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Discrete-time controllability of Cartesian product networks

Bo Liu, Mengjie Hu, Junjie Huang, Qiang Zhang, Yin Chen, Housheng Su

Abstract—**This work studies the discrete-time controllability of a composite network formed by factor networks via Cartesian products. Based on the Popov-Belevitch-Hautus test and properties of Cartesian products, we derive the algebra-theoretic necessary and sufficient conditions for the controllability of the Cartesian product network (CPN), which is devoted to carry out a comprehensive study of the intricate interplay between the node-system dynamics, network topology and the controllability of the CPN, especially the intrinsic connection between the CPN and its factors. This helps us enrich and perfect the theoretical framework of controllability of complex networks, and gives new insight into designing a valid control scheme for larger-scale composite networks.**

Index Terms—**Controllability; Cartesian product networks; composite networks.**

I. INTRODUCTION

Controllability is a structural attribute of a dynamical system, describing the ability of a dynamic system, with an appropriate choice of inputs, to transfer from any initial state to desired final state within a finite time [1], [2]. Controllability is an essential and important problem in the coordination control of multi-agent networks (MANs), which has rapidly attracted scholars' attention and become a very hot multi-disciplinary research area (e.g., [3]–[7] and the references therein).

Today, the network is becoming the inseparable part of our life. The emergence of large-scale complex networks, such as the Internet, power grids, information networks, intelligent transportation networks, biological neural networks, social networks and gene regulation networks, makes one pay increasing attention to the topological structure and performance of networks [8]–[15]. With the development of real networks, the more the numbers of edges and nodes are, the more diversified the types of edges and nodes are, and the more complex the network topology (network structure) is, which together with multi-layer, multi-level, multi-time-scale, multi-dimension, multi-attribute and other characteristics [16]–[22] makes things have an abundance of challenging. The primary problem or task of controlling a network is to determine whether it is controllable, which is one of the preconditions and cornerstones for the deep study of system performance index, optimization control, etc. For a large-scale networked system with a considerable number of agents (nodes) and complex connections between agents, one has to not only focus on the network system models and related dynamics, but also pay more attention to the topological evolution among agents. This will be a job of high complexity and a limitation of understanding the network structure and mechanism and determining the controllability of the whole system in terms of the existing controllability criteria such as the Kalman rank criterion [1] and the Popov-Belevitch-Hautus (PBH) rank test [2]. Therefore, it is of theoretical, computational and realistic significance to study the controllability of large-scale composite networks from the perspective of the multiple subsystems themselves and the interactions that exist between them. In [16], the author explored the controllability and observability of a large-scale networked system from the perspective of subsystems with different dynamics and obtained some necessary and sufficient conditions depending only on transmission zeros of each subsystem and the subsystem connection matrix. The controllability of large-scale dynamical systems was investigated based upon communications among interconnected subsystems in [17].

Decomposition provides both analytical and calculational benefits for a large-scale composite network or system, which has great significance and application for understanding and analyzing the system performance. There are some very classical and practical decomposition methods in system theory, such as Jordan decomposition, Kalman decomposition and structural decomposition. In fact, many large-scale networks can usually be generated by two or more smaller pieces and subsystems [16], [17], such as motifs [23], clusters and communities [24], by graph products ways, for example, Cartesian product, Corona product, Kronecker product, direct product, strong product, among others [25]. Some important studies, such as synchronization, stability and consensus, in complex networks by Cartesian product graphs were developed. The authors [26] investigated the influences of graph operations on the consensus of complex networks. The stability of delayed coupled oscillators in CPNs was studied in [27]. In

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recent paper [28], the average-consensus problem was solved by a Cartesian product-based hierarchical scheme to design complex networks.

However, to our knowledge, there has been few results in the literature devoted to the controllability of a composite network by graph products [29]–[35]. The controllability was investigated for Cartesian product networks [29]–[32], Kronecker product networks [33] and Corona product networks [35], respectively. The central theme in the literature is to analyze the controllability of product networks by means of the properties of factors under composition of networks. In [29], the authors first explored the controllability and observability of network-of-networks by Cartesian products and established a necessary and sufficient condition for the composite network under the assumption with the diagonalizability for the system matrix. Guan et al. [30] discussed the structural controllability for the Cartesian product of two digraphs. Recently, we respectively investigated the controllability of signed and unsigned MANs via Cartesian product [31], [32]. So far, it remains elusive for a composite MAN to be controllable and the effects of factor networks on the entire network cannot be fully reflected. The controllability problem of a composite MAN bears new characteristics and challenges. As is known, the coordination behaviors of MANs, such as controllability, observability and consensus, are jointly determined by the system dynamics, evolutionary protocols and network topologies, which will cause more obstacles in the performance analysis even when only one factor changes. On the other hand, for the composite networks, it is also hard to reveal the intrinsic connection between the entire network and its factors, as well as their own controllability. Usually, it is not easy to find suitable analytical tools and checkable methods to solve the controllability problem of the entire network through the controllability of its factor networks, which makes the controllability of a composite MAN generated by Cartesian products a nontrivial new problem.

This work discusses the controllability of a discrete-time composite CPN. Compared to most of the existing literature, the advantage and novelty of this study lies in that the network model is constructed by using Cartesian graph products and the properties of Kronecker matrix products are adopted in developing the controllability conditions. The main contributions of this work are threefold. First, a discrete-time CPN with multiple leaders and weighted topology is considered. Second, some necessary and sufficient conditions on the controllability of the discrete-time CPN is established according to the (generalized) left eigenvectors of topology matrices of factor networks, which employs Jordan decomposition to remove the diagonalizability requirement in [29]. Finally, a study of the intricate interplay between the node dynamics, network topology and the controllability of the discrete-time CPN is delved, which provides new insight into the relation between the controllability and the connectivity of a composite network.

The remainder is arranged as follows. Some relevant important preliminaries and problem statement are presented in Section II. The main results are given in Section III. Simulation examples are given in Section IV. Finally, conclusions are drawn in Section V.

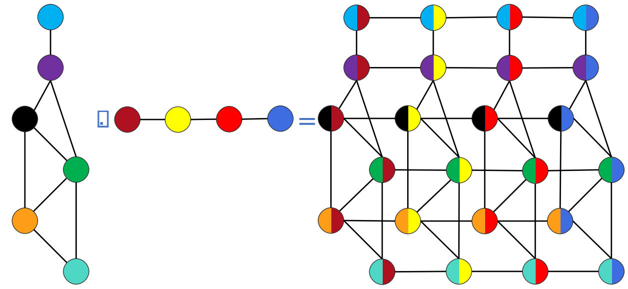

Fig. 1: Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and their Cartesian product $\mathcal{G}_1 \square \mathcal{G}_2$.

II. Preliminaries and Problem Formulation

A. Graph preliminaries

A triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ indicates a weighted graph with a node set $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$, an edge set $\mathcal{E} = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_k\} \in \mathcal{V} \times \mathcal{V}$ and the weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in R^{n\times n}$. Let $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}, i \neq j\}$, $\mathcal{D}$ and $L(\triangleq \mathcal{D} - \mathcal{A})$ be the neighborhood set of node i, the degree matrix and the Laplacian matrix of $\mathcal{G}$, respectively. A composite graph (a large-scale network) $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$ is generated by two small size graphs by Cartesian products, where $\mathcal{V} = \{(i,t) \in \mathcal{V}_1 \times \mathcal{V}_2 \mid i \in \mathcal{V}_1, t \in \mathcal{V}_2\}$ and $\mathcal{E} = \{((j,s),(i,t)) \mid s = t, (j,i) \in \mathcal{E}_1 \text{ or } j = i, (s,t) \in \mathcal{E}_2\}$ are the node and edge sets, respectively. The corresponding edge weight is $a_{(i,t),(j,s)} = \delta_{ij} a_{ts} + \delta_{ts} a_{ij}$, where $\delta_{pq} = 1$ if $p = q$ and if 0 otherwise. Obviously, $\mathcal{G}_1 \square \mathcal{G}_2$ is isomorphic with $\mathcal{G}_2 \square \mathcal{G}_1$, i.e., the operation $\square$ is commutative and associative. At the same time, it is easy to verify that $L(\mathcal{G}) = L(\mathcal{G}_1 \square \mathcal{G}_2) = L(\mathcal{G}_1) \oplus L(\mathcal{G}_2) = L(\mathcal{G}_1) \otimes I_{n_2} + I_{n_1} \otimes L(\mathcal{G}_2)$, where $\oplus$ is the Kronecker sum and $\otimes$ is Kronecker product; $L(\mathcal{G}) \in R^{n_1 n_2 \times n_1 n_2}$, $L(\mathcal{G}_1) \in R^{n_1 \times n_1}$ and $L(\mathcal{G}_2) \in R^{n_2 \times n_2}$ are the Laplacians of $\mathcal{G}$, $\mathcal{G}_1$, $\mathcal{G}_2$, respectively, when $|\mathcal{V}_1| = n_1$ and $|\mathcal{V}_2| = n_2$. An example of the Cartesian product graph is shown in Fig. 1.

B. Problem formulation

A discrete-time MAN with n agents under the leader-follower framework is governed by

$$\begin{cases} x_i(k+1) = x_i(k) - \sum\limits_{j \in \mathcal{N}_i} a_{ij}[x_i(k) - x_j(k)] + \nu_i(k), & i \in \mathcal{V}_l, \\ x_i(k+1) = x_i(k) - \sum\limits_{j \in \mathcal{N}_i} a_{ij}[x_i(k) - x_j(k)], & i \in \mathcal{V}_f, \end{cases} \tag{1}$$

where $x_i, \nu_i \in R$ are the state of agent i and the control signal received by leader i, respectively. $\mathcal{V}_l$ is the leaders' set and $\mathcal{V}_f$ is the followers' set, where $\mathcal{V}_l \cup \mathcal{V}_f = \mathcal{V}$, $\mathcal{V}_l \cap \mathcal{V}_f = \emptyset$.

Let $x = [x_1, x_2, \cdots, x_n]^T$, $u = [\nu_1, \nu_2, \cdots, \nu_m]^T$, then a compact form of the network (1) is

$$x(k+1) = Fx(k) + Bu(k),$$

where $F \triangleq I_n - L \in R^{n\times n}$, L is the Laplacian of $\mathcal{G}$ and $B = [e_1^T, e_2^T, \cdots, e_m^T] \in R^{n\times m}$.

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For simplicity, denote $L(\mathcal{G}) = L$, $F(\mathcal{G}) = F$, $L(\mathcal{G}_i) = L_i$ and $F(\mathcal{G}_i) = F_i$ for $i = 1, 2$. In this work, we will explore a composite theory for the controllability of factor networks

$$x_i(k+1) = F_i x_i(k) + B_i u_i(k),\ \ i = 1, 2,$$

where $x_i \in R^{n_i}$, $u_i \in R^{m_i}$, $F_i = I_{n_i} - L_i \in R^{n_i \times n_i}$, $B_i = [e_1^T, e_2^T, \cdots, e_{m_i}^T] \in R^{n_i \times m_i}$ for $i = 1, 2$. Compose together to form the CPN

$$\begin{aligned} x(k{+}1) &= F\left(\mathcal{G}\right) x(k) + Bu(k) \\ &= F\left(\prod_{\square} \mathcal{G}_i\right) x(k) + \left(\prod_{\otimes} B_i\right) u(k) \\ &= \left(2I_{n_1 n_2} - L\left(\prod_{\square} \mathcal{G}_i\right)\right) x(k) + \left(\prod_{\otimes} B_i\right) u(k), i = 1, 2, \end{aligned} \quad (2)$$

where $x \in R^{n_1 n_2}$, $u \in R^{m_1 m_2}$ denote the stacked system states and the total external control input, respectively; $L(\prod_{\square} \mathcal{G}_i) = L_1 \oplus L_2 = L_1 \otimes I_{n_2} + I_{n_1} \otimes L_2$ and $B = \prod_{\otimes} B_i = B_1 \otimes B_2$.

Of particular note is $F_1 \oplus F_2 = 2I_{n_1 n_2} - L_1 \oplus L_2 = 2I_{n_1} \otimes I_{n_2} - (L_1 \otimes I_{n_2} + I_{n_1} \otimes L_2)$, which is crucial for exploring the controllability of CPN (2) through its factors. As in its usual sense in systems theory, if the continuous system can be controllable, the controllability of its corresponding discretization depends on its sampling period. When the sampling period is small enough, the controllability of the continuous system can be well maintained. However, the continuous-time version of the composite MAN (2) is governed by

$$\dot{x}(t) = -L(G)x(t) + Bu(t), \quad (3)$$

investigated in [29], aiming at matrix pair $(L_1 \oplus L_2, B)$, in which the eigenvectors of $2I_{n_1 n_2} - L_1 \oplus L_2$ cannot be directly obtained from those of $L_1 \oplus L_2$ and, especially, as $L_1 \oplus L_2$ is non-diagonalizable, the case is more complicated (see Table I later). It is therefore to investigate that the controllable conditions in our study are not merely direct analogy or parallel preserving. Contrarily, it is very difficult to obtain the controllability conditions due to the appearance of the unit matrix in the system matrix pair $(2I_{n_1 n_2} - L_1 \oplus L_2, B)$. Controllability of a system refers to the possibility of controlling the state and output of the controlled system, which mainly has three equivalent statements, namely, the Gram matrix criterion, Kalman rank criterion and PBH test. In the analysis of network systems, Gram matrix criterion is usually associated to the minimum control energy problem, which is mainly applicable to the theoretical derivation stage, so that there is still a gap in the use due to the complex construction. PBH test is also applicable to the theoretical derivation including PBH rank test and PBH eigenvector test, where the PBH rank test, combined with the special structure of the network system, draws on the properties of the rank of the matrix (such as the properties of the rank of the block matrix) to realize the simplification of this test and, meanwhile, the PBH eigenvector test is more used to further reveal the relationship between eigenvalues, eigenvectors and controllability. In contrast, the Kalman rank criterion looks much better and is also the most commonly used judgment way. It only relies on the state matrix and the input matrix and has nothing to do with the length of time, but it has a larger computational cost due to the large number of agents in the network system and high dimension of the system. In general, each criterion has its own use scenarios and research questions, which can be chosen to use according to different research objectives and expectations. In what follows, we will delve into the controllability for the discrete-time CPN (2) by analyzing the eigenvalue-(left) eigenvector relationship between $2I_{n_1 n_2} - L_1 \oplus L_2$ and $L_1 \oplus L_2$.

C. Left eigenvalue-eigenvector pairs analysis

PBH test is one of the cornerstones for analyzing the controllability of a LTI system, which states the fact that the matrix pair (A, B) is uncontrollable if and only if there is a left eigenvalue-eigenvector pair (λ, v) of A such that $v^T B = 0$. This implies that it is important and necessary to explore the controllability of the discrete-time CPN (2) by seeking the eigenvalues and eigenvectors of Kronecker products, i.e., the matrix pair $(2I_{n_1 n_2} - L_1 \oplus L_2, B_1 \otimes B_2)$ is controllable if and only if $2I_{n_1 n_2} - L_1 \oplus L_2$ does not allow for all left eigenvectors orthogonal to $B_1 \otimes B_2$.

Definition 1: $J = diag\{J_1, \cdots, J_\varrho\}_{n \times n}$ is the Jordan form of $A \in C^{n \times n}$, if $\exists |P| \neq 0$, s.t. $A = PJP^{-1}$ with the Jordan block $\{J_{i_1}, J_{i_2}, \cdots, J_{i_{k_i}}\}_{d_i \times d_i}$ and the geometric multiplicity k_i corresponding to the eigenvalue ξ_i of A. The row vectors $\sigma_{i_l}^1, \sigma_{i_l}^2, \cdots, \sigma_{i_l}^{\alpha_i^l} (l = \{1, 2, \cdots, k_i\})$ are said to be the generalized left eigenvectors, if

$$\begin{cases} \sigma_{i_l}^1 (\xi_i I_n - A) = 0, \\ \sigma_{i_l}^2 (\xi_i I_n - A) = -\sigma_{i_l}^1, \\ \qquad \vdots \\ \sigma_{i_l}^{\alpha_i^l} (\xi_i I_n - A) = -\sigma_{i_l}^{\alpha_i^l - 1}, \end{cases}$$

where ξ_i is A's eigenvalue and α_i^l is the length of l-th Jordan chain generated by eigenvalue ξ_i.

In the meantime as discussed in the following, due to the same diagonalization of A and $I_n - A$, the (generalized) left eigenvectors of $I_n - A$ are of the form stated as the following proposition by the (generalized) left eigenvectors of A, which is the basis of the controllability analysis later.

Proposition 1: Let $\Lambda = \{\xi_1, \xi_2, \cdots, \xi_s\}$ and $\{\sigma_{i_l}^1, \sigma_{i_l}^2, \cdots, \sigma_{i_l}^{\alpha_i^l}\}$ be the eigenvalue set and the generalized left eigenvector set associated with ξ_{i_l} of $A \in R^{n \times n}$ for $i \in \underline{s} \triangleq \{1, \cdots, s\}$ and $l \in \underline{k_i} \triangleq \{1, \cdots, k_i\}$ respectively, where k_i is the geometric multiplicity of the eigenvalue ξ_i and α_i^l is the length of the l-th Jordan chain. Then the left eigenvalue-eigenvector pair of $I_n - A$ can be express as

(*i*). $(1 - \xi_i, \sigma_{i_l})$, if A is diagonalizable, for $i \in \underline{s}$ and $l \in \underline{k_i}$.

(*ii*). $(1 - \xi_i, \{\sigma_{i_l}^1, \sigma_{i_l}^2, \cdots, \sigma_{i_l}^{\alpha_i^l}\})$, if A is non-diagonalizable, for $i \in \underline{s}$ and $l \in \underline{k_i}$.

Proposition 1 can be simplified as shown in Table **I**.

Proposition 2: The controllability of pair (A, B) is equivalent to that of pair $(I_n - A, B)$.

Based on Propositions 1–2 and PBH test, the eigenvalues and generalized eigenspace of the low-dimensional factors are

TABLE I: Eigenvalues and left eigenvectors of A and $I_n - A$.

	diagonalization	eigenvalues	(generalized) left eigenvectors
A	Y	ξ_i	σ_{i_l}
$I_n - A$	Y	$1-\xi_i$	σ_{i_l}
A	N	ξ_i	$\sigma_{i_l}^1, \sigma_{i_l}^2, \sigma_{i_l}^3, \cdots, \sigma_{i_l}^{\alpha_i^l}$
$I_n - A$	N	$1-\xi_i$	$\sigma_{i_l}^1, -\sigma_{i_l}^2, \sigma_{i_l}^3, \cdots, (-1)^{\alpha_i^l - 1}\sigma_{i_l}^{\alpha_i^l}$

used to represent those of CPN (2), which is the key to get its controllable conditions.

Lemma 1: Let $\Psi = \{\psi_1, \psi_2, \cdots, \psi_s\}$ and $\Phi = \{\phi_1, \phi_2, \cdots, \phi_t\}$ be the eigenvalue sets of L_1 and L_2; $k_i(k_j)$ and $d_i(d_j)$ be the geometric and algebraic multiplicity of $\psi_i(\phi_j)$; $\alpha_i^l(\alpha_j^w)$ be the length of l-th (w-th) Jordan chain generated by $\psi_i(\phi_j)$; $g_{j_l} = \{g_{i_l}^1, g_{i_l}^2, \cdots, g_{i_l}^{\alpha_i^l}\}$ and $h_{j_w} = \{h_{j_w}^1, h_{j_w}^2, \cdots, h_{j_w}^{\alpha_j^w}\}$ be the generalized left eigenvectors corresponding to ψ_{i_l} and ϕ_{i_w}, respectively. Then the left eigenvalue-eigenvector pair of $F = F_1 \oplus F_2 = 2_{n_1 n_2} - L_1 \oplus L_2$ can be expressed as

(i). $(2 - \psi_i - \phi_j, \tilde{p}_{i_l j_w})$, if L_1 and L_2 are both diagonalizable, where $\tilde{p}_{i_l j_w} = g_{i_l} \otimes h_{j_w}$, $i \in \underline{s} \triangleq \{1, \cdots, s\}$, $l \in \underline{k_i} \triangleq \{1, \cdots, k_i\}$, $j \in \underline{t} \triangleq \{1, \cdots, t\}$, $w \in \underline{k_j} \triangleq \{1, \cdots, k_j\}$;

(ii). $(2 - \psi_i - \phi_j, \tilde{p}_{ij_w}^1)$, if L_1 is diagonalizable without repeated roots, L_2 is non-diagonalizable, and $\phi_j = \phi_{j_w}^1 = \phi_{j_w}^2 = \cdots = \phi_{j_w}^{\alpha_j^w}$, where $\tilde{p}_{ij_w}^1 = g_i \otimes h_{j_w}^1$, $i \in \underline{n_1}$, $j \in \underline{t}$, $w \in \underline{k_j}$;

(iii). $(2 - \psi_i - \phi_j, \tilde{p}_{i_l j_w}^1)$, if L_1 is diagonalizable with repeated roots (i.e. $\psi_i = \psi_{i_1} = \psi_{i_2} = \cdots = \psi_{i_{k_i}}$, $d_i = k_i$), L_2 is non-diagonalizable, and $\phi_j = \phi_{j_w}^1 = \phi_{j_w}^2 = \cdots = \phi_{j_w}^{\alpha_j^w}$, where $\tilde{p}_{i_l j_w}^1 = g_{i_l} \otimes h_{j_w}^1$, $i \in \underline{s}$, $l \in \underline{k_i}$, $j \in \underline{t}$, $w \in \underline{k_j}$.

(iv). $(2 - \psi_i - \phi_j, \tilde{p}_{i_l j_w}^q)$, if L_1 and L_2 are non-diagonalizable, and $\psi_i + \phi_j = \psi_{i_l}^1 + \phi_{j_w}^1 = \psi_{i_l}^2 + \phi_{j_w}^2 = \cdots = \psi_{i_l}^r + \phi_{j_w}^r$, $\psi_i = \psi_{i_l}^1 = \psi_{i_l}^2 = \cdots = \psi_{i_l}^{\alpha_i^l}$, $\phi_j = \phi_{j_w}^1 = \phi_{j_w}^2 = \cdots = \phi_{j_w}^{\alpha_j^w}$, where

$$\begin{aligned}
\tilde{p}_{i_l j_w}^1 &= g_{i_l}^1 \otimes h_{j_w}^1, \\
\tilde{p}_{i_l j_w}^2 &= -(g_{i_l}^2 \otimes h_{j_w}^1 - g_{i_l}^1 \otimes h_{j_w}^2), \\
\tilde{p}_{i_l j_w}^3 &= g_{i_l}^3 \otimes h_{j_w}^1 - g_{i_l}^2 \otimes h_{j_w}^2 + g_{i_l}^1 \otimes h_{j_w}^3, \\
&\vdots \\
\tilde{p}_{i_l j_w}^r &= (-1)^{r-1}\{g_{i_l}^r \otimes h_{j_w}^1 - g_{i_l}^{r-1} \otimes h_{j_w}^2 \\
&\quad + \cdots + (-1)^{r-1} g_{i_l}^1 \otimes h_{j_w}^r\},
\end{aligned}$$

$i \in \underline{s}$, $l \in \underline{k_i}$, $j \in \underline{t}$, $w \in \underline{k_j}$, $q \in \{1, 2, \cdots, r\}$, $r \triangleq r(i, l, j, w) = min\{\alpha_i^l, \alpha_j^w\}$.

Lemma 1 gives the eigenvalues and eigenvectors of the CPN ($F_1 \oplus F_2$) from those of factors (L_1 and L_2), of which one of the analytical cornerstones in this work is PBH test together with features of Cartesian products. Furthermore, a parallel result of Lemma 1 in conjunction with Proposition 1 can be directly obtained for F_1 and F_2.

Lemma 2: Let $\Delta = \{\lambda_1, \cdots, \lambda_s\}$ and $\Theta = \{\mu_1, \cdots, \mu_t\}$ be eigenvalue sets of F_1 and F_2; $k_i(k_j)$ and $d_i(d_j)$ be the geometric and algebraic multiplicity of $\lambda_i(\mu_j)$; $\alpha_i^l(\alpha_j^w)$ be the length of l-th (w-th) Jordan chain generated by $\lambda_i(\mu_j)$; $v_{j_l} = \{v_{i_l}^1, v_{i_l}^2, \cdots, v_{i_l}^{\alpha_i^l}\}$ and $u_{j_w} = \{u_{j_w}^1, \sigma_{j_w}^2, \cdots, u_{j_w}^{\alpha_j^w}\}$ be the generalized left eigenvectors associated with ψ_{i_l} and ϕ_{i_w}, respectively. Then the left eigenvalue-eigenvector pair of $F = F_1 \oplus F_2$ can be expressed as

(i). $(\lambda_i + \mu_j, p_{i_l j_w})$, if F_1 and F_2 are both diagonalizable, where $p_{i_l j_w} = v_{i_l} \otimes u_{j_w}$, $i \in \underline{s}$, $l \in \underline{k_i}$, $j \in \underline{t}$, $w \in \underline{k_j}$.

(ii). $(\lambda_i + \mu_j, p_{ij_w}^1))$, if F_1 is diagonalizable without repeated roots, F_2 is non-diagonalizable, and $\mu_j = \mu_{j_w}^1 = \mu_{j_w}^2 = \cdots = \mu_{j_w}^{\alpha_j^w}$, where $p_{ij_w}^1 = v_i \otimes u_{j_w}^1$, $i \in \underline{n_1}$, $l \in \underline{k_i}$, $w \in \underline{k_j}$.

(iii). $(\lambda_i + \mu_j, p_{i_l j_w}^1)$, if F_1 is diagonalizable with repeated roots (i.e., $\lambda_i = \lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_{k_i}}$, $d_i = k_i$), and $\mu_j = \mu_{j_w}^1 = \mu_{j_w}^2 = \cdots = \mu_{j_w}^{\alpha_j^w}$, where $p_{i_l j_w}^1 = v_{i_l} \otimes q_{j_w}^1$, $i \in \underline{s}$, $l \in \underline{k_i}$, $j \in \underline{t}$, $w \in \underline{k_j}$.

(iv). $(\lambda_i + \mu_j, p_{ij}^q)$, if F_1 and F_2 are both non-diagonalizable, and $\lambda_i + \mu_j = \lambda_{i_l}^1 + \mu_{j_w}^1 = \lambda_{i_l}^2 + \mu_{j_w}^2 = \cdots = \lambda_{i_l}^r + \mu_{j_w}^r, \lambda_i = \lambda_{i_l}^1 = \lambda_{i_l}^2 = \cdots = \lambda_{i_l}^{\alpha_i^l}, \mu_j = \mu_{j_w}^1 = \mu_{j_w}^2 = \cdots = \mu_{j_w}^{\alpha_j^w}$, where

$$\begin{aligned}
p_{i_l j_w}^1 &= v_{i_l}^1 \otimes u_{j_w}^1, \\
p_{i_l j_w}^2 &= v_{i_l}^2 \otimes u_{j_w}^1 - v_{i_l}^1 \otimes u_{j_w}^2, \\
&\vdots \\
p_{i_l j_w}^r &= v_{i_l}^r \otimes u_{j_w}^1 - v_{i_l}^{r-1} \otimes u_{j_w}^2 + \cdots + (-1)^{r-1} v_{i_l}^1 \otimes u_{j_w}^r,
\end{aligned}$$

$i \in \underline{s}$, $l \in \underline{k_i}$, $j \in \underline{t}$, $w \in \underline{k_j}$, $q \in \{1, 2, \cdots, r\}$, $r \triangleq r(i, l, j, w) = min\{\alpha_i^l, \alpha_j^w\}$.

It is worth noting that Lemmas 1–2 explore the connection between the eigenvalues and eigenvectors of the composite network and those of its factors from the points of view of the system matrices and interaction topologies between agents, respectively, which are the foundation for studying the controllability of a large-scale composite network via Cartesian products. Eigenvalue decomposition is adopted in developing the controllable conditions of the composite network. Eigenvalue decomposition is to extract the most important features of a matrix, which is a common and effective method for solving linear algebra problems and is widely applied in signal processing, image processing, machine learning among others. The traditional eigenvalue decomposition algorithm has problems in numerical stability, accuracy and computational efficiency. If a matrix A can satisfy $A^T A = AA^T$, the eigenvalue decomposition of the matrix is not numerically sensitive. In practice, there are some actual systems satisfying this condition, such as the spring oscillator system. In general, eigenvalue decomposition is numerically sensitive. Indeed, with the dimension increment of a matrix, its eigenvalue decomposition usually becomes more numerically sensitive, and even for a low dimensional matrix, there is no guarantee that it does not meet numerical instability issues in its eigenvalue decomposition. In this paper, the controllability of the Cartesian network is theoretically discussed by its factors. In [16], an interesting work was to design an algorithm with lower computation complexity and more numerical stability, which provides considerable insight into the controllability of Cartesian networks with perturbations. As we all know, numerical stability is an unavoidable problem in specific applications, and it is also the limitation of our research, which will be further investigated in Cartesian networks in future studies.

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III. Controllability Analysis

In what follows, the controllability of the composite discrete-time CPN (2) can directly be deciphered from that of its factors based on Lemma 1.

Theorem 1: CPN (2) is controllable if and only if one of the following four conditions holds

(i). If L_1 and L_2 are diagonalizable, and $\psi_i, \psi_{\hat{i}} \in \Psi$, $\phi_j, \phi_{\hat{j}} \in \Phi$, $\Omega_{ij} = \{(\hat{i},\hat{j}) : \psi_{\hat{i}} + \phi_{\hat{j}} = \psi_i + \phi_j \in \sigma(L)\}$, where $i, \hat{i} \in \underline{s}$, $l \in \underline{k}_{\hat{i}}$, $j, \hat{j} \in \underline{t}$, $w \in \underline{k}_{\hat{j}}$, then for $\{c_{\hat{i}_l\hat{j}_w}\}$ (not all zero),

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{l=1}^{k_{\hat{i}}} \sum_{w=1}^{k_{\hat{j}}} c_{\hat{i}_l\hat{j}_w} \tilde{p}_{\hat{i}_l\hat{j}_w}\right)(B_1 \otimes B_2) \neq 0.$$

(ii). If L_1 is diagonalizable without repeated roots, L_2 is non-diagonalizable, and $\psi_{\hat{i}} + \phi^1_{\hat{j}_1} = \cdots = \psi_{\hat{i}} + \phi^{\alpha^1_{\hat{j}}}_{\hat{j}_1} = \cdots = \psi_{\hat{i}} + \phi^{\alpha^{k_{\hat{j}}}_{\hat{j}}}_{\hat{j}_{k_{\hat{j}}}}$, where $\psi_i, \psi_{\hat{i}} \in \Psi$, $\phi_j, \phi_{\hat{j}} \in \Phi$, $\Omega_{ij} = \{(\hat{i},\hat{j}) : \psi_{\hat{i}} + \phi_{\hat{j}} = \psi_i + \phi_j \in \sigma(L)\}$, $i, \hat{i} \in \underline{n_1}$, $l \in \underline{k}_{\hat{i}}$, $j, \hat{j} \in \underline{t}$, $w \in \underline{k}_{\hat{j}}$, then for $\{c_{\hat{i}\hat{j}_w}\}$ (not all zero),

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{w=1}^{k_{\hat{j}}} c_{\hat{i}\hat{j}_w} \tilde{p}^1_{\hat{i}\hat{j}_w}\right)(B_1 \otimes B_2) \neq 0.$$

(iii). If L_1 is diagonalizable with repeated roots (i.e., $\psi_{\hat{i}_1} = \psi_{\hat{i}_2} = \cdots = \psi_{\hat{i}_{k_{\hat{i}}}}$), L_2 is non-diagonalizable, and $\psi_{\hat{i}_1} + \phi^1_{\hat{j}_1} = \cdots = \psi_{\hat{i}_1} + \phi^{\alpha^1_{\hat{j}}}_{\hat{j}_1} = \cdots = \psi_{\hat{i}_1} + \phi^{\alpha^{k_{\hat{j}}}_{\hat{j}}}_{\hat{j}_{k_{\hat{j}}}} = \cdots = \psi_{\hat{i}_{k_{\hat{i}}}} + \phi^{\alpha^{k_{\hat{j}}}_{\hat{j}}}_{\hat{j}_{k_{\hat{j}}}}$, where $\psi_i, \psi_{\hat{i}} \in \Psi$, $\phi_j, \phi_{\hat{j}} \in \Phi$, $\Omega_{ij} = \{(\hat{i},\hat{j}) : \psi_{\hat{i}} + \phi_{\hat{j}} = \psi_i + \phi_j \in \sigma(L)\}$, $i, \hat{i} \in \underline{s}$, $l \in \underline{k}_{\hat{i}}$, $j, \hat{j} \in \underline{t}$, $w \in \underline{k}_{\hat{j}}$, then for $\{c_{\hat{i}_l\hat{j}_w}\}$ (not all zero),

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{l=1}^{k_{\hat{i}}} \sum_{w=1}^{k_{\hat{j}}} c_{\hat{i}_l\hat{j}_w} \tilde{p}^1_{\hat{i}_l\hat{j}_w}\right)(B_1 \otimes B_2) \neq 0.$$

(iv). If L_1 and L_2 are both non-diagonalizable, and $\psi^1_{\hat{i}_1} + \phi^1_{\hat{j}_1} = \cdots = {}^{r(\hat{i},1,\hat{j},1)}_{\hat{i}_1} + \phi^{r(\hat{i},1,\hat{j},1)}_{\hat{j}_1} = \cdots = \psi^1_{\hat{i}_l} + \phi^1_{\hat{j}_w} = \psi^2_{\hat{i}_l} + \phi^2_{\hat{j}_w} = \cdots = \psi^{r(\hat{i},l,\hat{j},w)}_{\hat{i}_l} + \phi^{r(\hat{i},l,\hat{j},w)}_{\hat{j}_w}$, where $\psi_i, \psi_{\hat{i}} \in \Psi$, $\phi_j, \phi_{\hat{j}} \in \Phi$, $\Omega_{ij} = \{(\hat{i},\hat{j}) : \psi_{\hat{i}} + \phi_{\hat{j}} = \psi_i + \phi_j \in \sigma(L)\}$, $i, \hat{i} \in \underline{s}$, $l \in \underline{k}_{\hat{i}}$, $j, \hat{j} \in \underline{t}$, $w \in \underline{k}_{\hat{j}}$, $q \in \{1, 2, \cdots, r(\hat{i}, l, \hat{j}, w)\}$, $r(\hat{i}, l, \hat{j}, w) = min\{\alpha^l_{\hat{i}}, \alpha^w_{\hat{j}}\}$, then for $\{c^q_{\hat{i}_l\hat{j}_w}\}$ (not all zero),

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{l=1}^{k_{\hat{i}}} \sum_{w=1}^{k_{\hat{j}}} \sum_{q=1}^{r(\hat{i},l,\hat{j},w)} c^q_{\hat{i}_l\hat{j}_w} \tilde{p}^q_{\hat{i}_l\hat{j}_w}\right)(B_1 \otimes B_2) \neq 0.$$

A parallel result of Theorem 1 based on Lemma 2 can be received for F_1 and F_2 substituting for L_1 and L_2 in the following.

Theorem 2: CPN (2) is controllable if and only if one of the following four conditions holds

(i). If F_1 and F_2 are diagonalizable, and $\lambda_i, \lambda_{\hat{i}} \in \Delta$, $\mu_j, \mu_{\hat{j}} \in \Theta$, $\Omega_{ij} = \{(\hat{i},\hat{j}) : \lambda_{\hat{i}} + \mu_{\hat{j}} = \lambda_i + \mu_j \in \sigma(F)\}$, $i, \hat{i} \in \underline{s}$, $l \in \underline{k}_{\hat{i}}$, $j, \hat{j} \in \underline{t}$, $w \in \underline{k}_{\hat{j}}$, then for $\{c_{\hat{i}_l\hat{j}_w}\}$ (not all zero),

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{l=1}^{k_{\hat{i}}} \sum_{w=1}^{k_{\hat{j}}} c_{\hat{i}_l\hat{j}_w} p_{\hat{i}_l\hat{j}_w}\right)(B_1 \otimes B_2) \neq 0.$$

(ii). If F_1 is diagonalizable without repeated roots, L_2 is non-diagonalizable, and $\lambda_{\hat{i}} + \mu^1_{\hat{j}_1} = \cdots = \lambda_{\hat{i}} + \mu^{\alpha^1_{\hat{j}}}_{\hat{j}_1} = \cdots = \lambda_{\hat{i}} + \mu^{\alpha^{k_{\hat{j}}}_{\hat{j}}}_{\hat{j}_{k_{\hat{j}}}}$, where $\lambda_i, \lambda_{\hat{i}} \in \Delta$, $\mu_j, \mu_{\hat{j}} \in \Theta$, $\Omega_{ij} = \{(\hat{i},\hat{j}) : \lambda_{\hat{i}} + \lambda_{\hat{j}} = \psi_i + \mu_j \in \sigma(F)\}$, $i, \hat{i} \in \underline{n_1}$, $l \in \underline{k}_{\hat{i}}$, $j, \hat{j} \in \underline{t}$, $w \in \underline{k}_{\hat{j}}$, then for $\{c_{\hat{i}\hat{j}_w}\}$ (not all zero),

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{w=1}^{k_{\hat{j}}} c_{\hat{i}\hat{j}_w} p^1_{\hat{i}\hat{j}_w}\right)(B_1 \otimes B_2) \neq 0.$$

(iii). If F_1 is diagonalizable with repeated roots (i.e., $\lambda_{\hat{i}_1} = \lambda_{\hat{i}_2} = \cdots = \lambda_{\hat{i}_{k_i}}$), L_2 is non-diagonalizable, and $\lambda_{\hat{i}_1} + \mu^1_{\hat{j}_1} = \cdots = \lambda_{\hat{i}_1} + \mu^{\alpha^1_{\hat{j}}}_{\hat{j}_1} = \cdots = \lambda_{\hat{i}_1} + \mu^{\alpha^{k_{\hat{j}}}_{\hat{j}}}_{\hat{j}_{k_{\hat{j}}}} = \cdots = \lambda_{\hat{i}_{k_{\hat{i}}}} + \mu^{\alpha^{k_{\hat{j}}}_{\hat{j}}}_{\hat{j}_{k_{\hat{j}}}}$, where $\lambda_i, \lambda_{\hat{i}} \in \Delta$, $\mu_j, \mu_{\hat{j}} \in \Theta$, $\Omega_{ij} = \{(\hat{i},\hat{j}) : \lambda_{\hat{i}} + \mu_{\hat{j}} = \lambda_i + \mu_j \in \sigma(F)\}$, $i, \hat{i} \in \underline{s}$, $l \in \underline{k}_{\hat{i}}$, $j, \hat{j} \in \underline{t}$, $w \in \underline{k}_{\hat{j}}$, then for $\{c_{\hat{i}_l\hat{j}_w}\}$ (not all zero),

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{l=1}^{k_{\hat{i}}} \sum_{w=1}^{k_{\hat{j}}} c_{\hat{i}_l\hat{j}_w} p^1_{\hat{i}_l\hat{j}_w}\right)(B_1 \otimes B_2) \neq 0.$$

(iv). If F_1 and F_2 are both non-diagonalizable, and $\lambda^1_{\hat{i}_1} + \mu^1_{\hat{j}_1} = \cdots = \lambda^{r(\hat{i},1,\hat{j},1)}_{\hat{i}_1} + \mu^{r(\hat{i},1,\hat{j},1)}_{\hat{j}_1} = \cdots = \lambda^1_{\hat{i}_l} + \mu^1_{\hat{j}_w} = \lambda^2_{\hat{i}_l} + \mu^2_{\hat{j}_w} = \cdots = \lambda^{r(\hat{i},l,\hat{j},w)}_{\hat{i}_l} + \mu^{r(\hat{i},l,\hat{j},w)}_{\hat{j}_w}$, where $\lambda_i, \lambda_{\hat{i}} \in \Delta$, $\mu_j, \mu_{\hat{j}} \in \Theta$, $\Omega_{ij} = \{(\hat{i},\hat{j}) : \lambda_{\hat{i}} + \mu_{\hat{j}} = \lambda_i + \mu_j \in \sigma(F)\}$, $i, \hat{i} \in \underline{s}$, $l \in \underline{k}_{\hat{i}}$, $j, \hat{j} \in \underline{t}$, $w \in \underline{k}_{\hat{j}}$, $q \in \{1, 2, \cdots, r(\hat{i}, l, \hat{j}, w)\}$, $r(\hat{i}, l, \hat{j}, w) = min\{\alpha^l_{\hat{i}}, \alpha^w_{\hat{j}}\}$, then for $\{c^q_{\hat{i}_l\hat{j}_w}\}$ (not all zero),

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{l=1}^{k_{\hat{i}}} \sum_{w=1}^{k_{\hat{j}}} \sum_{q=1}^{r(\hat{i},l,\hat{j},w)} c^q_{\hat{i}_l\hat{j}_w} p^q_{\hat{i}_l\hat{j}_w}\right)(B_1 \otimes B_2) \neq 0.$$

Corollary 1: If CPN (2) is controllable, then factor networks $\mathcal{G}_1$ and $\mathcal{G}_2$ are both controllable.

Remark 1: Note from Theorem 1 and Theorem 2, that the controllability of the discrete-time CPN (2) is determined by the controllability and connectivity of its factors, different from the controllability of the corresponding continuous-time network (3) of version (2) only depending on the interaction topology between agents, which is not only related to the interaction topology between agents, but also associated with the dynamics of the agent itself. Fortunately, we have found the relationship between them.

This paper aims to propose controllability analysis methods of discrete-time MANs via Cartesian products, whose topologies are different from the general controllability problems for dynamical systems based on simple connected graphs in the literature. The Cartesian product is one of the effective methods to generate composite networks or hierarchical systems by using prime networks and preserves many properties of small-scale ones, through which some common networks can be built by simpler networks, including some important topology structures such as regular grids, meshes, cubes and hypercubes. CPNs have recently emerged in many real-world scenarios, such as the parallel and distributed systems (multi-core systems and multiprocessor SoC [36] and interconnection NoCs [37]), networked dynamical systems (sensor networks and power grids [38]), social networks (interactions among families [29]) and bio-evolutionary networks (evolutionary dynamics of organism-environment coupling networks [30]). Product networks are still in the primary stage of study, and most of research results aim at the network parameters, such as node degree, diameter and network size [39]. In addition, almost all of relevant studies (e.g. [29]–[33]) considered the continuous-time Cartesian product MANs, while our work considers the discrete-time Cartesian product MANs. Our results can be applied to control systems with the underlying hierarchical structure of producing a Cartesian product, such as layered man-made structures (infrastructure networks and quantum computing networks), physical systems (monitoring and control of fluids and heat flows) [29] and the control of UAV formation [31].

This work is concentrated on controllability that is a fundamental property of networks and the core of it is to investigate the controllability relationship between the entire network and its factors, and obtain controllable conditions consisting of the four mutually exclusive assertions. There exist some difficulties in solving the controllability problem for CPNs lack of the appropriate analytical tools. Therefore, this work has made a useful attempt, in order to completely solve the characterizations of the controllable matrices' eigenvalues and eigenvectors, and then obtain the controllability conditions of the CPN, although the controllability proof is a bit lengthy.

Besides, it is noteworthy that one of the main motivations in this study is to develop computationally feasible conditions for controllability verification of a large scale network. From the aforementioned studies, the controllability of the entire network completely depends on the structure and feature of its factor networks. Thus, as long as the factor networks are given, the controllability of the entire network can be judged by means of the new criteria established here. Theoretically, the obtained conditions are scalable with the dimension of the network and the number of factor networks. Moreover, the computational complexity of directly computing eigenvalues and eigenvectors of the composite CPN is $O(n_1^3 n_2^3)$, whereas the complexity of computing those from its factors' eigenvectors and generalized eigenvectors is no more than $O((\min\{n_1,n_2\})^2 n_1 n_2 + n_1^3 + n_2^3)$. Therefore, the computational cost of the new conditions is much lower. Undoubtedly, as the number and dimension of the factor networks increase, the amount of calculation will enlarge greatly and all the agents' motion trajectories are more complicated, as shown in the later examples.

IV. Simulation Examples

The controllability of MANs refers that in the case of interaction between agents, all follower agents can be steered to the specified target state from any given initial state in a limited time through regulating a few agents (leaders), which reflects the restriction ability of system input on the state. For network (2), its solution is $x(k) = F^k x(0) + \sum_{i=0}^{k-1} F^{k-i-1} Bu(i)$, where $x(0)$ is the any given initial state. Several simulation examples are provided to illustrate the theoretical results of Theorems 1–2 and describe the trajectories of agents.

Example 1: Suppose that CPN (2) with two factor networks $\mathcal{G}_1$ and $\mathcal{G}_2$ is depicted in Fig. 2(a). The system matrices are given as follows

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

By computing,

$$F_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & -2 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$rank(\mathcal{Q}_1) = rank[B_1 \ \ L_1B_1 \ \ L_1^2B_1] = 3$, $rank(\mathcal{Q}_2) = rank[B_2 \ \ L_2B_2 \ \ L_2^2B_2 \ \ L_2^3B_2] = 4$, $\psi(L_1) = \{0,1,3\}$, $\phi(L_2) = \{1,1,1,0\}$, $\lambda(F_1) = \{1,0,-2\}$, $\mu(F_2) = \{0,0,0,1\}$, respectively. It is easy to see that (L_1, B_1), (L_2, B_2), (F_1, B_1) and (F_2, B_2) are all controllable and L_1, F_1 are diagonalizable and have no repeated roots, and the (generalized) left eigenvectors of L_1, L_2, F_1 and F_2 can be respectively calculated as

$$\begin{cases} g_1 = e_1, \\ g_2 = e_1 - e_2, \\ g_3 = e_2 - e_3, \end{cases} \quad \begin{cases} h_{1_1}^1 = e_3 - e_4, \\ h_{1_1}^2 = -e_2 + e_4, \\ h_{1_1}^3 = e_1 - e_4, \\ h_{2_1}^1 = e_4, \end{cases}$$

$$\begin{cases} v_1 = e_1, \\ v_2 = e_1 - e_2, \\ v_3 = e_2 - e_3, \end{cases} \quad \begin{cases} u_{1_1}^1 = e_3 - e_4, \\ u_{1_1}^2 = e_2 - e_4, \\ u_{1_1}^3 = e_1 - e_4, \\ u_{2_1}^1 = e_4. \end{cases}$$

Furthermore,

$$\begin{cases} \psi_1 + \phi_{2_1}^1 = 0, \\ \psi_1 + \phi_{1_1}^1 = \psi_1 + \phi_{1_1}^2 = \psi_1 + \phi_{1_1}^3 = \psi_2 + \phi_{2_1}^1 = 1, \\ \psi_2 + \phi_{1_1}^1 = \psi_2 + \phi_{1_1}^2 = \psi_2 + \phi_{1_1}^3 = 2, \\ \psi_3 + \phi_{2_1}^1 = 3, \\ \psi_3 + \phi_{1_1}^1 = \psi_3 + \phi_{1_1}^2 = \psi_3 + \phi_{1_1}^3 = 4, \end{cases}$$

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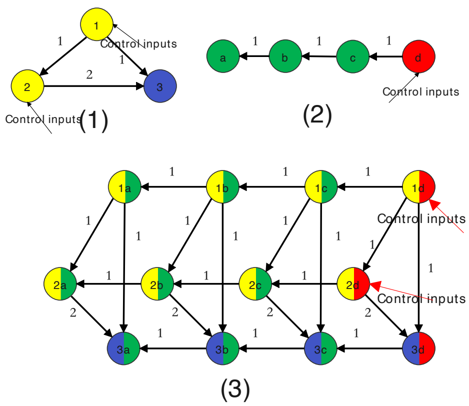
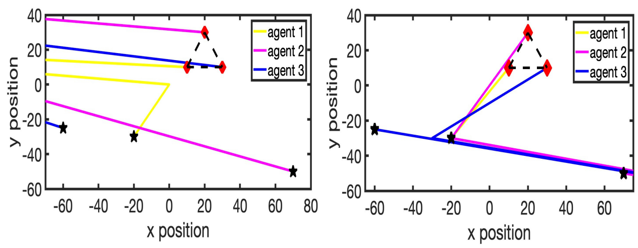
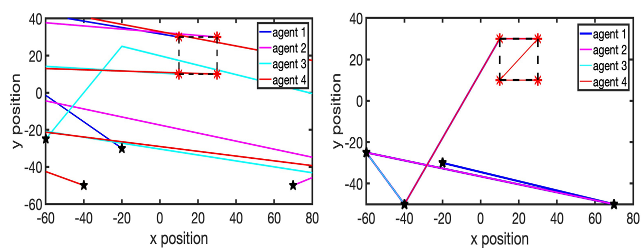


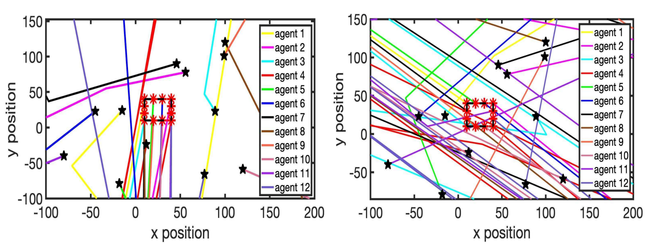
(a) Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and their composite graph $\mathcal{G}_1\square\mathcal{G}_2$ of Example 1.



(b) A triangle configuration for $\mathcal{G}_1$.



(c) A square configuration for $\mathcal{G}_2$.



(d) A rectangle configuration for $\mathcal{G}$.

Fig. 2: Topologies and the agents' motion trajectories of Example 1, where the corresponding matrices of left and right sub-figures are L and F, respectively.

$$\begin{cases}\lambda_1+\mu_{2_1}=2,\\ \lambda_1+\mu_{1_1}^1=\lambda_1+\mu_{1_1}^2=\lambda_1+\mu_{1_1}^3=\lambda_2+\mu_{2_1}=1,\\ \lambda_2+\mu_{1_1}^1=\lambda_2+\mu_{1_1}^2=\lambda_2+\mu_{1_1}^3=0,\\ \lambda_3+\mu_{2_1}=-1,\\ \lambda_3+\mu_{1_1}^1=\lambda_3+\mu_{1_1}^2=\lambda_3+\mu_{1_1}^3=-2,\end{cases}$$

then for any $[c_1,c_2]\neq 0$, we can have

$$\begin{cases}\tilde{p}_{12_1}^1(B_1\otimes B_2)=[1,0]\neq 0,\\ (c_1\tilde{p}_{11_1}^1+c_2\tilde{p}_{22_1}^1)(B_1\otimes B_2)=[-c_1+c_2,-c_2]\neq 0,\\ \tilde{p}_{21_1}^1(B_1\otimes B_2)=[-1,1]\neq 0,\\ \tilde{p}_{32_1}^1(B_1\otimes B_2)=[0,1]\neq 0,\\ \tilde{p}_{31_1}^1(B_1\otimes B_2)=[0,-1]\neq 0,\end{cases}$$

$$\begin{cases}p_{12_1}^1(B_1\otimes B_2)=[1,0]\neq 0,\\ (c_1p_{11_1}^1+c_2p_{22_1}^1)(B_1\otimes B_2)=[-c_1+c_2,-c_2]\neq 0,\\ p_{21_1}^1(B_1\otimes B_2)=[-1,1]\neq 0,\\ p_{32_1}^1(B_1\otimes B_2)=[0,1]\neq 0,\\ p_{31_1}^1(B_1\otimes B_2)=[0,-1]\neq 0,\end{cases}$$

which is consistent with Theorem 1 (ii) and Theorem 2 (ii), therefore, CPN (2) is controllable.

All the agents' motion trajectories are exhibited in Figs. $2(b)$–$2(d)$ with final triangle, square and rectangle configuration corresponding to matrices L (left sub-figures) and $F=I-L$ (right sub-figures) respectively, where arbitrary initial state and final desired configuration are denoted by "$\star$" and "$*$". By respectively comparing the left and right sub-figures for Figs. $2(b)$–$2(d)$, it is not difficult to find that for the same initial state, the agents' motion trajectories are different, but eventually they all converge to the same final state for matrices L and $F=I-L$. In other words, the controllability of the discrete-time CPN (2) is completely determined by the interaction topologies of its factor networks.

Example 2: Suppose that CPN (2) with two factor networks $\mathcal{G}_1$ and $\mathcal{G}_2$ is depicted in Fig. $3(a)$. The system matrices are given as follows

$$L_1=\begin{bmatrix}1&-1&0&0\\0&0&0&0\\0&-1&2&-1\\0&0&0&0\end{bmatrix},L_2=\begin{bmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&0\end{bmatrix};$$

$$B_1=\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix},B_2=\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}.$$

By computing,

$$F_1=\begin{bmatrix}0&1&0&0\\0&1&0&0\\0&1&-1&1\\0&0&0&1\end{bmatrix},\ F_2=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&1\end{bmatrix};$$

$rank(\mathcal{Q}_1)=[B_1\ L_1B_1\ L_1^2B_1\ L_1^3B_1]=4$, $rank(\mathcal{Q}_2)=[B_2\ L_2B_2\ L_2^2B_2\ L_2^3B_2]=4$, $\psi(L_1)=\{0,0,1,2\}$, $\phi(L_2)=\{1,1,1,0\}$, $\lambda(F_1)=\{1,1,0,-1\}$, $\mu(F_2)=\{0,0,0,1\}$, respectively. It is easy to see that (L_1,B_1), (L_2,B_2), (F_1,B_1)

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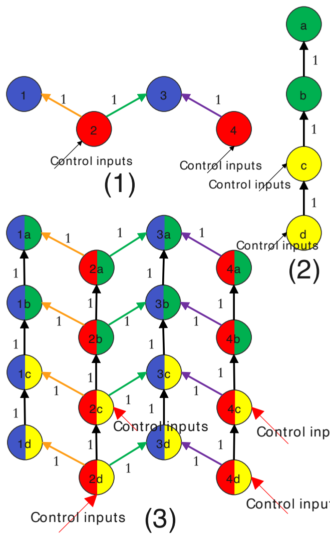
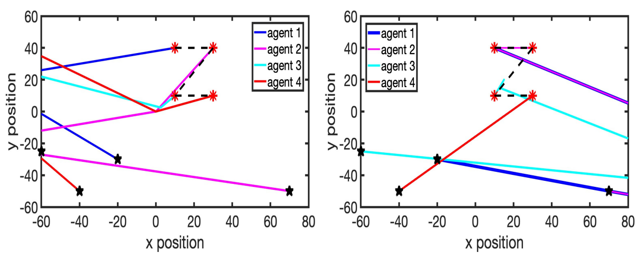
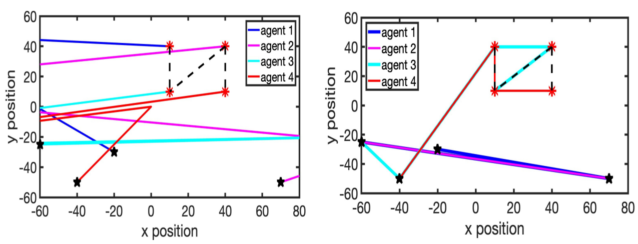


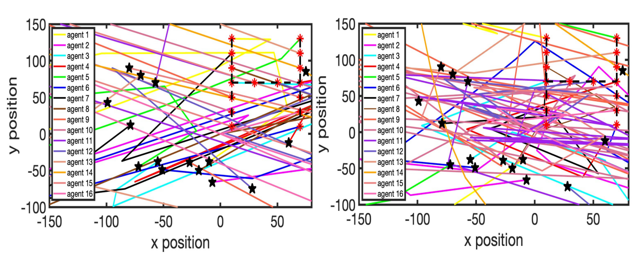
(a) Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and their composite graph $\mathcal{G}_1 \square \mathcal{G}_2$ of Example 2.



(b) A letter "Z" configuration for $\mathcal{G}_1$.



(c) A letter "N" configuration for $\mathcal{G}_2$.



(d) A letter "H" configuration for $\mathcal{G}$.

Fig. 3: Topologies and the agents' motion trajectories of Example 2, where the corresponding matrices of left and right sub-figures are L and F, respectively.

and (F_2, B_2) are all controllable and L_1, F_1 are diagonalizable and have repeated roots, and the (generalized) left eigenvectors of L_1, L_2, F_1 and F_2 can be respectively given by

$$\begin{cases} g_{1_1} = e_4, \\ g_{1_2} = e_2, \\ g_{2_1} = e_1 - e_2, \\ g_{3_1} = e_2 - 2e_3 + e_4, \end{cases} \quad \begin{cases} h^1_{1_1} = e_3 - e_4, \\ h^2_{1_1} = -e_2 + e_4, \\ h^3_{1_1} = e_1 - e_4, \\ h^1_{2_1} = e_4, \end{cases}$$

$$\begin{cases} v_{1_1} = e_4, \\ v_{1_2} = e_2, \\ v_{2_1} = e_1 - e_2, \\ v_{3_1} = e_2 - 2e_3 + e_4, \end{cases} \quad \begin{cases} u^1_{1_1} = e_3 - e_4, \\ u^2_{1_1} = e_2 - e_4, \\ u^3_{1_1} = e_1 - e_4, \\ u^1_{2_1} = e_4. \end{cases}$$

Furthermore,

$$\begin{cases} \psi_{1_1} + \phi^1_{2_1} = \psi_{1_2} + \phi^1_{2_1} = 0, \\ \psi_{1_1} + \phi^1_{1_1} = \psi_{1_1} + \phi^2_{1_1} = \psi_{1_1} + \phi^3_{1_1} = \psi_{1_2} + \phi^1_{1_1} = \psi_{1_2} + \phi^2_{1_1} \\ \qquad = \psi_{1_2} + \phi^3_{1_1} = \psi_2 + \phi^1_{2_1} = 1, \\ \psi_2 + \phi^1_{1_1} = \psi_2 + \phi^2_{1_1} = \psi_2 + \phi^3_{1_1} = \psi_3 + \phi^1_{2_1} = 2, \\ \psi_3 + \phi^1_{1_1} = \psi_3 + \phi^2_{1_1} = \psi_3 + \phi^3_{1_1} = 3, \end{cases}$$

$$\begin{cases} \lambda_{1_1} + \mu^1_{2_1} = \lambda_{1_2} + \mu^1_{2_1} = 2, \\ \lambda_{1_1} + \mu^1_{1_1} = \lambda_{1_1} + \mu^2_{1_1} = \lambda_{1_1} + \mu^3_{1_1} = \lambda_{1_2} + \mu^1_{1_1} = \lambda_{1_2} + \mu^2_{1_1} \\ \qquad = \lambda_{1_2} + \mu^3_{1_1} = \lambda_2 + \mu^1_{2_1} = 1, \\ \lambda_2 + \mu^1_{1_1} = \lambda_2 + \mu^2_{1_1} = \lambda_2 + \mu^3_{1_1} = \lambda_3 + \mu^1_{2_1} = 0, \\ \lambda_3 + \mu^1_{1_1} = \lambda_3 + \mu^2_{1_1} = \lambda_3 + \mu^3_{1_1} = -1, \end{cases}$$

then for any $[c_1, c_2] \neq 0$, we can have

$$\begin{cases} (c_1 \tilde{p}^1_{1_1 2_1} + c_2 \tilde{p}^1_{1_2 2_1})(B_1 \otimes B_2) = [0, c_2, 0, c_1] \neq 0, \\ (c_1 \tilde{p}^1_{2 1_1} + c_2 \tilde{p}^1_{3 2_1})(B_1 \otimes B_2) = [-c_1, c_1 + c_2, 0, c_2] \neq 0, \\ \tilde{p}^1_{3 1_1}(B_1 \otimes B_2) = [1, -1, 1, -1] \neq 0, \end{cases}$$

$$\begin{cases} (c_1 p^1_{1_1 2_1} + c_2 p^1_{1_2 2_1})(B_1 \otimes B_2) = [0, c_2, 0, c_1] \neq 0, \\ (c_1 p^1_{2 1_1} + c_2 p^1_{3 2_1})(B_1 \otimes B_2) = [-c_1, c_1 + c_2, 0, c_2] \neq 0, \\ p^1_{3 1_1}(B_1 \otimes B_2) = [1, -1, 1, -1] \neq 0, \end{cases}$$

and for any $[c_1, c_2, c_3] \neq 0$, we can have

$$\begin{aligned} &(c_1 \tilde{p}^1_{1_1 1_1} + c_2 \tilde{p}^1_{1_2 1_1} + c_3 \tilde{p}^1_{2 2_1})(B_1 \otimes B_2) \\ =&[c_2, -c_2 - c_3, c_1, -c_1] \neq 0, \\ &(c_1 p^1_{1_1 1_1} + c_2 p^1_{1_2 1_1} + c_3 p^1_{2 2_1})(B_1 \otimes B_2) \\ =&[c_2, -c_2 - c_3, c_1, -c_1] \neq 0, \end{aligned}$$

which is consistent with Theorem 1 (iii) and Theorem 2 (iii), therefore, CPN (2) is controllable.

All the agents' motion trajectories are exhibited in Figs. 3(b)–3(d) with final "Z", "N" and "H" configuration corresponding to matrices L (left sub-figures) and $F = I - L$ (right sub-figures) respectively, where arbitrary initial state and final desired configuration are denoted by "$\star$" and "$*$". By respectively comparing the left and right sub-figures for Figs. 3(b)–3(d), it is not difficult to find that for the same initial state, the agents' motion trajectories are different, but eventually they all converge to the same final state for matrices L and $F = I - L$.

Example 3: Suppose that CPN (2) with two factor networks $\mathcal{G}_1$ and $\mathcal{G}_2$ is depicted in Fig. 4(a). The system matrices are given as follows

$$L_1=\begin{bmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}, L_2=\begin{bmatrix}1&-1&0&0&0\\0&1&-1&0&0\\0&0&1&-1&0\\0&0&0&1&-1\\0&0&0&-1&1\end{bmatrix};$$

$$B_1=\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix},\quad B_2=\begin{bmatrix}0&0\\0&0\\1&0\\0&0\\0&1\end{bmatrix}.$$

By computing,

$$F_1=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&1\end{bmatrix},\ F_2=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&1&0\end{bmatrix};$$

$rank(\mathcal{Q}_1) = [B_1\ L_1B_1\ L_1^2B_1\ L_1^3B_1] = 4$, $rank(\mathcal{Q}_2) = [B_2\ L_2B_2\ L_2^2B_2\ L_2^3B_2\ L_2^4B_2] = 5$, $\psi(L_1) = \{1,1,1,0\}$, $\phi(L_2) = \{1,1,1,0,2\}$, $\lambda(F_1) = \{0,0,0,1\}$, $\mu(F_2) = \{0,0,0,1,-1\}$, respectively. (L_1,B_1), (L_2,B_2), (F_1,B_1) and (F_2,B_2) are all controllable and L_1, L_2, F_1 and F_2 are all undiagonalizable, and the generalized left eigenvectors of L_1, L_2, F_1 and F_2 can be respectively calculated as

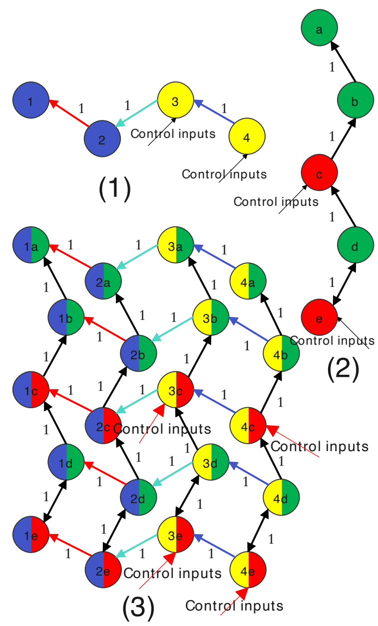
$$\begin{cases}g_{1_1}^1=e_3-e_4,\\g_{1_1}^2=-e_2+e_4,\\g_{1_1}^3=e_1-e_4,\\g_{2_1}^1=e_4,\end{cases}\quad\begin{cases}h_{1_1}^1=e_3-e_5,\\h_{1_1}^2=-e_2+e_3+e_4-e_5,\\h_{1_1}^3=e_1-e_2+e_4-e_5,\\h_{2_1}^1=e_4+e_5,\\h_{3_1}^1=e_4-e_5,\end{cases}$$

and

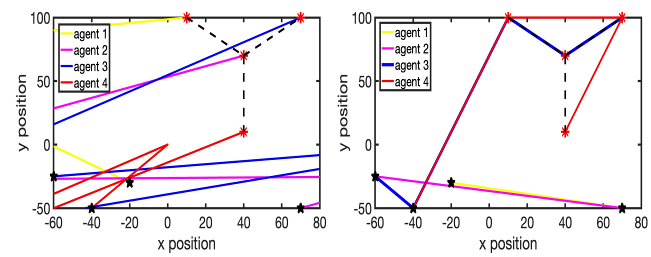
$$\begin{cases}v_{1_1}^1=e_3-e_4,\\v_{1_1}^2=e_2-e_4,\\v_{1_1}^3=e_1-e_4,\\v_{2_1}^1=e_4,\end{cases}\quad\begin{cases}u_{1_1}^1=e_3-e_5,\\u_{1_1}^2=e_2-e_3-e_4+e_5,\\u_{1_1}^3=e_1-e_2+e_4-e_5,\\u_{2_1}^1=e_4+e_5,\\u_{3_1}^1=e_4-e_5.\end{cases}$$

Furthermore,

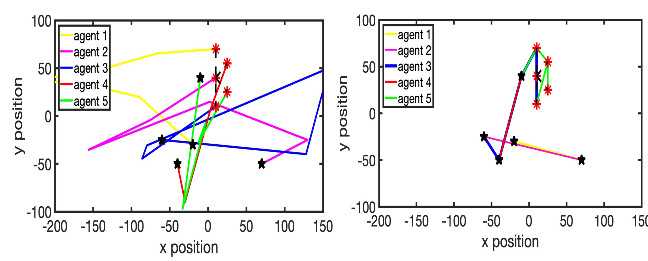
$$\begin{cases}\psi_{2_1}^1+\phi_{2_1}^1=0,\\\psi_{1_1}^1+\phi_{2_1}^1=\psi_{1_1}^2+\phi_{2_1}^1=\psi_{1_1}^3+\phi_{2_1}^1=\psi_{2_1}^1+\phi_{1_1}^1=\psi_{2_1}^1+\phi_{1_1}^2\\\qquad=\psi_{2_1}^1+\phi_{1_1}^3=1,\\\psi_{1_1}^1+\phi_{1_1}^1=\psi_{1_1}^1+\phi_{1_2}^1=\psi_{1_1}^1+\phi_{1_3}^1=\psi_{1_1}^2+\phi_{1_1}^2=\psi_{1_1}^2+\phi_{1_2}^2\\\qquad=\psi_{1_1}^2+\phi_{1_3}^2=\psi_{1_1}^3+\phi_{1_1}^3=\psi_{1_1}^3+\phi_{1_2}^3=\psi_{1_1}^3+\phi_{1_3}^3\\\qquad=\psi_{2_1}^1+\phi_{3_1}^1=2,\\\psi_{1_1}^1+\phi_{3_1}^1=\psi_{1_1}^2+\phi_{3_1}^1=\psi_{1_1}^3+\phi_{3_1}^1=3,\end{cases}$$



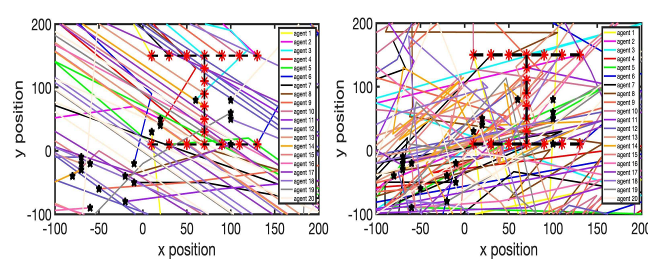
(a) Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and their composite graph $\mathcal{G}_1\square\mathcal{G}_2$ of Example 3.



(b) A letter "Y" configuration for $\mathcal{G}_1$.



(c) A letter "K" configuration for $\mathcal{G}_2$.



(d) A letter "I" configuration for $\mathcal{G}$.

Fig. 4: Topologies and the agents' motion trajectories of Example 3, where the corresponding matrices of left and right sub-figures are L and F, respectively.

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$$\begin{cases} \lambda_{2_1}^1+\mu_{2_1}^1=2,\\ \lambda_{1_1}^1+\mu_{2_1}^1=\lambda_{1_1}^2+\mu_{2_1}^1=\lambda_{1_1}^3+\mu_{2_1}^1=\lambda_{2_1}^1+\mu_{1_1}^1=\lambda_{2_1}^1+\mu_{1_1}^2\\ \qquad\qquad =\lambda_{2_1}^1+\mu_{1_1}^3=1,\\ \lambda_{1_1}^1+\mu_{1_1}^1=\lambda_{1_1}^1+\mu_{1_2}^1=\lambda_{1_1}^1+\mu_{1_3}^1=\lambda_{1_1}^2+\mu_{1_1}^2=\lambda_{1_1}^2+\mu_{1_2}^2\\ \qquad\qquad =\lambda_{1_1}^2+\mu_{1_3}^2=\lambda_{1_1}^3+\mu_{1_1}^3=\lambda_{1_1}^3+\mu_{1_2}^3=\lambda_{1_1}^3+\mu_{1_3}^3\\ \qquad\qquad =\lambda_{2_1}^1+\mu_{3_1}^1=0,\\ \lambda_{1_1}^1+\mu_{3_1}^1=\lambda_{1_1}^2+\mu_{3_1}^1=\lambda_{1_1}^3+\mu_{3_1}^1=-1, \end{cases}$$

then for any $[c_1,c_2]\neq 0$, we can have

$$\begin{cases} \tilde{p}_{2_12_1}^1(B_1\otimes B_2)=[0,0,0,1]\neq 0,\\ (c_1\tilde{p}_{1_12_1}^1+c_2\tilde{p}_{2_11_1}^1)(B_1\otimes B_2)=[0,c_1,c_2,-c_1-c_2]\neq 0,\\ \tilde{p}_{1_13_1}^1(B_1\otimes B_2)=[0,-1,0,1]\neq 0, \end{cases}$$

$$\begin{cases} p_{2_12_1}^1(B_1\otimes B_2)=[0,0,0,1]\neq 0,\\ (c_1 p_{1_12_1}^1+c_2 p_{2_11_1}^1)(B_1\otimes B_2)=[0,c_1,c_2,-c_1-c_2]\neq 0,\\ p_{1_13_1}^1(B_1\otimes B_2)=[0,-1,0,1]\neq 0, \end{cases}$$

and for any $[c_1,c_2,c_3,c_4]\neq 0$, we can have

$$\begin{aligned} &(c_1\tilde{p}_{1_11_1}^1+c_2\tilde{p}_{1_11_1}^2+c_3\tilde{p}_{1_11_1}^3+c_4\tilde{p}_{2_13_1}^1)(B_1\otimes B_2)\\ =&[c_1-c_2,-c_1+c_2-c_3,-c_1+2c_2-2c_3,c_1-2c_2+3c_3-c_4]\neq 0,\\ &(c_1 p_{1_11_1}^1+c_2 p_{1_11_1}^2+c_3 p_{1_11_1}^3+c_4 p_{2_13_1}^1)(B_1\otimes B_2)\\ =&[c_1-c_2,-c_1+c_2-c_3,-c_1+2c_2-2c_3,c_1-2c_2+3c_3-c_4]\neq 0, \end{aligned}$$

which is consistent with Theorem 1 (iv) and Theorem 2 (iv), therefore, CPN (2) is controllable.

All the agents' motion trajectories are exhibited in Figs. $4(b)$–$4(d)$ with final "Y", "K" and "I" configuration corresponding to matrices L (left sub-figures) and $F=I-L$ (right sub-figures) respectively, where arbitrary initial state and final desired configuration are denoted by "$\star$" and "$*$". By respectively comparing the left and right sub-figures for Figs. $4(b)$–$4(d)$, it is not difficult to find that for the same initial state, the agents' motion trajectories are different, but eventually they all converge to the same final state for matrices L and $F=I-L$.

V. Conclusion

We have studied the controllability of a discrete-time CPN formed by smaller-scale factor networks. The algebra-theoretic necessary and sufficient conditions for the controllability have been obtained by the eigenvalues and (generalized) left eigenvectors of its factor networks, which provides insights into the controllability of larger-scale composite control schemes.

The main theme in this study is to explore the controllability of a large-scale MAN model generated by the Cartesian product of smaller ones, which can effectively reduce the complexity of the calculation and analysis to verify the controllability of the MAN by checking some properties of the smaller factor ones and has aroused great interest in generative networks recently. It is not only important for building 'large' networks out of 'small' ones, but also useful to get insights about the properties of 'larger' composite networks from the 'smaller' factor ones. This would be a great hint and insight into how to effectively generate a realistic network with a mathematically tractable model, preserving some ideal properties of the basic graphs and allowing for a rigorous theoretical analysis of the network properties. Conversely, an important inverse problem is whether a given network can be decomposed in a product of two factor graphs and how to decompose efficiently. Our current model is based on the Kronecker product matrix operation for the Cartesian network, which is not a specific matrix that can be decomposed. It is very challenging for decomposing a specific matrix into some lower-dimensional small matrices with good properties. Indeed, to our knowledge, this issue is rarely reported in the literature, which would be interesting to study in our future work. In addition, the controllability of MANs with respect to other types of graph products, such as the strong product, direct product and Corona product, will be considered in future research directions. Future work of particular interest includes extending current results to matrix-weighted and / or signed hybrid composite MANs containing continuous-time and discrete-time dynamic agents.

Appendix. Proof of some technical results

Proof of Proposition 1. Case (i). If A is diagonalizable, then I_n-A is diagonalizable and $\sigma_{i_l}A=\xi_i\sigma_{i_l}$, and then

$$\sigma_{i_l}(I_n-A)=\sigma_{i_l}-\sigma_{i_l}A=\sigma_{i_l}-\xi_i\sigma_{i_l}=(1-\xi_i)\sigma_{i_l},$$

therefore, $(1-\xi_i,\sigma_{i_l})$ is th left eigenvalue-eigenvector pair of I_n-A.

Case (ii). If A is non-diagonalizable, let λ_i and $\delta_{i_l}^1,\delta_{i_l}^2,\cdots,\delta_{i_l}^{\alpha_i^l}$ be the eigenvalues and corresponding generalized left eigenvectors of I_n-A, from Definition 1, then

$$\begin{cases} \sigma_{i_l}^1(\xi_i I_n-A)=0,\\ \delta_{i_l}^1[\lambda_i I_n-(I_n-A)]=0 \end{cases} \Rightarrow \begin{cases} \delta_{i_l}^1=\sigma_{i_l}^1,\\ \lambda_i=1-\xi_i, \end{cases}$$

$$\begin{cases} \sigma_{i_l}^2(\xi_i I_n-A)=-\sigma_{i_l}^1,\\ \delta_{i_l}^2[\lambda_i I_n-(I_n-A)]=-\delta_{i_l}^1 \end{cases} \Rightarrow \begin{cases} \delta_{i_l}^2=-\sigma_{i_l}^2,\\ \lambda_i=1-\xi_i, \end{cases}$$

$$\begin{cases} \sigma_{i_l}^3(\xi_i I_n-A)=-\sigma_{i_l}^2,\\ \delta_{i_l}^3[\lambda_i I_n-(I_n-A)]=-\delta_{i_l}^2 \end{cases} \Rightarrow \begin{cases} \delta_{i_l}^3=\sigma_{i_l}^3,\\ \lambda_i=1-\xi_i, \end{cases}$$

$$\vdots$$

$$\begin{cases} \sigma_{i_l}^{\alpha_i^l}(\xi_i I_n-A)=-\sigma_{i_l}^{\alpha_i^l-1},\\ \delta_{i_l}^{\alpha_i^l}[\lambda_i I_n-(I_n-A)]=-\delta_{i_l}^{\alpha_i^l-1} \end{cases} \Rightarrow \begin{cases} \delta_{i_l}^{\alpha_i^l}=(-1)^{\alpha_i^l-1}\sigma_{i_l}^{\alpha_i^l},\\ \lambda_i=1-\xi_i. \end{cases}$$

Therefore, I_n-A is also non-diagonalizable, and $(1-\xi_i,\{\sigma_{i_l}^1,\sigma_{i_l}^2,\cdots,\sigma_{i_l}^{\alpha_i^l}\})$ is the generalized left eigenvalue-eigenvector pair. □

Proof of Lemma 1. (i). Because L_1 and L_2 are both diagonalizable, $g_{i_l}L_1=\psi_i g_{i_l}$ and $h_{j_w}L_2=\phi_j h_{j_w}$, $i\in\underline{s}$, $l\in\underline{k_i}$, $j\in\underline{t}$, $w\in\underline{k_j}$, then

$$\begin{aligned} &\tilde{p}_{i_lj_w}(F_1\oplus F_2)\\ &=(g_{i_l}\otimes h_{j_w})(2I_{n_1}\otimes I_{n_2}-L_1\otimes I_{n_2}-I_{n_1}\otimes L_2)\\ &=2(g_{i_l}\otimes h_{j_w})-(\psi_i g_{i_l})\otimes h_{j_w}-g_{i_l}\otimes(\phi_j h_{j_w})\\ &=(2-\psi_i-\phi_j)(g_{i_l}\otimes h_{j_w})\\ &=(2-\psi_i-\phi_j)\tilde{p}_{i_lj_w}. \end{aligned}$$

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Thus, the left eigenvalue-eigenvector pair of $F_1 \oplus F_2$ is $(2-\psi_i-\phi_j, \tilde{p}_{i_l j_w})$.

(*ii*). Because L_2 is not necessarily diagonalizable, we can know that $h_{j_w}^1 L_2 = \phi_j h_{j_w}^1$, $j \in \underline{t}$, $w \in \underline{k_j}$. From Definition (1) and the fact that L_1 is diagonalizable, then for $i \in \underline{n_1}$,

$$\begin{aligned}
&\tilde{p}_{ij_w}^1(F_1 \oplus F_2) \\
&= (g_i \otimes h_{j_w}^1)(2I_{n_1} \otimes I_{n_2} - L_1 \otimes I_{n_2} - I_{n_1} \otimes L_2) \\
&= 2(g_i \otimes h_{j_w}^1) - (\psi_i g_i) \otimes h_{j_w}^1 - g_i \otimes (\phi_j h_{j_w}^1) \\
&= (2-\psi_i-\phi_j)(g_i \otimes h_{j_w}^1) \\
&= (2-\psi_i-\phi_j)\tilde{p}_{ij_w}^1.
\end{aligned}$$

Thus, the left eigenvalue-eigenvector pair of $F_1 \oplus F_2$ is $(2-\psi_i-\phi_j, \tilde{p}_{ij_w}^1)$.

(*iii*). From Definition (1), we can have that $g_{i_l} L_1 = \psi_i g_{i_l}$ and $h_{j_w}^1 L_2 = \phi_j h_{j_w}^1$, $i \in \underline{s}$, $l \in \underline{k_i}$, $j \in \underline{t}$, $w \in \underline{k_j}$, then

$$\begin{aligned}
&\tilde{p}_{i_l j_w}^1(F_1 \oplus F_2) \\
&= (g_{i_l} \otimes h_{j_w}^1)(2I_{n_1} \otimes I_{n_2} - L_1 \otimes I_{n_2} - I_{n_1} \otimes L_2) \\
&= 2(g_{i_l} \otimes h_{j_w}^1) - (\psi_i g_{i_l}) \otimes h_{j_w}^1 - g_{i_l} \otimes (\phi_j h_{j_w}^1) \\
&= (2-\psi_i-\phi_j)(g_{i_l} \otimes h_{j_w}^1) \\
&= (2-\psi_i-\phi_j)\tilde{p}_{i_l j_w}^1.
\end{aligned}$$

Thus, the left eigenvalue-eigenvector pair of $F_1 \oplus F_2$ is $(2-\psi_i-\phi_j, \tilde{p}_{i_l j_w}^1)$.

(*iv*). From Definition (1), as $i \in \underline{s}$, $l \in \underline{k_i}$, $j \in \underline{t}$, $w \in \underline{k_j}$, $r = r(i,l,j,w) = min\{\alpha_i^l, \alpha_j^w\}$, then

$$\begin{aligned}
&\tilde{p}_{i_l j_w}^r(F_1 \oplus F_2) \\
&= (-1)^{r-1}\{g_{i_l}^r \otimes h_{j_w}^1 - g_{i_l}^{r-1} \otimes h_{j_w}^2 + \cdots + \\
&\quad (-1)^{r-1} g_{i_l}^1 \otimes h_{j_w}^r\}(2I_{n_1} \otimes I_{n_2} - L_1 \otimes I_{n_2} - I_{n_1} \otimes L_2) \\
&= (-1)^{r-1}\{(g_{i_l}^r \otimes h_{j_w}^1)(2I_{n_1} \otimes I_{n_2}) - (g_{i_l}^r \otimes h_{j_w}^1)(L_1 \otimes I_{n_2}) \\
&\quad - (g_{i_l}^r \otimes h_{j_w}^1)(I_{n_1} \otimes L_2)\} - (-1)^{r-1}\{(g_{i_l}^{r-1} \otimes h_{j_w}^2)(2I_{n_1} \otimes I_{n_2}) \\
&\quad - (g_{i_l}^{r-1} \otimes h_{j_w}^2)(L_1 \otimes I_{n_2}) - (g_{i_l}^{r-1} \otimes h_{j_w}^2)(I_{n_1} \otimes L_2)\} \\
&\quad + \cdots + (-1)^{r-1}(-1)^{r-1}\{(g_{i_l}^1 \otimes h_{j_w}^r)(2I_{n_1} \otimes I_{n_2}) \\
&\quad - (g_{i_l}^1 \otimes h_{j_w}^r)(L_1 \otimes I_{n_2}) - (g_{i_l}^1 \otimes h_{j_w}^r)(I_{n_1} \otimes L_2)\} \\
&= (-1)^{r-1}\{2(g_{i_l}^r I_{n_1}) \otimes (h_{j_w}^1 I_{n_2}) - (g_{i_l}^r L_1) \otimes (h_{j_w}^1 I_{n_2}) \\
&\quad - (g_{i_l}^r I_{n_1}) \otimes (h_{j_w}^1 L_2)\} - (-1)^{r-1}\{2(g_{i_l}^{r-1} I_{n_1}) \otimes (h_{j_w}^2 I_{n_2}) \\
&\quad - (g_{i_l}^{r-1} L_1) \otimes (h_{j_w}^2 I_{n_2}) - (g_{i_l}^{r-1} I_{n_1}) \otimes (h_{j_w}^2 L_2)\} + \cdots \\
&\quad + (-1)^{2(r-1)}\{2(g_{i_l}^1 I_{n_1}) \otimes (h_{j_w}^r I_{n_2}) - (g_{i_l}^1 L_1) \otimes (h_{j_w}^r I_{n_2}) \\
&\quad - (g_{i_l}^1 I_{n_1}) \otimes (h_{j_w}^r L_2)\} \\
&= (-1)^{r-1}\{2(g_{i_l}^r \otimes h_{j_w}^1) - (g_{i_l}^r L_1) \otimes h_{j_w}^1 - g_{i_l}^r \otimes (h_{j_w}^1 L_2)\} - \\
&\quad (-1)^{r-1}\{2(g_{i_l}^{r-1} \otimes h_{j_w}^2) - (g_{i_l}^{r-1} L_1) \otimes h_{j_w}^2 - g_{i_l}^{r-1} \otimes (h_{j_w}^2 L_2)\} \\
&\quad + \cdots + (-1)^{2(r-1)}\{2(g_{i_l}^1 \otimes h_{j_w}^r) - (g_{i_l}^1 L_1) \otimes h_{j_w}^r - \\
&\quad g_{i_l}^1 \otimes (h_{j_w}^r L_2)\} \\
&= (-1)^{r-1}\{2(g_{i_l}^r \otimes h_{j_w}^1) - \psi_i(g_{i_l}^r \otimes h_{j_w}^1) - g_{i_l}^{r-1} \otimes h_{j_w}^1 \\
&\quad - \phi_j(g_{i_l}^r \otimes h_{j_w}^1)\} - (-1)^{r-1}\{2(g_{i_l}^{r-1} \otimes h_{j_w}^2) - \psi_i(g_{i_l}^{r-1} \otimes h_{j_w}^2) \\
&\quad - g_{i_l}^{r-2} \otimes h_{j_w}^2 - \phi_j(g_{i_l}^{r-1} \otimes h_{j_w}^2) - g_{i_l}^{r-1} \otimes h_{j_w}^1\} + \cdots + (-1)^{2(r-1)} \\
&\quad \{2(g_{i_l}^1 \otimes h_{j_w}^r) - \psi_i(g_{i_l}^1 \otimes h_{j_w}^r) - \phi_j(g_{i_l}^1 \otimes h_{j_w}^r) - g_{i_l}^1 \otimes h_{j_w}^{r-1}\} \\
&= (-1)^{r-1}(2-\psi_i-\phi_j)(g_{i_l}^r \otimes h_{j_w}^1) - (-1)^{r-1}(2-\psi_i-\phi_j) \\
&\quad (g_{i_l}^{r-1} \otimes h_{j_w}^2) + \cdots + (-1)^{r-1}(2-\psi_i-\phi_j)(g_{i_l}^1 \otimes h_{j_w}^r) \\
&= (-1)^{r-1}(2-\psi_i-\phi_j)\{(g_{i_l}^r \otimes h_{j_w}^1) - (g_{i_l}^{r-1} \otimes h_{j_w}^2) \\
&\quad + \cdots + (g_{i_l}^1 \otimes h_{j_w}^r)\} \\
&= (-1)^{r-1}(2-\psi_i-\phi_j)\tilde{p}_{i_l j_w}^r.
\end{aligned}$$

Thus, the left eigenvalue-eigenvector pair of $F_1 \oplus F_2$ are $(2-\psi_i-\phi_j, \tilde{p}_{i_l j_w}^q)$ for $q \in \{1,2,\cdots,r\}$. □

Proof of Theorem 1. (Sufficiency): (*i*). Consider the case that L_1 and L_2 are diagonalizable. According to PBH test and Lemma 1(*i*), and

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{l=1}^{k_{\hat{i}}} \sum_{w=1}^{k_{\hat{j}}} c_{\hat{i}_l \hat{j}_w} \tilde{p}_{\hat{i}_l \hat{j}_w}\right)(B_1 \otimes B_2) \neq 0$$

holds, for $\{c_{\hat{i}_l \hat{j}_w}\}$ (not all zero), therefore, CPN (2) is controllable.

(*ii*). Consider the case that L_1 is diagonalizable without repeated roots, and $\psi_{\hat{i}} + \phi_{\hat{j}_1}^1 = \cdots = \psi_{\hat{i}} + \phi_{\hat{j}_1}^{\alpha_{\hat{j}}^1} = \cdots = \psi_{\hat{i}} + \phi_{\hat{j}_{k_{\hat{j}}}}^{\alpha_{\hat{j}}^{k_{\hat{j}}}}$. From PBH test and Lemma 1(*ii*), and

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{w=1}^{k_{\hat{j}}} c_{\hat{i}\hat{j}_w} \tilde{p}_{\hat{i}\hat{j}_w}^1\right)(B_1 \otimes B_2) \neq 0$$

holds, for $\{c_{\hat{i}\hat{j}_w}\}$ (not all zero), therefore, CPN (2) is controllable.

(*iii*). Consider the case that L_1 is diagonalizable with repeated roots (i.e., $\psi_{\hat{i}_1} = \psi_{\hat{i}_2} = \cdots = \psi_{\hat{i}_{k_{\hat{i}}}}$) and $\psi_{\hat{i}_1} + \phi_{\hat{j}_1}^1 = \cdots = \psi_{\hat{i}_1} + \phi_{\hat{j}_1}^{\alpha_{\hat{j}}^1} = \cdots = \psi_{\hat{i}_1} + \phi_{\hat{j}_{k_{\hat{j}}}}^{\alpha_{\hat{j}}^{k_{\hat{j}}}} = \cdots = \psi_{\hat{i}_{k_{\hat{i}}}} + \phi_{\hat{j}_{k_{\hat{j}}}}^{\alpha_{\hat{j}}^{k_{\hat{j}}}}$, where $\psi_i, \psi_{\hat{i}} \in \Psi$, $\phi_j, \phi_{\hat{j}} \in \Phi$. From PBH test and Lemma 1(*iii*), and

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{l=1}^{k_{\hat{i}}} \sum_{w=1}^{k_{\hat{j}}} c_{\hat{i}_l \hat{j}_w} \tilde{p}_{\hat{i}_l \hat{j}_w}^1\right)(B_1 \otimes B_2) \neq 0$$

holds, for $\{c_{\hat{i}_l \hat{j}_w}\}$ (not all zero), therefore, CPN (2) is controllable.

(*iv*). Consider the case that L_1 and L_2 are non-diagonalizable, $\psi_{\hat{i}_1}^1 + \phi_{\hat{j}_1}^1 = \cdots = {}_{\hat{i}_1}^{r(\hat{i},1,\hat{j},1)} + \phi_{\hat{j}_1}^{r(\hat{i},1,\hat{j},1)} = \cdots = \psi_{\hat{i}_l}^1 + \phi_{\hat{j}_w}^1 = \psi_{\hat{i}_l}^2 + \phi_{\hat{j}_w}^2 = \cdots = {}_{\hat{i}_l}^{r(\hat{i},l,\hat{j},w)} + \phi_{\hat{j}_w}^{r(\hat{i},l,\hat{j},w)}$, for $i, \hat{i} \in \underline{s}$, $l \in \underline{k_{\hat{i}}}$, $j, \hat{j} \in \underline{t}$, $w \in \underline{k_{\hat{j}}}$, $q \in \{1,2,\cdots,r(\hat{i},l,\hat{j},w)\}$, $r(\hat{i},l,\hat{j},w) = min\{\alpha_{\hat{i}}^l, \alpha_{\hat{j}}^w\}$. From PBH test and Lemma 1(*iv*), and

$$\left(\sum_{(\hat{i},\hat{j})\in\Omega_{ij}} \sum_{l=1}^{k_{\hat{i}}} \sum_{w=1}^{k_{\hat{j}}} \sum_{q=1}^{r(\hat{i},l,\hat{j},w)} c_{\hat{i}_l \hat{j}_w}^q \tilde{p}_{\hat{i}_l \hat{j}_w}^q\right)(B_1 \otimes B_2) \neq 0$$

holds, for $\{c_{\hat{i}_l \hat{j}_w}^q\}$ (not all zero), therefore, CPN (2) is controllable.

(Necessity): The controllability of CPN (2) together with Lemma 1 implies that the linear span of left eigenvectors associated with the given eigenvalue of $\mathcal{L}(\mathcal{G}_1) \oplus \mathcal{L}(\mathcal{G}_2)$ and $B_1 \otimes B_2$ are un-orthogonal. □

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