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METHODS

Scalable Analytic Eigenvalue Extraction Algorithm

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ABSTRACT Broadband sensor array problems can be formulated using parahermitian polynomial matrices, and the optimal solution to these problems can be based on the eigenvalue decomposition (EVD) of these matrices. An algorithm has been proposed in the past to extract analytic eigenvalues of parahermitian matrices, but it does not scale well with the temporal and spatial dimensions of the parahermitian matrix. This paper introduces a scalable analytical eigenvalue extraction algorithm for parahermitian polynomial matrices. The proposed algorithm operates in the discrete Fourier transform (DFT) domain, where an EVD is computed in each bin. Associations across bins are established based on properties of the analytic eigenvectors. The need to avoid problems with non-trivial algebraic multiplicities and control time-domain aliasing leads to an iterative algorithm that increases the DFT size until a suitable error criterion is satisfied. The algorithm can be shown to converge. Benchmarked against the existing algorithm, it performs accurately and with lower cost, and can successfully decompose matrices with dimensions much larger than previously had been feasible.

INDEX TERMS Analytic functions, algebraic multiplicities, space-time covariance, discrete Fourier transform, eigenvalue decomposition, parahermitian matrix, scalability.

I. INTRODUCTION

A parahermitian matrix $\boldsymbol{R}(z) : \mathbb{C} \to \mathbb{C}^{M\times M}$ arises for example as a space-time covariance matrix that captures the second order statistics in multichannel broadband problems [1]. It satisfies $\boldsymbol{R}(z) = \boldsymbol{R}^{\mathrm{P}}(z)$, where the parahermitian operator $\{\cdot\}^{\mathrm{P}}$ implies a Hermitian transposition and time reversal, such that $\boldsymbol{R}^{\mathrm{P}}(z) = \{\boldsymbol{R}(1/z^*)\}^{\mathrm{H}}$. In a number of applications [2], it is important to diagonalise such a matrix by means of an EVD of the type

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z), \tag{1}$$

with a diagonal parahermitian matrix $\boldsymbol{\Lambda}(z)$ containing the eigenvalues and a paraunitary $\boldsymbol{Q}(z)$ such that $\boldsymbol{Q}^{-1}(z) = \boldsymbol{Q}^{\mathrm{P}}(z)$. If $\boldsymbol{R}(z)$ is analytic in $z \in \mathbb{C}$, it has been proven that in almost all cases it is possible to achieve the decomposition in (1) with equality and with analytic factors $\boldsymbol{Q}(z)$ and $\boldsymbol{\Lambda}(z)$ [3], [5]. Further, analyticity implies smoothness and therefore can lead to factors with minimum order, motivating algorithms for such analytic decompositions.

While polynomial EVD algorithms [6], [7] will generally pursue non-analytic factors and coarse approximations of (1), methods specifically targetting smooth solutions have been reported in [8] and [9]. The approach in [8] aims for smoothness, without explicitly relying on or having awareness of analyticity. It is a clever DFT domain approach, but comes without adjustment of the DFT order, has no convergence proof, and can fail in the case of overlapping eigenvalues [9]. The approach in [9] and [10] has proven convergence towards (1), but its cost for extracting analytic eigenvalues grows factorially with the spatial dimension M, such that it is only viable for relatively small matrices, i.e. it is not scalable.

Finding decompositions of the type in (1) is important for a number of problems. This includes, for example, beamforming [11], coding, compression, and denoising [12], [13], angle of arrival estimation [14], [15], and subspace decompositions for detection of weak transient signals [9]

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for voice activity detection of weak speakers [16], [17] or for detecting primary users in a cognitive radio context [18]. These applications often require decompositions for spatial dimensions M that are higher than is currently possible with the algorithms in [9] and [10]. The feasibility of such applications depends on the precision with which (1) is approximated, the computational cost of the decomposition, as well as the order of the factors in (1), which translates into filter lengths and, therefore, the computational complexity of, e.g., a subspace projection. While the time domain approaches in [6], [7], and [12] are relatively low-cost, their accuracy is limited. They also tend to converge to spectrally majorised eigenvalues, which lead to a perturbation of subspaces [3]. These can result in poor performance and factors that require high approximation order.

The aim of this paper is therefore to present a scalable eigenvalue extraction approach based on a combination of the DFT-domain methods in [8], [9], and [10]. The challenge of operating in the DFT domain is to re-establish coherence across frequency bins. To avoid the relatively costly maximum likelihood sequence estimation in [9] based on only the eigenvalues, we utilise the idea in [8] to exploit the smoothness of the eigenvectors to align eigenvalues. However, different from [8], we formally exploit analyticity of the targeted solution for both an eigenvector-based alignment as well as for an iterative adjustment of the DFT length.

Below, Sec. II reviews the relation between the analytic EVD and a bin-wise solution. Based on an eigenvector-based bin-alignment in Sec. III, our proposed iterative scheme is outlined in Sec. IV. Sec. V demonstrates and benchmarks the algorithm, with conclusions being drawn in Sec. VI.

II. ANALYTIC VERSUS BIN-WISE EVD

For the analytic EVD in (1), the analytic eigenvalues $\lambda_m(z)$, $m = 1, \ldots, M$ in $\mathbf{\Lambda}(z) = \mathrm{diag}\{\lambda_1(z), \ldots, \lambda_M(z)\}$ are unique up to some arbitrary ordering [3]. Provided that there are M distinct eigenvalues, then their corresponding eigenvectors in $\boldsymbol{Q}(z) = [\boldsymbol{q}_1(z), \ldots, \boldsymbol{q}_M(z)]$ are also unique up to the multiplication by an arbitrary allpass $\phi_m(z)$, such that $\phi_m(z)\boldsymbol{q}_m(z)$ would also be a valid mth analytic eigenvector.

Evaluating $\boldsymbol{R}(z)$ on the unit circle at K equispaced frequency points yields $\mathbf{R}_k = \boldsymbol{R}(z)_{|z=\mathrm{e}^{\mathrm{j}\Omega_k}}$, $\Omega_k = \frac{2\pi k}{K}$, with $k = 0, \ldots, (K-1)$. Due to the parahermitian symmetry of $\boldsymbol{R}(z)$, the resulting $\mathbf{R}_k$ is a Hermitian matrix whose EVD is given as

$$\mathbf{R}_k = \mathbf{Q}_k \mathbf{\Lambda}_k \mathbf{Q}_k^{\mathrm{H}}, \qquad \text{for } k = 0, \ldots, (K-1)\,. \tag{2}$$

The diagonal matrix $\mathbf{\Lambda}_k = \mathrm{diag}\{d_{1,k}, \ldots, d_{M,k}\}$ holds the eigenvalues $d_{m,k}$, $m = 1, \ldots, M$ for the kth bin. The corresponding eigenvectors $\mathbf{q}_{m,k}$ can be selected to be mutually orthogonal, such that $\mathbf{Q}_k = [\mathbf{q}_{1,k}, \ldots, \mathbf{q}_{M,k}]$ is a unitary matrix.

The EVD in (2) is not unique. There is an ambiguity with respect to ordering of the eigenvalues and their corresponding eigenvectors. Conventionally the eigenvalues in (2) are assumed to be majorised such that $d_{i,k} \geq d_{i+1,k}, i = 1, \ldots, (M-1)$ [19]. In the case of a C-fold algebraic multiplicity, where $\lambda_{i,k} = \lambda_{i+1,k} = \ldots = \lambda_{i+C-1,k}$, if $[\mathbf{q}_{i,k}, \ldots, \mathbf{q}_{i+C-1,k}]$ are valid eigenvectors, then so are

$$[\mathbf{q}'_{i,k}, \ldots, \mathbf{q}'_{i+C-1,k}] = [\mathbf{q}_{i,k}, \ldots, \mathbf{q}_{i+C-1,k}]\mathbf{V}_{i,k}\,, \tag{3}$$

where $\mathbf{V}_{i,k} \in \mathbb{C}^{C\times C}$ is an arbitrary unitary matrix, i.e. the eigenvectors define a unique C-fold subspace but the basis is arbitrary. In the case of a trivial multiplicity with $C = 1$, this reduces to the well-known phase ambiguity of eigenvectors [19], such that $\mathrm{e}^{\mathrm{j}\phi}\mathbf{q}_{m,k}$ with an arbitrary phase $\phi \in \mathbb{R}$ would also be a valid mth eigenvector of $\mathbf{R}_k$.

Based on the above ambiguities, the analytic EVD in (1) with $z = \mathrm{e}^{\mathrm{j}\Omega_k}$ relates to the kth bin-wise EVD in (2) as

$$\mathbf{\Lambda}_k = \mathbf{P}_k \mathbf{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega_k})\mathbf{P}_k^{\mathrm{T}}\,, \tag{4}$$

$$\mathbf{Q}_k = \boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega_k})\mathbf{\Phi}_k\mathbf{P}_k^{\mathrm{T}}\mathbf{V}_k\,, \tag{5}$$

where $\mathbf{P}_k$ is a permutation matrix that reorders the eigenvalues in $\mathbf{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega_k})$ to match those of $\mathbf{\Lambda}_k$. The block-diagonal unitary matrix $\mathbf{V}_k$ contains the terms $\mathbf{V}_{i,k}$ of (3) as subblocks [9], and aligns 1−dimensional subspaces across non-trivial algebraic multiplicities; if $\mathbf{\Lambda}_k$ has no repeated eigenvalues, $\mathbf{V}_k$ is an identity [9]. Finally, $\mathbf{\Phi}_k$ is a diagonal matrix of phase shifts that establishes phase coherence between adjacent bin-wise eigenvectors [8], [9].

III. RESOLVING BIN-WISE PERMUTATIONS

A. CHALLENGE OUTLINE

In order to extract analytic eigenvalues, the task is to determine the permutation matrices $\mathbf{P}_k, k = 0, \ldots, (K-1)$, in (4) for a suitable DFT size K. We first illustrate how these permutation matrices affect the smooth associations across bins by way of an example.

Example 1: For the case of a simple $\boldsymbol{R}(z)$, Fig. 1(a) shows the analytic eigenvalues $\lambda_1(\mathrm{e}^{\mathrm{j}\Omega}) = 1 + \frac{1}{2}\sin\Omega$ and $\lambda_{2,3}(\mathrm{e}^{\mathrm{j}\Omega}) = \frac{1}{2}(1 \pm \cos\Omega)$ evaluated at $K = 16$ equispaced frequency points along the unit circle, i.e. $\lambda_m(\mathrm{e}^{\mathrm{j}\Omega_k})$ with $m = 1, 2, 3$, $\Omega_k = 2\pi k/K$, and $k = 0, \ldots, (K-1)$. The bin-wise majorised eigenvalues $\mathbf{\Lambda}_k = \mathrm{diag}\{d_{1,k}, d_{2,k}, d_{3,k}\}$ are illustrated in Fig. 1(b). The Dirichlet interpolations $d_\mu(\mathrm{e}^{\mathrm{j}\Omega})$, $\mu = 1, 2, 3$, through these sample points, which provide the shortest possible support [9], are also shown. Because of the non-trivial algebraic multiplicities in bins $k = \{4, 8, 12\}$ we have incorrect permutations for various pairs of eigenvalues w.r.t. analytic solution in bins $k = \{4, \ldots, 16\}$, and the functions $d_\mu(\mathrm{e}^{\mathrm{j}\Omega})$ lack the smoothness of the original analytic eigenvalues $\lambda_m(\mathrm{e}^{\mathrm{j}\Omega})$ in Fig. 1(a). △

B. INNER PRODUCT OF EIGENVECTORS

For distinct analytic eigenvalues, the corresponding analytic eigenvectors reside in one-dimensional eigenspaces which must evolve smoothly [10]. This has consequence for the inner product between analytic eigenvectors, which will aid us in ordering the bin-wise eigenvalues to establish a smooth association across frequency bins. With support from the

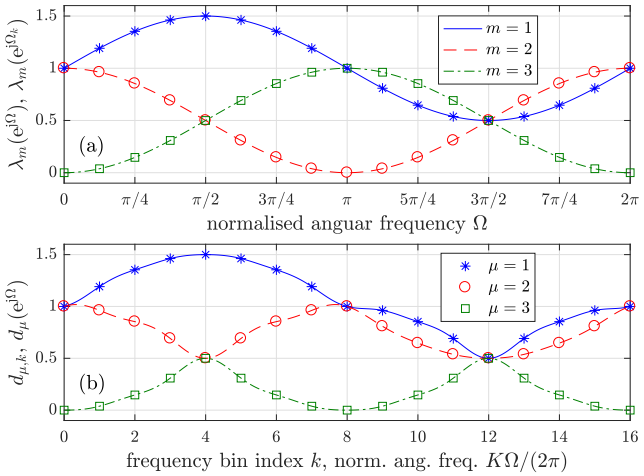

FIGURE 1. (a) Analytic eigenvalues evaluated on the unit circle in 16 frequency bins, and (b) eigenvalues obtained from the bin-wise EVD and their Dirichlet interpolations.

uniqueness theorem for analytic functions in [21], we state the following lemma.

Lemma 1 (Inner Product of Analytic Eigenvectors): If $\boldsymbol{R}(z)$ possesses M distinct analytic eigenvalues, then their corresponding analytic eigenvectors $\boldsymbol{q}_m^{\mathrm{H}}(z)$ satisfy on the unit circle

$$|\boldsymbol{q}_m^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{q}_m(\mathrm{e}^{\mathrm{j}(\Omega+\Delta\Omega)})| > |\boldsymbol{q}_m^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{q}_n(\mathrm{e}^{\mathrm{j}(\Omega+\Delta\Omega)})| \,, \tag{6}$$

for $m, n = 1, \ldots, M$ with $n \neq m$ and $|\Delta\Omega| < \Delta\Omega_{\max}$, where $\Delta\Omega_{\max}$ is some suitable upper limit.

Proof. Let us define

$$f_{m,n,\Delta\Omega}(z) = \boldsymbol{q}_m^{\mathrm{H}}(z)\boldsymbol{q}_n(z\mathrm{e}^{\mathrm{j}\Delta\Omega}) \,, \tag{7}$$

which, as a sum of products of analytic functions, is itself analytic in Ω. For $\Delta\Omega = 0$ we have $f_{m,n,0}(\mathrm{e}^{\mathrm{j}\Omega}) = \delta[m-n]$ independent of the allpass ambiguity of the analytic eigenvectors. Hence for $m \neq n$ we have $f_{m,n,0} = 0$ but $f_{m,m,0} = 1 > f_{m,n,0}$. Because of analyticity, $f_{m,n,\Delta\Omega}(\mathrm{e}^{\mathrm{j}\Omega})$ must evolve smoothly with Ω; therefore in an interval $[\Omega - \Delta\Omega;\ \Omega + \Delta\Omega]$ for a sufficiently small but non-vanishing range $|\Delta\Omega| < \Delta\Omega_{\max}$, the above inequality must be satisfied. ■

To estimate the range of values $\Delta\Omega$ over which we can apply Lemma 1, note that $f_{m,n,\Delta\Omega}(\mathrm{e}^{\mathrm{j}\Omega})$ will change very little for frequency shifts smaller than its coherence bandwidth $\Omega_{\mathrm{c}} \approx \frac{2\pi}{L_f}$ [20], where L_f is the support of $f_{m,n,\Delta\Omega}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{m,n,\Delta\Omega}(\mathrm{e}^{\mathrm{j}\Omega})\mathrm{e}^{\mathrm{j}\Omega n}\mathrm{d}\Omega$.

Therefore, the coherence bandwidth forms an approximate lower bound for $\Delta\Omega_{\max}$, such that $\Omega_{\mathrm{c}} < \Delta\Omega_{\max}$ Since $f_{m,n,\Delta\Omega}[n]$ is the convolution of two eigenvectors of support L_q, we have $L_f \leq 2L_q$. To apply Lemma 1 in a K-point DFT domain across neighbouring bins where $\Delta\Omega = \frac{2\pi}{K}$, we can ensure that

$$\Delta\Omega = \frac{2\pi}{K} \leq \frac{2\pi}{2L_q} \leq \frac{2\pi}{L_f} < \Delta\Omega_{\max} \tag{8}$$

by setting $K \geq 2L_q$, i.e. the DFT size should exceed twice the support of the analytic eigenvectors. Due to the at least exponential decay of the analytic eigenvectors [21], and for practical purposes due to e.g. quantisation noise or finite word length, there exists a finite L_q and therefore a finite K. Note that L_q can exceed the support of $\boldsymbol{R}(z)$ [3], and is not known a priori.

Theorem 1 (Inner Product of Bin-Wise Eigenvectors): Let $\mathbf{q}_{m,k}$, $m = 1, \ldots, M$ and $k = 0, \ldots, (K-1)$, be the bin-wise eigenvectors of $\boldsymbol{R}(z)$ with a sufficiently large DFT size K, such that $\frac{2\pi}{K}$ is smaller than the coherence bandwidth. Then $\mathbf{q}_{m',k}$ is associated with the same analytic eigenvector as $\mathbf{q}_{m,k-1}$ via

$$m' = \arg\max_n |\mathbf{q}_{n,k}^{\mathrm{H}}\mathbf{q}_{m,k-1}| \,, \tag{9}$$

as long as bins $k-1$ and k only contain eigenvalues with trivial algebraic multiplicities.

Proof. Assuming that eigenvalues only possess trivial algebraic multiplicities ensures that according to (5) with $\mathbf{V}_k = \mathbf{I}$, the bin-wise eigenvectors are, up to permutations and phase shifts, identical to the analytic eigenvectors. Due to the modulus operation in (9), the phase ambiguity is eliminated, and the reasoning in (8) ensures Lemma 1 applies. Hence (9) will retrieve the permutation between bin $(k-1)$ and bin k. ■

In the case that the eigenvalues have a C-fold algebraic multiplicity in a bin, the bin-wise eigenvectors can form an arbitrary basis within a C-dimensional subspace, and the simple equivalence exploited in Theorem 1 between the analytic and bin-wise eigenvectors is lost. Thus, in order to apply Theorem 1, we need to find a way to avoid bins that contain non-trivial algebraic multiplicities of eigenvalues.

C. AVOIDING BINS WITH NON-TRIVIAL ALGEBRAIC MULTIPLICITIES

In order to be able to use Theorem 1 we have to sample the analytic eigenvalues and eigenvectors at frequencies where there are no non-trivial algebraic multiplicities. Up to now we have assumed the sampling has been done via the DFT as this is computationally efficient. Since two distinct analytic functions can only have the same value at a finite number of points [21] it is likely that there are only a few frequencies where non-trivial algebraic multiplicities exist. Thus we can retain the computational efficiency of the DFT provided we deal with the set of problematic DFT bins, which we here accomplish by shifting the frequencies of these bins. This of course leads to a nonuniform DFT (NDFT) [22] albeit with a lot of uniformly distributed bins.

Let $\mathcal{S}$ be the set of bin indices where the eigenvalues possess a non-trivial algebraic multiplicity. Then for $k \in \mathcal{S}$, we find offsets $0 < q_k \ll 1$ — iteratively by small increments — such that for the evaluation of the cross-spectral density matrix at the modified frequency,

$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}(2\pi(k+q_k)/K)}) = \mathbf{Q}_k\boldsymbol{\Lambda}_k\mathbf{Q}_k^{\mathrm{H}} \tag{10}$$

possesses sufficiently distinct eigenvalues in $\boldsymbol{\Lambda}_k$.

With $|\cdot|$ the cardinality of a set, we have $|\mathcal{S}|$ sample points off the usual uniform grid yielded by a DFT matrix $\mathbf{W}_K \in \mathbb{C}^{K\times K}$. Let $\bar{\mathbf{W}}_K$ be the NDFT matrix that contains in its rows sampled complex exponentials at frequencies corresponding to all K bins. Then to reconstruct time domain quantities from such non-uniform sample points requires a inverse NDFT (INDFT) matrix $\bar{\mathbf{W}}_K^{-1}$ as opposed to a standard inverse DFT matrix $\mathbf{W}_K^{\mathrm{H}}$. Such an INDFT matrix allows a reconstruction as long as $\bar{\mathbf{W}}_K$ includes K distinct frequencies [22], [23], which is guaranteed by the condition $0 < q_k \ll 1$.

Although fast Fourier transform-related algorithms exist [22] for the efficient implementation of the NIDFT, we take a different approach. Since in practice, $|\mathcal{S}| \ll K$, we suggest the following method. Introducing a nugatory term $\mathbf{W}_K^{\mathrm{H}}\mathbf{W}_K = \mathbf{I}$, we have

$$\bar{\mathbf{W}}_K^{-1} = \left(\bar{\mathbf{W}}_K\mathbf{W}_K^{\mathrm{H}}\mathbf{W}_K\right)^{-1} = \mathbf{W}_K^{\mathrm{H}}\left(\bar{\mathbf{W}}_K\mathbf{W}_K^{\mathrm{H}}\right)^{-1}. \quad (11)$$

By introducing a permutation matrix $\mathbf{P}$ that can subdivide $\bar{\mathbf{W}}_K$ and $\mathbf{W}_K$ such that

$$\mathbf{P}\bar{\mathbf{W}}_K = \begin{bmatrix}\mathbf{W}_{\mathrm{u}}\\ \mathbf{W}_{\mathrm{n}}\end{bmatrix}, \qquad \mathbf{P}\mathbf{W}_K = \begin{bmatrix}\mathbf{W}_{\mathrm{u}}\\ \mathbf{W}_{\mathrm{u},\perp}\end{bmatrix}, \quad (12)$$

with $\mathbf{W}_{\mathrm{n}}$ containing rows corresponding to bins with non-uniform spacing whose indices belong to $\mathcal{S}$, and $\mathbf{W}_{\mathrm{u}}$ retaining rows of unmodified bin frequencies of $\bar{\mathbf{W}}_k$ that match an ordinary, uniform DFT. The matrix $\mathbf{W}_{\mathrm{u},\perp}$ contains those rows of a uniform DFT whose frequencies have been shifted with respect to $\bar{\mathbf{W}}_K$. Inserted into (11), we obtain

$$\bar{\mathbf{W}}_K^{-1} = \mathbf{W}_K^{\mathrm{H}}\left(\mathbf{P}^{\mathrm{T}}\begin{bmatrix}\mathbf{W}_{\mathrm{u}}\\ \mathbf{W}_{\mathrm{n}}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{W}_{\mathrm{u}}^{\mathrm{H}} & \mathbf{W}_{\mathrm{u},\perp}^{\mathrm{H}}\end{bmatrix}\mathbf{P}\right)^{-1} \quad (13)$$

$$= \mathbf{W}_K^{\mathrm{H}}\mathbf{P}^{\mathrm{T}}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{A} & \mathbf{B}\end{bmatrix}^{-1}\mathbf{P} \quad (14)$$

$$= \mathbf{W}_K^{\mathrm{H}}\mathbf{P}^{\mathrm{T}}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1}\end{bmatrix}\mathbf{P} = \mathbf{W}_K^{\mathrm{H}}\mathbf{C}, \quad (15)$$

with $\mathbf{A} = \mathbf{W}_{\mathrm{n}}\mathbf{W}_{\mathrm{u}}^{\mathrm{H}}$ and $\mathbf{B} = \mathbf{W}_{\mathrm{n}}\mathbf{W}_{\mathrm{u},\perp}^{\mathrm{H}}$. For the inversion of the matrix in (14), see Theorem 1.1 (p. 106) in [24]. Note that $\mathbf{B}$ is only $|\mathcal{S}| \times |\mathcal{S}|$ and hence relatively small; it is invertible if the shifted sample points are distinct.

When applying the INDFT operation in (15) to unequally spaced Fourier coefficients in some vector $\mathbf{f} \in \mathbb{C}^K$, i.e. $\mathbf{W}_k^{\mathrm{H}}\mathbf{C}\mathbf{f}$, the operation $\mathbf{C}\mathbf{f}$ only affects coefficients of bins whose indices belong to $\mathcal{S}$, and finds interpolations at the uniformly spaced bin frequencies. The subsequent multiplication with the standard IDFT matrix $\mathbf{W}_K^{\mathrm{H}}$, implementable via an inverse fast Fourier transform, then returns the time domain equivalent of $\mathbf{f}$.

D. DETERMINING BIN-WISE EIGENVALUE PERMUTATIONS

Using the approach of Sec. III-C to exclude non-trivial algebraic multiplicities and assuming a sufficiently large DFT length K, we are now free to apply Theorem 1 and in particular (9) in order to determine the permutation matrices $\mathbf{P}_k$ in (4) and (5). This is accomplished by initialising $\mathbf{P}_0 = \mathbf{I}$ for bin $k = 0$, and then recursively applying (9) for all subsequent bins $k = 1, \ldots, (K-1)$. The resulting permutations $\mathbf{P}_k$ can then be applied to the bin-wise eigenvalues according to (4). For illustration purposes, the outcome of this scheme is provided in the following example with and without avoiding bins that contain eigenvalues with non-trivial algebraic multiplicities.

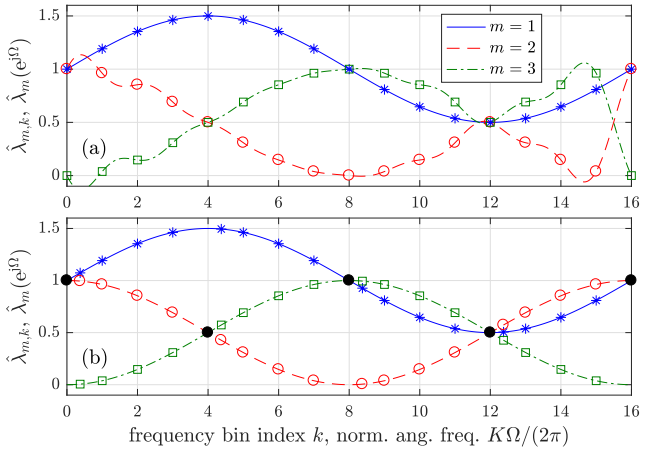

FIGURE 2. Permuted bin-wise eigenvalues of Example 1 (a) with non-trivial algebraic multiplicities in DFT bins, and (b) avoiding non-trivial algebraic multiplicities (marked as •) in modified DFT bins.

Example 2: For $\boldsymbol{R}(z)$ in Example 1, the bin-wise eigenvalues are permuted in each bin at a DFT size of $K = 16$ via the bin-wise inner product approach in (9). If only a uniform DFT is used, which is equivalent to the approach in [8], the premise of Theorem 1 is violated due to three bins containing non-trivial algebraic multiplicities. Subsequently, the method fails to extract the correct eigenvalues as seen in Fig. 2(a), due to an incorrect permutation at bin $k = 12$. Subsequently a discontinuity arises between bin $k = 15$ and $k = 16$ which is the same as $k = 0$ due to the 2π periodicity, with resulting oscillations in the Dirichlet interpolation for the shortest time-domain equivalent [9]. If the bins are modified to avoid non-trivial multiplicities via (9), the resulting re-ordered bin-wise eigenvalues $\hat{\lambda}_{m,k}^{(16)}$, shown in Fig. 2(b), are correctly permuted, and the Dirichlet interpolations $\hat{\lambda}_m^{(16)}(e^{j\Omega})$ based on $K = 16$ sample points match the exact eigenvalues $\lambda_m(e^{j\Omega})$ from Example 1. △

IV. ITERATIVE DFT ADJUSTMENT

A. SUFFICIENT DFT SIZE

A DFT size K might be sufficient to accurately approximate an analytic eigenvalue with a minimum amount of time domain aliasing, but may not result in a sufficient frequency resolution $\Delta\Omega = 2\pi/K$ for Theorem 1 to hold and vice versa. To satisfies both conditions, for an initial value of K, we compare Dirichlet interpolations based on two sets of bin-wise eigenvalues at DFT sizes $K/2$ and K ordered according

to (9). If the two interpolations are not sufficiently similar, we increase K and repeat similar to [9].

Let $\hat{\lambda}_{m,k}^{(K)}$ be the eigenvalues in the kth bin permuted according to (9) obtained from a bin-wise EVD of a K-point DFT of $\boldsymbol{R}(z)$. Assuming that K is even, the pth derivative of the Dirichlet interpolation can be determined via [9] as

$$\frac{\partial^p}{\partial\Omega^p}\hat{\lambda}_m^{(K)}(\mathrm{e}^{\mathrm{j}\Omega}) = \frac{1}{\sqrt{K}}\mathbf{e}_K^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\mathbf{D}_K^p\mathbf{O}_K\mathbf{T}_K\bar{\mathbf{W}}_K^{-1}\boldsymbol{\lambda}_m^{(K)}, \quad (16)$$

where

$$\begin{aligned}
\mathbf{e}_K^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega}) &= [\mathrm{e}^{\mathrm{j}\Omega\frac{K}{2}},\ \mathrm{e}^{\mathrm{j}\Omega(\frac{K}{2}-1)},\ldots,\mathrm{e}^{-\mathrm{j}\Omega\frac{K}{2}}]\,,\\
\mathbf{D}_K &= \mathrm{diag}\{-\mathrm{j}\tfrac{K}{2},\ldots,0,\ldots,\mathrm{j}\tfrac{K}{2}\}\,,\\
\mathbf{O}_K &= \mathrm{diag}\{\tfrac{1}{2},\underbrace{1,\ldots,1}_{K-1},\tfrac{1}{2}\}\,,\\
\mathbf{T}_K &= \begin{bmatrix}\mathbf{0}_{K/2\times K/2} & \mathbf{I}_{K/2}\\ \mathbf{I}_{K/2+1} & \mathbf{0}_{(K/2+1)\times(K/2-1)}\end{bmatrix}\,,\\
\boldsymbol{\lambda}_m^{(K)} &= [\lambda_{m,0},\ldots,\lambda_{m,(K-1)}]^{\mathrm{T}}\,.
\end{aligned}$$

Note the formulation from [9] is altered to include the INDFT $\bar{\mathbf{W}}_k^{-1}$ from (15). Based on the power in the pth derivative of the difference between Dirichlet interpolations with DFT sizes K and $K/2$, the metric is

$$\xi^p = \frac{1}{2\pi}\sum_{m=1}^{M}\int_{-\pi}^{\pi}\left|\frac{\partial^p}{\partial\Omega^p}\left(\hat{\lambda}_m^{(K)}(\mathrm{e}^{\mathrm{j}\Omega}) - \hat{\lambda}_m^{(\frac{K}{2})}(\mathrm{e}^{\mathrm{j}\Omega})\right)\right|\mathrm{d}\Omega. \quad (17)$$

With (17), we can state the following theorem:

Theorem 2 (Convergence): The permutations according to Theorem 1 will be correct iff ξ_p, for $p\to\infty$ and a sufficiently large K, falls below some sufficiently small threshold ϵ_ξ.

Proof. Please see the proof to Theorem 4 in [9], which is not affected by the introduction of the INDFT. ■

Intuitively, pursuing analytic functions that are thus infinitely differentiable motivates the inclusion of higher order derivatives. Reconsidering Example 2, derivatives will emphasise oscillations such as in Fig. 2 that occur due to incorrect bin-wise permutations. Theorem 2 guarantees that the sample points of analytic eigenvalues have been correctly assigned, and that time domain aliasing, when reconstructing a time domain solution via an INDFT, falls below the threshold ϵ_ξ [9]. The practical calculation of (17) reduces to the evaluation of a simple vector norm by exploiting Parseval's theorem [9], [10].

B. ITERATIVE PROCEDURE

The overall eigenvalue extraction can now be based on an iteration that grows the DFT length until a sufficient value of K is reached. Given the order of $\boldsymbol{R}(z)$, $\mathcal{O}\{\boldsymbol{R}(z)\}$, we start with an initial value K such that $\log_2 K = \lceil\log_2\mathcal{O}\{\boldsymbol{R}(z)\}\rceil + 1$. At each iteration, we perform a bin-wise EVD according to (2), and amend the bin frequency using (10) to ensure sufficiently distinct eigenvalues by increments of a small quantity q, $0 < q \ll \frac{2\pi}{K}$, such that the minimum eigenvalue distance is at least ϵ_λ. Permutations are evaluated via (9) in subsequent bins. At the end of each iteration, we can determine via (17) if a further iteration is required. The algorithm terminates once $\xi^p < \epsilon$, or if a maximum DFT size $K_{\max}$ has been reached. This extraction procedure is summarised in Algorithm 1. Note that it is efficient to avoid recalculating an EVD in bins where one already has been evaluated previously. Hence after the first iteration and K being powers of two, bin-wise EVDs only need to be calculated in odd-indexed bins.

Algorithm 1 Extraction of Analytic Eigenvalues

Input: $\boldsymbol{R}(z)$, p, $K_{\max}$, ϵ_λ, ϵ_ξ, q;
initialisation: $K = 2^{\lceil\log_2(\mathcal{O}[\boldsymbol{R}(z)])\rceil+1}$, $\mathbf{P}_0 = \mathbf{I}$;
determine EVDs in $K/2$ even bins with modified bin frequency if $\min_m |d_{m,k} - d_{m+1,k}| < \epsilon_\lambda$;
repeat
  determine EVDs in $K/2$ odd bins with modified bin frequency if $\min_m |d_{m,k} - d_{m+1,k}| < \epsilon_\lambda$;
  for $k = 1 : (K-1)$
    determine permutation $\mathbf{P}_k$ via (9);
  end
  calculate ξ^p for permuted eigenvalues via (17) ;
  $K \leftarrow 2K$;
until $(\xi^p < \epsilon_\xi) \vee (K > K_{\max})$;
Output: eigenvalues via INDFT of $\{\mathbf{P}_k^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{P}_k\}$.

V. SIMULATIONS AND RESULTS

A. NUMERICAL EXAMPLE

In order to demonstrate the failings of existing time domain algorithms for the problems mentioned in the introduction, we first provide a numerical subspace decomposition example for a relatively simple $\boldsymbol{R}(z) : \mathbb{C} \to \mathbb{C}^{4\times4}$. We consider a parahermitian matrix that is constructed from a diagonal matrix of eigenvalues $\boldsymbol{\Sigma}(z) = \mathrm{blockdiag}\{\boldsymbol{\Sigma}'(z), 0\} + \sigma_v^2\mathbf{I}_4$, where $\boldsymbol{\Sigma}'(z)$ contains the three eigenvalues of Example 1 as characterised in Figure 1(a), and $\sigma_v^2 = \frac{1}{4}$ is a small noise variance term. Further, a fourth order random paraunitary matrix $\boldsymbol{Q}(z)$ assembled from random first order elementary paraunitary matrices [1] is used to construct $\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Sigma}(z)\boldsymbol{Q}^{\mathrm{P}}(z) : \mathbb{C}\to\mathbb{C}^{4\times4}$.

As a benchmark, we utilise the sequential best rotation (SMD) algorithm [7], which yields a decomposition $\boldsymbol{R}(z) = \boldsymbol{U}(z)\boldsymbol{D}(z)\boldsymbol{U}^{\mathrm{P}}(z)$ with a paraunitary $\boldsymbol{U}(z)$ and eigenvalues estimates in the approximately diagonal matrix $\boldsymbol{D}(z) \approx \mathrm{diag}\{d_1(z),\ldots,d_4(z)\}$. Another time domain method, the second order sequential best rotation (SBR2) algorithm, has been proven to converge to a spectrally majorised solution [25]; while no such proof currently exists for the more advanced SMD algorithm, it is well-known to encourage spectral majorisation such that $d_m(\mathrm{e}^{\mathrm{j}\Omega}) \geq d_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})$, $\forall\Omega$ and $m = 1,\ldots,3$, as evident from the extracted eigenvalues shown in Fig. 3(a). In contrast, the proposed method yields the exact, spectrally unmajorised eigenvalues. This solution is the same as the one obtained using [9], but is achieved with significantly fewer computations, which we will quantify

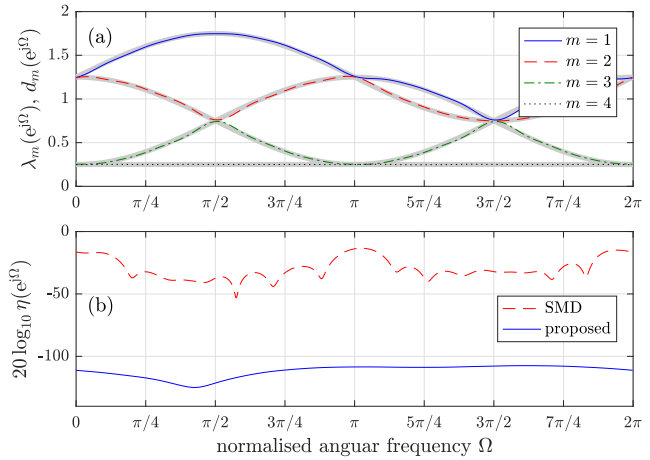

FIGURE 3. (a) Spectrally majorised eigenvalues approximated by the SMD algorithm [7], and (b) subspace leakage incurred by SMD and by the proposed algorithm with a subsequent eigenvector extraction according to [10].

in a subsequent simulation. Here, in terms of the extracted eigenvalues, the SMD algorithm's result $\boldsymbol{D}(z)$ has order 24, while proposed method yields the polynomial order 2 of the ground truth. The difference stems from the SMD in Fig. 3(a) approximating a piece-wise analytic solution. Where the joints between the segments are non-differentiable, SMD has to use a much higher order polynomial to achieve a good approximation.

To illustrate the impact of extracting spectrally majorised versus analytic eigenvalues for applications, we consider the effect of subspace leakage. For the above example, we define a subspace decomposition

$$\boldsymbol{R}(z) = \begin{bmatrix}\boldsymbol{Q}_{||}(z)\boldsymbol{Q}_{\perp}(z)\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}'(z)+\sigma_v^2\mathbf{I}_3 & \\ & \sigma_v^2\end{bmatrix}\begin{bmatrix}\boldsymbol{Q}^{\mathrm{P}}_{||}(z)\\ \boldsymbol{Q}^{\mathrm{P}}{}_{\perp}(z)\end{bmatrix} \quad (18)$$

where with the subspace partitioning, $\boldsymbol{Q}_{||}(z) : \mathbb{C} \to \mathbb{C}^{4\times 3}$ and $\boldsymbol{Q}_{\perp}(z) : \mathbb{C} \to \mathbb{C}^{4\times 1}$. While $\boldsymbol{Q}^{\mathrm{P}}{}_{||}(z)\boldsymbol{Q}_{\perp}(z) = \underline{0}$, with SMD we find that $\eta(e^{j\Omega}) = \|\boldsymbol{U}^{\mathrm{P}}{}_{||}(e^{j\Omega})\boldsymbol{Q}_{\perp}(e^{j\Omega})\|_2 \neq 0$. The subspace mismatch or subspace leakage $\eta(e^{j\Omega})$ for the SMD algorithm is characterised in Fig. 3(b). The piece-wise analytic solution to which SMD converges means that eigenvectors aim to approximate discontinuous functions, leading to a high approximation order of 1070 despite trimming [26] and to large subspace leakage, which peaks at frequencies where a small eigenvalue distance can cause large subspace perturbations [27]. In contrast, feeding the eigenvalues extracted with the proposed method into an eigenvector extraction algorithm [10] yields very close approximations $\hat{\boldsymbol{Q}}(z)$ of the eigenvector matrices with the correct ground-truth order of 4. In this case, the leakage error $e(e^{j\Omega}) = \|\hat{\boldsymbol{Q}}^{\mathrm{P}}{}_{||}(e^{j\Omega})\boldsymbol{Q}_{\perp}(e^{j\Omega})\|_2 \neq 0$ is much smaller and controlled by algorithm-internal thresholds [10].

Both in terms of accuracy and computational complexity of an implementation, where SMD yields an order for $\boldsymbol{U}(z)$ of 1070 while the proposed method gives an order of 4 for $\hat{\boldsymbol{Q}}(z)$, the analytic eigenvalue extraction as in [9] and the proposed approach significantly outperform time domain methods such as SMD. Thus, DFT domain methods are better suited where subspace-based applications such as [9], [17], and [18] rely on a precise identification of signal or noise subspaces. For this reason, in the following, we drop comparisons to time domain methods and will restrict our comparison of the proposed method to the analytic eigenvalue extraction in [9].

B. ENSEMBLE TEST

We evaluate the proposed algorithm using parahermitian matrices of dimensions $M = \{2^2, \ldots, 2^7\}$. These are constructed via a source model discussed in [10], which consists of eigenvalues of order $\mathcal{O}\{\boldsymbol{\Lambda}(z)\} = \{2^1, \cdots, 2^6\}$ that are spectrally unmajorised and based on innovation filters [28], and of random paraunitary matrices [1] that match the order of the innovation filters. By constructing $\boldsymbol{R}(z)$ via (1), we know its ground truth analytic EVD. For each parameter setting, we generate 100 randomised para-Hermitian matrices, and, where possible, compare the results against the state-of-the-art algorithm in [9] as a benchmark. The proposed approach adjusts bin frequencies to satisfy a minimum eigenvalue distance $\epsilon_\lambda = 10^{-6}$. In all cases, both algorithms were terminated by the precision criterion, i.e. by ξ^p in (17) or by its uniform DFT-based equivalent in [9], falling below a threshold of $\epsilon_\xi = 10^{-5}$ for $p = 3$; a maximum DFT size of $K_{\max} = 2^{11}$ was not reached due to the finite support of both $\boldsymbol{R}(z)$ and $\boldsymbol{\Lambda}(z)$. The accuracy of the extracted eigenvalues was measured through

$$\zeta = \sum_{\tau} \|\boldsymbol{\Lambda}[\tau] - \hat{\boldsymbol{\Lambda}}[\tau]\|_{\mathrm{F}}^2 / \sum_{\tau} \|\boldsymbol{\Lambda}[\tau]\|_{\mathrm{F}}^2 .$$

For every set of parameters i.e. M and $\mathcal{O}\{\boldsymbol{\Lambda}(z)\}$, we observe a mean $\bar{\zeta} < 10^{-30}$ which indicates that both algorithms produce the correct eigenvalues.

In terms of scalability, we measure the execution time under Matlab R2024a on an Intel Core i5-8250U CPU @ 1.60GHz, with results shown in Fig. 4. For the benchmark approach in [9], where the algorithm's complexity grows with $M!$, it is only practical to evaluate matrices of spatial dimension $M = 4$. Fitting a straight line to the curve (as underlaid in grey) suggests a proportionality to $(\mathcal{O}\{\boldsymbol{\Lambda}(z)\})^{1.5}$. In contrast, the execution time of the proposed method, despite operating with much higher values of M, is much lower compared to the benchmark [9]. The straight line fits, underlaid in grey in Fig. 4, possess a slope equivalent to a proportionality to $\mathcal{O}\{\boldsymbol{\Lambda}(z)\}$. Since in terms of the spatial dimension the complexity of the proposed algorithm will be dominated by EVD calculations of the order of M^3 for large values of M, overall our method seems to have a complexity proportionate to $M^3(\mathcal{O}\{\boldsymbol{\Lambda}(z)\})$, whereas the benchmark is approximately proportionate to $M!(\mathcal{O}\{\boldsymbol{\Lambda}(z)\})^{1.5}$. Hence compared to the benchmark, without sacrificing accuracy, the computational cost of the proposed method scales much slower in terms of both spatial and temporal dimensions.

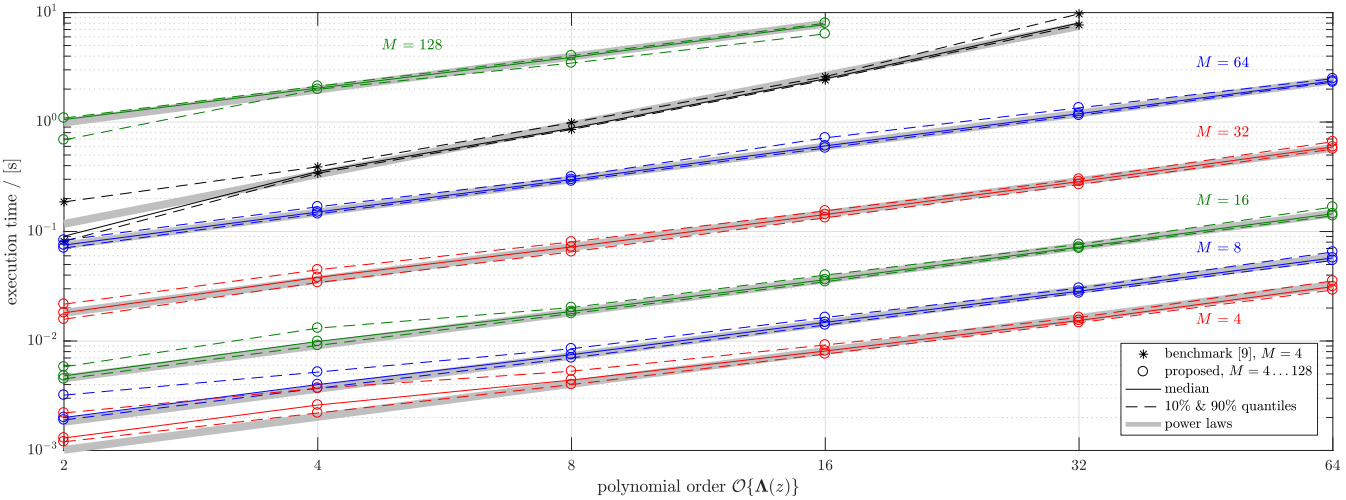

FIGURE 4. Execution time statistics for matrices with various spatial dimensions M and polynomial orders of ground truth eigenvalues for the method and the benchmark [9].

VI. CONCLUSION

The current state-of-the-art algorithm for extracting analytic eigenvalues scales poorly particularly with the spatial dimension of the matrix to be factorised, hence limiting its applicability to systems of very modest dimensions. We have therefore proposed a new, efficient analytic eigenvalue extraction algorithm that scales well in relation to both the spatial and temporal dimensions of the parahermitian matrix to be factorised. Our proposed method operates in the DFT domain, whereby bin frequencies are shifted if eigenvalues are not sufficiently separated. In the — potentially non-equispaced — resulting bins, any ambiguity in the bin-wise eigenvectors is reduced to a permutation which can be resolved based on the the analyticity of the pursued solution. This then drives an algorithm that iteratively increases the DFT size until a criterion on the difference between subsequent extractions is sufficiently minimised. This method has been proven to converge to the analytic eigenvalues, and in simulations has demonstrated a significant reduction in execution time and scalability to previously unfeasible matrix dimensions.

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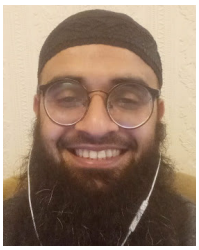

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