Stationary distribution and extinction of an HCV transmission model with protection awareness and environmental fluctuations

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Abstract

We propose a stochastic HCV model with protection awareness and latency-acute-chronic phases in this study. First of all, we show that the stochastic HCV model admits a unique global positive solution for any given positive initial values. Then, we verify that HCV model has a unique stationary distribution under the sufficient criterion $\mathcal{R}_0^s > 1$, which indicates that HCV transmission undergoes the persistence in the long term. Furthermore, we derive the sufficient conditions for HCV extinction under the condition $\mathcal{R}_0^e < 1$. As a consequence, we derive the relationships among the stochastic persistence index $\mathcal{R}_0^s$, the stochastic extinction index $\mathcal{R}_0^e$ and the threshold (the basic reproduction number $\mathcal{R}_0$) of the model without fluctuations. The condition $\mathcal{R}_0 > \mathcal{R}_0^s > 1$ reveals that the existence of the white noises triggers the less stochastic persistence index. While, the condition $\mathcal{R}_0 < \mathcal{R}_0^e < 1$ reveals that, when the intensities of the white noises are controlled, the value $\mathcal{R}_0^e$ triggers the stochastic extinction.

Keywords: HCV, Protection awareness, Stationary distribution, Extinction

1. Introduction

Hepatitis C (HCV), caused by the hepatitis C virus, is an inflammation of the liver. The symptoms of HCV range from moderate to severe and include fever, fatigue, loss of appetite, nausea, vomiting, abdominal pain, dark urine and yellowing of the skin or eyes (jaundice) [1]. Although the deterministic models have been widely used, they fail to describe the uncertainties that exist in the real world. Within the recent contributions, some studies have introduced stochastic fluctuations into their models, including parameter fluctuations [2, 3, 4, 5, 6] and environmental fluctuations [7, 8, 9, 10, 11, 12]. Therein, a noteworthy recent contribution was the work by Zhai et al. [9], who focused on the susceptible population with protection awareness in a stochastic model and found that detailed publicity was important for controlling the infection scale of HIV/AIDS. In the population level, the recent contributions have explored HCV transmission dynamics with stochastic fluctuations. Alnafisah et al. [13] conducted a stochastic model of HCV transmission across diverse viral genomes and concluded that smaller white noise ensured HCV persistence, while the larger white noises led to its extinction. It was found by Rajasekar et al. [14] that the persistence and extinction of HCV depended on the fluctuations in some key parameters of the chronically infected population. Later, Qi et al. [15] took the latent period and nonlinear incidence rates into account, and proposed an improved HCV stochastic model, motivating by Cui's model [16].

In this study, we are motivated by the transmission mechanisms of SACTR models in [16, 17], the role of the exposed population during the HCV spread in [15], the protection awareness of the susceptible population in [9], and the environmental fluctuations in [7, 8], we now propose a new HCV transmission model with protection awareness in the environmental fluctuations. Next, we assume that the total population of HCV is separated into eight compartments: S_u, the susceptible population without protection awareness; S_a, the susceptible population with protection awareness; E, the number of the exposed population; A and C, the numbers of the actively infected population and the chronically infected population; T, the number of treated population; R_1 and R_2, the numbers of self-cured population and cured population respectively. The equations for a new HCV transmission model with

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protection awareness in the environmental fluctuations are written as follows:

$$\begin{cases} \mathrm{d}S_u(t) = [\Lambda - \beta_e S_u(t)E(t) - \beta_a S_u(t)A(t) - \beta_c S_u(t)C(t) - \lambda S_u(t) - \mu S_u(t)]\,\mathrm{d}t + \sigma_1 S_u(t)\mathrm{d}B_1(t), \\ \mathrm{d}S_a(t) = [\lambda S_u(t) - \beta(E,A,C)S_a(t) - \mu S_a(t)]\,\mathrm{d}t + \sigma_2 S_a(t)\mathrm{d}B_2(t), \\ \mathrm{d}E(t) = [\beta(E,A,C)S_a(t) + (\beta_e E(t) + +\beta_a A(t) + \beta_c C)S_u(t) - \nu E(t) - \zeta E(t) - \mu E(t)]\,\mathrm{d}t + \sigma_3 E(t)\mathrm{d}B_3(t), \\ \mathrm{d}A(t) = [\zeta E(t) - \delta A(t) - \mu A(t)]\,\mathrm{d}t + \sigma_4 A(t)\mathrm{d}B_4(t), \\ \mathrm{d}C(t) = [\alpha\delta A(t) - \varepsilon C(t) - \eta C(t) - \mu C(t)]\,\mathrm{d}t + \sigma_5 C(t)\mathrm{d}B_5(t), \\ \mathrm{d}T(t) = [\varepsilon C(t) - \gamma T(t) - \mu T(t)]\,\mathrm{d}t + \sigma_6 T(t)\mathrm{d}B_6(t), \\ \mathrm{d}R_1(t) = [\nu E(t) + (1-\alpha)\delta A(t) - \mu R_1(t)]\,\mathrm{d}t + \sigma_7 R_1(t)\mathrm{d}B_7(t), \\ \mathrm{d}R_2(t) = [\gamma T(t) - \mu R_2(t)]\,\mathrm{d}t + \sigma_8 R_2(t)\mathrm{d}B_8(t), \end{cases} \tag{1}$$

with $\beta(E,A,C) = (1-k_e)\beta_e E(t) - (1-k_a)\beta_a A(t) - (1-k_c)\beta_c C(t)$. Here, Λ is the constant recruitment rate; β_i $(i = e, a, c)$ are the contact rates of HCV infected populations (exposed population, acutely infected population, and chronically infected population), respectively; λ is the conversion rate from S_u to S_a through education and publicity; k_i $(i = e, a, c)$ are the protection efficiencies of S_a for three HCV infected populations; ν is the rate of self-healing for exposed population; $1/\zeta$ represents the average time removing out from E; $1/\delta$ represents the average time removing out from A; α is the transfer proportion from acutely infected population to chronically infected population; ε is the transfer proportion from chronically infected population to treated population; η is the induced-death rate of the chronically infected population; $1/\gamma$ denotes the average treatment time; μ is the natural death rate of the total population. Meanwhile, the protection efficiencies k_i of S_a towards three HCV infected populations are described as three distinct values, which improves the same protection efficiencies in [9]. Let $B_i(t)\,(i = 1,2,3,4,5,6,7,8)$ be mutually independent standard Brownian motions defined on a complete probability space $\left\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P}\right\}$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all P-null sets). Let $\langle x(t)\rangle = \frac{1}{t}\int_0^t x(s)\mathrm{d}s$.

2. Positive global solution of model (1)

Theorem 1. Model (1) has a unique global positive solution $Z(t) = (S_u(t), S_a(t), E(t), A(t), C(t), T(t), R_1(t), R_2(t))^\top \in \mathbb{R}^8_+$ initiated with $Z(0) \in \mathbb{R}^8_+$ for any $t \geq 0$.

Proof. It is obvious to verify that model (1) satisfy the local Lipschitz condition. There is a unique local solution $Z(t)$ for $t \in [0, \tau_e)$. Next, we will prove that the $\tau_e = \infty$ holds almost surely. Let, $n_0 > 1$ be a sufficiently large number such that each component of $Z(t)$ all lying in $\left[\frac{1}{n_0}, n_0\right]$. For each integer $n \geqslant n_0$, define the stopping time

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{Z_i(t)\} \leq \frac{1}{n} \text{ or } \max\{Z_i(t)\} \geq n\right\} \text{for } i = 1,2,3,4,5,6,7,8,$$

where $\inf\varnothing = \infty$. Obviously, τ_n is monotonically increasing as $n \to \infty$. We denote $\tau_\infty = \lim_{n\to\infty}\tau_n$. If we show that $\tau_\infty = \infty$ holds almost surely, then the proof is complete. By contradiction, there exits a pair of constants $M > 0$ and $\varepsilon \in (0,1)$, such that $\mathbb{P}\{\tau_n \leq M\} \geq \varepsilon$ for $n \geqslant n_0$. Now, we define a C^2-function $V_1 : \mathbb{R}^8_+ \to \mathbb{R}_+$ for $m \in \mathbb{R}_+$ as follows:

$$\begin{aligned} V_1 \;=\;& S_u - m - m\ln\frac{S_u}{m} + S_a - m - m\ln\frac{S_a}{m} + E - 1 - \ln E + A - 1 - \ln A \\ &+ C - 1 - \ln C + T - 1 - \ln T + R_1 - 1 - \ln R_1 + R_2 - 1 - \ln R_2, \end{aligned}$$

by the Itô formula, we derive

$$\begin{aligned} \mathcal{L}V_1 \;\leq\;& \Lambda + (m\beta + m(1-k)\beta - \mu)(E + A + C) + 2m\mu + 6\mu + m\lambda + \nu + \zeta + \delta + \varepsilon + \eta + \gamma \\ &+ \frac{1}{2}(m\sigma_1^2 + m\sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2 + \sigma_7^2 + \sigma_8^2) \\ \leq\;& \Lambda + 2m\mu + 6\mu + m\lambda + \nu + \zeta + \delta + \varepsilon + \eta + \gamma + \frac{1}{2}(m\sigma_1^2 + m\sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2 + \sigma_7^2 + \sigma_8^2) := G, \end{aligned}$$

where $\beta = \max\{\beta_e, \beta_a, \beta_c\}$, $k = \min\{k_e, k_a, k_c\}$ and we let $m = \frac{\mu}{\beta(2-k)}$, the remaining part of the proof follows the approaches in [3, 9] and we omit them here.

3. Stationary distribution

Theorem 2. Model (1) admits a unique stationary distribution, which has the ergodic property, if

$$\mathcal{R}_0^s = \mathcal{R}_0^{s-e} + \mathcal{R}_0^{s-a} + \mathcal{R}_0^{s-c} > 1,$$

holds, the expressions of $\mathcal{R}_0^{s-e}, \mathcal{R}_0^{s-a}, \mathcal{R}_0^{s-c}$ are written as follows:

$$\mathcal{R}_0^{s-e} = \frac{\Lambda\beta_e(\mu + \frac{1}{2}\sigma_2^2 + \lambda(1-k_e))}{b_1 b_2 b_3}, \mathcal{R}_0^{s-a} = \frac{\Lambda\beta_a\zeta(\mu + \frac{1}{2}\sigma_2^2 + \lambda(1-k_a))}{b_1 b_2 b_3 b_4}, \mathcal{R}_0^{s-c} = \frac{\Lambda\beta_c\alpha\delta\zeta(\mu + \frac{1}{2}\sigma_2^2 + \lambda(1-k_c))}{b_1 b_2 b_3 b_4 b_5},$$

with $b_1 = \lambda + \mu + \frac{1}{2}\sigma_1^2, b_2 = \mu + \frac{1}{2}\sigma_2^2, b_3 = \zeta + \nu + \mu + \frac{1}{2}\sigma_3^2, b_4 = \delta + \mu + \frac{1}{2}\sigma_4^2, b_5 = \varepsilon + \eta + \mu + \frac{1}{2}\sigma_5^2$.
Proof. The diffusion matrix of model (1) is re-organized as follows

$$A = \mathrm{diag}\{\sigma_1^2 S_u^2, \sigma_2^2 S_a^2, \sigma_3^2 E^2, \sigma_4^2 A^2, \sigma_5^2 C^2, \sigma_6^2 T^2, \sigma_7^2 R_1^2, \sigma_8^2 R_2^2\},$$

we have that matrix A is positive definite. Now, we define a C^2-function K such that $\mathcal{L}K \leq -1$ for any $Z(t) \in \mathbb{R}_+^8 \backslash D$. We find a C^2-Lyapunov function $\widetilde{K} : \mathbb{R}_+^8 \to \mathbb{R}_+$:

$$\widetilde{K} = FV_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8 + V_9 + V_{10},$$

where

$$\begin{aligned}
V_2 &= -(\ln E + (c_1 + c_3 + c_4 + c_7 + c_9 + c_{13}) \ln S_u + (c_2 + c_6 + c_{12}) \ln S_a + (c_5 + c_8 + c_{10} + c_{14}) \ln A \\
&\quad + (c_{11} + c_{15}) \ln C) + \frac{Q_3}{\varepsilon + \eta + \mu} C + \left(Q_2 + \frac{Q_3 \alpha\delta}{\varepsilon + \eta + \mu}\right) \frac{A}{\delta + \mu}, \\
V_3 &= (S_u + S_a + E + A + C + T + R_1 + R_2)^{\theta+1}, V_4 = -\ln S_u, V_5 = -\ln S_a, V_6 = -\ln A, \\
V_7 &= -\ln C, V_8 = -\ln T, V_9 = -\ln R_1, V_{10} = -\ln R_2, \\
c_1 &= \frac{\Lambda\beta_e}{b_1^2}, c_2 = \frac{\Lambda\lambda\beta_e(1-k_e)}{b_1 b_2^2}, c_3 = \frac{\Lambda\lambda\beta_e(1-k_e)}{b_1^2 b_2}, c_4 = \frac{\Lambda\beta_a\zeta}{b_1^2 b_4}, c_5 = \frac{\Lambda\beta_a\zeta}{b_1 b_4^2}, c_6 = \frac{\Lambda\lambda\beta_a\zeta(1-k_a)}{b_1 b_2^2 b_4}, \\
c_7 &= \frac{\Lambda\lambda\beta_a\zeta(1-k_a)}{b_1^2 b_2 b_4}, c_8 = \frac{\Lambda\lambda\beta_a\zeta(1-k_a)}{b_1 b_2 b_4^2}, c_9 = \frac{\Lambda\beta_c\alpha\delta\zeta}{b_1^2 b_4 b_5}, c_{10} = \frac{\Lambda\beta_c\alpha\delta\zeta}{b_1 b_4^2 b_5}, c_{11} = \frac{\Lambda\beta_c\alpha\delta\zeta}{b_1 b_4 b_5^2}, c_{12} = \frac{\Lambda\lambda\beta_c(1-k_c)\alpha\delta\zeta}{b_1 b_2^2 b_4 b_5}, \\
c_{13} &= \frac{\Lambda\lambda\beta_c(1-k_c)\alpha\delta\zeta}{b_1^2 b_2 b_4 b_5}, c_{14} = \frac{\Lambda\lambda\beta_c(1-k_c)\alpha\delta\zeta}{b_1 b_2 b_4^2 b_5}, c_{15} = \frac{\Lambda\lambda\beta_c(1-k_c)\alpha\delta\zeta}{b_1 b_2 b_4 b_5^2}.
\end{aligned}$$

Here, $F > 0$ is a sufficiently large positive constant satisfying $-F\left(\nu + \zeta + \mu + \frac{1}{2}\sigma_3^2\right)(\mathcal{R}_0^s - 1) + G \leq -2$, with

$$\begin{aligned}
G &= P + \lambda + \delta + \varepsilon + \eta + \gamma + 7\mu + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2 + \sigma_7^2 + \sigma_8^2\right) + e_1 + e_2, \\
P &= \sup_{N \in \mathbb{R}_+} \left\{(\theta+1)\left(N^\theta \Lambda - \frac{1}{2}\widetilde{M} N^{\theta+1}\right)\right\} < \infty, \\
e_1 &= \sup_{A \in \mathbb{R}_+} \left\{-\frac{1}{2}(\theta+1)\widetilde{M} A^{\theta+1} + (2-k_a)\beta_a A\right\}, e_2 = \sup_{C \in \mathbb{R}_+} \left\{-\frac{1}{2}(\theta+1)\widetilde{M} C^{\theta+1} + (2-k_c)\beta_c C\right\}, \\
e_3 &= \sup_{E \in \mathbb{R}_+} \left\{-\frac{1}{2}(\theta+1)\widetilde{M} E^{\theta+1} + FQ_4 E + (2-k_e)\beta_e E\right\}, e_4 = \sup_{E \in \mathbb{R}_+} \left\{-\frac{1}{4}(\theta+1)\widetilde{M} E^{\theta+1} + FQ_4 E + (2-k_e)\beta_e E\right\},
\end{aligned}$$

and $\theta > 0$ is a sufficiently small positive constant satisfying

$$\widetilde{M} = \mu - \frac{\theta}{2}\max\{\sigma_i^2\} > 0 \text{ for } i = 1,2,3,4,5,6,7,8.$$

Obviously, $\widetilde{K}$ is a continuous function and takes minimum at the point $\overline{Z}$. We define a non-negative C^2-function

$$K = \widetilde{K} - \widetilde{K}(\overline{Z}).$$

Applying the Itô formula on V_2 :

$$\begin{aligned}
\mathcal{L}V_2 &= \Big(-\beta_e S_u - c_1\frac{\Lambda}{S_u}\Big) + c_1b_1 + \Big(-\beta_e(1-k_e)S_a - c_2\lambda\frac{S_u}{S_a} - c_3\frac{\Lambda}{S_u}\Big) + c_2b_2 + c_3b_1 + \Big(-\beta_a\frac{S_uA}{E} - c_4\frac{\Lambda}{S_u} - c_5\zeta\frac{E}{A}\Big)\\
&\quad + c_4b_1 + c_5b_4 + \Big(-\beta_a(1-k_a)\frac{S_aA}{E} - c_6\lambda\frac{S_u}{S_a} - c_7\frac{\Lambda}{S_u} - c_8\zeta\frac{E}{A}\Big) + c_6b_2 + c_7b_1 + c_8b_4\\
&\quad + \Big(-\beta_c\frac{S_uC}{E} - c_9\frac{\Lambda}{S_u} - c_{10}\zeta\frac{E}{A} - c_{11}\alpha\delta\frac{A}{C}\Big) + c_9b_1 + c_{10}b_4 + c_{11}b_5 + \Big(-\beta_c(1-k_c)\frac{S_aC}{E} - c_{12}\lambda\frac{S_u}{S_a} - c_{13}\frac{\Lambda}{S_u}\\
&\quad - c_{14}\zeta\frac{E}{A} - c_{15}\alpha\delta\frac{A}{C}\Big) + c_{12}b_2 + c_{13}b_1 + c_{14}b_4 + c_{15}b_5 + b_3 + QE\\
&\le -2(c_1\Lambda\beta_e)^{\frac{1}{2}} + c_1b_1 - 3(c_2c_3\Lambda\lambda\beta_e(1-k_e))^{\frac{1}{3}} + c_2b_2 + c_3b_1 - 3(c_4c_5\Lambda\zeta\beta_a)^{\frac{1}{3}} + c_4b_1 + c_5b_4\\
&\quad - 4(c_6c_7c_8\Lambda\lambda\zeta\beta_a(1-k_a))^{\frac{1}{4}} + c_6b_2 + c_7b_1 + c_8b_4 - 4(c_9c_{10}c_{11}\Lambda\alpha\delta\zeta\beta_c)^{\frac{1}{4}} + c_9b_1 + c_{10}b_4 + c_{11}b_5\\
&\quad - 5(c_{12}c_{13}c_{14}c_{15}\Lambda\alpha\delta\zeta\beta_c(1-k_c))^{\frac{1}{5}} + c_{12}b_2 + c_{13}b_1 + c_{14}b_4 + c_{15}b_5 + b_3 + QE\\
&= -b_3(\mathcal{R}_0^{s-e} + \mathcal{R}_0^{s-a} + \mathcal{R}_0^{s-c} - 1) + QE,
\end{aligned}$$

where

$$Q_i = (c_1 + c_2(1-k_i) + c_3 + c_4 + c_6(1-k_i) + c_7 + c_9 + c_{12}(1-k_i) + c_{13})\beta_i \ \text{ for } i = E, A, C,$$

$$Q = Q_E + \Big(Q_A + \frac{Q_C\alpha\delta}{\varepsilon+\eta+\mu}\Big)\frac{\zeta}{\delta+\mu}.$$

Similarly, we obtain that

$$\begin{aligned}
\mathcal{L}K &\le -Fb_3(\mathcal{R}_0^s - 1) + (FQ + (2-k_e)\beta_e)E + \lambda + \delta + \varepsilon + \eta + \gamma + 7\mu + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2 + \sigma_7^2 + \sigma_8^2)\\
&\quad + P - \frac{1}{2}(\theta+1)\widetilde{M}(S_u^{\theta+1} + S_a^{\theta+1} + E^{\theta+1} + A^{\theta+1} + C^{\theta+1} + T^{\theta+1} + R_1^{\theta+1} + R_2^{\theta+1}) - \frac{\Lambda}{S_u} + \beta_aA + \beta_cC\\
&\quad - \lambda\frac{S_u}{S_a} + (1-k_a)\beta_aA + (1-k_c)\beta_cC - \zeta\frac{E}{A} - \alpha\delta\frac{A}{C} - \varepsilon\frac{C}{T} - (1-\alpha)\delta\frac{A}{R_1} - \nu\frac{E}{R_1} - \gamma\frac{T}{R_2},
\end{aligned} \tag{2}$$

Now we construct a bounded set D as follows:

$$\begin{aligned}
D = \Big\{&\varepsilon_1 \le S_u \le \frac{1}{\varepsilon_1},\ \varepsilon_1^2 \le S_a \le \frac{1}{\varepsilon_1^2},\ \varepsilon_1 \le E \le \frac{1}{\varepsilon_1},\ \varepsilon_1^2 \le A \le \frac{1}{\varepsilon_1^2},\\
&\varepsilon_1^3 \le C \le \frac{1}{\varepsilon_1^3},\ \varepsilon_1^4 \le T \le \frac{1}{\varepsilon_1^4},\ \varepsilon_1^3 \le R_1 \le \frac{1}{\varepsilon_1^3},\ \varepsilon_1^5 \le R_2 \le \frac{1}{\varepsilon_1^5}\Big\},
\end{aligned}$$

where ε_1 is a sufficiently small constant satisfying the following conditions:

$$-\frac{H}{\varepsilon_1} + e_3 + G \le -1 \ \text{ with } H = \min\{\Lambda, \lambda, \zeta, \alpha\delta, \varepsilon, (1-\alpha)\delta + \nu, \gamma\}, \tag{3}$$

$$-F\Big(\zeta + k + \mu + \frac{1}{2}\sigma_3^2\Big)(\mathcal{R}_0^s - 1) + (FQ_4 + \beta_e + (1-k)\beta_e)\varepsilon_1 + G \le -1, \tag{4}$$

$$-\frac{1}{2}(\theta+1)\widetilde{M}\frac{1}{\varepsilon_1^{i(\theta+1)}} + e_3 + G \le -1 \ \text{ for } i = 1,2,3,4,5, \tag{5}$$

$$-\frac{1}{4}(\theta+1)\widetilde{M}\frac{1}{\varepsilon_1^{\theta+1}} + e_4 + G \le -1. \tag{6}$$

Next, we define sixteen bounded subregions to prove the assertion $\mathcal{L}K \le -1$ as follows:

$$\begin{aligned}
&D_1 = \Big\{Z(t) \in \mathbb{R}_+^8, 0 < S_u < \varepsilon_1\Big\},\ D_2 = \Big\{Z(t) \in \mathbb{R}_+^8, 0 < S_a < \varepsilon_1^2, S_u > \varepsilon_1\Big\}, D_3 = \Big\{Z(t) \in \mathbb{R}_+^8, 0 < E < \varepsilon_1\Big\},\\
&D_4 = \Big\{Z(t) \in \mathbb{R}_+^8, 0 < A < \varepsilon_1^2,\ E > \varepsilon_1\Big\}, D_5 = \Big\{Z(t) \in \mathbb{R}_+^8, 0 < C < \varepsilon_1^3,\ A > \varepsilon_1^2\Big\}, D_6 = \Big\{Z(t) \in \mathbb{R}_+^8, 0 < T < \varepsilon_1^4, C > \varepsilon_1^3\Big\},\\
&D_7 = \Big\{Z(t) \in \mathbb{R}_+^8, 0 < R_1 < \varepsilon_1^3, E > \varepsilon_1, A > \varepsilon_1^2,\Big\}, D_8 = \Big\{Z(t) \in \mathbb{R}_+^8, 0 < R_2 < \varepsilon_1^5, T > \varepsilon_1^4\Big\},\ D_9 = \Big\{Z(t) \in \mathbb{R}_+^8, S_u > \tfrac{1}{\varepsilon_1}\Big\},\\
&D_{10} = \Big\{Z(t) \in \mathbb{R}_+^8, S_a > \tfrac{1}{\varepsilon_1^2}\Big\}, D_{11} = \Big\{Z(t) \in \mathbb{R}_+^8, E > \tfrac{1}{\varepsilon_1}\Big\},\ D_{12} = \Big\{Z(t) \in \mathbb{R}_+^8, A > \tfrac{1}{\varepsilon_1^2}\Big\}, D_{13} = \Big\{Z(t) \in \mathbb{R}_+^8, C > \tfrac{1}{\varepsilon_1^3}\Big\},\\
&D_{14} = \Big\{Z(t) \in \mathbb{R}_+^8, T > \tfrac{1}{\varepsilon_1^4}\Big\}, D_{15} = \Big\{Z(t) \in \mathbb{R}_+^8, R_1 > \tfrac{1}{\varepsilon_1^3}\Big\},\ D_{16} = \Big\{Z(t) \in \mathbb{R}_+^8, R_2 > \tfrac{1}{\varepsilon_1^5}\Big\}.
\end{aligned}$$

Case 1. When $Z(t) \in D_1, D_2, D_4, D_5, D_6, D_7, D_8$ by (2), (3), we obtain that $\mathcal{L}K \leq -1$.
Case 2. When $Z(t) \in D_3$, by (2), (4), we derive that $\mathcal{L}K \leq -1$.
Case 3. When $Z(t) \in D_9, D_{10}, D_{12}, D_{13}, D_{14}, D_{15}, D_{16}$ by (2), (5), one finds that $\mathcal{L}K \leq -1$.
Case 4. When $Z(t) \in D_{11}$ by (2), (6), one can derive that $\mathcal{L}K \leq -1$.

Remark 1. It is easy to check that model (1) without white noises admits the basic reproduction number $\mathcal{R}_0$. Let $\sigma_i = 0$ $(i = 1, 2, 3, 4, 5, 6, 7, 8)$. Then,

$$\mathcal{R}_0^s = \mathcal{R}_0 = \frac{\Lambda\beta_e(\mu + \lambda(1-k_e))}{(\lambda+\mu)\mu(\zeta+\nu+\mu)} + \frac{\Lambda\beta_a\zeta(\mu+\lambda(1-k_a))}{(\lambda+\mu)\mu(\zeta+\nu+\mu)(\delta+\mu)} + \frac{\Lambda\beta_c\alpha\delta\zeta(\mu+\lambda(1-k_c))}{(\lambda+\mu)\mu(\zeta+\nu+\mu)(\delta+\mu)(\varepsilon+\eta+\mu)}.$$

4. Extinction

Theorem 3. For any initial value $Z(0) \in \mathbb{R}_+^8$, if

$$\mathcal{R}_0^e = (\beta_e + \beta_a + \beta_c)\Big(\frac{\Lambda}{\lambda+\mu} + \frac{(1-k)\lambda\Lambda}{\mu(\lambda+\mu)}\Big)\Big(\frac{\check{\sigma}}{3} + \mu\Big)^{-1} < 1 \text{ and } \max\{\sigma_i^2\} < 2\mu \text{ for } i = 1,2,3,4,5,6,7,8$$

are valid, then HCV undergoes the extinction with probability one, and the solution of model (1) has the following property:

$$\lim_{t\to\infty} E(t) = 0, \lim_{t\to\infty} A(t) = 0, \lim_{t\to\infty} C(t) = 0, \lim_{t\to\infty} T(t) = 0, \lim_{t\to\infty} R_1(t) = 0, \lim_{t\to\infty} R_2(t) = 0 \text{ a.s..}$$

Proof. Integrating the first and second equations of model (1) from 0 to t, and then divided by t, we give

$$\lim_{t\to\infty}\langle S_u(t)\rangle \leq \frac{\Lambda}{\lambda+\mu}, \lim_{t\to\infty}\langle S_a(t)\rangle \leq \frac{\lambda\Lambda}{\mu(\lambda+\mu)} \text{ a.s..}$$

We continue the proof using $W = E + A + C$, by the generalized Itô formula on W, we obtain

$$\begin{aligned}
\mathcal{L}\ln W &= \frac{1}{W}(\beta_e E(S_u + (1-k_e)S_a) + \beta_a A(S_u + (1-k_a)S_a) + \beta_c C(S_u + (1-k_c)S_a) \\
&\quad -\nu E - (1-\alpha)\delta A - \varepsilon C - \eta C - \mu W) - \frac{1}{2W^2}(\sigma_3^2 E^2 + \sigma_4^2 A^2 + \sigma_5^2 C^2) \\
&\leq (\beta_e + \beta_a + \beta_c)(S_u + (1-k)S_a) - \mu - \frac{1}{W^2}\Big(\nu E^2 + (1-\alpha)\delta A^2 + (\varepsilon+\eta)C^2 + \frac{\sigma_3^2}{2}E^2 + \frac{\sigma_4^2}{2}A^2 + \frac{\sigma_5^2}{2}C^2\Big) \\
&\leq (\beta_e + \beta_a + \beta_c)(S_u + (1-k)S_a) - \mu - \frac{\check{\sigma}}{3},
\end{aligned}$$

where $k = \min\{k_e, k_a, k_c\}$, $\check{\sigma} = \min\{\nu + \frac{\sigma_3^2}{2}, (1-\alpha)\delta + \frac{\sigma_4^2}{2}, \varepsilon + \eta + \frac{\sigma_5^2}{2}\}$, therefore, we have

$$\frac{1}{t}(\ln W(t) - \ln W(0)) \leq (\beta_e + \beta_a + \beta_c)[\,\langle S_u(t)\rangle + (1-k)\langle S_a(t)\rangle\,] - \mu - \frac{\check{\sigma}}{3} + \frac{M_1}{t} + \frac{M_2}{t} + \frac{M_3}{t},$$

an application of strong law of large numbers for local martingale [18], we derive

$$\lim_{t\to\infty}\frac{M_i(t)}{t} = 0 \quad \text{for } i = 1,2,3 \text{ a.s.,}$$

where $M_1 = \sigma_3\int_0^t \frac{E(s)}{W(s)}\,\mathrm{d}B_3(s)$, $M_2 = \sigma_4\int_0^t \frac{A(s)}{W(s)}\,\mathrm{d}B_4(s)$, $M_3 = \sigma_5\int_0^t \frac{C(s)}{W(s)}\,\mathrm{d}B_5(s)$. Taking the upper limit, by $\mathcal{R}_0^e < 1$, we gain

$$\limsup_{t\to\infty}\frac{\ln W(t)}{t} \leq (\beta_e + \beta_a + \beta_c)\Big(\frac{\Lambda}{\lambda+\mu} + (1-k)\frac{\lambda\Lambda}{\mu(\lambda+\mu)}\Big) - \mu - \frac{\check{\sigma}}{3} = \Big(\mathcal{R}_0^e - 1\Big)\Big(\mu + \frac{1}{3}\check{\sigma}\Big) < 0 \text{ a.s..}$$

Hence,

$$\lim_{t\to\infty} E(t) = 0, \lim_{t\to\infty} A(t) = 0, \lim_{t\to\infty} C(t) = 0 \text{ a.s..}$$

Then, we also get

$$\lim_{t\to\infty} T(t) = 0, \lim_{t\to\infty} R_1(t) = 0, \lim_{t\to\infty} R_2(t) = 0 \text{ a.s..}$$

Remark 2. In this study, we identify that the following relationships hold:

$$\mathcal{R}_0^e > \mathcal{R}_0^s,\ \mathcal{R}_0 > \mathcal{R}_0^s.$$

Moreover, we have $\mathcal{R}_0^e > \mathcal{R}_0$ when $\check{\sigma} < 3\min\{\nu, (1-\alpha)\delta, \varepsilon+\eta\}$ is valid. If $\mathcal{R}_0 > \mathcal{R}_0^s > 1$ holds, then model (1) has a unique ergodic stationary distribution. If the intensity of white noise is enough large, then the value of $\mathcal{R}_0^e$ is below one, further the extinction of HCV might occur.

Acknowledgements

F. Wei was supported by Natural Science Foundation of Fujian Province of China (2021J01621). Z. Jin was supported by National Natural Science Foundation of China (12231012). X. Mao was supported by Royal Society of Edinburgh (RSE1832) and Engineering and Physical Sciences Research Council (EP/W522521/1).

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