 3) = -l_0 \lambda(t) \exp\{-l_0 \Lambda(t)\} \left[\frac{1}{3!}(l_0 - 1)(l_0 - 2)(l_0 - 3)(\exp\{\Lambda(t)\} - 1)^3 - \frac{1}{2!}(l_0 - 1)(l_0 - 2)(\exp\{\Lambda(t)\} - 1)^2\right]$,

$\vdots$

such that

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = 0) + \frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = 1) = -l_0 \lambda(t) \exp\{-l_0 \Lambda(t)\} [(l_0 - 1)(\exp\{\Lambda(t)\} - 1)]$,

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = 0) + \frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = 1) + \frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = 2) = -l_0 \lambda(t) \exp\{-l_0 \Lambda(t)\} \left[\frac{1}{2!}(l_0 - 1)(l_0 - 2)(\exp\{\Lambda(t)\} - 1)^2\right]$,

$\vdots$

$\sum_{j=0}^{k-1} \frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = j) = -l_0 \lambda(t) \exp\{-l_0 \Lambda(t)\} \left[\frac{1}{(k-1)!}(l_0 - 1)(l_0 - 2)\dots(l_0 - k + 1)(\exp\{\Lambda(t)\} - 1)^{k-1}\right]$.

Therefore,

$$\begin{aligned} f_{S_k}(t) &= -\sum_{j=0}^{k-1} \frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = j) \\ &= \frac{l_0!}{(k-1)!\,(l_0 - k)!} \lambda(t) \exp\{-l_0 \Lambda(t)\} (\exp\{\Lambda(t)\} - 1)^{k-1}. \end{aligned}$$

Appendix III

Alternative derivation for the joint pdf of (S_m, S_k):

The joint pdf of (S_m, S_k) can be derived from first principals using the fact that

$$f_{S_m, S_k}(t, u) = f_{S_m | S_k}(t \mid u) f_{S_k}(u)$$

where,

$$f_{S_m | S_k}(t \mid u) = -\sum_{j=k}^{m-1} \frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = j \mid S_k = u).$$

Now,

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = k \mid S_k = u) = -(l_0 - k) \lambda(t) \exp(-W(t, u))$,

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = k + 1 \mid S_k = u) = -(l_0 - k) \lambda(t) \exp(-W(t, u)) [W(t, u) - 1]$,

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = k + 2 \mid S_k = u) = -(l_0 - k) \lambda(t) \exp(-W(t, u)) \left[\frac{1}{2!} W(t, u)^2 - W(t, u)\right]$,

$\frac{\mathrm{d}}{\mathrm{d}t} P(N(t) = k + 3 \mid S_k = u) = -(l_0 - k) \lambda(t) \exp(-W(t, u)) \left[\frac{1}{3!} W(t, u)^3 - \frac{1}{2!} W(t, u)^2\right]$,

$\vdots$

such that

$\frac{\mathrm{d}}{\mathrm{d}t}P\left(N\left(t\right)=k \mid S_{k}=u\right)+\frac{\mathrm{d}}{\mathrm{d}t}P\left(N\left(t\right)=k+1 \mid S_{k}=u\right)=-\left(l_{0}-k\right)\lambda\left(t\right)\exp\left(-W\left(t,u\right)\right)W\left(t,u\right),$

$\frac{\mathrm{d}}{\mathrm{d}t}P\left(N\left(t\right)=k \mid S_{k}=u\right)+\frac{\mathrm{d}}{\mathrm{d}t}P\left(N\left(t\right)=k+1 \mid S_{k}=u\right)+\frac{\mathrm{d}}{\mathrm{d}t}P\left(N\left(t\right)=k+2 \mid S_{k}=u\right)=-\left(l_{0}-k\right)\lambda\left(t\right)\exp\left(-W\left(t,u\right)\right)\left[\frac{1}{2!}W\left(t,u\right)^{2}\right],$

$\vdots$

$\sum_{j=k}^{m-1}\frac{\mathrm{d}}{\mathrm{d}t}P\left(N\left(t\right)=j \mid S_{k}=u\right)=-\left(l_{0}-k\right)\lambda\left(t\right)\exp\left(-W\left(t,u\right)\right)\left[\frac{1}{(m-k-1)!}\left(W\left(t,u\right)\right)^{m-k-1}\right].$

Therefore,

$$\begin{aligned}
f_{S_{m}|S_{k}}\left(t \mid u\right) &= -\sum_{j=k}^{m-1}\frac{\mathrm{d}}{\mathrm{d}t}P\left(N\left(t\right)=j \mid S_{k}=u\right)\\
&= \frac{l_{0}-k}{(m-k-1)!}\lambda\left(t\right)\exp\left(-W\left(t,u\right)\right)\left(W\left(t,u\right)\right)^{m-k-1}.
\end{aligned}$$

Finally,

$$\begin{aligned}
f_{S_{m},S_{k}}\left(t,u\right) =& f_{S_{m}|S_{k}}\left(t \mid u\right)f_{S_{k}}\left(u\right)\\
=& \frac{l_{0}!}{(k-1)!\left(l_{0}-k-1\right)!\left(m-k-1\right)!}\lambda\left(u\right)\lambda\left(s\right)\left(W\left(s,u\right)\right)^{m-k-1}\exp\left\{-W\left(s,u\right)\right\}\\
&\times\exp\left\{-l_{0}\Lambda\left(u\right)\right\}\left(\exp\left\{\Lambda\left(u\right)\right\}-1\right)^{k-1}.
\end{aligned}$$

Appendix IV

Derivation of $E[N(t)]$ for Model 2:

$$
\begin{aligned}
&E[N(t)]\\
=&\sum_{n=0}^{k} n \begin{pmatrix} l_0 \\ n \end{pmatrix} (1-\exp\{-\Lambda(t)\})^{n} (\exp\{-\Lambda(t)\})^{l_0-n}\\
&+\sum_{n=k+1}^{m} n \int_0^t \frac{(W(t,u))^{n-k}}{(n-k)!} \exp\{-W(t,u)\} \frac{l_0!}{(k-1)!\,(l_0-k)!} \lambda(u) \exp\{-l_0\Lambda(u)\} (\exp\{\Lambda(u)\}-1)^{k-1}\, \mathrm{d}u\\
&+\sum_{n=m+1}^{\infty} n \int_0^t \int_0^s \left[\frac{\Gamma\left(\frac{1}{\alpha}+n-m\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n-m)!} \exp\{-W(t,s)\} (1-\exp\{-\alpha W(t,s)\})^{n-m} \frac{l_0!}{(k-1)!\,(l_0-k-1)!\,(m-k-1)!} \right.\\
&\left. \times \lambda(u)\lambda(s) (W(s,u))^{m-k-1} \exp\{-W(s,u)\} \exp\{-l_0\Lambda(u)\} (\exp\{\Lambda(u)\}-1)^{k-1} \right] \mathrm{d}u\mathrm{d}s\\
=&\sum_{n=0}^{k} n \begin{pmatrix} l_0 \\ n \end{pmatrix} (1-\exp\{-\Lambda(t)\})^{n} (\exp\{-\Lambda(t)\})^{l_0-n}\\
&+\sum_{n=k+1}^{m} n \int_0^t \frac{(W(t,u))^{n-k}}{(n-k)!} \exp\{-W(t,u)\} \frac{l_0!}{(k-1)!\,(l_0-k)!} \lambda(u) \exp\{-l_0\Lambda(u)\} (\exp\{\Lambda(u)\}-1)^{k-1}\, \mathrm{d}u\\
&+\int_0^t \int_0^s \left[\left(\sum_{n=m}^{\infty} n \frac{\Gamma\left(\frac{1}{\alpha}+n-m\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n-m)!} \exp\{-W(t,s)\} (1-\exp\{-\alpha W(t,s)\})^{n-m} - m\exp\{-W(t,s)\} \right) \frac{l_0!}{(k-1)!\,(l_0-k-1)!\,(m-k-1)!} \right.\\
&\left. \times \lambda(u)\lambda(s) (W(s,u))^{m-k-1} \exp\{-W(s,u)\} \exp\{-l_0\Lambda(u)\} (\exp\{\Lambda(u)\}-1)^{k-1} \right] \mathrm{d}u\mathrm{d}s\\
=&\sum_{n=0}^{k} n \begin{pmatrix} l_0 \\ n \end{pmatrix} (1-\exp\{-\Lambda(t)\})^{n} (\exp\{-\Lambda(t)\})^{l_0-n}\\
&+\sum_{n=k+1}^{m} n \int_0^t \frac{(W(t,u))^{n-k}}{(n-k)!} \exp\{-W(t,u)\} \frac{l_0!}{(k-1)!\,(l_0-k)!} \lambda(u) \exp\{-l_0\Lambda(u)\} (\exp\{\Lambda(u)\}-1)^{k-1}\, \mathrm{d}u\\
&+\int_0^t \int_0^s \left[\left(\sum_{n=0}^{\infty} (n+m) \frac{\Gamma\left(\frac{1}{\alpha}+n\right)}{\Gamma\left(\frac{1}{\alpha}\right) n!} \exp\{-W(t,s)\} (1-\exp\{-\alpha W(t,s)\})^{n} - m\exp\{-W(t,s)\} \right) \frac{l_0!}{(k-1)!\,(l_0-k-1)!\,(m-k-1)!} \right.\\
&\left. \times \lambda(u)\lambda(s) (W(s,u))^{m-k-1} \exp\{-W(s,u)\} \exp\{-l_0\Lambda(u)\} (\exp\{\Lambda(u)\}-1)^{k-1} \right] \mathrm{d}u\mathrm{d}s\\
=&\sum_{n=0}^{k} n \begin{pmatrix} l_0 \\ n \end{pmatrix} (1-\exp\{-\Lambda(t)\})^{n} (\exp\{-\Lambda(t)\})^{l_0-n}\\
&+\sum_{n=k+1}^{m} n \int_0^t \frac{(W(t,u))^{n-k}}{(n-k)!} \exp\{-W(t,u)\} \frac{l_0!}{(k-1)!\,(l_0-k)!} \lambda(u) \exp\{-l_0\Lambda(u)\} (\exp\{\Lambda(u)\}-1)^{k-1}\, \mathrm{d}u\\
&+\int_0^t \int_0^s \left[\left(\frac{1}{\alpha} (\exp\{\alpha W(t,s)\}-1) + m - m\exp\{-W(t,s)\} \right) \frac{l_0!}{(k-1)!\,(l_0-k-1)!\,(m-k-1)!} \right.\\
&\left. \times \lambda(u)\lambda(s) (W(s,u))^{m-k-1} \exp\{-W(s,u)\} \exp\{-l_0\Lambda(u)\} (\exp\{\Lambda(u)\}-1)^{k-1} \right] \mathrm{d}u\mathrm{d}s
\end{aligned}
$$