Stabilisation in Distribution of Periodic Hybrid Systems by Discrete-time State Feedback Control

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Abstract

Periodic hybrid stochastic differential equations (SDEs) have been widely used to model systems in many branches of science and industry which are subject to the following natural phenomena: (a) uncertainty and environmental noise, (b) abrupt changes in their structure and parameters, (c) periodicity. In many situations, it is inappropriate to study whether the solutions of periodic hybrid SDEs will converge to an equilibrium state (say, 0 or the trivial solution) but more appropriate to discuss whether the probability distributions of the solutions will converge to a stationary distribution, known as stability in distribution. Given a periodic hybrid SDE, which is not stable in distribution, can we design a periodic feedback control in the shift term based on state observations at discrete times so that the controlled SDE becomes stable in distribution? We will refer to this problem as stabilisation in distribution by periodic feedback control. There is little known on this problem so far. This paper initiates the study in this direction.

Key Words: Brownian motion, Markov chain, Periodic SDEs, Stabilisation in distribution, Periodic feedback control.

1 Introduction

Systems in many branches of science and industry are often subject to the following natural phenomena: (a) uncertainty and environmental noise, (b) abrupt changes in their structure and parameters, (c) periodicity. Stochastic differential equations (SDEs) have been widely used to model systems with phenomenon (a) (see, e.g., [2, 19, 21, 27]). Modelling phenomenon (b) by Markov chains (see, e.g., [1]), SDEs with Markov switching (also known as hybrid SDEs) have been developed to model systems with phenomena (a) and (b) (see, e.g., [4, 15, 16, 37, 39, 47, 48]). Furthermore, taking the periodicity (e.g., seasonal changes) into account, we arrive at periodic hybrid SDEs of the form

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{1.1}$$

(see, e.g., [3, 8, 13, 24, 25, 36, 42, 43, 54]). Here $x(t)$ takes values in $\mathbb{R}^n$ and is referred to as the state, $r(t)$ is regarded as the mode and is modelled by a Markov chain with a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$, $B(t)$ is an m-dimensional Brownian motion, and the coefficients $f(\cdot, \cdot, t)$ and $g(\cdot, \cdot, t)$ are periodic in t with period h, that is, $f(x, i, t) = f(x, i, h + t)$ and $g(x, i, t) = g(x, i, h + t)$

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for $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$. (The notation used in this section will be explained in more detail in the next section).

In the study of hybrid SDEs, stability is one of the most popular topics. On the asymptotic stability, there are two fundamental concepts: (i) asymptotic stability of an equilibrium state; (ii) asymptotic stability in distribution. Concept (i) is to study whether the solution $x(t)$ of equation (1.3) will tend to the equilibrium state (e.g., 0 as in most papers) in various senses, e.g., $\mathbb{E}|x(t)|^p \to 0$ or $x(t) \to 0$ a.s. or $x(t) \to 0$ in probability, and there is an intensive literature on it (see, e.g., [11, 20, 21, 28, 29, 33, 35] and many others). Concept (ii) is to study whether the probability distribution, denoted by $\mathcal{L}(x(t))$, of the solution $x(t)$ of equation (1.3) will converge to a probability distribution μ, known as a stationary distribution. The literature on concept (ii) is much less than (i) but has been growing quickly for the past 10 years (see, e.g., [45, 51, 52]) and several recent papers [6, 10, 22, 23, 50]. The reason why there are fewer papers on concept (ii) than (i) is because it is much more difficult to study whether $\mathcal{L}(x(t)) \to \mu$ than to study whether $\mathbb{E}|x(t)|^p \to 0$ etc. but certainly not because concept (ii) is less important. In fact, it is inappropriate to study concept (i) for many systems in the real world but more appropriate to study concept (ii). For example, for many population systems under random environment, the stochastic permanence is a more desired control objective than the extinction (see, e.g., [5, 12, 17]). In this situation it is useful to investigate concept (ii) but not to concept (i) (see, e.g., [17, 29, 40]).

Suppose that the given system (1.3) is not asymptotically stable in the sense of concept (i). It is a normal practice that a feedback control $u(x(t), r(t), t)$ is used to make the controlled system

$$dX(t) = [f(X(t), r(t), t) + u(X(t), r(t), t)]dt + g(X(t), r(t), t)dB(t) \tag{1.2}$$

to be stable, where we have replaced $x(t)$ by $X(t)$ to highlight that the state in this controlled system differs from that in the given system. There is already an intensive literature in this area (see, e.g., [11, 16, 20, 38, 46]). Note that in practice, the state $X(t)$ cannot be observed continuously but only at discrete times, say $0,\ \tau,\ 2\tau,\ \cdots$, where τ is a positive number and stands for the duration between two consecutive observations. Accordingly, the feedback control $u(X(t), r(t), t)$ should be replaced with $u(X(\lfloor t/\tau \rfloor \tau), r(t), t)$ to arrive at another more practically used controlled system

$$dX(t) = [f(X(t), r(t), t) + u(X(\lfloor t/\tau \rfloor \tau), r(t), t)]dt + g(X(t), r(t), t)dB(t), \tag{1.3}$$

where $\lfloor t/\tau \rfloor$ is the integer part of t/τ. This stabilisation problem was initiated by [31] and has since become popular (see, e.g., [14, 32, 49]).

However, if the aim of feedback control is to make the controlled system (1.3) to be stable in distribution, a little has been known. To the best of our knowledge, there is only one paper [23] in the case when the given SDE (1.1) is time-homogeneous (i.e., $f(x, i, t) = f(x, i)$ and $g(x, i, t) = g(x, i)$). In the time-homogeneous case, it should be mentioned that [22, 50] have recently investigated the stabilisation in distribution by the delay feedback control (different from our control). In the periodic case, [6, 10] should also be mentioned though they are not concerned with the feedback control based on discrete-time state observations. Both papers investigated the stability in distribution of the periodic SDE $dX(t) = f(X(t), r(t), t)dt + g(X(t), r(t), t)dB(t)$, though $g(X(t), r(t), t)dB(t)$ may be regarded as a stochastic feedback control based on continuous-time state observation in [6] (different from our control once again). In other words, there is so far no result on the stability in distribution of the proposed controlled system (1.3). In this paper we will address this open problem.

Mathematically speaking, our controlled system (1.3) is a stochastic differential delay equation (SDDE). It is known that analysis of SDDEs is much difficult than that of SDEs. Moreover, the continuous-time $X(t)$ and discrete-time $X(\lfloor t/\tau \rfloor \tau)$ which are mixed with each other in (1.3) makes the analysis even harder than general SDDEs. Therefore the mathematical analysis in our present paper is much more complicated than that in [6, 10]. Moreover, we need the time-homogeneous Markov property to study the stability in distribution (see, e.g., [1]). However, the coefficients f, g and the control function u are all periodic so they are not time-homogeneous. Although both [6, 10] explained how to identify time-homogeneous Markov processes by making use of the periodicity, there were no rigorous proofs of the Markov property. In this paper, we do not only identify proper time-homogeneous Markov processes but also present rigorous proofs of them (see Lemmas 2.3 and 2.4 below). To close this section, we highlight the main features of this paper:

- This is the first paper to show that given a *periodic* unstable nonlinear hybrid SDE (1.1), a periodic feedback control can be designed based on discrete-time state observations for the stochastically controlled system (1.3) to be stable in distribution.
- New techniques will be developed to tackle the challenges due to the *time-inhomogeneous coefficients and periodic feedback control based on discrete-time state observations*. These are significantly different from the existing papers mentioned above in the area of stabilisation in distribution.

2 Notation and Definition

The notations used in this paper are essentially standard (see, e.g., [6, 10, 23, 33]) but we define them here for the convenience of the reader. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathcal{B}(\mathbb{R}^n)$ be the family of all Borel measurable sets in $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then $|x|$ is its Euclidean norm. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. By $A > 0$ and $A \geq 0$, we mean A is positive and non-negative definite, respectively. If a is a real number, its integer part is denoted by $\lfloor a \rfloor$. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $\mathbb{N}_+$ denote the set of nonnegative integers. If G is a set, $I_G(\cdot)$ denotes its indicator function, that is $I_G(x) = 1$ for $x \in G$ and 0 otherwise. We set $\inf \emptyset = \infty$, where $\emptyset$ denotes the empty set. Moreover, $x := y$ means x is defined by y while $y =: x$ means y is denoted by x.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). If $\Omega_1 \subset \Omega$, its complement is denoted by Ω_1^c. Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous irreducible Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$, where $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$ though they are both $\mathcal{F}_t$-adapted. Denote by $L^2_t(\mathbb{R}^n)$ the family of $\mathcal{F}_t$-measurable $\mathbb{R}^n$-valued random variables $\hat{x}$ such that $\mathbb{E}|\hat{x}|^2 < \infty$. Denote by $\mathbb{S}_t$ the family of $\mathcal{F}_t$-measurable $\mathbb{S}$-valued random variables.

For a positive number h, denote by $\mathcal{K}_h$ the family of càdlàg (right continuous with left limits) periodic functions κ from $\mathbb{R}_+$ to $[0, 1]$ with period h. If $\kappa \in \mathcal{K}_h$, we set $\bar{\kappa} = (1/h)\int_0^h \kappa(s)ds$. That is, $\bar{\kappa}$ is the average value of $\kappa(\cdot)$ per period. Denote by $\mathcal{C}_h$ the family of continuous functions ξ from $[0, h]$ to $\mathbb{R}^n$ with norm $\|\xi\|_h = \sup_{s\in[0,h]} |\xi(s)|$. Denote by $\mathcal{B}(\mathcal{C}_h)$ the family of all Borel measurable sets in $\mathcal{C}_h$. Denote by $\mathcal{P}(\mathcal{C}_h)$ the family of probability measures on $\mathcal{C}_h$. A coupling of P_1 and P_2 in $\mathcal{P}(\mathcal{C}_h)$ is a probability measure π on the product space $\mathcal{C}_h \times \mathcal{C}_h$ such that the marginals of π coincide with P_1 and P_2, i.e., $\pi(A \times \mathcal{C}_h) = P_1(A)$ and $\pi(\mathcal{C}_h \times A) = P_2(A)$ for any $A \in \mathcal{B}(\mathcal{C}_h)$ (see, e.g., [1]). Let $C(P_1, P_2)$ denote the family of all couplings of P_1 and P_2. For $p \geq 1$, denote by $\mathcal{P}_p(\mathcal{C}_h)$ the family of probability measures on $\mathcal{C}_h$ with finite pth-moments. The Wasserstein p-distance between $P_1, P_2 \in \mathcal{P}_p(\mathcal{C}_h)$ is

$$W_p(P_1, P_2) = \inf_{\pi \in C(P_1, P_2)} \Big(\int_{\mathcal{C}_h \times \mathcal{C}_h} \|\xi_1 - \xi_2\|_h^p \pi(d\xi_1, d\xi_2) \Big)^{1/p}.$$

In particular, if X and Y are two $\mathcal{C}_h$-valued random variables such that $\mathbb{E}\|X\|_h^p + \mathbb{E}\|Y\|_h^p < \infty$. Let $\mathcal{L}_{X,Y}(\cdot, \cdot)$ be the joint probability distribution of random vector (X, Y). Let $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ denote the probability measures on $\mathcal{C}_h$ generated by X and Y, respectively (see, e.g., [19], for more details about probability measures generated by random variables). It is then known (see, e.g., [41]) that

$$W_p(\mathcal{L}(X), \mathcal{L}(Y)) \leq \Big(\int_{\mathcal{C}_h \times \mathcal{C}_h} \|\xi_1 - \xi_2\|_h^p \mathcal{L}_{X,Y}(d\xi_1, d\xi_2) \Big)^{1/p} = (\mathbb{E}\|X - Y\|_h^p)^{1/p}. \tag{2.1}$$

Let us consider the given SDE (1.1) and its controlled SDE (1.3), where f, u are Borel measurable functions from $\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$ to $\mathbb{R}^n$ and g from $\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$ to $\mathbb{R}^{n \times m}$. For both of equations (1.1) and (1.3) to be well defined, we impose the following assumptions.

Assumption 2.1 *The coefficients $f(x, i, t)$ and $g(x, i, t)$ are periodic in t with period h (> 0). There are periodic functions $\kappa_1, \kappa_2 \in \mathcal{K}_h$ and non-negative constants a_i, b_i ($i \in \mathbb{S}$) such that*

$$|f(x, i, t) - f(y, i, t)| \leq a_i \kappa_1(t) |x - y|,$$
$$|g(x, i, t) - g(y, i, t)| \leq \sqrt{b_i \kappa_2(t)} |x - y|$$

for all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$. Moreover,

$$\sup_{t \in [0,h]} |f(0, i, t)| \vee |g(0, i, t)| < \infty, \quad \forall i \in \mathbb{S}.$$

To make the design of the feedback control simpler, we will seek a linear form of the feedback control which is described in the following assumption.

Assumption 2.2 *The control function $u(x, i, t)$ has the form*

$$u(x, i, t) = \kappa_3(t) A_i x \tag{2.2}$$

for $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $\kappa_3 \in \mathcal{K}_h$ and all A_i's are non-positive definite symmetric matrices (i.e., $\lambda_{\max}(A_i) \leq 0$). These imply

$$|u(x, i, t) - u(y, i, t)| \leq c_0 |x - y|, \tag{2.3}$$
$$(x - y)^T (u(x, i, t) - u(y, i, t)) \leq -c_i \kappa_3(t) |x - y|^2 \tag{2.4}$$

for all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $c_0 = \max_{i \in \mathbb{S}} \|A_i\| > 0$ and $c_i = -\lambda_{\max}(A_i) \geq 0$.

We will explain how to design A_i's and κ_3 in Section 4 but we simply assume that we have them in this section. Although f, g, u are all periodic in t with the same period h, we need to choose τ, the duration between two consecutive state observations, carefully in order to make sure the controlled SDE (1.3) is periodic with the same period h. Consequently, we will set $\tau = h/M$ for some positive integer M from now on (i.e., τ is a divisor of h). When we ask $\tau < c$ for some positive number, we of course mean $M > h/c$.

Set $K_1 := \sup_{(i,t)\in\mathbb{S}\times[0,h]}(|f(0,i,t)| \vee |g(0,i,t)|)$, $\check{a} = \max_{i\in\mathbb{S}} a_i$ and $\check{b} = \max_{i\in\mathbb{S}} b_i$. It then follows from Assumptions 2.1 and 2.2 that

$$|f(x,i,t)| \le K_1 + \check{a}|x|, \quad |g(x,i,t)| \le K_1 + \sqrt{\check{b}}|x|, \quad |u(x,i,t)| \le c_0|x|. \tag{2.5}$$

To see the controlled system (1.3) is well defined, we will re-write it as a stochastic differential delay equation (SDDE). In fact, defining the variable time delay function $\zeta : \mathbb{R}_+ \to [0,\tau]$ by

$$\zeta(t) = t - \lfloor t/\tau \rfloor \tau, \quad t \ge 0, \tag{2.6}$$

we see the controlled system (1.3) is the same as the following SDDE

$$dX(t) = [f(X(t), r(t), t) + u(t - \zeta(t), r(t), t)]dt + g(X(t), r(t), t)dB(t). \tag{2.7}$$

Noting that $t - \zeta(t) \ge 0$ for all $t \ge 0$, we only need initial data $X(0)$ and $r(0)$ at time 0 to solve this SDDE (see, e.g., [18, 28, 33]). More precisely, under Assumptions 2.1 and 2.2, for any given initial data $X(0) = \hat{x} \in \mathbb{R}^n$ and $r(0) = \hat{i} \in \mathbb{S}$ at time 0, the SDDE (2.7), namely the controlled system (1.3), has a unique global solution on $t \ge 0$, which will be denoted by $X_{\hat{x},\hat{i}}(t)$ in this paper in order to highlight the role of the initial data, though we often write it as $X(t)$ for convenience. We also denote by $r_{\hat{i}}(t)$ the Markov chain starting from $\hat{i}$ at time 0. It is also known ([33, p. 99, Theorem 3.24]) that

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X_{\hat{x},\hat{i}}(t)|^2 \Big) \le C_T(1 + |\hat{x}|^2) \tag{2.8}$$

for all $T \ge 0$, where C_T is a positive number dependent on T but independent of $(\hat{x}, \hat{i})$.

To discuss the stability in distribution, we need the time-homogeneous Markov property (see, e.g., [1]). However, from the general theory of SDDEs (see, e.g., [27, 33]), the joint process $(X_{\hat{x},\hat{i}}(t), r_{\hat{i}}(t))$ is not a Markov process on $t \ge 0$. Fortunately, the coefficients and the control function are periodic with period h while τ is designed to be a divisor of h (i.e., $\tau = h/M$ for some positive integer) and the delay function ζ takes its special form of (2.6). These enable us to form at least two time-homogeneous Markov processes for the use of this paper. The following lemma identifies the first Markov process used in this paper.

Lemma 2.3 *The process $\{(X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in\mathbb{N}_+}$ is a time-homogeneous strong Markov process with its state space $\mathbb{R}^n \times \mathbb{S}$. We will define its k-step transition probabilities by*

$$P(k, \hat{x}, \hat{i}; A \times S) := \mathbb{P}((X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh)) \in A \times S). \tag{2.9}$$

Proof. To prove this, we need a number of more complicated notations. For $k \in \mathbb{N}_+$ and $(x_k, i_k) \in L^2_{kh}(\mathbb{R}^n) \times \mathbb{S}_{kh}$, denote by $\{r^k_{i_k}(t)\}_{t\ge kh}$ the Markov chain starting from $r(kh) = i_k$ at time kh and by $\{X^k_{x_k,i_k}(t)\}_{t\ge kh}$ the unique global solution of the controlled system (1.3) with the initial data $X(kh) = x_k$ and $r(kh) = i_k$ at time kh. Please note that $r^0_{\hat{i}}(t) = r_{\hat{i}}(t)$ and $X^0_{\hat{x},\hat{i}}(t) = X_{\hat{x},\hat{i}}(t)$. It is also easy to see that if we set $(X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh)) =: (\hat{x}_k, \hat{i}_k)$, then $(X_{\hat{x},\hat{i}}(t), r_{\hat{i}}(t)) = (X^k_{\hat{x}_k,\hat{i}_k}(t), r^k_{\hat{i}_k}(t))$ for all $t \ge kh$. In particular, $(X_{\hat{x},\hat{i}}(\bar{k}h), r_{\hat{i}}(\bar{k}h)) = (X^k_{\hat{x}_k,\hat{i}_k}(\bar{k}h), r^k_{\hat{i}_k}(\bar{k}h))$ for any integer $\bar{k} > k$. Note that for any $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times \mathbb{S}$, $\{(X^k_{\hat{x},\hat{i}}(\bar{k}h), r^k_{\hat{i}}(\bar{k}h))\}_{\bar{k}>k}$ is independent of $\mathcal{F}_{kh}$. We can then apply [27, Lemma 9.2 on page 87] to derive that for $A \in \mathcal{B}(\mathbb{R}^n)$, $S \subset \mathbb{S}$ and $\bar{k} > k$,

$$\begin{aligned} &\mathbb{P}((X_{\hat{x},\hat{i}}(\bar{k}h), r_{\hat{i}}(\bar{k}h)) \in A \times S|\mathcal{F}_{kh}) = \mathbb{E}(I_{A\times S}(X_{\hat{x},\hat{i}}(\bar{k}h), r_{\hat{i}}(\bar{k}h))|\mathcal{F}_{kh}) \\ =&\mathbb{E}(I_{A\times S}(X^k_{\hat{x}_k,\hat{i}_k}(\bar{k}h), r^k_{\hat{i}_k}(\bar{k}h))|\mathcal{F}_{kh}) = \mathbb{E}(I_{A\times S}(X^k_{\hat{x},\hat{i}}(\bar{k}h), r^k_{\hat{i}}(\bar{k}h)))\big|_{(\hat{x},\hat{i})=(\hat{x}_k,\hat{i}_k)}. \end{aligned} \tag{2.10}$$

Noting also that for any $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times \mathbb{S}$, $\{(X^k_{\hat{x},\hat{i}}(\bar{k}h),\ r^k_{\hat{i}}(\bar{k}h))\}_{\bar{k}>k}$ is independent of the σ-algebra generated by $(\hat{x}_k, \hat{i}_k)$, we can show in the same way as above that

$$\mathbb{P}((X_{\hat{x},\hat{i}}(\bar{k}h), r_{\hat{i}}(\bar{k}h)) \in A \times S|(\hat{x}_k, \hat{i}_k)) = \mathbb{E}(I_{A\times S}(X^k_{\hat{x},\hat{i}}(\bar{k}h), r^k_{\hat{i}}(\bar{k}h)))\big|_{(\hat{x},\hat{i})=(\hat{x}_k,\hat{i}_k)}. \tag{2.11}$$

We hence have

$$\mathbb{P}((X_{\hat{x},\hat{i}}(\bar{k}h), r_{\hat{i}}(\bar{k}h)) \in A \times S|\mathcal{F}_{kh}) = \mathbb{P}((X_{\hat{x},\hat{i}}(\bar{k}h), r_{\hat{i}}(\bar{k}h)) \in A \times S|(\hat{x}_k, \hat{i}_k)).$$

This proves that $\{(X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in\mathbb{N}_+}$ is a Markov process. Moreover, in light of the periodic property of f, g, u as well as the time-homogeneous property of the Markov chain $\{r(t)\}_{t\geq 0}$, we see

$$\mathbb{P}((X^k_{\hat{x},\hat{i}}(\bar{k}h), r^k_{\hat{i}}(\bar{k}h)) \in A \times S) = \mathbb{P}((X_{\hat{x},\hat{i}}((\bar{k}-k)h), r_{\hat{i}}((\bar{k}-k)h)) \in A \times S).$$

That is, the Markov process is time-homogeneous. Finally, it is well-known that any discrete-time Markov process has the strong Markov property. In summary, $\{(X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in\mathbb{N}_+}$ is a time-homogeneous strong Markov process. □

To form the second time-homogeneous Markov process, we define $\bar{X}_{\hat{x},\hat{i}}(kh) = \{X_{\hat{x},\hat{i}}(kh+s) : -h \leq s \leq 0\}$ for $k \in \mathbb{N}_+$, where we set $X_{\hat{x},\hat{i}}(s) = \hat{x}$ for $-h \leq s \leq 0$.

Lemma 2.4 *The process $\{(\bar{X}_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in\mathbb{N}_+}$ forms a time-homogeneous strong Markov process with its state space $\mathcal{C}_h \times \mathbb{S}$.*

Proof. The time-homogeneous property once again follows from the periodic property of f, g, u as well as the time-homogeneous property of the Markov chain $\{r(t)\}_{t\geq 0}$, while the strong Markov property follows as long as we can show the Markov property as the process is of discrete-time. We will use the same notations defined in the last paragraph, e.g., $(X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh)) =: (\hat{x}_k, \hat{i}_k)$, and set $\bar{X}^k_{\hat{x}_k,\hat{i}_k}(\bar{k}h) := \{X^k_{\hat{x}_k,\hat{i}_k}(\bar{k}h+s) : -h \leq s \leq 0\}$ for $\bar{k} > k \geq 0$. With these notations, we see clearly that

$$(\bar{X}_{\hat{x},\hat{i}}(\bar{k}h), r_{\hat{i}}(\bar{k}h)) = (\bar{X}^k_{\hat{x}_k,\hat{i}_k}(\bar{k}h), r^k_{\hat{i}_k}(\bar{k}h)). \tag{2.12}$$

Noting that for any $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times \mathbb{S}$, $\{(\bar{X}^k_{\hat{x},\hat{i}}(\bar{k}h), r^k_{\hat{i}}(\bar{k}))\}_{\bar{k}>k}$ is independent of $\mathcal{F}_{kh}$ and of the σ-algebra generated by $(\hat{x}_k, \hat{i}_k)$, we can show in the same way as (2.10) and (2.11) were proved that for $D \in \mathcal{B}(\mathcal{C}_h)$, $S \subset \mathbb{S}$ and $\bar{k} > k$,

$$\begin{aligned}\mathbb{P}((\bar{X}_{\hat{x},\hat{i}}(\bar{k}h), r_{\hat{i}}(\bar{k}h)) \in D \times S|\mathcal{F}_{kh}) &= \mathbb{E}(I_{D\times S}(\bar{X}^k_{\hat{x},\hat{i}}(\bar{k}h), r^k_{\hat{i}}(\bar{k}h)))\big|_{(\hat{x},\hat{i})=(\hat{x}_k,\hat{i}_k)} \\ &= \mathbb{P}((\bar{X}_{\hat{x},\hat{i}}(\bar{k}h), r_{\hat{i}}(\bar{k}h)) \in D \times S|(\hat{x}_k, \hat{i}_k)). \end{aligned} \tag{2.13}$$

This proves the Markov property and hence the lemma. □

We will not use the k-step transition probabilities of this Markov process in this paper. Denote by $\mathcal{L}(\bar{X}_{\hat{x},\hat{i}}(kh))$ the probability measure on $\mathcal{C}_h$ generated by $\bar{X}_{\hat{x},\hat{i}}(kh)$. We can now give the definition of the stability in distribution under the Wasserstein p-distance.

Definition 2.5 *Let $p \geq 1$. The controlled SDE (1.3) is said to be asymptotically stable in distribution under the Wasserstein p-distance if $\mathcal{L}(\bar{X}_{\hat{x},\hat{i}}(kh)) \in \mathcal{P}_p(\mathcal{C}_h)$ for all $k \in \mathbb{N}_+$ and there exists a unique probability measure $\mu \in \mathcal{P}_p(\mathcal{C}_h)$ such that*

$$\lim_{k\to\infty} W_p(\mathcal{L}(\bar{X}_{\hat{x},\hat{i}}(kh)), \mu) = 0$$

for all $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times \mathbb{S}$.

3 Asymptotic Stability in Distribution

In this section, we shall study the asymptotic stability in distribution of the controlled SDE (1.3). We need to impose a couple of additional conditions but will only explain in Section 4 how to design the control function $u(x,i,t)$ (namely A_i's and κ_3) to meet these additional conditions given that f and g satisfy Assumption 2.1. The following assumption is the first technical condition.

Assumption 3.1 *Assume that*

$$\mathcal{A} := \text{diag}(2c_1 - 2a_1 - b_1, \cdots, 2c_N - 2a_N - b_N) - \Gamma$$

is a nonsingular M-matrix.

We need a number of new notations associated with this M-matrix. Set

$$(\theta_1, \cdots, \theta_N)^T = \mathcal{A}^{-1}(1, \cdots, 1)^T. \tag{3.1}$$

By the theory of M-matrices (see, e.g., [33, Theorem 2.10 on page 68]), $\theta_i > 0$ for all $i \in \mathbb{S}$. Set

$$\hat{\theta} = \min_{1\le i\le N} \theta_i, \quad \check{\theta} = \max_{1\le i\le N} \theta_i. \tag{3.2}$$

To state a new assumption, we set

$$\hat{a} = \min_{1\le i\le N} a_i, \quad \hat{b} = \min_{1\le i\le N} b_i, \quad \check{c} = \max_{1\le i\le N} c_i, \tag{3.3}$$

$$\bar{\varphi} = 2\hat{a}(1-\bar{\kappa}_1) + \hat{b}(1-\bar{\kappa}_2) - 2\check{c}(1-\bar{\kappa}_3), \quad \alpha = 2\hat{a} + \hat{b} + 1/\check{\theta}. \tag{3.4}$$

Assumption 3.2 *Assume that*

$$\frac{1}{\check{\theta}} + \bar{\varphi} > 0. \tag{3.5}$$

It is useful to observe that if we can design $\kappa_3(\cdot) \equiv 1$, then $\bar{\varphi} \ge 0$ and Assumption 3.2 holds. However, we may not be able to make it sometimes in practice, e.g., when an intermittent control has to be used. We will illustrate this in Section 5.

In what follows, we also need a number of key functions and parameters. To develop the paper more clearly, we state them here. For $v \in (0, 1/c_0)$, define

$$H_1(v) = (v(0.5\check{a} + c_0) + 0.5\sqrt{\check{b}v})/(1 - vc_0) \quad \text{and} \quad H_2(v) = 0.5\big(\check{a} + \sqrt{\check{b}/v}\big)/(1 - vc_0). \tag{3.6}$$

Let $\tau^* \in (0, 1/c_0)$ be the unique number for

$$\frac{1}{\check{\theta}} + \bar{\varphi} = \frac{2c_0\check{\theta}}{\hat{\theta}}[H_1(\tau^*) + \tau^* e^{\alpha\tau^*} H_2(\tau^*)]. \tag{3.7}$$

Noting that $H_1(v) + ve^{\alpha v}H_2(v)$ is a continuously increasing function of $v \in (0, 1/c_0)$ and tends to 0 and ∞ as $v \to 0$ and $1/c_0$, respectively, we see that τ^* is well defined. For $\tau \in (0, \tau^*)$, set

$$\varepsilon_0 := \frac{2c_0\check{\theta}}{\hat{\theta}}[H_1(\tau) + \tau e^{\alpha\tau} H_2(\tau)] \quad \text{and} \quad \gamma := 0.5(1/\check{\theta} + \bar{\varphi} - \varepsilon_0). \tag{3.8}$$

Of course, both ε_0 and γ depend on τ but we do not make them explicitly without causing any confusion. By the definition of τ^*, we see $0 < \varepsilon_0 < 1/\check{\theta} + \bar{\varphi}$ and $\gamma > 0$. Please note that we will use these notations without further explanation unless necessary. Let us present a number of lemmas in order to establish our main theorems.

Lemma 3.3 *Let Assumptions 2.1 and 2.2 hold. Define*

$$\varphi(t) = 2\hat{a}(1-\kappa_1(t)) + \hat{b}(1-\kappa_2(t)) - 2\check{c}(1-\kappa_3(t))$$

for $t \geq 0$. *Then*

$$\Big|\int_0^t \varphi(s)ds - \bar{\varphi}t\Big| \leq (2\hat{a} + \hat{b} + 2\check{c})h, \quad \forall t \geq 0. \tag{3.9}$$

This lemma can be proved in a similar way as [6, Lemma 3.4] was proved. The proof is hence omitted. From now on we will set $\lfloor t/\tau \rfloor \tau =: \delta_t$ for $t \geq 0$ in order to make the notation shorter.

Lemma 3.4 *Let Assumptions 2.1 and 2.2 hold. Write* $X_{\hat{x},\hat{\imath}}(t) = X(t)$ *for* $(\hat{x}, \hat{\imath}) \in \mathbb{R}^n \times \mathbb{S}$ *and* $t \geq 0$. *If* $\tau < 1/c_0$, *then*

$$\begin{aligned} \mathbb{E}(|X(t)||X(t) - X(\delta_t)|) \leq H_1(\tau)\mathbb{E}|X(t)|^2 + H_2(\tau)\mathbb{E}\int_{\delta_t}^t |X(s)|^2 ds \\ + H_3(\tau) + H_4(\tau)\mathbb{E}|X(t)| + H_5(\tau)\mathbb{E}\int_{\delta_t}^t |X(s)|ds, \end{aligned} \tag{3.10}$$

where $H_1(\cdot)$ *and* $H_2(\cdot)$ *have been defined by (3.6) while* $H_3(\tau) = 0.5K_1^2\sqrt{\tau/\check{b}}/(1-\tau c_0)$, $H_4(\tau) = \tau K_1/(1-\tau c_0)$, $H_5(\tau) = (K_1/\sqrt{\tau})/(1-\tau c_0)$.

Proof. Fix $t \geq 0$ arbitrarily and set $k = \lfloor t/\tau \rfloor$. Then $k\tau \leq t < (k+1)\tau$ and $\delta_t = k\tau$. By (2.5), it is straightforward to show from (1.3) that

$$\begin{aligned} &\mathbb{E}(|X(t)||X(t) - X(\delta_t)|) = \mathbb{E}(|X(t)||X(t) - X(k\tau)|) \\ \leq &\mathbb{E}\Big(|X(t)|\int_{k\tau}^t \big[K_1 + \check{a}|X(s)| + c_0|X(k\tau)|\big]ds + |X(t)||M_k(t)|\Big), \end{aligned} \tag{3.11}$$

where $M_k(t) = \int_{k\tau}^t g(X(s), r(s), s)dB(s)$. But

$$\begin{aligned} &\mathbb{E}\Big(|X(t)|\int_{k\tau}^t \big[K_1 + \check{a}|X(s)| + c_0|X(k\tau)|\big]ds\Big) \\ &\leq \tau K_1\mathbb{E}|X(t)| + \mathbb{E}\int_{k\tau}^t \big[\check{a}|X(t)||X(s)| + c_0|X(t)|(|X(t)| + |X(t) - X(k\tau)|\big]ds \\ &\leq \tau K_1\mathbb{E}|X(t)| + \tau(0.5\check{a} + c_0)\mathbb{E}|X(t)|^2 + 0.5\check{a}\mathbb{E}\int_{k\tau}^t |X(s)|^2 ds \\ &+ \tau c_0\mathbb{E}(|X(t)||X(t) - X(\delta_t)|). \end{aligned} \tag{3.12}$$

Moreover, by (2.5),

$$\begin{aligned} &\mathbb{E}(|X(t)||M_k(t)|) \leq 0.5\sqrt{\check{b}\tau}\mathbb{E}|X(t)|^2 + (0.5/\sqrt{\check{b}\tau})\mathbb{E}|M_k(t)|^2 \\ &\leq 0.5\sqrt{\check{b}\tau}\mathbb{E}|X(t)|^2 + (0.5/\sqrt{\check{b}\tau})\mathbb{E}\int_{k\tau}^t \big(K_1^2 + 2K_1\sqrt{\check{b}}|X(s)| + \check{b}|X(s)|^2\big)ds \\ &\leq 0.5\sqrt{\check{b}\tau}\mathbb{E}|X(t)|^2 + 0.5K_1^2\sqrt{\tau/\check{b}} + \mathbb{E}\int_{k\tau}^t \Big((K_1/\sqrt{\tau})|X(s)| + 0.5\sqrt{\check{b}/\tau}|X(s)|^2\Big)ds. \end{aligned} \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11) implies

$$\begin{aligned}&\mathbb{E}(|X(t)||X(t)-X(\delta_t)|)\le \big(\tau(0.5\breve{a}+c_0)+0.5\sqrt{\breve{b}\tau}\big)\mathbb{E}|X(t)|^2\\&+0.5\big(\breve{a}+\sqrt{\breve{b}/\tau}\big)\mathbb{E}\int_{k\tau}^t|X(s)|^2ds+0.5K_1^2\sqrt{\tau/\breve{b}}+\tau K_1\mathbb{E}|X(t)|\\&+K_1/\sqrt{\tau}\mathbb{E}\int_{k\tau}^t|X(s)|ds+\tau c_0\mathbb{E}(|X(t)||X(t)-X(\delta_t)|).\end{aligned}\tag{3.14}$$

This implies the required assertion (3.10) as $\tau c_0<1$ and $k\tau=\delta_t$. The proof is complete. □

Lemma 3.5 *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold. If $\tau<\tau^*$, then for any $(\hat{x},\hat{i})\in\mathbb{R}^n\times\mathbb{S}$,*

$$\mathbb{E}\|\bar{X}_{\hat{x},\hat{i}}(kh)\|_h^2\le C_1(1+|\hat{x}|^2),\quad \forall k\in\mathbb{N}_+.\tag{3.15}$$

where C_1 is a positive number dependent on $\tau,h,\hat{a},\breve{a},\hat{b},\breve{b},\breve{c},c_0,\breve{\theta},\hat{\theta},\bar{\varphi},K_1,K_2$ and its explicit form is defined in the proof below. From its explicit form, we see that C_1 is independent of $(\hat{x},\hat{i})$.

Proof. Fix $(\hat{x},\hat{i})\in\mathbb{R}^n\times\mathbb{S}$ arbitrarily and write $X_{\hat{x},\hat{i}}(t)=X(t)$ and $r_{\hat{i}}(t)=r(t)$ for convenience. It is easy to show from Assumption 2.1 that there is a positive constant K_2 such that

$$2x^Tf(x,i,t)+|g(x,i,t)|^2\le K_2(|x|+1)+[2a_i\kappa_1(t)+b_i\kappa_2(t)]|x|^2\tag{3.16}$$

for all $(x,i,t)\in\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}_+$. In the proof, we will use the following two parameters

$$\varepsilon_1=0.5\Big(\varepsilon_0+\frac{1}{\breve{\theta}}+\bar{\varphi}\Big)\ \text{ and }\ \varepsilon_2=(\varepsilon_1-\varepsilon_0)\hat{\theta}.\tag{3.17}$$

Recalling that $0<\varepsilon_0<1/\breve{\theta}+\bar{\varphi}$, we see that

$$\varepsilon_0<\varepsilon_1<1/\breve{\theta}+\bar{\varphi}\ \text{ and }\ \varepsilon_2>0.\tag{3.18}$$

Define a Lyapunov function $V(x,i,t)=\theta_i|x|^2e^{\lambda(t)}$ for $(x,i,t)\in\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}_+$, where

$$\lambda(t)=(1/\breve{\theta}-\varepsilon_1)t+\int_0^t\varphi(s)ds.\tag{3.19}$$

By the generalised Itô formula (see, e.g., [33, Theorem 1.45 on page 48]), it is easy to show that

$$\begin{aligned}&\mathbb{E}V(X(t),r(t),t)-\theta_{\hat{i}}|\hat{x}|^2\\&\le\mathbb{E}\int_0^te^{\lambda(s)}\Big([1+\varphi(s)\theta_{r(s)}-\varepsilon_1\hat{\theta}]|X(s)|^2+LV(X(s),X(\delta_s),r(s),s)\Big)ds\end{aligned}\tag{3.20}$$

for $t\ge0$, where $LV:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}_+\to\mathbb{R}$ is defined by

$$LV(x,y,i,s)=2\theta_ix^T[f(x,i,s)+u(y,i,s)]+\theta_i|g(x,i,s)|^2+\sum_{j=1}^N\gamma_{ij}\theta_j|x|^2.\tag{3.21}$$

Using Assumption 2.2 as well as conditions (3.1) and (3.16), we then derive that

$$\begin{aligned}
&LV(x,y,i,s) \le 2\theta_i x^T[f(x,i,s)+u(x,i,s)] + \theta_i|g(x,i,s)|^2\\
&+ 2\theta_i|x||u(x,i,s)-u(y,i,s)| + \sum_{j=1}^N \gamma_{ij}\theta_j|x|^2\\
&\le \Big(\theta_i[2a_i\kappa_1(s)+b_i\kappa_2(s)-2c_i\kappa_3(s)] + \sum_{j=1}^N \gamma_{ij}\theta_j\Big)|x|^2\\
&+ 2c_0\check{\theta}|x||x-y| + K_2(|x|+1)\\
&= \Big(\theta_i[2a_i+b_i-2c_i] + \sum_{j=1}^N \gamma_{ij}\theta_j\Big)|x|^2 +\\
&- \theta_i[2a_i(1-\kappa_1(s)) + b_i(1-\kappa_2(s)) - 2c_i(1-\kappa_3(s))]|x|^2\\
&+ 2c_0\check{\theta}|x||x-y| + K_2(|x|+1)\\
&\le -|x|^2 - \theta_i\varphi(s)|x|^2 + 2c_0\check{\theta}|x||x-y| + K_2(|x|+1).
\end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.20), we obtain that

$$\begin{aligned}
&\mathbb{E}V(X(t),r(t),t) - \check{\theta}|\hat{x}|^2\\
&\le \mathbb{E}\int_0^t e^{\lambda(s)}\Big(-\varepsilon_1\hat{\theta}|X(s)|^2 + 2c_0\check{\theta}|X(s)||X(s)-X(\delta_s)| + K_2(|X(s)|+1)\Big)ds.
\end{aligned} \tag{3.23}$$

On the other hand, by Lemma 3.4,

$$\begin{aligned}
&\mathbb{E}\int_0^t e^{\lambda(s)}\Big(2c_0\check{\theta}|X(s)||X(s)-X(\delta_s)| + K_2(|X(s)|+1)\Big)ds\\
\le&\mathbb{E}\int_0^t e^{\lambda(s)}\Big(K_2 + 2c_0\check{\theta}H_3(\tau) + [K_2+2c_0\check{\theta}H_4(\tau)]|X(s)| + 2c_0\check{\theta}H_1(\tau)|X(s)|^2\\
&+2c_0\check{\theta}H_2(\tau)\int_{\delta_s}^s |X(v)|^2dv + 2c_0\check{\theta}H_5(\tau)\int_{\delta_s}^s |X(v)|dv\Big)ds.
\end{aligned} \tag{3.24}$$

Note that

$$\int_0^t e^{\lambda(s)}\Big(\int_{\delta_s}^s |X(v)|dv\Big)ds \le \int_0^t e^{\lambda(s)}\Big(\int_{(s-\tau)\vee 0}^s |X(v)|dv\Big)ds \le \int_0^t |X(v)|\Big(\int_v^{v+\tau} e^{\lambda(s)}ds\Big)dv.$$

Recalling (3.4), we see from (3.19) that $\lambda(s) \le \lambda(v) + \alpha\tau$ for $v \le s \le v+\tau$. Hence

$$\int_0^t e^{\lambda(s)}\Big(\int_{\delta_s}^s |X(v)|dv\Big)ds \le \tau e^{\alpha\tau}\int_0^t e^{\lambda(v)}|X(v)|dv. \tag{3.25}$$

Similarly,

$$\int_0^t e^{\lambda(s)}\Big(\int_{\delta_s}^s |X(v)|^2dv\Big)ds \le \tau e^{\alpha\tau}\int_0^t e^{\lambda(v)}|X(v)|^2dv. \tag{3.26}$$

Substituting these inequalities into (3.24) yields

$$\mathbb{E}\int_0^t e^{\lambda(s)}\Big(2c_0|X(s)||X(s)-X(\delta_s)|+K_2(|X(s)|+1)\Big)ds$$
$$\leq \mathbb{E}\int_0^t e^{\lambda(s)}\Big(K_2+2c_0\check{\theta}H_3(\tau)+[K_2+2c_0\check{\theta}H_4(\tau)+2c_0\check{\theta}\tau e^{\alpha\tau}H_5(\tau)]|X(s)|$$
$$+2c_0\check{\theta}[H_1(\tau)+\tau e^{\alpha\tau}H_2(\tau)]|X(s)|^2\Big)ds. \tag{3.27}$$

However

$$K_2+2c_0\check{\theta}H_3(\tau)+[K_2+2c_0\check{\theta}H_4(\tau)+2c_0\tau e^{\alpha\tau}H_5(\tau)]|X(s)|$$
$$=\varepsilon_2|X(s)|^2-\varepsilon_2|X(s)|^2+K_2+2c_0\check{\theta}H_3(\tau)+[K_2+2c_0\check{\theta}H_4(\tau)+2c_0\check{\theta}\tau e^{\alpha\tau}H_5(\tau)]|X(s)|$$
$$\leq \varepsilon_2|X(s)|^2+K_3,$$

where $K_3 := \max_{u\geq 0}\{-\varepsilon_2 u^2+K_2+2c_0\check{\theta}H_3(\tau)+[K_2+2c_0\check{\theta}H_4(\tau)+2c_0\check{\theta}\tau e^{\alpha\tau}H_5(\tau)]u\}$. It then follows from (3.23) and (3.27) that

$$\mathbb{E}V(X(t),r(t),t)-\check{\theta}|\hat{x}|^2\leq K_3\int_0^t e^{\lambda(s)}ds$$
$$+\mathbb{E}\int_0^t e^{\lambda(s)}\big(-\varepsilon_1\hat{\theta}+\varepsilon_2+2c_0\check{\theta}[H_1(\tau)+\tau e^{\alpha\tau}H_2(\tau)]\big)|X(s)|^2ds. \tag{3.28}$$

Recelling (3.8) and (3.17), we have

$$-\varepsilon_1\hat{\theta}+\varepsilon_2+2c_0\check{\theta}[H_1(\tau)+\tau e^{\alpha\tau}H_2(\tau)]=-\varepsilon_1\hat{\theta}+(\varepsilon_1-\varepsilon_0)\hat{\theta}+\varepsilon_0\hat{\theta}=0.$$

It then follows from (3.28) that

$$\hat{\theta}e^{\lambda(t)}\mathbb{E}|X(t)|^2\leq\check{\theta}|\hat{x}|^2+K_3\int_0^t e^{\lambda(s)}ds. \tag{3.29}$$

Noting that $1/\check{\theta}+\bar{\varphi}-\varepsilon_1=\gamma$, we can easily apply Lemma 3.3 to show

$$\int_0^t e^{\lambda(s)}ds\leq K_4e^{\gamma t}\quad\text{and}\quad K_5e^{\gamma t}\leq e^{\lambda(t)}, \tag{3.30}$$

where $K_4=e^{(2\hat{a}+\hat{b}+2\check{c})h}/\gamma$ and $K_5=e^{-(\hat{a}+\hat{b}+2\check{c})h}$. It then follows from (3.29) that

$$\mathbb{E}|X(t)|^2\leq K_6(1+|\hat{x}|^2),\quad\forall t\geq 0, \tag{3.31}$$

where $K_6=(\check{\theta}+K_3K_4)/(\hat{\theta}K_5)$. Finally, we are in the position to show assertion (3.15). Clearly, it holds when $k=0$. For any $k\geq 1$, by the Hölder inequality, the Doob martingale inequality as well as (2.5), it is a routine to show that

$$\mathbb{E}\|\bar{X}(kh)\|_h^2\leq 4\mathbb{E}|X((k-1)h)|^2+4c_0^2\int_{(k-1)h}^{kh}\mathbb{E}|X((k-1)h)|^2dt$$
$$+4\int_{(k-1)h}^{kh}\Big[2\big(K_1^2+\check{a}^2\mathbb{E}|X(t)|^2\big)+8\big(K_1^2+\check{b}\mathbb{E}|X(t)|^2\big)\Big]dt. \tag{3.32}$$

Making use of (3.31), we get

$$\mathbb{E}\|\bar{X}(kh)\|_h^2\leq C_1(1+|\hat{x}|^2), \tag{3.33}$$

where $C_1=4K_6(1+h[4c_0^2+10K_1^2+2\check{a}^2+8\check{b}])$. That is, assertion (3.15) holds for any $k\geq 1$ too. The proof is complete. □

Lemma 3.6 *Let Assumptions 2.1 and 2.2 hold. Set $Z(t) = X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t)$ for $(\hat{x},\hat{y},\hat{i}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$ and $t \geq 0$. If $\tau < 1/c_0$, then*

$$\mathbb{E}(|Z(t)||Z(t) - Z(\delta_t)|) \leq H_1(\tau)\mathbb{E}|Z(t)|^2 + H_2(\tau)\mathbb{E}\int_{k\tau}^t |Z(s)|^2 ds. \tag{3.34}$$

Proof. Fix $t \geq 0$ arbitrarily and set $k = \lfloor t/\tau \rfloor$. Then $k\tau \leq t < (k+1)\tau$ and $\delta_t = k\tau$. By Assumptions 2.1 and 2.2, it is straightforward to show from (1.3) that

$$\mathbb{E}(|Z(t)||Z(t) - Z(\delta_t)|) \leq \mathbb{E}\Big(|Z(t)|\int_{k\tau}^t \big[\breve{a}|Z(s)| + c_0|Z(k\tau)|\big]ds + |X(t)||\bar{M}_k(t)|\Big), \tag{3.35}$$

where $\bar{M}_k(t) = \int_{k\tau}^t [g(X_{\hat{x},\hat{i}}(s), r(s), s) - g(X_{\hat{y},\hat{i}}(s), r(s), s)]dB(s)$. But

$$\begin{aligned}
&\mathbb{E}\Big(|Z(t)|\int_{k\tau}^t \big[\breve{a}|Z(s)| + c_0|Z(k\tau)|\big]ds\Big) \\
&\leq \mathbb{E}\int_{k\tau}^t \breve{a}|Z(t)||Z(s)|ds + \tau c_0\mathbb{E}[|Z(t)|(|Z(t)| + |Z(t) - Z(k\tau)|)] \\
&\leq \tau(0.5\breve{a} + c_0)\mathbb{E}|Z(t)|^2 + 0.5\breve{a}\mathbb{E}\int_{k\tau}^t |Z(s)|^2 ds \\
&+ \tau c_0\mathbb{E}(|Z(t)||Z(t) - Z(\delta_t)|).
\end{aligned} \tag{3.36}$$

Moreover, by Assumption 2.1

$$\begin{aligned}
&\mathbb{E}(|Z(t)||\bar{M}_k(t)|) \leq 0.5\sqrt{\breve{b}\tau}\mathbb{E}|Z(t)|^2 + (0.5/\sqrt{\breve{b}\tau})\mathbb{E}|\bar{M}_k(t)|^2 \\
&\leq 0.5\sqrt{\breve{b}\tau}\mathbb{E}|Z(t)|^2 + 0.5\sqrt{\breve{b}/\tau}\mathbb{E}\int_{k\tau}^t |Z(s)|^2 ds.
\end{aligned} \tag{3.37}$$

Substituting (3.36) and (3.37) into (3.35) implies

$$\begin{aligned}
&\mathbb{E}(|Z(t)||Z(t) - Z(\delta_t)|) \leq \big(\tau(0.5\breve{a} + c_0) + 0.5\sqrt{\breve{b}\tau}\big)\mathbb{E}|Z(t)|^2 \\
&+ 0.5\big(\breve{a} + \sqrt{\breve{b}/\tau}\big)\mathbb{E}\int_{\delta_t}^t |Z(s)|^2 ds + \tau c_0\mathbb{E}(|Z(t)||Z(t) - Z(\delta_t)|).
\end{aligned} \tag{3.38}$$

This implies the required assertion (3.34) as $\tau c_0 < 1$. The proof is complete. □

Lemma 3.7 *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold. If $\tau < \tau^*$, then for any $(\hat{x},\hat{y},\hat{i}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$,*

$$\mathbb{E}\|\bar{X}_{\hat{x},\hat{i}}(kh) - \bar{X}_{\hat{y},\hat{i}}(kh)\|_h^2 \leq C_2|\hat{x} - \hat{y}|^2 e^{-\gamma kh} \tag{3.39}$$

for all $k \in \mathbb{N}_+$, where C_2 is a positive number dependent on $h, \hat{a}, \breve{a}, \hat{b}, \breve{b}, \check{c}, c_0, \breve{\theta}, \hat{\theta}, \gamma$ and its explicit form is defined in the proof below. From its explicit form, we see that C_2 is independent of the initial data $(\hat{x},\hat{y},\hat{i})$.

Proof. Fix $(\hat{x},\hat{y},\hat{i}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$ arbitrarily. Write $X_{\hat{x},\hat{i}}(t) = X(t)$, $X_{\hat{y},\hat{i}}(t) = Y(t)$ and $r_{\hat{i}}(t) = r(t)$ simply. Set $Z(t) = X(t) - Y(t)$. We will use ε_1 and ε_2 defined by (3.17). Define a

Lyapunov function $U(z,i,t)=\theta_i|z|^2e^{\lambda(t)}$ for $(z,i,t)\in\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}_+$, where $\lambda(t)$ has been defined by (3.19). Applying the generalised Itô formula, we can show that

$$\begin{aligned}&\mathbb{E}U(Z(t),r(t),t)-\theta_{\hat{i}}|\hat{x}-\hat{y}|^2\\&\le\mathbb{E}\int_0^t e^{(\lambda(s)}\Big([1+\varphi(s)\theta_{r(s)}-\varepsilon_1\hat{\theta}]|Z(s)|^2+LU(X(s),Y(s),X(\delta_s),Y(\delta_s),r(s),s)\Big)ds\end{aligned}\tag{3.40}$$

for $t\ge0$, where $LU:\mathbb{R}^{n\times4}\times\mathbb{S}\times\mathbb{R}_+\to\mathbb{R}$ is defined by

$$LU(x,y,\bar{x},\bar{y},i,s)=2\theta_iz^T(\tilde{f}+\tilde{u})+\theta_i|\tilde{g}|^2+\sum_{j=1}^N\gamma_{ij}\theta_j|z|^2,\tag{3.41}$$

in which $z=x-y$, $\tilde{f}=f(x,i,s)-f(y,i,s)$, $\tilde{u}=u(\bar{x},i,s)-u(\bar{y},i,s)$ and $\tilde{g}=g(x,i,s)-g(y,i,s)$. Setting $\bar{z}=\bar{x}-\bar{y}$, we can derive by Assumptions 2.1 and 2.2 as well as (3.1) that

$$\begin{aligned}&LU(x,y,\bar{x},\bar{y},i,s)\\&\le\Big(\theta_i[2a_i\kappa_1(t)+b_i\kappa_2(t)-2c_i\kappa_3(s)]+\sum_{j=1}^N\gamma_{ij}\theta_j\Big)|z|^2+2c_0\check{\theta}|z||z-\bar{z}|\\&=\Big(\theta_i[2a_i+b_i-2c_i]+\sum_{j=1}^N\gamma_{ij}\theta_j\Big)|z|^2+\\&-\theta_i[2a_i(1-\kappa_1(t))+b_i(1-\kappa_2(t))-2c_i(1-\kappa_3(s))]|z|^2+2c_0\check{\theta}|z||z-\bar{z}|\\&\le-|z|^2-\theta_i\varphi(s)|z|^2+2c_0\check{\theta}|z||z-\bar{z}|.\end{aligned}$$

Substituting this into (3.40) and then apply Lemma 3.6, we derive

$$\begin{aligned}&\mathbb{E}U(Z(t),r(t),t)-\theta_{\hat{i}}|\hat{x}-\hat{y}|^2\\&\le\int_0^t e^{\lambda(s)}\Big(-\varepsilon_1\hat{\theta}\mathbb{E}|Z(s)|^2+2c_0\check{\theta}\mathbb{E}(|Z(s)||Z(s)-Z(\delta_s)|)\Big)ds\\&\le\int_0^t e^{\lambda(s)}\Big(\big(-\varepsilon_1\hat{\theta}+2c_0\check{\theta}H_1(\tau)\big)\mathbb{E}|Z(s)|^2+2c_0\check{\theta}\int_{\delta_s}^s\mathbb{E}|Z(v)|^2dv\Big)ds.\end{aligned}\tag{3.42}$$

But, in the save way as (3.25) was proved, we can show that

$$\int_0^t e^{\lambda(s)}\Big(\int_{\delta_s}^s|Z(v)|^2dv\Big)ds\le\tau e^{\alpha\tau}\int_0^t e^{\lambda(v)}|Z(v)|dv.$$

Hence

$$\begin{aligned}&\mathbb{E}U(Z(t),r(t),t)-\theta_{\hat{i}}|\hat{x}-\hat{y}|^2\\&\le\int_0^t e^{\lambda(s)}\big(-\varepsilon_1\hat{\theta}+2c_0\check{\theta}[H_1(\tau)+\tau e^{\alpha\tau}H_2(\tau)]\big)\mathbb{E}|Z(s)|^2ds.\end{aligned}\tag{3.43}$$

Recelling (3.8) and (3.18), we have

$$-\varepsilon_1\hat{\theta}+2c_0\check{\theta}[H_1(\tau)+\tau e^{\alpha\tau}H_2(\tau)]=-\varepsilon_1\hat{\theta}+\varepsilon_0\hat{\theta}<0.$$

Consequently

$$\mathbb{E}U(Z(t),r(t),t)-\theta_{\hat{i}}|\hat{x}-\hat{y}|^2\le0,$$

which yields

$$\hat{\theta}e^{\lambda(t)}\mathbb{E}|Z(t)|^2 \le \check{\theta}|\hat{x}-\hat{y}|^2.$$

By (3.30), we then obtain

$$\mathbb{E}|Z(t)|^2 \le K_7|\hat{x}-\hat{y}|^2 e^{-\gamma t} \tag{3.44}$$

for all $t \ge 0$, where $K_7 = \check{\theta}/(\hat{\theta}K_5)$. Finally we are in the position to show assertion (3.39). It holds clearly for $k=0$. For any $k \ge 1$, set $\bar{Z}(kh) = \{Z(kh+s) : 0 \le s \le h\}$. By the Hölder inequality, the Doob martingale inequality as well as Assumptions 2.1 and 2.2, it is a routine to show that

$$\begin{aligned}\mathbb{E}\|\bar{Z}(kh)\|_h^2 \le 4\mathbb{E}|Z((k-1)h)|^2 + 4c_0^2 h \int_{(k-1)h}^{kh} \mathbb{E}|Z(\delta_t)|^2 dt \\ + 4(\check{a}^2 h + 4\check{b}^2)\int_{(k-1)h}^{kh} \mathbb{E}|Z(t)|^2 dt.\end{aligned} \tag{3.45}$$

Making use of (3.44), we get

$$\mathbb{E}\|\bar{Z}(kh)\|_h^2 \le C_2|\hat{x}-\hat{y}|^2 e^{-\gamma kh}, \tag{3.46}$$

where $C_2 = 4K_7 e^{\gamma h}(1 + (\check{a}^2 h + 4\check{b}^2)h + c_0^2 h^2)$. This is, assertion (3.39) holds for any $k \ge 1$ as well. The proof is complete. $\square$

Lemma 3.8 *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold. If $\tau < \tau^*$, there is positive number ρ such that for any $(\hat{x}, \hat{y}, \hat{\imath}, \hat{\jmath}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{S}$,*

$$\mathbb{E}\|\bar{X}_{\hat{x},\hat{\imath}}(kh) - \bar{X}_{\hat{y},\hat{\jmath}}(kh)\|_h \le C_3(1+|\hat{x}|+|\hat{y}|)e^{-\rho kh} \tag{3.47}$$

for all $k \in \mathbb{N}_+$, where C_3 is a positive number independent of the initial data $(\hat{x}, \hat{y}, \hat{\imath}, \hat{\jmath})$ and k but dependent on $\tau, h, \hat{a}, \check{a}, \hat{b}, \check{b}, \check{c}, c_0, \check{\theta}, \hat{\theta}, \bar{\varphi}, \gamma, K_1, K_2$.

Proof. Note that $\{r(kh)\}_{k\in\mathbb{N}_+}$ is a discrete-time irreducible Markov chain with its one-step transition probability matrix $e^{h\Gamma}$. Define the stopping time

$$\sigma_{\hat{\imath}\hat{\jmath}} = \inf\{kh : r_{\hat{\imath}}(kh) = r_{\hat{\jmath}}(kh),\ k \in \mathbb{N}_+\}.$$

It is known (see, e.g., [1, p.260]) that there is a positive number $\bar{\rho}$ such that

$$\mathbb{P}(\sigma_{\hat{\imath}\hat{\jmath}} > kh) \le e^{-\bar{\rho}kh}, \quad \forall k \in \mathbb{N}_+. \tag{3.48}$$

In the remaining of the proof, we let C be the generic positive numbers that are independent of the initial data $(\hat{x}, \hat{y}, \hat{\imath}, \hat{\jmath})$ and k but may depend on $\tau, h, \hat{a}, \check{a}, \hat{b}, \check{b}, \check{c}, c_0, \check{\theta}, \hat{\theta}, \bar{\varphi}, \gamma, K_1, K_2$. (The explicit form of C is of no use in the proof but makes the proof look more complicated. Please also note that C may change from line to line as it is generic.)

Fix $k \in \mathbb{N}_+$ arbitrarily from now on and let $\bar{k} = \lfloor k/2 \rfloor$. Obviously,

$$\begin{aligned}&\mathbb{E}\|\bar{X}_{\hat{x},\hat{\imath}}(kh) - \bar{X}_{\hat{y},\hat{\jmath}}(kh)\|_h \\ =&\mathbb{E}\Big(I_{\{\sigma_{\hat{\imath}\hat{\jmath}} > \bar{k}h\}}\|\bar{X}_{\hat{x},\hat{\imath}}(kh) - \bar{X}_{\hat{y},\hat{\jmath}}(kh)\|_h\Big) + \mathbb{E}\Big(I_{\{\sigma_{\hat{\imath}\hat{\jmath}} \le \bar{k}h\}}\|\bar{X}_{\hat{x},\hat{\imath}}(kh) - \bar{X}_{\hat{y},\hat{\jmath}}(kh)\|_h\Big).\end{aligned} \tag{3.49}$$

But, by the Hölder inquality and Lemma 3.5 as well inequality (3.48), we derive

$$\begin{aligned}&\mathbb{E}\Big(I_{\{\sigma_{\hat{i}\hat{j}}>\bar{k}h\}}\|\bar{X}_{\hat{x},\hat{i}}(kh)-\bar{X}_{\hat{y},\hat{j}}(kh)\|_h\Big)\le\big[\mathbb{P}(\sigma_{\hat{i}\hat{j}}>\bar{k}h)\big]^{1/2}\big[\mathbb{E}\|\bar{X}_{\hat{x},\hat{i}}(kh)-\bar{X}_{\hat{y},\hat{j}}(kh)\|_h^2\big]^{1/2}\\&\le Ce^{-0.5\bar{\rho}\bar{k}h}\big[\mathbb{E}\|\bar{X}_{\hat{x},\hat{i}}(kh)\|_h^2+\mathbb{E}\|\bar{X}_{\hat{y},\hat{j}}(kh)\|_h^2\big]^{1/2}\le Ce^{-0.5\bar{\rho}kh}(1+|\hat{x}|+|\hat{y}|).\end{aligned}\tag{3.50}$$

Moreover, by Lemmas 2.3 and 2.4 , we have

$$\begin{aligned}&\mathbb{E}\Big(I_{\{\sigma_{\hat{i}\hat{j}}\le\bar{k}h\}}\|\bar{X}_{\hat{x},\hat{i}}(kh)-\bar{X}_{\hat{y},\hat{j}}(kh)\|_h\Big)=\mathbb{E}\Big[I_{\{\sigma_{\hat{i}\hat{j}}\le\bar{k}h\}}\mathbb{E}\Big(\|\bar{X}_{\hat{x},\hat{i}}(kh)-\bar{X}_{\hat{y},\hat{j}}(kh)\|_h|\mathcal{F}_{\sigma_{\hat{i}\hat{j}}}\Big)\Big]\\&=\mathbb{E}\Big[I_{\{\sigma_{\hat{i}\hat{j}}\le\bar{k}h\}}\mathbb{E}\|\bar{X}_{w,l}(kh-\sigma_{\hat{i}\hat{j}})-\bar{X}_{z,l}(kh-\sigma_{\hat{i}\hat{j}})\|_h\Big],\end{aligned}$$

where $w=X_{\hat{x},\hat{i}}(\sigma_{\hat{i}\hat{j}})$, $z=X_{\hat{y},\hat{j}}(\sigma_{\hat{i}\hat{j}})$ and $l=r_{\hat{i}}(\sigma_{\hat{i}\hat{j}})=r_{\hat{j}}(\sigma_{\hat{i}\hat{j}})$. Applying Lemmas 3.7 and 3.5, we further derive

$$\begin{aligned}&\mathbb{E}\Big(I_{\{\sigma_{\hat{i}\hat{j}}\le\bar{k}h\}}\|\bar{X}_{\hat{x},\hat{i}}(kh)-\bar{X}_{\hat{y},\hat{j}}(kh)\|_h\Big)\le\mathbb{E}\Big[I_{\{\sigma_{\hat{i}\hat{j}}\le\bar{k}h\}}\big(\mathbb{E}\|\bar{X}_{w,l}(kh-\sigma_{\hat{i}\hat{j}})-\bar{X}_{z,l}(kh-\sigma_{\hat{i}\hat{j}})\|_h^2\big)^{1/2}\Big]\\&\le C\mathbb{E}\Big[I_{\{\sigma_{\hat{i}\hat{j}}\le\bar{k}h\}}|w-z|e^{-0.5\gamma(kh-\sigma_{\hat{i}\hat{j}})}\Big]\le Ce^{-0.25\gamma kh}\mathbb{E}\Big[I_{\{\sigma_{\hat{i}\hat{j}}\le\bar{k}h\}}|X_{\hat{x},\hat{i}}(\sigma_{\hat{i}\hat{j}})-X_{\hat{y},\hat{j}}(\sigma_{\hat{i}\hat{j}})|\Big]\\&\le Ce^{-0.25\gamma kh}\mathbb{E}\Big[\max_{0\le\kappa\le\bar{k}}(|X_{\hat{x},\hat{i}}(\kappa h)|+|X_{\hat{y},\hat{j}}(\kappa h)|)\Big]\le Ce^{-0.25\gamma kh}\sum_{\kappa=0}^{\bar{k}}(\mathbb{E}|X_{\hat{x},\hat{i}}(\kappa h)|+\mathbb{E}|X_{\hat{y},\hat{j}}(\kappa h)|)\Big]\\&\le Ce^{-0.25\gamma kh}(1+\bar{k})(1+|\hat{x}|+|\hat{y}|)\le Ce^{-0.2\gamma kh}(1+|\hat{x}|+|\hat{y}|).\end{aligned}\tag{3.51}$$

Substituting (3.50) and (3.51) into (3.49) yields the required assertion (3.47), where $\rho=(0.5\bar{\rho})\wedge(0.2\gamma)$ while C_3 is in terms of the last two C's in (3.50) and (3.51), respectively, and is hence independent of $(\hat{x},\hat{y},\hat{i},\hat{j})$ and k. The proof is complete. □

We are in the position to establish our main theorem in this paper.

Theorem 3.9 *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold. If $\tau<\tau^*$, then there exists a unique probability measure $\mu\in\mathcal{P}_1(\mathcal{C}_h)$ such that*

$$\limsup_{k\to\infty}\frac{1}{kh}\log(W_1(\mu,\mathcal{L}(\bar{X}_{\hat{x},\hat{i}}(kh)))\le-\rho,\tag{3.52}$$

for all $(\hat{x},\hat{i})\in\mathbb{R}^n\times\mathbb{S}$, where ρ is a positive number specified in Lemma 3.8. In other words, the SDE (1.3) is asymptotically stable in distribution under W_1-distance.

Proof. For any positive integers k and k_1, by Lemmas 2.3, 2.4, 3.5 and 3.8 while recalling (2.1) and (2.9) , we derive

$$\begin{aligned}&W_1(\mathcal{L}(\bar{X}_{0,1}((k_1+k)h)),\mathcal{L}(\bar{X}_{0,1}(kh)))\\\le&\mathbb{E}\|\bar{X}_{0,1}((k_1+k)h)-\bar{X}_{0,1}(kh))\|_h\\=&\sum_{j\in\mathbb{S}}\int_{\mathbb{R}^n}\mathbb{E}\|\bar{X}_{y,j}(kh)-\bar{X}_{0,1}(kh))\|_hP(k_1,0,1;dy\times\{j\})\\\le&\sum_{j\in\mathbb{S}}\int_{\mathbb{R}^n}C_3(1+1+|y|)e^{-\rho kh}P(k_1,0,1;dy\times\{j\})\\\le&C_3e^{-\rho kh}(2+\mathbb{E}\|\bar{X}_{0,1}(k_1h)\|_h)\le C_3e^{-\rho kh}(2+\sqrt{C_1}).\end{aligned}\tag{3.53}$$

This shows that $\{\mathcal{L}(\bar{X}_{0,1}(kh))\}_{k\in\mathbb{N}_+}$ is a Cauchy sequence in $\mathcal{P}_1(\mathcal{C}_h)$ with W_1-distance. There is hence a unique $\mu \in \mathcal{P}_1(\mathcal{C}_h)$ such that

$$\lim_{k\to\infty} W_1(\mathcal{L}(\bar{X}_{0,1}(kh)), \mu) = 0. \tag{3.54}$$

Letting $k_1 \to \infty$ in (3.53) gives

$$W_1(\mu, \mathcal{L}(\bar{X}_{0,1}(kh))) \le C_3 e^{-\rho kh}(2 + \sqrt{C_1}). \tag{3.55}$$

Now, for any $(\hat{x}, \hat{\imath}) \in \mathbb{R}^n \times \mathbb{S}$, it follows this and Lemma 3.8 that

$$\begin{aligned} W_1(\mu, \mathcal{L}(\bar{X}_{\hat{x},\hat{\imath}}(kh))) \le& W_1(\mu, \mathcal{L}(\bar{X}_{0,1}(kh))) + W_1(\mathcal{L}(\bar{X}_{0,1}(kh)), \mathcal{L}(\bar{X}_{\hat{x},\hat{\imath}}(kh))) \\ \le& C_3 e^{-\rho kh}(2 + \sqrt{C_1}) + C_3 e^{-\rho kh}(1 + |\hat{x}|). \end{aligned}$$

This implies the required assertion (3.52). The proof is complete. $\Box$

Note that $\mathcal{C}_h$ is an infinite space and $\mathcal{P}(\mathcal{C}_h)$ is huge. It is very hard to obtain its probability distribution both theoretically and numerically and this is a challenge for our research in the future. On the other hand, in practice, we are more concerned with the probability distribution of $X_{\hat{x},\hat{\imath}}(t)$ in long term. For $p \ge 1$, denote by $\mathcal{P}_p(\mathbb{R}^n)$ the family of probability measures on $\mathbb{R}^n$ with finite pth-moments. The Wasserstein p-distance between $P_1, P_2 \in \mathcal{P}_p(\mathbb{R}^n)$ is

$$\mathcal{W}_p(P_1, P_2) = \inf_{\pi\in C(P_1,P_2)} \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^p \pi(dx,dy)\Big)^{1/p},$$

where $C(P_1, P_2)$ is the family of all couplings of P_1 and P_2. Denote by $\mathcal{L}(X_{\hat{x},\hat{\imath}}(t))$ the probability measure on $\mathbb{R}^n$ generated by $X_{\hat{x},\hat{\imath}}(t)$. Given that the controlled SDE (1.3) is periodic, it is unreasonable to expect that $\mathcal{L}(X_{\hat{x},\hat{\imath}}(t))$ will converge in $\mathcal{P}_p(\mathbb{R}^n)$ under the Wasserstein p-distance as $t \to \infty$. The question is: will some subsequence of $\mathcal{L}(X_{\hat{x},\hat{\imath}}(t))$ converge? The following theorem answers it very positively.

Theorem 3.10 *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold while $\tau < \tau^*$. Then, for every $\bar{h} \in [0, h)$, there exists a unique probability measure $\mu_{\bar{h}} \in \mathcal{P}_1(\mathbb{R}^n)$ such that*

$$\limsup_{k\to\infty} \frac{1}{kh} \log(\mathcal{W}_1(\mu_{\bar{h}}, \mathcal{L}(X_{\hat{x},\hat{\imath}}(kh-\bar{h})))) \le -\rho, \tag{3.56}$$

for all $(\hat{x}, \hat{\imath}) \in \mathbb{R}^n \times \mathbb{S}$, where ρ is a positive number specified in Lemma 3.8.

This theorem can be proved in the same way as Theorem 3.9 was proved by noting, for example, from Lemmas 3.5 and 3.8 that $\mathbb{E}|X_{\hat{x},\hat{\imath}}(kh-\bar{h})|^2 \le C_1(1+|\hat{x}|^2)$ and $\mathbb{E}|X_{\hat{x},\hat{\imath}}(kh-\bar{h}) - \bar{X}_{\hat{y},\hat{\jmath}}(kh-\bar{h})| \le C_3(1 + |\hat{x}| + |\hat{y}|)e^{-\rho kh}$ for all $k \in \mathbb{N}_+$ and $(\hat{x}, \hat{y}, \hat{\imath}, \hat{\jmath}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{S}$.

Let us make a useful remark on our Assumption 2.1 to close this section.

Remark 3.11 *By Assumption 2.1, we impose the global Lipschitz condition with respect to x on f and g. Although it covers some useful periodic hybrid SDEs including the linear ones, it is somehow restrictive. We are fully aware that it does not need this global Lipschitz condition for the hybrid SDE (1.1) to be stable in distribution (see, e.g., [33, Section 5.6 on pages 210–226]). On the other hand, in this paper, we are concerned with the stability in distribution of the controlled SDE (1.3), which is a special SDDE (2.7). To prove its stability in distribution, we present several lemmas. Some of these lemmas do not need the the global Lipschitz condition. For example, Lemma 3.4 only needs the weaker local Lipschitz condition plus the linear growth condition. However, Lemma 3.6, and hence Lemmas 3.7 and 3.8, require the global Lipschitz condition. We do not know at this moment if Lemma 3.6 can be proved by a different way without the global Lipschitz condition. We will tackle this problem in the future.*

4 Design of Feedback Control

In this section, we will explain how to design the matrices A_i and the periodic function $\kappa_3(t)$ in the control function (2.3).

4.1 Design of A_i's

We will first explain how to design A_i's for Assumption 3.1 to hold. It is sufficient if we explain how to determine non-negative numbers c_i ($i \in \mathbb{S}$) for Assumption 3.1 to hold. In fact, once these c_i are determined, we can easily find non-positive definite symmetric matrices A_i for $c_i = -\lambda_{\max}(A_i)$.

To determine c_i's, we will use the following classical Minkowski theorem (see, e.g., [28, 34]), which is stated as a lemma here.

Lemma 4.1 *(Minkowski theorem, 1907). For a square matrix $Q = (q_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$, if $q_{ij} \le 0$ for all $i \ne j$ and $\sum_{j=1}^d q_{ij} > 0$ for all j, then* $\det A > 0$.

We also need another known result (see, e.g., [7, 28]).

Lemma 4.2 *Assumption 3.1 holds if and only if all determinants of the leading principal minors of $\mathcal{A}$ are positive.*

For convenience, we denote $\mathcal{A} =: (a_{ij})_{N\times N}$, where $a_{ii} = 2c_i - 2a_i - b_i + |\gamma_{ii}|$ and $a_{ij} = -\gamma_{ij}$ for $i, j \in \mathbb{S}$ and $i \ne j$. For $1 \le k \le N$, the determinant of the leading principal minor of order k of $\mathcal{A}$ is denoted by D_k, namely

$$D_k := \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}.$$

Lemma 4.2 means that $\mathcal{A}$ is a nonsingular M-matrix if and only if $D_k > 0$ for all $k \in \mathbb{S}$. Let us begin to describe a couple of methods to determine c_i's.

Method 1. It is simplest to choose $c_i > a_i + 0.5b_i$ for all $i \in \mathbb{S}$. By Lemma 4.1, it is straightforward to verify that all determinants of the leading principal minors of $\mathcal{A}$ are positive and hence, by Lemma 4.2, Assumption 3.1 holds.

Method 2. This is slightly more careful method than last one. We illustrate the method by considering the case where there is a state, say N without loss of generality, such that the Markov chain can jump to N directly from any other state. In terms of mathematics, we assume that

$$\gamma_{iN} > 0, \quad 1 \le i \le N-1. \tag{4.1}$$

Choose c_i for $1 \le i \le N-1$ such that

$$2c_i - 2a_i - b_i + \gamma_{iN} > 0. \tag{4.2}$$

By Lemma 4.1, it is easy to verify that

$$D_i > 0, \quad 1 \le i \le N-1. \tag{4.3}$$

To determine c_N, we note

$$D_N = D_{N-1}(2c_N - 2a_N - b_N + |\gamma_{NN}|) + \alpha_N, \tag{4.4}$$

where α_N is a real number independent of c_N. We can finally choose c_N for $D_N > 0$. By Lemma 4.2, Assumption 3.1 holds.

This method reveals an interesting case. We observe from (4.2) that if for some i ($\le N-1$), we have $\gamma_{iN} > 2a_i + b_i$, we may choose $c_i = 0$ and hence $A_i = 0$. Similarly, if $|\gamma_{NN}| > 2a_N - b_N$ and $D_{N-1}(|\gamma_{NN}| - 2a_N - b_N) > -C_N$, we may choose $c_N = 0$ and $A_N = 0$. This means, in practice, we may not need to use the feedback control in some mode(s).

Method 3. This is possibly a more useful method. It will determine c_i one by one. First, choose c_1 for

$$D_1 = a_{11} = 2c_1 - 2a_1 - b_1 + |\gamma_{11}| > 0. \tag{4.5}$$

Once again, we observe that if $|\gamma_{11}| > 2a_1 + b_1$ we may set $c_1 = 0$. We next choose c_2 for

$$\begin{aligned} D_2 &= \begin{vmatrix} a_{11}, & -\gamma_{12} \\ -\gamma_{21}, & 2c_2 - 2a_2 - b_2 + |\gamma_{22}| \end{vmatrix} \\ &= D_1(2c_2 - 2a_2 - b_2 + |\gamma_{22}|) - \gamma_{12}\gamma_{21} > 0. \end{aligned} \tag{4.6}$$

Clearly, this is always possible. In particular, if $|\gamma_{22}| > 2a_2 + b_2$ and $D_1(|\gamma_{22}| - 2a_2 - b_2) > \gamma_{12}\gamma_{21}$, we may set $c_2 = 0$. Repeating this procedure, we can further determine $c_3, \cdots, c_N$.

4.2 Design of $\kappa_3(\cdot)$

Let us begin to explain how $\kappa_3(\cdot)$ can be designed for Assumption 3.2 to be satisfied, namely for the following inequality to hold:

$$\frac{1}{\breve{\theta}} + 2\hat{a}(1 - \bar{\kappa}_1) + \hat{b}(1 - \bar{\kappa}_2) > 2\check{c}(1 - \bar{\kappa}_3). \tag{4.7}$$

Method 4. It is simplest to let $\kappa_3(t) = 1$ for all $t \ge 0$. In this case, $\bar{\kappa}_3 = 1$ and (4.7) holds.

Recalling definition (2.2) of the control function, we can regard $\kappa_3(t)$ as the intensity of the feedback control $A_{r(t)}X(\delta_t)$ at time t. For example, the feedback control acts fully when $\kappa_3(t) = 1$ but does not act when $\kappa_3(t) = 0$. Taking into account that the control cost is in general proportional to $\kappa_3(t)|A_{r(t)}X(\delta_t)|$, we realise that it is not very wise to let $\kappa_3(t) \equiv 1$ though it is simple. The following two methods show how we could reduce the control cost by designing $\kappa_3(\cdot)$ more wisely.

Method 5. Let

$$\kappa_3(t) = \sum_{k=0}^{\infty} I_{[kh,(k+\nu)h)}(t), \quad t \ge 0, \tag{4.8}$$

where $\nu \in (0, 1]$. In operation, the feedback control is switched on during time periods $[0, \nu h))$, $[h, (1+\nu)h)$, $[2h, (2+\nu)h)$, $\cdots$, while off during $[\nu h, h)$, $[(1+\nu)h, 2h)$, $[(2+\nu)h, 3h)$, $\cdots$. Such a control is known as an intermittent control (see, e.g., [26, 53]). One of the practical reasons to use an intermittent control is because a controller needs a rest periodically. The parameter ν is the proportion of the duration in one period of h during which the feedback control acts fully. Hence, $1 - \nu$ is the proportion of the duration in one period of h during which the feedback control rests. In the case when $\nu = 1$, $\kappa_3(t) \equiv 1$ so the feedback control acts without any rest and this is Method 4. In this intermittent case, we have $\bar{\kappa}_3 = \nu$. It then follows from (4.7) that we need to choose $\nu \in (0, 1]$ for

$$1 - \nu < \frac{1}{2\check{c}}\Big(\frac{1}{\breve{\theta}} + 2\hat{a}(1 - \bar{\kappa}_1) + \hat{b}(1 - \bar{\kappa}_2)\Big). \tag{4.9}$$

Method 6. It is sufficient to design $\kappa_3(\cdot) \in \mathcal{C}_h$ such that (4.7) holds. There are lots of choices for $\kappa_3(\cdot)$. For example, let $\nu \in (0, h/2)$ and define

$$\kappa_3(t) = \begin{cases} t/\nu, & \text{if } 0 \le t \le \nu, \\ 1, & \text{if } \nu < t \le h - \nu, \\ (h-t)/\nu, & \text{if } h - \nu < t \le h; \end{cases} \tag{4.10}$$

and $\kappa_3(kh + t) = \kappa_3(t)$ for $t \in (0, h]$ and $k = 1, 2, \cdots$. Noting that $\bar{\kappa}_3 = 1 - \nu/h$, we need to choose ν sufficiently small for

$$\frac{1}{\check{\theta}} + 2\hat{a}(1 - \bar{\kappa}_1) + \hat{b}(1 - \bar{\kappa}_2) > \frac{2\check{c}\nu}{h}. \tag{4.11}$$

5 An Example

Due to the page limit, we will only discuss an example to illustrate our theory. Consider an n-dimensional semi-linear SDE

$$dx(t) = \kappa_1(t)[Q_{r(t)}x(t) + U_{r(t)}w_{r(t)}(x(t)) + \xi_{r(t)}]dt + \kappa_2(t)\sigma_{r(t)}dB(t). \tag{5.1}$$

Here $\kappa_1(t), \kappa_2(t), B(t)$ are the same as before, while for each $i \in \mathbb{S}$, $\xi_i \in \mathbb{R}^n$, $Q_i, U_i \in \mathbb{R}^{n\times n}$, $\sigma_i \in \mathbb{R}^{n\times m}$, $w_i(x) = (w_{i1}(x_1), \cdots, w_{in}(x_n))^T$ with $w_{ij}(v)$ being sigmoidal, saturating at ± 1 with maximum slope at $v = 0$. To be more precise, $w_{ij}(v)$ is a nondecreasing Lipschitz continuous function with properties that

$$vw_{ij}(v) \ge 0, \quad |w_{ij}(v)| \le 1, \quad |w_{ij}(v) - w_{ij}(\bar{v})| \le \eta_{ij}|v - \bar{v}|$$

for $v, \bar{v} \in \mathbb{R}$ and $\eta_{ij} > 0$. Please note that $\kappa_1(t)$ represents the common factor of the periodic change of the system matices/parameters Q_i, U_i etc. while $\kappa_2(t)$ stands for the periodic change of the external additive noise. Such SDEs appear frequently in various applications, for example, in the area of stochastic neural networks (see, e.g., [9, 33, 44]).

Defining $f(x, i, t) = \kappa_1(t)[-Q_i x + U_i w_i(x) + \xi_i]$ and $g(x, i, t) = \kappa_2(t)\sigma_i$ for $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, we see that equation (5.1) is a special form of our underlying SDE (1.1). Moreover, f and g satisfy Assumption 2.1 with

$$a_i = \max_{1\le j\le n}(\|Q_i\| + \|U_i\|\eta_{ij}) \text{ and } b_i = 0, \ i \in \mathbb{S}.$$

In the case when the given SDE (5.1) is not stable in distribution, we can therefore apply our theory to design a periodic state feedback control $\kappa_3(t)A_{r(t)}x(\delta_t)$ based on state observations at discrete times so that the corresponding controlled SDE

$$\begin{aligned} dX(t) = \Big(&\kappa_1(t)[Q_{r(t)}X(t) + U_{r(t)}w_{r(t)}(X(t)) + \xi_{r(t)}] \\ &+ \kappa_3(t)A_{r(t)}X(\delta(t))\Big)dt + \kappa_2(t)\sigma_{r(t)}dB(t) \end{aligned} \tag{5.2}$$

becomes stable in distribution.

To illustrate our theory more clearly, we will discuss a special case of the SDE (5.1) by specifying the followings: $n = 2$, $m = 1$, $\mathbb{S} = \{1, 2\}$,

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.05 \end{pmatrix}, Q_2 = \begin{pmatrix} 0.05 & 0 \\ -0.1 & 0.1 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{pmatrix}, U_2 = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix},$$

$$\xi_1 = (2,1)^T,\ \xi_2 = (1,2)^T,\ \sigma_1 = (0.1, 0.2)^T,\ \sigma_2 = (0.2, 0.1)^T,\ h = 1,$$

$$\kappa_1(t) = \kappa_2(t) = 0.75 + 0.25\sin(2\pi t),$$

$$w_{11}(v) = w_{12}(v) = w_{21}(v) = w_{21}(v) = \tanh(v).$$

If this specified SDE (5.1) is asymptotically stable in distribution under W_1-distance, then for any initial data $(\hat{x}, \hat{\imath}) \in \mathbb{R}^2 \times \mathbb{S}$, the probability distribution of the solution $x(t)$ should converge to a stationary probability distribution $\hat{\mu} \in \mathcal{P}_1(\mathbb{R}^2)$, which has its finite mean. By the ergodic theory (see, e.g., [21]), the time average of almost every (a.e.) trajectory of the solution should converge to the finite mean of $\hat{\mu}$, no matter whatever the initial data $(\hat{x}, \hat{\imath})$ are. We perform the computer simulation of the trajectories of two solutions of the specified SDE (5.1) with 2 different initial values $(10, -10)^T$ and $(0, 0)^T$ for $x(0)$ but the same initial value 1 for $r(0)$, which are corresponding to Sample 1 and 2 in Fig.1, respectively. It shows that their time averages are all tending to infinity. This indicates that the given specified SDE (5.1) is not stable in distribution.

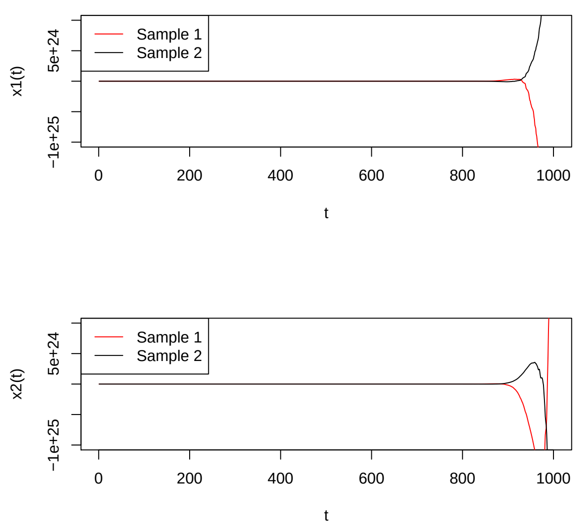

Figure 1: Two trajectories of the given SDE (5.1) using the Euler-Maruyama method (see, e.g., [33]) with step-size 0.001.

We now apply Method 3 in Section 4 to design matrices A_1, A_2 etc. so that the controlled SDE (5.2) is stable in distribution. It is easy to compute that $a_1 = a_2 = 0.429$. Based on (4.5), we need choose c_1 for $2c_1 + 1.142 > 0$. It is sufficient to let $c_1 = 0.5$. Based on (4.6), c_2 should satisfy $1.142(2c_2 - 0.142) - 2 > 0$ and we let $c_2 = 0.8$. Moreover, we let $A_1 = -c_1 I_2$ and $A_2 = -c_2 I_2$, where I_2 is the 2×2 identity matrix. Consequently, we have

$$\mathcal{A} = \begin{pmatrix} 2.142 & -2 \\ -1 & 1.742 \end{pmatrix},\ \mathcal{A}^{-1} = \begin{pmatrix} 1.006 & 1.155 \\ 0.578 & 1.237 \end{pmatrix},$$

and $\theta_1 = 2.161$, $\theta_2 = 1.815$. Let us now design $\kappa_3(t)$ in the form of (4.8). That is, we need to determine ν. This can be done by (4.9) of course, but we will determine ν for $\bar{\varphi} = 0$, which guarantees Assumption 3.2. Noting that $\bar{\kappa}_1 = \bar{\kappa}_2 = 0.75$ and $\bar{\kappa}_2 = \nu$, by (3.4), we have $\bar{\varphi} = 0.2145 - 0.8\nu = 0$, which yields $\nu = 0.731875$. So far we have already seen that Assumptions 2.1, 2.2, 3.1 and 3.2 hold. To apply our Theorems 3.9 and 3.10, we still need to determine τ^*. Given the parameters specified above, (3.7) becomes

$$0.242911 = \frac{1.0145\tau^* + 0.2145\tau^* e^{1.32075\tau^*}}{1 - 0.8\tau^*}$$

for $\tau^* \in (0, 1.25)$. Solving the equation above numerically we obtain $\tau^* = 0.1646$. By Theorem 3.9 we can conclude that the controlled SDE (5.2) is asymptotically stable in distribution if $A_1, A_2, \kappa_3(t)$ are designed as above while the integer M is sufficiently large for $\tau = 1/M < 0.1646$, namely $M \geq 7$. Letting $\tau = 0.01$ (i.e., $M = 100$), we perform the computer simulation of three trajectories of the solution of the controlled SDE (5.2) with 3 different initial values $(0,0)^T, (10,-10)^T$ and $(-10,10)^T$ for $X(0)$ but the same initial value 1 for $r(0)$, which are corresponding to Sample 1, 2 and 3 in Fig.2, respectively. The simulation does not only show that three trajectories from different initial values approach to each other very quickly but also that they look as stationary sequences we would observe in practice. These are what we would expect when the controlled SDE (5.2) is asymptotically stable in distribution under W_1-distance. In other words, the simulation illustrates that the periodic state feedback control based on the state observations at discrete times stabilises the given unstable SDE (5.1) in distribution. It should be pointed out that the simulation is only for illustration and the stability in distribution of the controlled SDE (5.2) is based on our new theory but not based on the simulation.

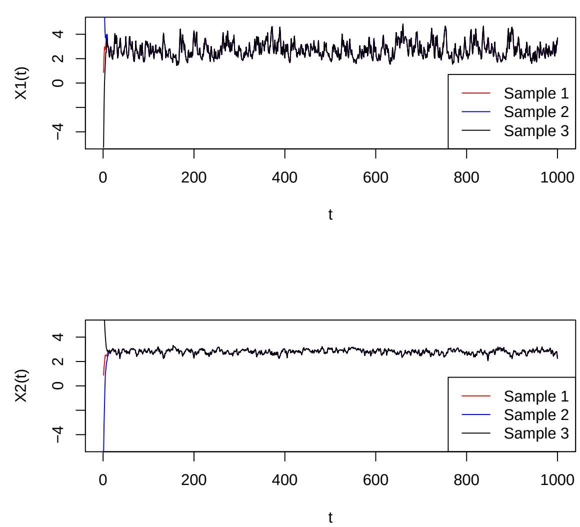

Figure 2: Three trajectories of the controlled SDE (5.2) using the Euler-Maruyama method (see, e.g., [33]) with step-size 0.001.

6 Conclusion

In this paper we proposed a new problem of stabilisation in distribution: Given a periodic hybrid SDE, which is not stable in distribution, can we design a periodic state feedback control in the shift term based on the state observations at discrete times so that the controlled SDE becomes stable in distribution? We did not only investigate the problem successfully but also demonstrated how a periodic stochastic feedback control based on the state observations at discrete times could be designed to stabilise the given SDE in distribution. We demonstrated how our new theory could be applied to a class of unstable semi-linear SDEs, which appear frequently in various applications, for example, in the area of stochastic neural networks.

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