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Stochastic Comparisons for Finite Mixtures from Location-scale Family of Distributions

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Abstract
In this study, we consider two finite mixture models (FMMs) with location-scale family distributed components, in which ordering results are established in various stochastic senses. For heterogeneity in one parameter, the comparisons are obtained with respect to usual stochastic order, hazard rate order, reversed hazard rate order and likelihood ratio order. Further, for heterogeneity in two parameters, we derive sufficient conditions for the stochastic comparison of FMMs with respect to usual stochastic order and hazard rate order. Various examples and counterexamples are presented to illustrate the proposed results.

Keywords Finite mixture models · Location-scale family · Stochastic orders · Majorization · T-transform matrix

Mathematics Subject Classification (2010) 60E15

1 Introduction

In different fields of research, such as biology, reliability and survival analysis, finite mixture models (FMMs) have been widely employed, and thus have received a considerable interest from both theorists and practitioners. Mixture models (MM) allow to model heterogeneous data, whose pattern can not be designed by a single parametric distribution. To model such heterogeneity, a number of homogeneous subpopulations are mixed via some latent, unknown parameter, which is considered as a random variable. Throughout this paper, we call the corresponding distribution the mixing proportions. There are various situations, where FMMs

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appear naturally. Below, we present few of them. One may refer to Lindsay (1995); McLachlan and Peel (2000) and Finkelstein (2008) for various applications of FMMs.

- In biological science, assume that a biological population made of a single species has reached equilibrium. The random variations between individuals are completely attributed to cumulative effect of many minor factors. Then, according to the classical central limit theorem, the resulting uncertainty can be well approximated by normal distribution. However, if the data sets show non-normality, a FMM can be considered as a natural alternative.
- In reliability theory, MMs are also useful in analyzing failure rate data. There are situations where some components of a product are produced over a period of time by collecting items from different vendors, using different raw materials, machines, and manpower. In such situations, mixtures of distributions provide a useful and effective tool for modeling reliability data that come from these mixed populations.
- Furthermore, population failure rate can exhibit different from subpopulations failure rates monotonicity properties. It is well known, e.g., that the mixture of supbopulations with constant failure rates is decreasing, whereas the mixture of subpopulations with increasing power (Wibull) failure rates first increases and then decreasing. See more details in Finkelstein (2008).

In this paper, we have considered two FMMs for location-scale family of distributions. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a random vector with n number of components assuming that the ith component is from ith subpopulation. Denote the survival function (sf), cumulative distribution function (cdf) and probability density function (pdf) of the ith random variable (RV) X_i, $i = 1, \ldots, n$ by $\bar{F}_{X_i}(.)$, $F_{X_i}(.)$, and $f_{X_i}(.)$, respectively. Then, the sf and the pdf of a mixture random variable (MRV) for $(X_1, \ldots, X_n)$ are

$$\bar{F}_M(x) = \sum_{i=1}^{n} r_i \bar{F}_{X_i}(x) \text{ and } f_M(x) = \sum_{i=1}^{n} r_i f_{X_i}(x), \tag{1}$$

respectively, where r_i's are known as the mixing proportions such that $\sum_{i=1}^{n} r_i = 1$. We consider the location-scale family of distributions as baseline for our study. A RV X belongs to the location-scale family of distributions if its cdf is given by

$$K(x) = F\left(\frac{x-\sigma}{\lambda}\right), \quad x > \sigma, \tag{2}$$

where $\sigma \in \mathbb{R}$ is known as the location parameter and λ is known as the scale parameter. Here, $F(.)$ is the baseline cdf of X. In this paper, we have considered nonnegative RVs, thus σ must be greater than or equal to 0. It is worth mentioning that introduction of location parameter in a family of distributions has various interpretations. For example, in reliability and life testing studies, failure of an aging item practically becomes non-negligible only after a certain time $t = \sigma > 0$. In the area of supply chain management, we have a non-zero threshold value, from which the lead time of a product starts. Further, in insurance, for a particular medical insurance policy, the claim by an individual may not be started just after its activation. The claim is usually initiated after a certain time, for example one year.

Stochastic comparisons between two RVs described by the relevant mixture distributions were discussed in quite a number of publications. For convenience, in this paper, we will call it comparisons between the corresponding FMMS. For example, two FMMs for different semiparametric families of distributions have been compared in various stochastic senses by Hazra and Finkelstein (2018). Nadeb and Torabi (2022) compared two FMMs using usual stochastic order when the subpopulations follow a general class of distributions. Barmalzan et al. (2021) considered two finite α-MMs and proposed sufficient conditions for the comparison of two α-MRVs. We recall that the α MM is a generalization of the classical FMM. For some properties of the α-MM, please refer to Asadi et al. (2018). Sattari et al. (2021) considered MMs with generalized Lehmann distributed components and proposed various ordering results. Two FMMs with location-scale family distributed components have been considered by Barmalzan et al. (2022) and some stochastic comparison results between them have been established with respect to usual stochastic order and reversed hazard rate order. Kayal et al. (2023) considered two finite mixture models with general distributed components and obtained various ordering results. Panja et al. (2022) established several ordering results between two finite MRVs, where the mixing components follow proportional odds, proportional hazards, and proportional reversed hazards models.

Along this line of research, our paper focuses on further development of ordering properties for the FMMs with location-scale family distributed components. Various new ordering results between two FMMs have been obtained for heterogeneity in one and two parameters. We note that, some of the established ordering results (for two parameters) are in the same vein as in Barmalzan et al. (2022). However, our sufficient conditions for these results are different (assumptions on the baseline distribution and the corresponding majorization), which obviously requires different proofs and analysis. Moreover, we had discussed also relevant orderings between the corresponding MRVs in the sense of the likelihood ratio, which was not studied in this paper.

The rest of the paper is rolled out as follows. The next section recalls basic definitions of stochastic orders and preliminary lemmas. Section 3 presents the main results of this paper. It is divided into two subsections. In Section 3.1, we have established usual stochastic order, hazard rate order, reversed hazard rate order, and likelihood ratio order. In Section 3.2, ordering results have been established by considering heterogeneity in two parameters with respect to usual stochastic order and hazard rate order. Finally, in Section 4, some concluding remarks are added.

2 Preliminaries

Throughout the paper, the terms "increasing" is used for "nondecreasing" and "decreasing" is used for "nonincreasing". Also, we use "$\overset{sign}{=}$" to denote that both sides of the equality have the same sign. For any differentiable function $\xi(\cdot)$, we write $\xi'(x)$ to represent the first order derivative of $\xi(x)$ with respect to x. Let X and Y be two nonnegative absolutely continuous RVs with (i) cdfs $F_X(\cdot)$ and $G_Y(\cdot)$, (ii) pdfs $f_X(\cdot)$ and $g_Y(\cdot)$, (iii) sfs $\bar{F}_X(\cdot) \equiv 1 - F_X(\cdot)$ and $\bar{G}_Y(\cdot) \equiv 1 - G_Y(\cdot)$, (iv) right continuous inverses (quantile functions) $F_X^{-1}(\cdot)$ and $G_Y^{-1}(\cdot)$, (v) hazard rate functions $h_X(\cdot) \equiv f_X(\cdot)/\bar{F}_X(\cdot)$ and $h_Y(\cdot) \equiv g_Y(\cdot)/\bar{G}_Y(\cdot)$, and (vi) reversed hazard rate functions $\tilde{h}_X(\cdot) \equiv f_X(\cdot)/F_X(\cdot)$ and $\tilde{h}_Y(\cdot) \equiv g_Y(\cdot)/G_Y(\cdot)$, respectively. Below, we introduce some notations which will be used throughout the rest of this paper.

Notation 1
- $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$;
- $\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \ldots, x_n) : x_i \in \mathbb{R}, \forall\, i\} = n$-dimensional Euclidean space, $\mathbb{R}_+^n = [0, +\infty)^n$;
- $\mathcal{D}_n = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n\}$;
- $\mathcal{D}_n^+ = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n > 0\}$;
- $\mathcal{E}_n = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq \ldots \leq x_n\}$;
- $\mathcal{E}_n^+ = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_1 \leq \ldots \leq x_n\}$.

Next, we present definitions of various stochastic orders.

Definition 1 A RV X is said to be smaller than another RV Y in the sense of

- usual stochastic order (denoted by $X \leq_{st} Y$) if $F_X(x) \geq G_Y(x)$ for all $x \in \mathbb{R}_+$; or equivalently, $X \leq_{st} Y$ if $\bar{F}_X(x) \leq \bar{G}_Y(x)$ for all $x \in \mathbb{R}_+$;
- hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}_+$; or equivalently, $X \leq_{hr} Y$ if $r_X(x) \geq r_Y(x)$ for all $x \in \mathbb{R}_+$;
- reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $G_Y(x)/F_X(x)$ is increasing in $x \in \mathbb{R}_+$; or equivalently, $X \leq_{rh} Y$ if $\tilde{r}_X(x) \leq \tilde{r}_Y(x)$ for all $x \in \mathbb{R}+$;
- likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g_Y(x)/f_X(x)$ is increasing in $x \in \mathbb{R}_+$;

It is important to note that the following chains of implications hold:

$$X \leq_{st} Y \Leftarrow X \leq_{hr} Y \Leftarrow X \leq_{lr} Y \Rightarrow X \leq_{rh} Y \Rightarrow X \leq_{st} Y.$$

For various stochastic orders and their applications, one may refer to Shaked and Shanthikumar (2007). Next, we describe the concept of majorization and related orders. For any two real-valued vectors $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, assume that $u_{(1)} \leq \ldots \leq u_{(n)}$ and $v_{(1)} \leq \ldots \leq v_{(n)}$ are the respective increasing arrangements of the components.

Definition 2 The vector $\boldsymbol{u}$ is said to be

- majorized by the vector $\boldsymbol{v}$ (denoted by $\boldsymbol{u} \overset{m}{\preccurlyeq} \boldsymbol{v}$) if $\sum_{i=1}^{j} u_{(i)} \geq \sum_{i=1}^{j} v_{(i)}$, for all $j = 1, \ldots, n-1$, and $\sum_{i=1}^{n} u_{(i)} = \sum_{i=1}^{n} v_{(i)}$;
- weakly supermajorized by the vector $\boldsymbol{v}$ (denoted by $\boldsymbol{u} \overset{w}{\preccurlyeq} \boldsymbol{v}$) if $\sum_{i=1}^{j} u_{(i)} \geq \sum_{i=1}^{j} v_{(i)}$, for all $j = 1, \ldots, n$;
- weakly submajorized by the vector $\boldsymbol{v}$ (denoted by $\boldsymbol{u} \preccurlyeq_{w} \boldsymbol{v}$) if $\sum_{i=j}^{n} u_{(i)} \leq \sum_{i=j}^{n} v_{(i)}$, for all $j = 1, \ldots, n$;
- p-larger than the vector $\boldsymbol{v}$ (denoted by $\boldsymbol{u} \overset{p}{\succcurlyeq} \boldsymbol{v}$) if $\prod_{i=1}^{j} u_{(i)} \leq \prod_{i=1}^{j} v_{(i)}$, for all $j = 1, \ldots, n$;
- reciprocally majorized by the vector $\boldsymbol{v}$ (denoted by $\boldsymbol{u} \overset{rm}{\preccurlyeq} \boldsymbol{v}$) if $\sum_{i=1}^{j} \frac{1}{u_{(i)}} \leq \sum_{i=1}^{j} \frac{1}{v_{(i)}}$, for all $j = 1, \ldots, n$.

It is well known that

$$\boldsymbol{u} \preccurlyeq_{w} \boldsymbol{v} \Leftarrow \boldsymbol{u} \overset{m}{\preccurlyeq} \boldsymbol{v} \Rightarrow \boldsymbol{u} \overset{w}{\preccurlyeq} \boldsymbol{v} \Rightarrow \boldsymbol{u} \overset{p}{\preccurlyeq} \boldsymbol{v} \Rightarrow \boldsymbol{u} \overset{rm}{\preccurlyeq} \boldsymbol{v}.$$

For more details on majorization, related orders, and their applications, one may refer to Marshall et al. (2011). Next, we present a definition, which shows that the Schur-convex function preserves the ordering of majorization.

Definition 3 A real-valued function φ, defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$, is said to be Schur-convex (Schur-concave) on $\mathcal{A}$ if and only if $\boldsymbol{u} \overset{m}{\preccurlyeq} \boldsymbol{v}$ implies $\varphi(\boldsymbol{u}) \leq (\geq)\ \varphi(\boldsymbol{v})$, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}$.

Next, we present a lemma which will be used in proving the main results of this paper.

Lemma 1 (Theorem 3.A.8, Marshall et al. 2011) *A real valued function Ω on $\mathbb{R}^n$, satisfies*

$$\boldsymbol{u} \overset{w}{\preccurlyeq} \boldsymbol{v} \Rightarrow \Omega(\boldsymbol{u}) \leq (\geq)\, \Omega(\boldsymbol{v}),$$

if and only if Ω is decreasing and Schur-convex (Schur-concave) on $\mathbb{R}^n$. Similarly, Ω satisfies

$$\boldsymbol{u} \preccurlyeq_w \boldsymbol{v} \Rightarrow \Omega(\boldsymbol{u}) \leq (\geq)\, \Omega(\boldsymbol{v}),$$

if and only if Ω is increasing and Schur-convex (Schur-concave) on $\mathbb{R}^n$.

The following lemma will be used in order to established some results of this paper.

Lemma 2 (Hazra et al. 2017) *Let $\mathcal{T} \subseteq \mathbb{R}^n_+$. Further, let $\phi : \mathcal{T} \to \mathbb{R}$ be a function. Then, for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{T}$, $\boldsymbol{u} \overset{rm}{\succcurlyeq} \boldsymbol{v} \Rightarrow \phi(\boldsymbol{u}) \geq (\leq)\, \phi(\boldsymbol{v})$ if and only if the following two conditions hold:*

(i) $\phi(1/a_1, \ldots, 1/a_n)$ is Schur-convex (Schur-concave) in $(a_1, \ldots, a_n)$;
(ii) $\phi(1/a_1, \ldots, 1/a_n)$ is increasing (decreasing) in a_i for each $i \in \{1, \ldots, n\}$,

where $a_1 = 1/u_1, \ldots, a_n = 1/u_n$.

The following lemma is useful in proving few results of this paper.

Lemma 3 (Lemma 2, Hazra et al. 2017) *Let $\mathcal{A} \subseteq \mathbb{R}^n$. For $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathcal{A}$ and $\boldsymbol{v} = (u, \ldots, u) \in \mathcal{A}$, the following results hold true:*

(i) $\boldsymbol{u} \overset{m}{\succcurlyeq} \boldsymbol{v}$ if and only if $u = \sum_{i=1}^n u_i/n$;
(ii) $\boldsymbol{u} \overset{w}{\succcurlyeq} \boldsymbol{v}$ if and only if $u \geq \sum_{i=1}^n u_i/n$;
(iii) $\boldsymbol{u} \succcurlyeq_w \boldsymbol{v}$ if and only if $u \leq \sum_{i=1}^n u_i/n$;
(iv) $\boldsymbol{u} \overset{p}{\succcurlyeq} \boldsymbol{v}$ if and only if $u \geq (\prod_{i=1}^n u_i)^{\frac{1}{n}}$;
(v) $\boldsymbol{u} \overset{rm}{\succcurlyeq} \boldsymbol{v}$ if and only if $u \geq n/\sum_{i=1}^n 1/u_i$.

The following lemma is useful in proving few results in this paper.

Lemma 4 (Lemma 2, Hazra et al. 2017) *Let $\eta : \mathcal{D}_n \to \mathbb{R}$ be a function, continuously differentiable on the interior of $\mathcal{D}_n$. Then, for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{D}_n$,*

$$\boldsymbol{u} \overset{m}{\succcurlyeq} \boldsymbol{v} \Rightarrow \eta(\boldsymbol{u}) \geq (\leq)\, \eta(\boldsymbol{v}),$$

if and only if, $\eta_{(k)}(\boldsymbol{z})$ is decreasing (increasing) in $k \in \{1, \ldots, n\}$, where $\boldsymbol{z} = (z_1, \ldots, z_n)$ and $\eta_{(k)}(\boldsymbol{z}) = \partial\eta(\boldsymbol{z})/\partial z_k$ denotes the partial derivative of η with respect to its kth argument.

Next, we state a lemma which is useful to obtain some results of this paper.

Lemma 5 (Lemma 3, Hazra et al. 2017) *Let $\Xi : \mathcal{E}_n \to \mathbb{R}$ be a function, continuously differentiable on the interior of $\mathcal{E}_n$. Then, for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{E}_n$,*

$$\boldsymbol{u} \overset{m}{\succcurlyeq} \boldsymbol{v} \Rightarrow \Xi(\boldsymbol{u}) \geq (\leq)\, \Xi(\boldsymbol{v}),$$

if and only if, $\Xi_{(k)}(\boldsymbol{z})$ is increasing (decreasing) in $k \in \{1, \ldots, n\}$, where $\boldsymbol{z} = (z_1, \ldots, z_n)$ and $\Xi_{(k)}(\boldsymbol{z}) = \partial\Xi(\boldsymbol{z})/\partial z_k$ denotes the partial derivative of Ξ with respect to its kth argument.

The next lemma gives necessary and sufficient conditions for determining Schur-convex and Schur-concave functions on the spaces $\mathcal{D}_n$ and $\mathcal{E}_n$.

Lemma 6 (Lemma 1, Haidari et al. 2019)

(i) *Suppose the function $\xi : \mathcal{D}_n \to \mathbb{R}$ is continuous on $\mathcal{D}_n$ and continuously differentiable on the interior of $\mathcal{D}_n$. Then, ξ is Schur-convex (Schur-concave) on $\mathcal{D}_n$ if and only if $\xi_{(k)}(\boldsymbol{u})$ is decreasing (increasing) in $k \in \{1, \ldots, n\}$, for all $\boldsymbol{u}$ in the interior of $\mathcal{D}_n$, where $\xi_{(k)}(\boldsymbol{u}) = \partial\xi(\boldsymbol{u})/\partial u_k$.*

(ii) *Suppose the function $\zeta : \mathcal{E}_n \to \mathbb{R}$ is continuous on $\mathcal{E}_n$ and continuously differentiable on the interior of $\mathcal{E}_n$. Then, ζ is Schur-convex (Schur-concave) on $\mathcal{E}_n$ if and only if $\zeta_{(k)}(\boldsymbol{u})$ is increasing (decreasing) in $k \in \{1, \ldots, n\}$, for all $\boldsymbol{u}$ in the interior of $\mathcal{E}_n$.*

Remark 1 If the spaces $\mathcal{D}_n$ and $\mathcal{E}_n$ are replaced by the spaces $\mathcal{D}_n^+$ and $\mathcal{E}_n^+$, then the stated conclusions in Lemma 6 hold true.

Definition 4 Let $P = \{p_{ij}\}$ and $Q = \{q_{ij}\}$ be two $m \times n$ matrices. Further, let $p_1^R, \ldots, p_m^R$ and $q_1^R, \ldots, q_m^R$ be the rows of P and Q respectively in such a way that each of these quantities is a row vector of length n. Then, P is said to be

- chain majorized by Q (denoted by $P \gg Q$) if there exists a finite number of $n \times n$ T-transform matrices $T_{\omega_1}, \ldots, T_{\omega_k}$ such that $Q = PT_{\omega_1} \ldots T_{\omega_k}$.

A T-transform matrix has the form $T = \rho I + (1-\rho)\Pi$, where $0 \le \rho \le 1$, I is an identity matrix, and Π is a permutation matrix that just interchanges two coordinates, that is, row and column.

Set

$$\mathcal{M}_n = \left\{(\boldsymbol{u}, \boldsymbol{v}) = \begin{pmatrix} u_1 \ldots u_n \\ v_1 \ldots v_n \end{pmatrix} : u_i,\ v_j > 0 \text{ and } (u_i - u_j)(v_i - v_j) \le 0,\ i,\ j = 1, \ldots, n\right\}.$$

Lemma 7 (Marshall et al. 2011, Chapter 15) *A differentiable function $\varphi : \mathbb{R}_4^+ \to \mathbb{R}^+$ satisfies*

$$\varphi(P) \ge (\le)\ \varphi(Q) \text{ for all } P,\ Q \text{ such that } P \in \mathcal{M}_2, \text{ and } P \gg Q, \tag{3}$$

if and only if

(i) $\varphi(P) = \varphi(P\Pi)$ *for all permutation matrices Π, and for all $P \in \mathcal{M}_2$ and;*

(ii) $\sum_{i=1}^{2}(p_{ik} - p_{ij})[\varphi_{ik}(P) - \varphi_{ij}(P)] \ge (\le)\ 0$ *for all $j,\ k = 1, 2$ and for all $P \in \mathcal{M}_2$, where $\varphi_{ij}(P) = \frac{\partial\varphi(P)}{\partial p_{ij}}$.*

Proof The proof is similar to the proof of Theorem 2 of Balakrishnan et al. (2015), and thus it is omitted. □

3 Main Results

In this section, we present the main results of the paper. The aim of this section is two-fold: establishing results when there is heterogeneity in one parameter and in two parameters. Note that some comments on the three-parameters case will be given in Section 4.

3.1 Heterogeneity in One Parameter

In this subsection, we present various stochastic ordering results between two MRVs in the sense of usual stochastic, hazard rate, reversed hazard rate, and likelihood ratio orders.

Assume that the sub-populations are modelled by location-scale family of distributions. In the following theorem, we establish weakly supermajorization order-based sufficient conditions to show usual stochastic order between the MRVs $U_n(\boldsymbol{r};\boldsymbol{\lambda})$ and $V_n(\boldsymbol{r};\boldsymbol{\theta})$, where $\boldsymbol{r}=(r_1,\ldots,r_n)$, $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ and $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_n)$. Here, different scale parameters, same mixing proportions and fixed location parameters are considered.

Theorem 1 *Let $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)=\sum_{i=1}^n r_i\bar{F}(\frac{x-\sigma}{\lambda_i})$ and $\bar{F}_{V_n(\boldsymbol{r};\boldsymbol{\theta})}(x)=\sum_{i=1}^n r_i\bar{F}(\frac{x-\sigma}{\theta_i})$ be the sfs of two MRVs $U_n(\boldsymbol{r};\boldsymbol{\lambda})$ and $V_n(\boldsymbol{r};\boldsymbol{\theta})$, respectively, where $x\geq\sigma$. Further, suppose $tf(t)$ is decreasing in $t>0$. Then, for $\boldsymbol{r},\ \boldsymbol{\lambda}\in\mathcal{E}_n^+$ (or $\mathcal{D}_n^+$), and fixed $\sigma>0$, we have*

$$\frac{\mathbf{1}}{\boldsymbol{\lambda}}\overset{w}{\preccurlyeq}\frac{\mathbf{1}}{\boldsymbol{\theta}}\Rightarrow U_n(\boldsymbol{r};\boldsymbol{\lambda})\leq_{st}V_n(\boldsymbol{r};\boldsymbol{\theta}),$$

where $\frac{\mathbf{1}}{\boldsymbol{\lambda}}=\left(\frac{1}{\lambda_1},\ldots,\frac{1}{\lambda_n}\right)$ and $\frac{\mathbf{1}}{\boldsymbol{\theta}}=\left(\frac{1}{\theta_1},\ldots,\frac{1}{\theta_n}\right)$.

Proof The desired result will be established, if the conditions of Lemma 1 are satisfied. Under the setting, the sf of $U_n(\boldsymbol{r};\boldsymbol{\lambda})$ is given by

$$\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)=\sum_{i=1}^n r_i\bar{F}\left(\frac{x-\sigma}{\lambda_i}\right)=\Omega\left(\frac{\mathbf{1}}{\boldsymbol{\alpha}}\right),\ \text{(say)},\tag{4}$$

where $\alpha_i=\frac{1}{\lambda_i}, i=1,\ldots,n$ and $\frac{\mathbf{1}}{\boldsymbol{\alpha}}=\left(\frac{1}{\alpha_1},\ldots,\frac{1}{\alpha_n}\right)$. Now, differentiating (4) partially with respect to α_i, we obtain

$$\frac{\partial\Omega\left(\frac{\mathbf{1}}{\boldsymbol{\alpha}}\right)}{\partial\alpha_i}=-(x-\sigma)r_if\left((x-\sigma)\alpha_i\right)\leq 0,\tag{5}$$

since $x\geq\sigma$. Now, Eq. 5 implies that $\Omega\left(\frac{\mathbf{1}}{\boldsymbol{\alpha}}\right)$ is decreasing with respect to α_i, $i=1,\ldots,n$. Thus, to complete the remaining part of the proof, it is sufficient to show that $\Omega\left(\frac{\mathbf{1}}{\boldsymbol{\alpha}}\right)$ is Schur-convex with respect to $\boldsymbol{\alpha}$. In this process, we consider

$$\Delta_{11}=\frac{\partial\Omega\left(\frac{\mathbf{1}}{\boldsymbol{\alpha}}\right)}{\partial\alpha_i}-\frac{\partial\Omega\left(\frac{\mathbf{1}}{\boldsymbol{\alpha}}\right)}{\partial\alpha_j}=\frac{r_j}{\alpha_j}(x-\sigma)\alpha_jf\left((x-\sigma)\alpha_j\right)-\frac{r_i}{\alpha_i}(x-\sigma)\alpha_if\left((x-\sigma)\alpha_i\right).\tag{6}$$

Let $1\leq i\leq j\leq n$ and consider $\boldsymbol{r}\in\mathcal{E}_n^+$ i.e., $r_i\leq r_j$ and $\boldsymbol{\lambda}\in\mathcal{E}_n^+$, implies $\boldsymbol{\alpha}\in\mathcal{D}_n^+$ i.e., $\alpha_i\geq\alpha_j$. Thus, clearly $(x-\sigma)\alpha_i\geq(x-\sigma)\alpha_j$. Using this and from the assumptions made, it can be shown that

$$(x-\sigma)\alpha_if\left((x-\sigma)\alpha_i\right)\leq(x-\sigma)\alpha_jf\left((x-\sigma)\alpha_j\right).\tag{7}$$

Combining Eq. 7 with $r_i\leq r_j$, from Eq. 6, one can easily establish that Δ_{11} is nonnegative, giving that $\frac{\partial\Omega(\frac{\mathbf{1}}{\boldsymbol{\alpha}})}{\partial\alpha_k}$ is decreasing in $k\in\{1,\ldots,n\}$. Thus, from the result in Lemma 4, $\Omega(\frac{\mathbf{1}}{\boldsymbol{\alpha}})$ is Schur-convex with respect to $\boldsymbol{\alpha}\in\mathcal{D}_n^+$. Further, when $\boldsymbol{r}\in\mathcal{D}_n^+$ and $\boldsymbol{\lambda}\in\mathcal{D}_n^+$, i.e., $\boldsymbol{\alpha}\in\mathcal{E}_n^+$, employing similar arguments, it can be shown that, Δ_{11} also takes nonpositive values, implying that $\frac{\partial\Omega(\frac{\mathbf{1}}{\boldsymbol{\alpha}})}{\partial\alpha_k}$ is increasing in $k\in\{1,\ldots,n\}$. Thus, from Lemma 5, $\Omega(\frac{\mathbf{1}}{\boldsymbol{\alpha}})$ is Schur-convex with respect to $\boldsymbol{\alpha}\in\mathcal{E}_n^+$. Hence the theorem is proved. □

To illustrate the result in Theorem 1, we consider the following numerical example.

Example 1 Consider the baseline distribution as Pareto distribution with pdf $f(t) = \frac{1}{t^2}, 1 \leq t < \infty$. Clearly, $tf(t)$ is decreasing with respect to t in its domain. Take $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (0.1, 0.4, 0.8) \in \mathcal{E}_3^+$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) = (0.2, 0.5, 0.8) \in \mathcal{E}_3^+$. Here, it can be shown that $\frac{\mathbf{1}}{\boldsymbol{\lambda}} \overset{w}{\preccurlyeq} \frac{\mathbf{1}}{\boldsymbol{\theta}}$. Further, assume $\boldsymbol{r} = (r_1, r_2, r_3) = (0.2, 0.3, 0.5) \in \mathcal{E}_3^+$ and $\sigma = 0.1$. The sfs of $U_3(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_3(\boldsymbol{r}; \boldsymbol{\theta})$ can be obtained using the concept of distribution theory, hence are omitted for brevity. The difference between the sfs of $U_3(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_3(\boldsymbol{r}; \boldsymbol{\theta})$ has been plotted in Fig. 1(a), from which, we can easily observe that $U_3(\boldsymbol{r}; \boldsymbol{\lambda}) \leq_{st} V_3(\boldsymbol{r}; \boldsymbol{\theta})$, confirming the result in Theorem 1.

The following implication between various stochastic orders is well known.

$$\boldsymbol{x} \overset{m}{\succcurlyeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{w}{\succcurlyeq} \boldsymbol{y}.$$

Thus, under the same setting, with decreasing $tf(t)$ in $t > 0$, the result in Theorem 1 also holds if we replace the weakly supermajorization order between $\frac{\mathbf{1}}{\boldsymbol{\lambda}}$ and $\frac{\mathbf{1}}{\boldsymbol{\theta}}$ by majorization order. From Lemma 3, we have

$$(x_1, \ldots, x_n) \overset{w}{\preccurlyeq} (x, \ldots, x), \text{ if } x \leq \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Using this fact, the following corollary immediately follows from Theorem 1.

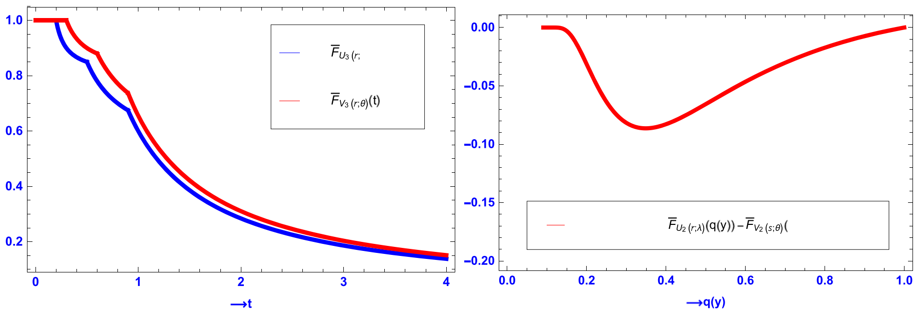
Corollary 1 *In* Theorem 1, *consider* $(\theta_1, \ldots, \theta_n) = (\theta, \ldots, \theta)$. *Suppose* $tf(t)$ *is decreasing in* $t > 0$. *Then, for* $\boldsymbol{r}, \boldsymbol{\lambda} \in \mathcal{E}_n^+$ *(or* $\mathcal{D}_n^+$*) and fixed* $\sigma > 0$, *we have* $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) \leq \bar{F}\left(\frac{x-\sigma}{\theta}\right)$, *if* $\theta \geq n / \sum_{i=1}^{n} \frac{1}{\lambda_i}$.

Next, we present sufficient conditions, under which the usual stochastic order holds between two MRVs. Note that the conditions are different from that of Theorem 1.

Theorem 2 *Consider* $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ *and* $\bar{F}_{V_n(\boldsymbol{r};\boldsymbol{\theta})}(x)$ *as in* Theorem 1. *Let* $t^2 f(t)$ *be increasing in* $t > 0$. *Then, for* $\boldsymbol{r} \in \mathcal{E}_n^+$ *(or* $\mathcal{D}_n^+$*) and* $\boldsymbol{\lambda} \in \mathcal{D}_n^+$ *(or* $\mathcal{E}_n^+$*), and fixed* $\sigma > 0$, *we have*

$$\boldsymbol{\lambda} \preccurlyeq_w \boldsymbol{\theta} \Rightarrow U_n(\boldsymbol{r}; \boldsymbol{\lambda}) \geq_{st} V_n(\boldsymbol{r}; \boldsymbol{\theta}).$$

Proof In order to establish the result, we need to show that $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) = \sum_{i=1}^{n} r_i \bar{F}(\frac{x-\sigma}{\lambda_i})$ is increasing and Schur-concave with respect to $\boldsymbol{\lambda}$. The increasing property can be checked easily. The Schur-concavity of $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ can be shown using Lemmas 4 and 5. □



(a) (b)

Fig. 1 (a) Plots of the sfs of MRVs $U_3(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_3(\boldsymbol{r}; \boldsymbol{\theta})$ as in Example 1 for $\sigma = 0.1$. The blue-colour graph represents the sf of $U_3(\boldsymbol{r}; \boldsymbol{\lambda})$ and the red colour graph represents the sf of $V_3(\boldsymbol{r}; \boldsymbol{\theta})$. (b) Plot of the difference between the sfs of the MRVs $U_2(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_2(\boldsymbol{s}; \boldsymbol{\theta})$ as in Example 7 for $\sigma = 0.1$ and $\beta = 2$

Remark 2 There are various lifetime distributions such as inverted exponential with pdf $f(t) = \frac{\lambda}{t^2} e^{-\frac{\lambda}{t}},\ t > 0,\ \lambda > 0$ and Burr type-III with pdf $f(t) = \frac{1}{(t+1)^2},\ t > 0$, which satisfy the condition "$t^2 f(t)$ is increasing in t" as considered in Theorem 2.

In the following result, we provide sufficient condition based on reciprocally majorization order for the existence of the usual stochastic order between two MRVs. Here, we have considered that the baseline pdf $f(t)$ is increasing with respect to $t > 0$. It is important to mention that this condition is satisfied only by the distributions with bounded support. Sometimes, the continuous models with finite support are useful in reliability theory to describe lifetime data. This is often motivated by considering physical reasons such as the finite lifetime of a component or the bounded signals occurring in industrial systems (for details see Jiang 2013 and Dedecius and Ettler 2013). Besides this, when the reliability is computed as percentage or ratio, it is important to have models defined on the unit interval in order to have some reasonable results.

Theorem 3 *Consider $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ and $\bar{F}_{V_n(\boldsymbol{r};\boldsymbol{\theta})}(x)$ as in* Theorem 1. *Further, suppose $f(t)$ is increasing in $t > 0$. Then, for $\boldsymbol{r} \in \mathcal{D}_n^+$ (or $\mathcal{E}_n^+$), $\boldsymbol{\lambda} \in \mathcal{E}_n^+$ (or $\mathcal{D}_n^+$), and fixed $\sigma > 0$, we have*

$$\boldsymbol{\lambda} \overset{rm}{\succcurlyeq} \boldsymbol{\theta} \Rightarrow U_n(\boldsymbol{r};\boldsymbol{\lambda}) \leq_{st} V_n(\boldsymbol{r};\boldsymbol{\theta}).$$

Proof Making use of Lemma 2, the proof of this theorem follows from Lemmas 4 and 5. Hence, it is omitted. □

Remark 3 We note that the condition "$f(t)$ is increasing in $t > 0$" is satisfied by the power distribution ($f(t) = pt^{p-1},\ p > 1$) and beta distribution ($f(t) = \frac{t^{\nu-1}(1-t)^{\omega-1}}{\mathcal{B}(\nu,\omega)},\ 0 < t < 1,\ \nu > 1,\ \omega < 1$, $\mathcal{B}(\nu,\omega)$ is the complete beta function).

In Theorem 1, we have proposed sufficient conditions for the usual stochastic order between two FMMs. In this sequel, it is a natural query "can we strengthen the usual stochastic order to other stronger stochastic order?" The next result establishes that it is possible, but under different sufficient conditions and different settings. Next, we investigate hazard rate order between MRVs. Denote $\boldsymbol{s} = (s_1, \ldots, s_n)$.

Theorem 4 *Let $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) = \sum_{i=1}^n r_i \bar{F}(\frac{x-\sigma}{\lambda_i})$ and $\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\lambda})}(x) = \sum_{i=1}^n s_i \bar{F}(\frac{x-\sigma}{\lambda_i})$ be the sfs of two MRVs $U_n(\boldsymbol{r};\boldsymbol{\lambda})$ and $V_n(\boldsymbol{s};\boldsymbol{\lambda})$, respectively. Suppose $f(t)$ is increasing in $t > 0$. Then, for $\boldsymbol{r} \in \mathcal{D}_n^+$ (or $\mathcal{E}_n^+$), $\boldsymbol{\lambda} \in \mathcal{E}_n^+$ (or $\mathcal{D}_n^+$), and for fixed $\sigma > 0$, we have*

$$\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s} \Rightarrow U_n(\boldsymbol{r};\boldsymbol{\lambda}) \geq_{hr} V_n(\boldsymbol{s};\boldsymbol{\lambda}).$$

Proof Under the given setting, the hazard rate function of $U_n(\boldsymbol{r};\boldsymbol{\lambda})$ is given by

$$h_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) = \frac{\rho_1(x;\boldsymbol{r},\boldsymbol{\lambda})}{\rho_2(x;\boldsymbol{r},\boldsymbol{\lambda})}, \tag{8}$$

where $\rho_1(x;\boldsymbol{r},\boldsymbol{\lambda}) = \sum_{i=1}^n r_i \frac{1}{\lambda_i} f(\frac{x-\sigma}{\lambda_i}) \geq 0$ and $\rho_2(x;\boldsymbol{r},\boldsymbol{\lambda}) = \sum_{i=1}^n r_i \bar{F}(\frac{x-\sigma}{\lambda_i}) \geq 0$. To get the desired result by using Definition 3, it suffices to show that $h_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ is Schur-convex with respect to $\boldsymbol{r}$. Differentiating $h_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ with respect to r_i partially, we obtain as

$$\frac{\partial h_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_i} \overset{sign}{=} \rho_2(x;\boldsymbol{r},\boldsymbol{\lambda}) \frac{1}{\lambda_i} f\left(\frac{x-\sigma}{\lambda_i}\right) - \rho_1(x;\boldsymbol{r},\boldsymbol{\lambda}) \bar{F}\left(\frac{x-\sigma}{\lambda_i}\right). \tag{9}$$

Further, consider

$$\Delta_{12} = \frac{\partial h_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_i} - \frac{\partial h_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_j} \overset{sign}{=} \rho_1(x;\boldsymbol{r},\boldsymbol{\lambda})\tau_1 + \rho_2(x;\boldsymbol{r},\boldsymbol{\lambda})\tau_2, \tag{10}$$

where $\tau_1 = F(\frac{x-\sigma}{\lambda_i}) - F(\frac{x-\sigma}{\lambda_j})$ and $\tau_2 = \frac{1}{\lambda_i} f(\frac{x-\sigma}{\lambda_i}) - \frac{1}{\lambda_j} f(\frac{x-\sigma}{\lambda_j})$. Consider $1 \leq i \leq j \leq n$. Let $\boldsymbol{r} \in \mathcal{D}_n^+$ $(or\ \mathcal{E}_n^+)$ and $\boldsymbol{\lambda} \in \mathcal{E}_n^+$ $(or\ \mathcal{D}_n^+)$. That is, $r_i \geq$ $(or\ \leq)$ r_j and $\lambda_i \leq$ $(or\ \geq)$ λ_j. We present the proof when $r_i \geq r_j$ and $\lambda_i \leq \lambda_j$, since the other case is quite similar. For $\lambda_i \leq \lambda_j$, we obtain $\frac{1}{\lambda_i} \geq \frac{1}{\lambda_j}$ and $\frac{x-\sigma}{\lambda_i} \geq \frac{x-\sigma}{\lambda_j}$ implies $F(\frac{x-\sigma}{\lambda_i}) \geq F(\frac{x-\sigma}{\lambda_j})$, that is, $\tau_1 \geq 0$. Further, $f(t)$ is increasing with respect to $t > 0$, implies $f(\frac{x-\sigma}{\lambda_i}) \geq f(\frac{x-\sigma}{\lambda_j})$, that is, $\tau_2 \geq 0$. Thus, from Eq. 10, we see that $\Delta_{12} \geq 0$, giving that $\frac{\partial h_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_k}$ is decreasing in $k \in \{1,\ldots,n\}$. Thus, from Lemma 4, $h_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ is Schur-convex with respect to $\boldsymbol{r} \in \mathcal{D}_n^+$. Thus, the proof is finished. □

The following counterexample describes that the result stated in Theorem 4 is not necessarily true if $\boldsymbol{r} \notin \mathcal{D}_3^+$ $(or \notin \mathcal{E}_3^+)$, $\boldsymbol{\lambda} \notin \mathcal{E}_3^+$ $(or \notin \mathcal{D}_3^+)$ and $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s}$.

Counterexample 1 Consider power distribution with pdf $f(t) = 2t$, $0 < t \leq 1$ and sf $\bar{F}(t) = 1 - t^2$, $0 < t \leq 1$ as the baseline distribution for the MMs of location-scale family of distributions. The pdf $f(t)$ is increasing with respect to t in its domain. Now, we assume that $\boldsymbol{r} = (r_1, r_2, r_3) = (0.2, 0.5, 0.3) \notin \mathcal{D}_3^+$ $(or \notin \mathcal{E}_3^+)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (0.4, 0.5, 0.1) \notin \mathcal{E}_3^+$ $(or \notin \mathcal{D}_3^+)$. Further, take $\boldsymbol{s} = (s_1, s_2, s_3) = (0.1, 0.6, 0.3) \notin \mathcal{D}_3^+$ $(or \notin \mathcal{E}_3^+)$ and $\sigma = 0.1$. Clearly, we obtain $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s}$. Writing $K_1(t) = h_{U_3(\boldsymbol{r};\boldsymbol{\lambda})}(t) - h_{V_3(\boldsymbol{s};\boldsymbol{\lambda})}(t)$, where $h_{U_3(\boldsymbol{r};\boldsymbol{\lambda})}(t)$ and $h_{V_3(\boldsymbol{s};\boldsymbol{\lambda})}(t)$ are not presented here to maintain brevity. Now, if we choose $t = 0.52$, then the function $K_1(t)$ gives the value -5.32907×10^{-15} (< 0), giving that $h_{U_3(\boldsymbol{r};\boldsymbol{\lambda})}(t) < h_{V_3(\boldsymbol{s};\boldsymbol{\lambda})}(t)$. Again, if we choose $t = 0.55$, then $K_1(t)$ gives the value 2.84217×10^{-14} (> 0), giving that $h_{U_3(\boldsymbol{r};\boldsymbol{\lambda})}(t) > h_{V_3(\boldsymbol{s};\boldsymbol{\lambda})}(t)$. Hence, it is easy to say that $K_1(t)$ changes in sign. Thus, $U_3(\boldsymbol{r};\boldsymbol{\lambda}) \ngeq_{hr} V_3(\boldsymbol{s};\boldsymbol{\lambda})$.

The following counterexample illustrates that the result in Theorem 4 does not hold if $\boldsymbol{r} \in \mathcal{E}_3^+$, $\boldsymbol{\lambda} \in \mathcal{D}_3^+$ and $\boldsymbol{r} \overset{m}{\not\preccurlyeq} \boldsymbol{s}$.

Counterexample 2 Considering the same distribution as in the above Counterexample 1, we have $f(t)$ is increasing with respect to t in its domain. Assume that $\boldsymbol{r} = (r_1, r_2, r_3) = (0.1, 0.2, 0.7) \in \mathcal{E}_3^+$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (0.9, 0.8, 0.6) \in \mathcal{D}_3^+$. Furthermore, take $\boldsymbol{s} = (s_1, s_2, s_3) = (0.2, 0.2, 0.6) \in \mathcal{E}_3^+$ and $\sigma = 0.1$. Clearly, we obtain $\boldsymbol{r} \overset{m}{\not\preccurlyeq} \boldsymbol{s}$. Also, let the difference between two hazard rate functions of the MRVs $U_3(\boldsymbol{r};\boldsymbol{\lambda})$ and $V_3(\boldsymbol{s};\boldsymbol{\lambda})$ be $K_2(t)$ as in the above Counterexample 1. If we take $t = 0.903$, then $K_2(0.903) = 2.66454 \times 10^{-14}$ (> 0) and if we choose $t = 0.990$, then $K_2(0.990) = -2.50111 \times 10^{-12}$ (< 0), which together imply that $K_2(t)$ changes in sign. Thus, $U_3(\boldsymbol{r};\boldsymbol{\lambda}) \ngeq_{hr} V_3(\boldsymbol{s};\boldsymbol{\lambda})$.

One may rise a question whether the result stated in Theorem 4 could hold if "$f(t)$ is decreasing with respect to t in its domain"? The answer is given in the following counterexample whenever $\boldsymbol{r} \notin \mathcal{D}_3^+$ $(or \notin \mathcal{E}_3^+)$, $\boldsymbol{\lambda} \notin \mathcal{E}_3^+$ $(or \notin \mathcal{D}_3^+)$ and $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s}$.

Counterexample 3 Consider Pareto distribution with pdf $f(t) = t^{-2}$, $1 \leq t < \infty$ and sf $\bar{F}(t) = t^{-1}$, $1 \leq t < \infty$ as the baseline distribution. Here, $f(t)$ is decreasing with respect to t in $t \geq 1$. Assume that $\boldsymbol{r} = (r_1, r_2, r_3) = (0.2, 0.5, 0.3) \notin \mathcal{D}_3^+$ $(or \notin \mathcal{E}_3^+)$

and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (0.6, 0.2, 0.3) \notin \mathcal{E}_3^+$ (*or* $\notin \mathcal{D}_3^+$). Further, take $\boldsymbol{s} = (s_1, s_2, s_3) = (0.1, 0.6, 0.3) \notin \mathcal{D}_3^+$ (*or* $\notin \mathcal{E}_3^+$) and $\sigma = 0.1$. It is clear that $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s}$. Writing $K_3(t) = h_{U_3(\boldsymbol{r};\boldsymbol{\lambda})}(t) - h_{V_3(\boldsymbol{s};\boldsymbol{\lambda})}(t)$. Now, if we choose $t = 0.80$, then the function $K_3(t)$ gives the value 2.22045×10^{-16} (> 0), giving that $h_{U_3(\boldsymbol{r};\boldsymbol{\lambda})}(t) > h_{V_3(\boldsymbol{s};\boldsymbol{\lambda})}(t)$. Again, if we take $t = 0.77$, then $K_3(t)$ gives -2.22045×10^{-16} (< 0), giving that $h_{U_3(\boldsymbol{r};\boldsymbol{\lambda})}(t) < h_{V_3(\boldsymbol{s};\boldsymbol{\lambda})}(t)$. Hence, it is easy to conclude that $K_3(t)$ changes in sign. Thus, $U_3(\boldsymbol{r};\boldsymbol{\lambda}) \not\geq_{hr} V_3(\boldsymbol{s};\boldsymbol{\lambda})$.

Below, we propose different sufficient conditions under which the hazard rate order between two MRVs holds.

Theorem 5 *Let $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) = \sum_{i=1}^n r_i \bar{F}(\frac{x-\sigma}{\lambda_i})$ and $\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\lambda})}(x) = \sum_{i=1}^n s_i \bar{F}(\frac{x-\sigma}{\lambda_i})$ be the sfs of two MRVs $U_n(\boldsymbol{r};\boldsymbol{\lambda})$ and $V_n(\boldsymbol{s};\boldsymbol{\lambda})$, respectively. Suppose $h(t) = f(t)/\bar{F}(t)$ is increasing in $t > 0$. Then, for $\boldsymbol{\lambda} \in \mathcal{D}_n^+$ $(\mathcal{E}_n^+)$, and for fixed $\sigma > 0$, we have*

$$U_n(\boldsymbol{r};\boldsymbol{\lambda}) \leq_{hr} (\geq_{hr})\ V_n(\boldsymbol{s};\boldsymbol{\lambda}).$$

Proof In order to obtain the required result, it suffices to show that $\frac{\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\lambda})}(x)}{\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)}$ is increasing (decreasing) in $x > 0$, where

$$\frac{\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\lambda})}(x)}{\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)} = \frac{\sum_{i=1}^n s_i \bar{F}(\frac{x-\sigma}{\lambda_i})}{\sum_{j=1}^n r_j \bar{F}(\frac{x-\sigma}{\lambda_j})} = \xi(x),\ (say). \tag{11}$$

Now, differentiating $\xi(x)$ in Eq. 11 with respect to x, and after some simplifications, we obtain

$$\begin{aligned}
\xi'(x) &\overset{sign}{=} \sum_{i=1}^n \sum_{j=1}^n \frac{s_i r_j}{\lambda_j} \bar{F}\left(\frac{x-\sigma}{\lambda_i}\right) f\left(\frac{x-\sigma}{\lambda_j}\right) - \sum_{j=1}^n \sum_{i=1}^n \frac{r_j s_i}{\lambda_i} \bar{F}\left(\frac{x-\sigma}{\lambda_j}\right) f\left(\frac{x-\sigma}{\lambda_i}\right) \\
&\overset{sign}{=} \sum_{i=1}^n \sum_{j=1}^n s_i r_j \bar{F}\left(\frac{x-\sigma}{\lambda_i}\right) \bar{F}\left(\frac{x-\sigma}{\lambda_j}\right) \left[\frac{1}{\lambda_j} h\left(\frac{x-\sigma}{\lambda_j}\right) - \frac{1}{\lambda_i} h\left(\frac{x-\sigma}{\lambda_i}\right)\right].
\end{aligned} \tag{12}$$

Consider $1 \leq i \leq j \leq n$. Let $\boldsymbol{\lambda} \in \mathcal{D}_n^+$ $(\mathcal{E}_n^+)$. That is, $\lambda_i \geq (\leq)\ \lambda_j$. Then, clearly, we obtain $\frac{1}{\lambda_i} \leq (\geq) \frac{1}{\lambda_j}$ and $\frac{x-\sigma}{\lambda_i} \leq (\geq) \frac{x-\sigma}{\lambda_j}$. Under the assumption made, we have

$$h\left(\frac{x-\sigma}{\lambda_i}\right) \leq (\geq)\ h\left(\frac{x-\sigma}{\lambda_j}\right). \tag{13}$$

Thus, from Eq. 12, we obtain $\xi'(x) \geq (\leq)\ 0$, which implies that $\xi(x)$ is increasing (decreasing) with respect to $x > 0$. Hence the result follows. □

Remark 4 There are many lifetime distributions which have increasing hazard rate function. For example, the Weibull distribution with pdf $f(t) = kt^{k-1}e^{-t^k}$, $0 < t < \infty$, $k > 1$ has increasing hazard rate.

Next, we prove reversed hazard rate order between two MRVs. Here, the location parameter vectors are considered to be common, mixing proportions are different and scale parameters are fixed.

Theorem 6 *Let $U_n(\boldsymbol{r};\boldsymbol{\sigma})$ and $V_n(\boldsymbol{s};\boldsymbol{\sigma})$ be two MRVs with respective mixing proportion vectors $\boldsymbol{r}$ and $\boldsymbol{s}$, constructed from the set of nonnegative RVs $\{X_1, \ldots, X_n\}$, where $X_i \sim F\left(\frac{x-\sigma_i}{\lambda}\right)$, $i = 1, \ldots, n$. Further, let $f(t)$ be decreasing with respect to $t > 0$. Then, for fixed $\lambda > 0$, if*

(i) $\boldsymbol{r} \in \mathcal{D}_n^+$ *and* $\boldsymbol{\sigma} \in \mathcal{D}_n^+$*, then* $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s} \Rightarrow U_n(\boldsymbol{r};\boldsymbol{\sigma}) \leq_{rh} V_n(\boldsymbol{s};\boldsymbol{\sigma})$*;*
(ii) $\boldsymbol{r} \in \mathcal{E}_n^+$ *and* $\boldsymbol{\sigma} \in \mathcal{D}_n^+$*, then* $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s} \Rightarrow U_n(\boldsymbol{r};\boldsymbol{\sigma}) \geq_{rh} V_n(\boldsymbol{s};\boldsymbol{\sigma})$*.*

Proof Here, we present the proof of the first part. The proof of the second part can be established using similar arguments. The reversed hazard rate function of the MRV $U_n(\boldsymbol{r};\boldsymbol{\sigma})$ is given by

$$\tilde{h}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x) = \begin{cases} \dfrac{r_n \frac{1}{\lambda} f\left(\frac{x-\sigma_n}{\lambda}\right)}{r_n F\left(\frac{x-\sigma_n}{\lambda}\right)} & ; \text{if} \quad \sigma_n < x \leq \sigma_{n-1} \\ \dfrac{r_n \frac{1}{\lambda} f\left(\frac{x-\sigma_n}{\lambda}\right) + r_{n-1} \frac{1}{\lambda} f\left(\frac{x-\sigma_{n-1}}{\lambda}\right)}{r_n F\left(\frac{x-\sigma_n}{\lambda}\right) + r_{n-1} F\left(\frac{x-\sigma_{n-1}}{\lambda}\right)} & ; \text{if} \quad \sigma_{n-1} < x \leq \sigma_{n-2} \\ \quad\vdots & \quad\vdots \\ \dfrac{r_n \frac{1}{\lambda} f\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + r_2 \frac{1}{\lambda} f\left(\frac{x-\sigma_2}{\lambda}\right)}{r_n F\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + r_2 F\left(\frac{x-\sigma_2}{\lambda}\right)} & ; \text{if} \quad \sigma_2 < x \leq \sigma_1 \\ \dfrac{r_n \frac{1}{\lambda} f\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + r_1 \frac{1}{\lambda} f\left(\frac{x-\sigma_1}{\lambda}\right)}{r_n F\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + r_1 F\left(\frac{x-\sigma_1}{\lambda}\right)} & ; \text{if} \quad \sigma_1 < x < \infty. \end{cases} \tag{14}$$

To prove the desired result in Part (i) of the theorem by using Definition 3, it is sufficient to show that $\tilde{h}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ given in Eq. 14 is Schur-concave with respect to $\boldsymbol{r}$. Consider $\sigma_1 < x < \infty$. Differentiating $\tilde{h}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ with respect to r_i partially, we obtain

$$\frac{\partial \tilde{h}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_i} \overset{sign}{=} \xi_2(x;\boldsymbol{r},\boldsymbol{\sigma}) f\left(\frac{x-\sigma_i}{\lambda}\right) - \xi_1(x;\boldsymbol{r},\boldsymbol{\sigma}) F\left(\frac{x-\sigma_i}{\lambda}\right), \tag{15}$$

where $\xi_1(x;\boldsymbol{r},\boldsymbol{\sigma}) = \sum_{i=1}^n r_i f\left(\frac{x-\sigma_i}{\lambda}\right) \geq 0$ and $\xi_2(x;\boldsymbol{r},\boldsymbol{\sigma}) = \sum_{i=1}^n r_i F\left(\frac{x-\sigma_i}{\lambda}\right) \geq 0$. Further, let us take

$$\begin{aligned} \Delta_{13} &= \frac{\partial \tilde{h}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_i} - \frac{\partial \tilde{h}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_j} \\ &\overset{sign}{=} \xi_1(x;\boldsymbol{r},\boldsymbol{\sigma})\zeta_1 + \xi_2(x;\boldsymbol{r},\boldsymbol{\sigma})\zeta_2, \end{aligned} \tag{16}$$

where $\zeta_1 = F\left(\frac{x-\sigma_j}{\lambda}\right) - F\left(\frac{x-\sigma_i}{\lambda}\right)$ and $\zeta_2 = f\left(\frac{x-\sigma_i}{\lambda}\right) - f\left(\frac{x-\sigma_j}{\lambda}\right)$. Consider $1 \leq i \leq j \leq n$. Without loss of generality, let $\boldsymbol{r} \in \mathcal{D}_n^+$ and $\boldsymbol{\sigma} \in \mathcal{D}_n^+$, that is, $r_i \geq r_j$ and $\sigma_i \geq \sigma_j$. Now, from $\sigma_i \geq \sigma_j$, we get $\frac{x-\sigma_i}{\lambda} \leq \frac{x-\sigma_j}{\lambda}$. Then, we obtain

$$F\left(\frac{x-\sigma_i}{\lambda}\right) \leq F\left(\frac{x-\sigma_j}{\lambda}\right), \tag{17}$$

that is, $\zeta_1 \geq 0$. Under the assumption made, it holds that

$$f\left(\frac{x-\sigma_i}{\lambda}\right) \geq f\left(\frac{x-\sigma_j}{\lambda}\right), \tag{18}$$

that is, $\zeta_2 \geq 0$. Thus, for $1 \leq i \leq j \leq n$, $\boldsymbol{r} \in \mathcal{D}_n^+$ and $\boldsymbol{\sigma} \in \mathcal{D}_n^+$, we have $\Delta_{13} \geq 0$, giving that $\frac{\partial \tilde{h}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_k}$ is decreasing in $k \in \{1, \ldots, n\}$. Hence, by using Part (i) of Lemma 6 with the help of Remark 1, $\tilde{h}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ is Schur-concave with respect to $\boldsymbol{r} \in \mathcal{D}_n^+$. Using similar arguments, it can be established that $\tilde{h}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ is also Schur-concave with respect to $\boldsymbol{r} \in \mathcal{D}_n^+$ when x belongs to the other subintervals, say $\sigma_2 < x \leq \sigma_1, \sigma_3 < x \leq \sigma_2, \ldots, \sigma_{n-1} < x \leq \sigma_{n-2}, \sigma_n < x \leq \sigma_{n-1}$. Thus, the theorem is proved. □

Remark 5 Part (i) and Part (ii) of Theorem 6 also hold if $\boldsymbol{r} \in \mathcal{D}_n^+$, $\boldsymbol{\sigma} \in \mathcal{E}_n^+$ and $\boldsymbol{r} \in \mathcal{E}_n^+$, $\boldsymbol{\sigma} \in \mathcal{E}_n^+$.

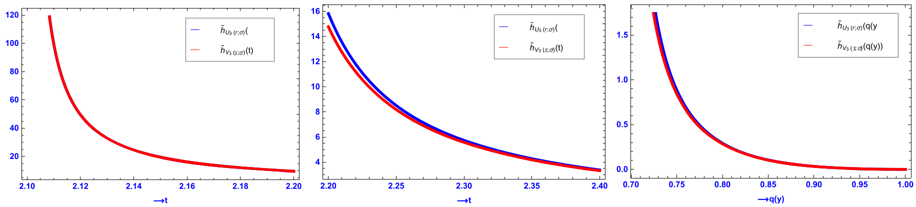
The following example provides an illustration of the result stated in Theorem 6.

Example 2 Consider Pareto distribution with cdf $F(t) = 1 - t^{-1}$, $1 \leq t < \infty$ and pdf $f(t) = t^{-2}$, $1 \leq t < \infty$ as the baseline distribution for the MMs of location-scale family of distributions. The pdf is decreasing with respect to t in its domain. Now, we assume that $\boldsymbol{r} = (r_1, r_2, r_3) = (0.2, 0.3, 0.5) \in \mathcal{E}_3^+$ and $\boldsymbol{s} = (s_1, s_2, s_3) = (0.1, 0.3, 0.6) \in \mathcal{E}_3^+$. Further, take $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (0.4, 0.2, 0.1) \in \mathcal{D}_3^+$ and $\lambda = 2$. Clearly, $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s}$. For the clear visualization of the graphs of reversed hazard rate functions of $U_3(\boldsymbol{r};\boldsymbol{\sigma})$ and $V_3(\boldsymbol{s};\boldsymbol{\sigma})$, we have plotted them in three different subintervals, say (2.1, 2.2), (2.2, 2.4), and $(2.4, \infty)$ in Fig. 2, which readily suggest that $U_3(\boldsymbol{r};\boldsymbol{\sigma}) \geq_{rh} V_3(\boldsymbol{s};\boldsymbol{\sigma})$. For the third graph, we consider $q(y) = \frac{y}{1-y}$, $\frac{12}{17} < y < 1$ to capture the line from 2.4 to ∞.

The following counterexample shows that the result in Theorem 6 does not hold if $\boldsymbol{r} \notin \mathcal{D}_3^+$, $\boldsymbol{\sigma} \notin \mathcal{D}_3^+$, and $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s}$.

Counterexample 4 Consider Pareto distribution with pdf $f(t) = 2t^{-3}$, $1 \leq t < \infty$ and cdf $F(t) = 1 - 1/t^2$, $1 \leq t < \infty$ as the baseline distribution. Here, $f(t)$ is decreasing with respect to t in its domain. Assume that $\boldsymbol{r} = (r_1, r_2, r_3) = (0.2, 0.6, 0.2) \notin \mathcal{D}_3^+$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (0.4, 0.6, 0.3) \notin \mathcal{D}_3^+$. Further, take $\boldsymbol{s} = (s_1, s_2, s_3) = (0.1, 0.2, 0.7) \in \mathcal{E}_3^+$ and $\lambda = 2$. It is then easy to check that $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s}$. Writing $k_4(t) = \tilde{h}_{U_3(\boldsymbol{r};\boldsymbol{\sigma})}(t) - \tilde{h}_{V_3(\boldsymbol{s};\boldsymbol{\sigma})}(t)$. Now, if we take $t = 5$, then the function $K_4(t)$ gives the value 0.0028828 (> 0), giving that $\tilde{h}_{U_3(\boldsymbol{r};\boldsymbol{\sigma})}(t) > \tilde{h}_{V_3(\boldsymbol{s};\boldsymbol{\sigma})}(t)$. Again, if we take $t = 50$, then $K_4(t)$ gives the value -3.44858×10^{-6} (< 0), giving that $\tilde{h}_{U_3(\boldsymbol{r};\boldsymbol{\sigma})}(t) < \tilde{h}_{V_3(\boldsymbol{s};\boldsymbol{\sigma})}(t)$. Thus, $K_4(t)$ changes in sign. Hence, $U_3(\boldsymbol{r};\boldsymbol{\sigma}) \ngeq_{rh} V_3(\boldsymbol{s};\boldsymbol{\sigma})$.

One question may arise whether the result in Theorem 6 holds if $\boldsymbol{r} \in \mathcal{E}_3^+$, $\boldsymbol{\sigma} \in \mathcal{D}_3^+$, and $\boldsymbol{r} \overset{m}{\npreceq} \boldsymbol{s}$? The following counterexample gives an answer.



(a) (b) (c)

Fig. 2 Plots of the reversed hazard rate functions of $U_3(\boldsymbol{r};\boldsymbol{\sigma})$ and $V_3(\boldsymbol{s};\boldsymbol{\sigma})$ in Example 2, when (a) $t \in (2.1, 2.2)$ (b) $t \in (2.2, 2.4)$, and (c) $y \in (\frac{12}{17}, 1)$. The red colour graph is for the reversed hazard rate of $V_3(\boldsymbol{s};\boldsymbol{\sigma})$ and the blue-colour graph is for the reversed hazard rate of $U_3(\boldsymbol{r};\boldsymbol{\sigma})$

Counterexample 5 Considering the same distribution as in the above Counterexample 4, we have $f(t)$ is decreasing with respect to $t \geq 1$. Assume that $\boldsymbol{r} = (r_1, r_2, r_3) = (0.1, 0.3, 0.6) \in \mathcal{E}_3^+$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (0.6, 0.4, 0.2) \in \mathcal{D}_3^+$. Further, take $\boldsymbol{s} = (s_1, s_2, s_3) = (0.2, 0.3, 0.5) \in \mathcal{E}_3^+$ and $\lambda = 2$. It is clear that $\boldsymbol{r} \overset{m}{\not\preccurlyeq} \boldsymbol{s}$. Also, let the difference between two reversed hazard rate functions of the MRVs $U_3(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_3(\boldsymbol{s}; \boldsymbol{\sigma})$ be $K_5(t)$ as in the above Counterexample 4, which is not presented here for brevity. If we choose $t = 2.2003$, then $K_5(2.2003) = -9.09495 \times 10^{-13}$ (< 0) and choose $t = 2.2083$, then $K_5(2.2083) = 1.42109 \times 10^{-14}$ (> 0). Thus, we have seen that $K_5(t)$ changes in sign. Hence, we conclude that $U_3(\boldsymbol{r}; \boldsymbol{\sigma}) \not\geq_{rh} V_3(\boldsymbol{s}; \boldsymbol{\sigma})$.

Remark 6 One may find various other distributions, for which the pdf $f(t)$ is decreasing with respect to $t > 0$. For example,

(i) Power distribution: Consider the power distribution with $pdf\ f(t) = at^{a-1}, 0 < t < 1$ and $0 < a < 1$. After differentiating $f(t)$, we get $f'(t) = a(a-1)t^{a-2} < 0$, for all $0 < t < 1$ and $0 < a < 1$, which implies that $f(t)$ is decreasing with respect to $t > 0$.

(ii) Exponential distribution: Assume the exponential distribution with $pdf\ f(t) = \lambda e^{-\lambda t}$, $0 < t < \infty$ and $\lambda > 0$. On differentiating $f(t)$, we obtain $f'(t) = -\lambda^2 e^{-\lambda t} < 0$, for all $0 < t < \infty$ and $\lambda > 0$. This clearly implies that $f(t)$ is decreasing in $t > 0$.

Next, we present sufficient conditions for the likelihood ratio ordering between two MRVs.

Theorem 7 *Let $U_n(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_n(\boldsymbol{r}; \boldsymbol{\mu})$ be two MRVs with a common mixing proportion vector $\boldsymbol{r}$, constructed from two sets of nonnegative RVs $\{X_{\sigma_1}, \ldots, X_{\sigma_n}\}$ and $\{X_{\mu_1}, \ldots, X_{\mu_n}\}$, respectively, such that $X_{\sigma_i} \sim F\left(\frac{x-\sigma_i}{\lambda}\right)$ and $X_{\mu_i} \sim F\left(\frac{x-\mu_i}{\lambda}\right)$, for $i = 1, \ldots, n$. Further, let $f(t)$ be log-concave (log-convex) with respect to $t > 0$. Then, for $\boldsymbol{\sigma}, \boldsymbol{\mu} \in \mathcal{D}_n^+$ and for fixed $\lambda > 0$, we have*

$$U_n(\boldsymbol{r}; \boldsymbol{\sigma}) \geq_{lr} (\leq_{lr})\ V_n(\boldsymbol{r}; \boldsymbol{\mu}),$$

provided that $\max\{\mu_1, \ldots, \mu_n\} \leq \min\{\sigma_1, \ldots, \sigma_n\}$.

Proof We prove the result when $f(t)$ is log-concave with respect to $t > 0$. The proof when $f(t)$ is log-convex is similar, and thus is not presented here. The pdfs of the MRVs $U_n(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_n(\boldsymbol{r}; \boldsymbol{\mu})$ are respectively given by

$$f_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x) = \begin{cases} l_1(x) = r_n \frac{1}{\lambda} f\left(\frac{x-\sigma_n}{\lambda}\right) & ; \text{if} \quad \sigma_n < x \leq \sigma_{n-1} \\ l_2(x) = r_n \frac{1}{\lambda} f\left(\frac{x-\sigma_n}{\lambda}\right) + r_{n-1} \frac{1}{\lambda} f\left(\frac{x-\sigma_{n-1}}{\lambda}\right) & ; \text{if} \quad \sigma_{n-1} < x \leq \sigma_{n-2} \\ \vdots & \vdots \\ l_{n-1}(x) = r_n \frac{1}{\lambda} f\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + r_2 \frac{1}{\lambda} f\left(\frac{x-\sigma_2}{\lambda}\right) & ; \text{if} \quad \sigma_2 < x \leq \sigma_1 \\ l_n(x) = r_n \frac{1}{\lambda} f\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + r_1 \frac{1}{\lambda} f\left(\frac{x-\sigma_1}{\lambda}\right) & ; \text{if} \quad \sigma_1 < x < \infty \end{cases} \tag{19}$$

and

$$f_{V_n(\boldsymbol{r};\boldsymbol{\mu})}(x) = \begin{cases} l_1^*(x) = r_n \frac{1}{\lambda} f\left(\frac{x-\mu_n}{\lambda}\right) & ;\text{if} \quad \mu_n < x \le \mu_{n-1} \\ l_2^*(x) = r_n \frac{1}{\lambda} f\left(\frac{x-\mu_n}{\lambda}\right) + r_{n-1}\frac{1}{\lambda} f\left(\frac{x-\mu_{n-1}}{\lambda}\right) & ;\text{if} \quad \mu_{n-1} < x \le \mu_{n-2} \\ \vdots & \vdots \\ l_{n-1}^*(x) = r_n \frac{1}{\lambda} f\left(\frac{x-\mu_n}{\lambda}\right) + \ldots + r_2\frac{1}{\lambda} f\left(\frac{x-\mu_2}{\lambda}\right) & ;\text{if} \quad \mu_2 < x \le \mu_1 \\ l_n^*(x) = r_n \frac{1}{\lambda} f\left(\frac{x-\mu_n}{\lambda}\right) + \ldots + r_1\frac{1}{\lambda} f\left(\frac{x-\mu_1}{\lambda}\right) & ;\text{if} \quad \mu_1 < x < \infty. \end{cases} \tag{20}$$

In order to obtain the required result, it suffices to show that $f_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)/f_{V_n(\boldsymbol{r};\boldsymbol{\mu})}(x)$ is increasing in $x \in (\sigma_n, \infty) \cup (\mu_n, \infty) = (\mu_n, \infty)$, since $\max\{\mu_1, \ldots, \mu_n\} \le \min\{\sigma_1, \ldots, \sigma_n\}$, where $f_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ and $f_{V_n(\boldsymbol{r};\boldsymbol{\mu})}(x)$ are given in Eqs. 19 and 20, respectively. First we have to establish that

$$\frac{l_n(x)}{l_n^*(x)} = \frac{\sum\limits_{i=1}^{n} r_i f\left(\frac{x-\sigma_i}{\lambda}\right)}{\sum\limits_{i=1}^{n} r_i f\left(\frac{x-\mu_i}{\lambda}\right)} = \chi_1(x),\ (say), \tag{21}$$

is increasing with respect to $x \in (\sigma_1, \infty)$. For this, differentiating $\chi_1(x)$ with respect to x, we obtain

$$\begin{aligned} \chi_1'(x) &\overset{sign}{=} \left[\sum_{i=1}^{n} r_i f\left(\frac{x-\mu_i}{\lambda}\right)\right]\left[\sum_{i=1}^{n} r_i f'\left(\frac{x-\sigma_i}{\lambda}\right)\right] - \left[\sum_{i=1}^{n} r_i f\left(\frac{x-\sigma_i}{\lambda}\right)\right]\left[\sum_{i=1}^{n} r_i f'\left(\frac{x-\mu_i}{\lambda}\right)\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} r_i r_j f\left(\frac{x-\mu_i}{\lambda}\right) f'\left(\frac{x-\sigma_j}{\lambda}\right) - \sum_{i=1}^{n}\sum_{j=1}^{n} r_i r_j f\left(\frac{x-\sigma_i}{\lambda}\right) f'\left(\frac{x-\mu_j}{\lambda}\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} r_i r_j \left[f\left(\frac{x-\mu_i}{\lambda}\right) f'\left(\frac{x-\sigma_j}{\lambda}\right) - f\left(\frac{x-\sigma_j}{\lambda}\right) f'\left(\frac{x-\mu_i}{\lambda}\right)\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} r_i r_j f\left(\frac{x-\mu_i}{\lambda}\right) f\left(\frac{x-\sigma_j}{\lambda}\right)\left[\frac{f'\left(\frac{x-\sigma_j}{\lambda}\right)}{f\left(\frac{x-\sigma_j}{\lambda}\right)} - \frac{f'\left(\frac{x-\mu_i}{\lambda}\right)}{f\left(\frac{x-\mu_i}{\lambda}\right)}\right]. \end{aligned} \tag{22}$$

Under the assumption made, $\max\{\mu_1, \ldots, \mu_n\} \le \min\{\sigma_1, \ldots, \sigma_n\}$ implies that $\mu_i \le \sigma_j$, $i,\ j = 1, \ldots, n$. After some calculations and using the log-concavity property of $f(t)$, it can be established that

$$\frac{x-\mu_i}{\lambda} \ge \frac{x-\sigma_j}{\lambda} \Rightarrow \frac{f'(\frac{x-\mu_i}{\lambda})}{f(\frac{x-\mu_i}{\lambda})} \le \frac{f'(\frac{x-\sigma_j}{\lambda})}{f(\frac{x-\sigma_j}{\lambda})}. \tag{23}$$

Using the inequality given by Eq. 23 in Eq. 22, we obtain $\chi_1'(x) \geq 0$, which implies that $\chi_1(x)$ is increasing with respect to $x \in (\sigma_1, \infty)$. Further, we have to establish that

$$\frac{l_{n-1}(x)}{l^*_{n-1}(x)} = \frac{\sum_{i=2}^{n} r_i f\left(\frac{x-\sigma_i}{\lambda}\right)}{\sum_{i=2}^{n} r_i f\left(\frac{x-\mu_i}{\lambda}\right)} = \chi_2(x),\ (say), \tag{24}$$

is increasing with respect to x in $(\sigma_2, \sigma_1]$. Now, differentiating $\chi_2(x)$ with respect to x and after some calculations, we obtain

$$\chi_2'(x) \stackrel{sign}{=} \sum_{i=2}^{n}\sum_{j=2}^{n} r_i r_j f\left(\frac{x-\mu_i}{\lambda}\right) f\left(\frac{x-\sigma_j}{\lambda}\right)\left[\frac{f'\left(\frac{x-\sigma_j}{\lambda}\right)}{f\left(\frac{x-\sigma_j}{\lambda}\right)} - \frac{f'\left(\frac{x-\mu_i}{\lambda}\right)}{f\left(\frac{x-\mu_i}{\lambda}\right)}\right]. \tag{25}$$

Under the assumption made, $\max\{\mu_2, \ldots, \mu_n\} \leq \min\{\sigma_2, \ldots, \sigma_n\}$ implies that $\mu_i \leq \sigma_j$, $i,\ j = 2, \ldots, n$. After some calculations and using the log-concavity property of $f(t)$, it can be established that

$$\frac{x-\mu_i}{\lambda} \geq \frac{x-\sigma_j}{\lambda} \Rightarrow \frac{f'(\frac{x-\mu_i}{\lambda})}{f(\frac{x-\mu_i}{\lambda})} \leq \frac{f'(\frac{x-\sigma_j}{\lambda})}{f(\frac{x-\sigma_j}{\lambda})}. \tag{26}$$

Substituting Eq. 26 in Eq. 25, we get $\chi_2'(x) \geq 0$, implying that $\chi_2(x)$ is increasing with respect to $x \in (\sigma_2, \sigma_1]$. Using similar arguments, it can be established that $\chi_3(x) = l_{n-2}(x)/l^*_{n-2}(x), \ldots, \chi_{n-1}(x) = l_2(x)/l^*_2(x)$, $\chi_n(x) = l_1(x)/l^*_1(x)$ are also increasing with respect to x belongs to the other subintervals, say $(\sigma_3, \sigma_2], \ldots, (\sigma_{n-1}, \sigma_{n-2}]$, $(\sigma_n, \sigma_{n-1}]$, respectively. Also, $0/l^*_n(x), \ldots, 0/l^*_2(x)$, $0/l^*_1(x)$ are increasing with respect to x belongs to the another subintervals, say $(\mu_1, \sigma_n], \ldots, (\mu_{n-1}, \mu_{n-2}]$, $(\mu_n, \mu_{n-1}]$, respectively. Hence, the theorem is proved. □

Remark 7 If we consider $\boldsymbol{\sigma} \in \mathcal{E}_n^+$ and $\boldsymbol{\mu} \in \mathcal{E}_n^+$ instead of $\boldsymbol{\sigma} \in \mathcal{D}_n^+$ and $\boldsymbol{\mu} \in \mathcal{D}_n^+$, then the established result in Theorem 7 also holds.

Consider Weibull distribution with pdf

$$f(t) = ct^{c-1}e^{-t^c},\ t > 0,\ c > 0. \tag{27}$$

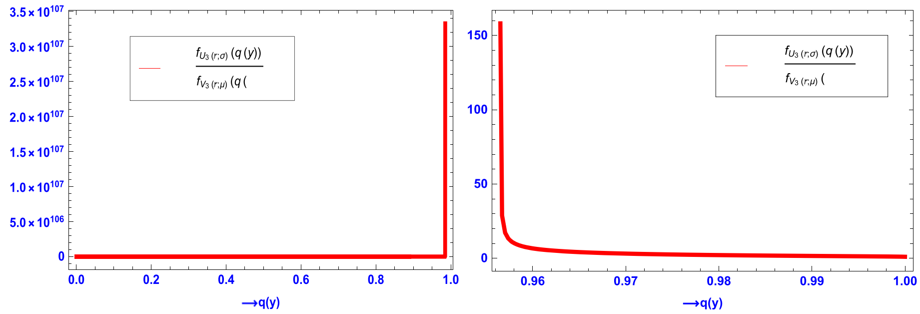
Taking logarithm both sides of Eq. 27 and differentiating twice with respect to t, we obtain

$$(\ln f(t))'' = -(c-1)\frac{1}{t^2}\left(1 + ct^c\right) = \begin{cases} \geq 0, \text{ if } 0 < c \leq 1 \\ < 0, \text{ if } c > 1. \end{cases} \tag{28}$$

Thus, clearly the pdf of Weibull distribution is log-convex when $c \in (0, 1]$ and log-concave when $c \in (1, \infty)$. To illustrate the result in Theorem 7, we consider the following example.

Example 3

(i) Consider Weibull distribution with $pdf\ f(t) = 2te^{-t^2}$, $0 < t < \infty$ as the baseline distribution. Here, $f(t)$ is log-concave with respect to $t > 0$. Assume that $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (22, 18, 16) \in \mathcal{D}_3^+$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) = (12, 8, 2) \in \mathcal{D}_3^+$. Further, take $\boldsymbol{r} = (r_1, r_2, r_3) = (0.1, 0.7, 0.2)$ and $\lambda = 2$. Clearly, all the assumptions of Theorem 7 are satisfied. Now, the ratio of pdfs of the MRVs $U_3(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_3(\boldsymbol{r}; \boldsymbol{\mu})$ is plotted in Fig. 3(a), from which we have $U_3(\boldsymbol{r}; \boldsymbol{\sigma}) \geq_{lr} V_3(\boldsymbol{r}; \boldsymbol{\mu})$. Here, we have considered $q(y) = \frac{y}{1-y}$, $0 < y < 1$ to capture the whole positive real axis.



(a) (b)

Fig. 3 (a) Graph of the ratio of pdfs of $U_3(\boldsymbol{r};\boldsymbol{\sigma})$ and $V_3(\boldsymbol{r};\boldsymbol{\mu})$ as in Example 3(i). (b) Plot of the ratio of pdfs of $U_3(\boldsymbol{r};\boldsymbol{\sigma})$ and $V_3(\boldsymbol{r};\boldsymbol{\mu})$ as in Example 3(ii)

(ii) We take Weibull distribution with pdf $f(t)=\frac{1}{2}t^{-1/2}e^{-t^{1/2}}$, $0<t<\infty$ as the baseline distribution, which is log-convex with respect to $t>0$. Considering the same numerical values of the parameters as in the previous case, the ratio of pdfs of $U_3(\boldsymbol{r};\boldsymbol{\sigma})$ and $V_3(\boldsymbol{r};\boldsymbol{\mu})$ is plotted in Fig. 3(b), which assures $U_3(\boldsymbol{r};\boldsymbol{\sigma})\leq_{lr}V_3(\boldsymbol{r};\boldsymbol{\mu})$. Similar to the previous case, here, we consider $q(y)=\frac{y}{1-y}$, $0<y<1$ to capture the whole positive real axis.

The following counterexample shows that the result in Theorem 7 may not be true for the likelihood ratio order if the sufficient condition is not satisfied.

Counterexample 6 Consider log-normal distribution as the baseline distribution with pdf $f(t)=\frac{1}{t\sqrt{2\pi}}e^{-(\ln t)^2/2}$, $t>0$. It can be easily seen that $f(t)$ is neither log-concave nor log-convex on its entire domain. Assume $\boldsymbol{r}=(r_1,r_2,r_3)=(0.2,0.7,0.1)$, $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)=(4.8,7.3,10.2)\in\mathcal{E}_3^+$, $\boldsymbol{\mu}=(\mu_1,\mu_2,\mu_3)=(1.2,2.5,3.9)\in\mathcal{E}_3^+$, and $\lambda=12$. Clearly, $\max\{\mu_1,\mu_2,\mu_3\}\leq\min\{\sigma_1,\sigma_2,\sigma_3\}$. The ratio of pdfs of the MRVs $U_3(\boldsymbol{r};\boldsymbol{\sigma})$ and $V_3(\boldsymbol{r};\boldsymbol{\mu})$, denoted as $f_{U_3(\boldsymbol{r};\boldsymbol{\sigma})}(x)/f_{V_3(\boldsymbol{r};\boldsymbol{\mu})}(x)$, is plotted in Fig. 6(b). It can be seen that the ratio function is nonmonotone in $t>0$, which means that $U_3(\boldsymbol{r};\boldsymbol{\sigma})\ngeq_{lr}V_3(\boldsymbol{r};\boldsymbol{\mu})$ and $U_3(\boldsymbol{r};\boldsymbol{\sigma})\nleq_{lr}V_3(\boldsymbol{r};\boldsymbol{\mu})$.

It is worth pointing that there are various other distributions, for which the pdfs satisfy log-convexity or log-concavity properties. In the following remark, we consider three such distributions.

Remark 8 (i) Pareto distribution: Consider Pareto distribution with pdf $f(t)=\beta t^{-\beta-1}$, $1\leq t<\infty$, $\beta>0$. After some calculations, we get $(\ln f(t))'=-\frac{\beta+1}{t}$ and $(\ln f(t))''=\frac{\beta+1}{t^2}>0$, for all $1\leq t<\infty$ and $\beta>0$, which implies that the density function $f(t)$ is log-convex with respect to $t\in[1,\infty)$.

(ii) Power distribution: Take power distribution with pdf $f(t)=ct^{c-1}$, $0<t\leq 1$, $0<c<1$. Some calculations lead to $(\ln f(t))'=\frac{c-1}{t}$ and $(\ln f(t))''=-\frac{c-1}{t^2}>0$, for all $0<c<1$. This observation proves that $f(t)$ is log-convex with respect to $t\in(0,1]$.

(iii) Gamma distribution: Assume gamma distribution with pdf $f(t)=\frac{1}{\Gamma(d)}t^{d-1}e^{-t}$, $t>0$, $d>0$, where $\Gamma(d)$ is an Eular gamma function. After differentiating $\ln f(t)$ with respect to t twice, we get $(\ln f(t))''=\frac{1-d}{t^2}$, which is ≥ 0 when $d\in(0,1]$ and <0 when $d\in(1,\infty)$. Thus, the density function $f(t)$ is log-convex and log-concave with respect to t, when $d\in(0,1]$ and $d\in(1,\infty)$, respectively.

3.2 Heterogeneity in Two Parameters

In the previous subsection, we have proposed sufficient conditions for comparing MRVs in various stochastic senses when there is heterogeneity in one model parameter. Here, we assume that heterogeneity exists in two model parameters. The next result shows that usual stochastic order between two MRVs $U_n(\boldsymbol{r};\boldsymbol{\lambda})$ and $V_n(\boldsymbol{s};\boldsymbol{\theta})$ holds when mixing proportions r_i, s_i and scale parameters λ_i, θ_i are varying.

Theorem 8 *Let* $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) = \sum_{i=1}^n r_i[1 - F(\frac{x-\sigma}{\lambda_i})]$ *and* $\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\theta})}(x) = \sum_{i=1}^n s_i[1 - F(\frac{x-\sigma}{\theta_i})]$ *be the sfs of two MRVs* $U_n(\boldsymbol{r};\boldsymbol{\lambda})$ *and* $V_n(\boldsymbol{s};\boldsymbol{\theta})$*, respectively. Further, suppose* $t^2 f(t)$ *is increasing in* $t > 0$*. Then, for* $\boldsymbol{r} \preccurlyeq_w \boldsymbol{s}$*,* $\boldsymbol{\lambda} \preccurlyeq_w \boldsymbol{\theta}$*,* $\boldsymbol{r}, \boldsymbol{s} \in \mathcal{E}_n^+$*,* $\boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathcal{D}_n^+$ *and for fixed* $\sigma > 0$*, we have*

$$U_n(\boldsymbol{r};\boldsymbol{\lambda}) \geq_{st} V_n(\boldsymbol{s};\boldsymbol{\theta}).$$

Proof To prove the result, we consider a MRV $W_n(\boldsymbol{s};\boldsymbol{\lambda})$ with sf

$$\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x) = \sum_{i=1}^n s_i \left[1 - F\left(\frac{x-\sigma}{\lambda_i}\right)\right],$$

where $F(.)$ is the cdf of the baseline distribution. Now, our aim is to show that

$$\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) \geq \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x) \geq \bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\theta})}(x). \tag{29}$$

To establish the first inequality in Eq. 29, we differentiate $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ with respect to r_i and obtain as

$$\frac{\partial \bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_i} = 1 - F\left(\frac{x-\sigma}{\lambda_i}\right) \geq 0, \tag{30}$$

implies $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ is increasing with respect to r_i, $i = 1, \ldots, n$. Further, consider

$$\begin{aligned}\nabla_{11} &= \frac{\partial \bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_i} - \frac{\partial \bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_j} \\ &= F\left(\frac{x-\sigma}{\lambda_j}\right) - F\left(\frac{x-\sigma}{\lambda_i}\right). \end{aligned} \tag{31}$$

Under the assumptions made, we have $\boldsymbol{r} \in \mathcal{E}_n^+$ and $\boldsymbol{\lambda} \in \mathcal{D}_n^+$. That is, $r_i \leq r_j$ and $\lambda_i \geq \lambda_j$ for $1 \leq i \leq j \leq n$. Then, clearly

$$F\left(\frac{x-\sigma}{\lambda_j}\right) \geq F\left(\frac{x-\sigma}{\lambda_i}\right). \tag{32}$$

Thus, for $1 \leq i \leq j \leq n$, $\boldsymbol{r} \in \mathcal{E}_n^+$ and $\boldsymbol{\lambda} \in \mathcal{D}_n^+$, using Eq. 32 in Eq. 31, we obtain $\nabla_{11} \geq 0$, giving that $\frac{\partial \bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_k}$ is decreasing in $k \in \{1, \ldots, n\}$. Hence, by using Part (ii) of Lemma 6 with the help of Remark 1, $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ is Schur-concave with respect to $\boldsymbol{r} \in \mathcal{E}_n^+$. Hence,

$$\boldsymbol{r} \preccurlyeq_w \boldsymbol{s} \Rightarrow \bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) \geq \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x). \tag{33}$$

Furthermore, to show the second inequality in Eq. 29, let us differentiate $\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x)$ with respect to λ_i and obtain

$$\frac{\partial \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x)}{\partial \lambda_i} = (x-\sigma)\frac{1}{\lambda_i^2} s_i f\left(\frac{x-\sigma}{\lambda_i}\right) \geq 0. \tag{34}$$

Thus, $\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x)$ is increasing with respect to λ_i, $i = 1, \ldots, n$. Now, let

$$\nabla_{12} = \frac{\partial \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x)}{\partial \lambda_i} - \frac{\partial \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x)}{\partial \lambda_j} \overset{sign}{=} s_i \left(\frac{x-\sigma}{\lambda_i}\right)^2 f\left(\frac{x-\sigma}{\lambda_i}\right) - s_j \left(\frac{x-\sigma}{\lambda_j}\right)^2 f\left(\frac{x-\sigma}{\lambda_j}\right). \tag{35}$$

For $1 \le i \le j \le n$, consider $\boldsymbol{s} \in \mathcal{E}_n^+$ and $\boldsymbol{\lambda} \in \mathcal{D}_n^+$, i.e., $s_i \le s_j$ and $\lambda_i \ge \lambda_j$. Then, clearly $\frac{x-\sigma}{\lambda_i} \le \frac{x-\sigma}{\lambda_j}$, and since $t^2 f(t)$ is increasing in $t > 0$, we have

$$\left(\frac{x-\sigma}{\lambda_i}\right)^2 f\left(\frac{x-\sigma}{\lambda_i}\right) \le \left(\frac{x-\sigma}{\lambda_j}\right)^2 f\left(\frac{x-\sigma}{\lambda_j}\right). \tag{36}$$

Substituting Eq. 36 in Eq. 35 with $s_i \le s_j$, we have $\nabla_{12} \le 0$, giving that $\frac{\partial \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x)}{\partial \lambda_k}$ is increasing in $k \in \{1, \ldots, n\}$. Thus, by using Part (i) of Lemma 6 with the help of Remark 1, $\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x)$ is Schur-concave with respect to $\boldsymbol{\lambda} \in \mathcal{D}_n^+$. As a result,

$$\boldsymbol{\lambda} \preccurlyeq_w \boldsymbol{\theta} \Rightarrow \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\lambda})}(x) \ge \bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\theta})}(x). \tag{37}$$

Thus, the desired result follows after combining Eqs. 33 and 37. The proof is completed. □

Remark 9 If we consider $\boldsymbol{r} \in \mathcal{D}_n^+$ and $\boldsymbol{\lambda} \in \mathcal{E}_n^+$ instead of $\boldsymbol{r} \in \mathcal{E}_n^+$ and $\boldsymbol{\lambda} \in \mathcal{D}_n^+$, then the established result in Theorem 8 also holds.

The following example provides an illustration of the result stated in Theorem 8.

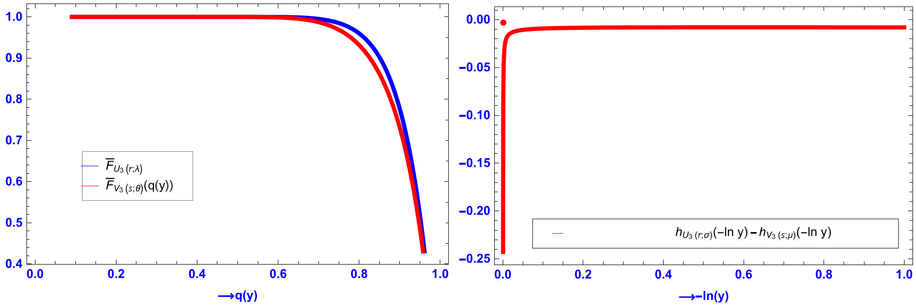
Example 4 Consider inverted exponential distribution with cdf $F(t) = e^{-\beta/t}$, $t > 0,\ \beta > 0$ and pdf $f(t) = \frac{\beta}{t^2} e^{-\beta/t}$, $t > 0,\ \beta > 0$ as the baseline distribution. Here, the function $t^2 f(t)$ is increasing with respect to t in its domain. Now, we assume that $\boldsymbol{r} = (r_1, r_2, r_3) = (0.2, 0.3, 0.5) \in \mathcal{E}_3^+$, $\boldsymbol{s} = (s_1, s_2, s_3) = (0.1, 0.3, 0.6) \in \mathcal{E}_3^+$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (10.2, 8.3, 5.2) \in \mathcal{D}_3^+$, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) = (12.5, 9.8, 4.3) \in \mathcal{D}_3^+$. Further, take $\sigma = 0.1$. Clearly, $\boldsymbol{r} \preccurlyeq_w \boldsymbol{s}$ and $\boldsymbol{\lambda} \preccurlyeq_w \boldsymbol{\theta}$. Now, the difference between the sfs of $U_3(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_3(\boldsymbol{s}; \boldsymbol{\theta})$ is plotted (for $\beta = 2$) in Fig. 4(a), from which we can readily observe that $U_3(\boldsymbol{r}; \boldsymbol{\lambda}) \ge_{st} V_3(\boldsymbol{s}; \boldsymbol{\theta})$, which confirms the result in Theorem 8. Here, we take $q(y) = \frac{y}{1-y}$, $0 < y < 1$ to capture the whole positive real axis.

In the previous theorem, we have considered different scale parameters and mixing proportions. In the following result, we deal with the models having different location parameters and mixing proportions.

Theorem 9 *Let $U_n(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_n(\boldsymbol{s}; \boldsymbol{\mu})$ be two MRVs with different mixing proportion vectors $\boldsymbol{r}$ and $\boldsymbol{s}$, constructed from two sets of nonnegative RVs $\{X_{\sigma_1}, \ldots, X_{\sigma_n}\}$ and $\{X_{\mu_1}, \ldots, X_{\mu_n}\}$, respectively, such that $X_{\sigma_i} \sim F\left(\frac{x-\sigma_i}{\lambda}\right)$ and $X_{\mu_i} \sim F\left(\frac{x-\mu_i}{\lambda}\right)$, for $i = 1, \ldots, n$. Further, let $tf(t)$ be decreasing in $t > 0$. Then, for $\boldsymbol{r} \overset{m}{\preccurlyeq} \boldsymbol{s}$, $\boldsymbol{\sigma} \preccurlyeq_w \boldsymbol{\mu}$, $\boldsymbol{r},\ \boldsymbol{\sigma} \in \mathcal{E}_n^+$ and for fixed $\lambda > 0$, we have*

$$U_n(\boldsymbol{r}; \boldsymbol{\sigma}) \le_{st} V_n(\boldsymbol{s}; \boldsymbol{\mu}),$$

provided that $\max\{\sigma_1, \ldots, \sigma_n\} \le \min\{\mu_1, \ldots, \mu_n\}$.



(a) (b)

Fig. 4 (a) Graphs of the sfs of $U_3(\boldsymbol{r};\boldsymbol{\lambda})$ (in blue colour) and $V_3(\boldsymbol{s};\boldsymbol{\theta})$ (in red colour) as in Example 4. (b) Graph of the difference between the hazard rate functions of $U_3(\boldsymbol{r};\boldsymbol{\sigma})$ and $V_3(\boldsymbol{s};\boldsymbol{\mu})$ as in Example 5

Proof We consider a MRV, denoted by $W_n(\boldsymbol{s};\boldsymbol{\sigma})$ with sf given by

$$\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x) = \begin{cases} 1 - s_n F\left(\frac{x-\sigma_n}{\lambda}\right) & ;\text{if} \quad \sigma_n < x \le \sigma_{n-1} \\ 1 - \left[s_n F\left(\frac{x-\sigma_n}{\lambda}\right) + s_{n-1} F\left(\frac{x-\sigma_{n-1}}{\lambda}\right)\right] & ;\text{if} \quad \sigma_{n-1} < x \le \sigma_{n-2} \\ \vdots & \vdots \\ 1 - \left[s_n F\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + s_2 F\left(\frac{x-\sigma_2}{\lambda}\right)\right] & ;\text{if} \quad \sigma_2 < x \le \sigma_1 \\ 1 - \left[s_n F\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + s_1 F\left(\frac{x-\sigma_1}{\lambda}\right)\right] & ;\text{if} \quad \sigma_1 < x < \infty. \end{cases} \tag{38}$$

Thus, to prove the result, it suffices to show that

$$\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x) \le \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x) \le \bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\mu})}(x). \tag{39}$$

To prove the first inequality in Eq. 39 by using Definition 3, our goal is to show that $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ is Schur-convex with respect to $\boldsymbol{r} \in \mathcal{E}_n^+$. Consider $\sigma_1 < x < \infty$. The derivative of $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ with respect to r_i, $i = 1, \ldots, n$ is given by

$$\frac{\partial \bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_i} = -F\left(\frac{x-\sigma_i}{\lambda}\right). \tag{40}$$

Now, to show $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ is Schur-convex with respect to $\boldsymbol{r}$, it is enough to prove that, for $1 \le i \le j \le n$

$$\begin{aligned} \nabla_{13} &= \frac{\partial \bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_i} - \frac{\partial \bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_j} \\ &= F\left(\frac{x-\sigma_j}{\lambda}\right) - F\left(\frac{x-\sigma_i}{\lambda}\right) \le 0, \end{aligned} \tag{41}$$

which holds since $\boldsymbol{r} \in \mathcal{E}_n^+$ and $\boldsymbol{\sigma} \in \mathcal{E}_n^+$ i.e., $r_i \le r_j$ and $\sigma_i \le \sigma_j$, and $F(.)$ is nondecreasing. Thus, for $1 \le i \le j \le n$, $\boldsymbol{r} \in \mathcal{E}_n^+$ and $\boldsymbol{\sigma} \in \mathcal{E}_n^+$, we obtain $\nabla_{13} \le 0$, giving that $\frac{\partial \bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_k}$

is increasing in $k \in \{1, \ldots, n\}$. Thus, from Part (ii) of Lemma 6 with the help of Remark 1, $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ is Schur-convex with respect to $\boldsymbol{r} \in \mathcal{E}_n^+$. Using similar arguments, it can be established that $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ is also Schur-convex with respect to $\boldsymbol{r} \in \mathcal{E}_n^+$, where x belongs to the other subintervals, say $\sigma_2 < x \leq \sigma_1, \ldots, \sigma_{n-1} < x \leq \sigma_{n-2}$ and $\sigma_n < x \leq \sigma_{n-1}$. Furthermore, similar to the first inequality, to establish $\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x) \leq \bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\mu})}(x)$ by using Lemma 1, it is required to show that $\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)$ is increasing and Schur-convex with respect to $\boldsymbol{\sigma}$, whenever $x \in (\sigma_n, \infty) \cup (\mu_n, \infty)$ i.e., $x \in (\sigma_n, \infty)$ (according to the given condition $\max\{\sigma_1, \ldots, \sigma_n\} \leq \min\{\mu_1, \ldots, \mu_n\}$). Consider $\sigma_1 < x < \infty$. In doing so, the first order derivative of $\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)$ with respect to σ_i, for $i = 1, \ldots, n$ is given by

$$\frac{\partial \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)}{\partial \sigma_i} = \frac{1}{\lambda} s_i f\left(\frac{x-\sigma_i}{\lambda}\right) \geq 0. \tag{42}$$

Again, consider

$$\begin{aligned}\nabla_{14} &= \frac{\partial \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)}{\partial \sigma_i} - \frac{\partial \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)}{\partial \sigma_j} \\ &= \left(\frac{s_i}{x-\sigma_i}\right)\left(\frac{x-\sigma_i}{\lambda}\right) f\left(\frac{x-\sigma_i}{\lambda}\right) - \left(\frac{s_j}{x-\sigma_j}\right)\left(\frac{x-\sigma_j}{\lambda}\right) f\left(\frac{x-\sigma_j}{\lambda}\right) \leq 0, \end{aligned} \tag{43}$$

since $\boldsymbol{s} \in \mathcal{E}_n^+$ and $\boldsymbol{\sigma} \in \mathcal{E}_n^+$ i.e., $s_i \leq s_j$ and $\sigma_i \leq \sigma_j$, and $tf(t)$ is decreasing in t. Thus, for $1 \leq i \leq j \leq n$, $\boldsymbol{s} \in \mathcal{E}_n^+$ and $\boldsymbol{\sigma} \in \mathcal{E}_n^+$, we obtain $\nabla_{14} \leq 0$, giving that $\frac{\partial \bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)}{\partial \sigma_k}$ is increasing in $k \in \{1, \ldots, n\}$. Thus, from Part (ii) of Lemma 6 with the help of Remark 1, $\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)$ is Schur-convex with respect to $\boldsymbol{\sigma} \in \mathcal{E}_n^+$. Using similar arguments, it can be established that $\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)$ is also Schur-convex with respect to $\boldsymbol{\sigma} \in \mathcal{E}_n^+$, where x belongs to the other subintervals, say $\sigma_2 < x \leq \sigma_1, \ldots, \sigma_{n-1} < x \leq \sigma_{n-2}$ and $\sigma_n < x \leq \sigma_{n-1}$. This completes the proof of the theorem. □

Remark 10 If we consider $\boldsymbol{r}, \boldsymbol{\sigma} \in \mathcal{D}_n^+$ instead of $\boldsymbol{r}, \boldsymbol{\sigma} \in \mathcal{E}_n^+$, then the established result in Theorem 9 also holds.

The preceding theorem provides sufficient conditions for the usual stochastic order between two MRVs, when location parameters and mixing proportions are different. Thus, it is natural to ask the question "is it possible to extend the usual stochastic order to a stronger stochastic order, say hazard rate order?" The following theorem answers this in affirmative way with different sufficient conditions.

Theorem 10 *Let $U_n(\boldsymbol{r};\boldsymbol{\sigma})$ and $V_n(\boldsymbol{s};\boldsymbol{\mu})$ be two MRVs with different mixing proportion vectors $\boldsymbol{r}$ and $\boldsymbol{s}$, constructed from two sets of nonnegative RVs $\{X_{\sigma_1}, \ldots, X_{\sigma_n}\}$ and $\{X_{\mu_1}, \ldots, X_{\mu_n}\}$, respectively, such that $X_{\sigma_i} \sim F\left(\frac{x-\sigma_i}{\lambda}\right)$ and $X_{\mu_i} \sim F\left(\frac{x-\mu_i}{\lambda}\right)$, for $i = 1, \ldots, n$. Further, let $f(t)$ be increasing in $t > 0$ and $\tilde{h}(t)$ be decreasing with respect to $t > 0$. Then, for $\boldsymbol{r} \overset{m}{\succcurlyeq} \boldsymbol{s}$, $\boldsymbol{r} \in \mathcal{D}_n^+$, $\boldsymbol{\sigma}$, $\boldsymbol{\mu} \in \mathcal{D}_n^+$ and for fixed $\lambda > 0$, we have*

$$U_n(\boldsymbol{r};\boldsymbol{\sigma}) \geq_{hr} V_n(\boldsymbol{s};\boldsymbol{\mu}),$$

provided that $\max\{\mu_1, \ldots, \mu_n\} \leq \min\{\sigma_1, \ldots, \sigma_n\}$.

Proof The hazard rate function of the MRV $U_n(\boldsymbol{r};\boldsymbol{\sigma})$ is given by

$$h_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)=\begin{cases}\frac{r_n\frac{1}{\lambda}f\left(\frac{x-\sigma_n}{\lambda}\right)}{1-r_nF\left(\frac{x-\sigma_n}{\lambda}\right)} & ;\text{ if } \quad \sigma_n<x\le\sigma_{n-1}\\ \frac{r_n\frac{1}{\lambda}f\left(\frac{x-\sigma_n}{\lambda}\right)+r_{n-1}\frac{1}{\lambda}f\left(\frac{x-\sigma_{n-1}}{\lambda}\right)}{1-\left[r_nF\left(\frac{x-\sigma_n}{\lambda}\right)+r_{n-1}F\left(\frac{x-\sigma_{n-1}}{\lambda}\right)\right]} & ;\text{ if } \quad \sigma_{n-1}<x\le\sigma_{n-2}\\ \vdots & \vdots\\ \frac{r_n\frac{1}{\lambda}f\left(\frac{x-\sigma_n}{\lambda}\right)+\ldots+r_2\frac{1}{\lambda}f\left(\frac{x-\sigma_2}{\lambda}\right)}{1-\left[r_nF\left(\frac{x-\sigma_n}{\lambda}\right)+\ldots+r_2F\left(\frac{x-\sigma_2}{\lambda}\right)\right]} & ;\text{ if } \quad \sigma_2<x\le\sigma_1\\ \frac{r_n\frac{1}{\lambda}f\left(\frac{x-\sigma_n}{\lambda}\right)+\ldots+r_1\frac{1}{\lambda}f\left(\frac{x-\sigma_1}{\lambda}\right)}{1-\left[r_nF\left(\frac{x-\sigma_n}{\lambda}\right)+\ldots+r_1F\left(\frac{x-\sigma_1}{\lambda}\right)\right]} & ;\text{ if } \quad \sigma_1<x<\infty.\end{cases}\tag{44}$$

Denote the hazard rate function of another MRV $W_n(\boldsymbol{s};\boldsymbol{\sigma})$ by $h_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)$. Note that the proof of theorem will be completed by establishing two inequalities. Consider $\sigma_1<x<\infty$. Differentiating $h_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ with respect to r_i partially, for $i=1,\ldots,n$, we obtain

$$\frac{\partial h_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_i}\overset{sign}{=}\zeta_2(x;\boldsymbol{r},\boldsymbol{\sigma})f\left(\frac{x-\sigma_i}{\lambda}\right)+\zeta_1(x;\boldsymbol{r},\boldsymbol{\sigma})F\left(\frac{x-\sigma_i}{\lambda}\right),\tag{45}$$

where $\zeta_1(x;\boldsymbol{r},\boldsymbol{\sigma})=\sum_{i=1}^n r_i f\left(\frac{x-\sigma_i}{\lambda}\right)\ge 0$ and $\zeta_2(x;\boldsymbol{r},\boldsymbol{\sigma})=1-\sum_{i=1}^n r_i F\left(\frac{x-\sigma_i}{\lambda}\right)\ge 0$. Further, let us take

$$\begin{aligned}\nabla_{15} &= \frac{\partial h_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_i}-\frac{\partial h_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_j}\\ &\overset{sign}{=}\zeta_1(x;\boldsymbol{r},\boldsymbol{\sigma})T_1+\zeta_2(x;\boldsymbol{r},\boldsymbol{\sigma})T_2,\end{aligned}\tag{46}$$

where $T_1=F\left(\frac{x-\sigma_i}{\lambda}\right)-F\left(\frac{x-\sigma_j}{\lambda}\right)$ and $T_2=f\left(\frac{x-\sigma_i}{\lambda}\right)-f\left(\frac{x-\sigma_j}{\lambda}\right)$. Consider $1\le i\le j\le n$. Without loss of generality, let $\boldsymbol{r}\in\mathcal{D}_n^+$ and $\boldsymbol{\sigma}\in\mathcal{D}_n^+$, that is, $r_i\ge r_j$ and $\sigma_i\ge\sigma_j$. Now, from $\sigma_i\ge\sigma_j$, we get $\frac{x-\sigma_i}{\lambda}\le\frac{x-\sigma_j}{\lambda}$. Then, we obtain

$$F\left(\frac{x-\sigma_i}{\lambda}\right)\le F\left(\frac{x-\sigma_j}{\lambda}\right),\tag{47}$$

that is, $T_1\le 0$. Under the assumptions made, it holds that

$$f\left(\frac{x-\sigma_i}{\lambda}\right)\le f\left(\frac{x-\sigma_j}{\lambda}\right),\tag{48}$$

that is, $T_2\le 0$. Thus, for $1\le i\le j\le n$, $\boldsymbol{r}\in\mathcal{D}_n^+$ and $\boldsymbol{\sigma}\in\mathcal{D}_n^+$, we have $\nabla_{15}\le 0$, giving that $\frac{\partial h_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)}{\partial r_k}$ is increasing in $k\in\{1,\ldots,n\}$. Hence, by using Part (i) of Lemma 6 with the help of Remark 1, $h_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ is Schur-concave with respect to $\boldsymbol{r}\in\mathcal{D}_n^+$. Thus, from Definition 3, we have

$$\boldsymbol{r}\overset{m}{\succcurlyeq}\boldsymbol{s}\Rightarrow h_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)\le h_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x).\tag{49}$$

Using similar arguments, it can be established that $h_{U_n(\boldsymbol{r};\boldsymbol{\sigma})}(x)$ is also Schur-concave with respect to $\boldsymbol{r} \in \mathcal{D}_n^+$ when x belongs to the other subintervals, say $\sigma_2 < x \leq \sigma_1$, $\sigma_3 < x \leq \sigma_2, \ldots, \sigma_{n-1} < x \leq \sigma_{n-2}, \sigma_n < x \leq \sigma_{n-1}$. Next, our target is to show that $h_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x) \leq h_{V_n(\boldsymbol{s};\boldsymbol{\mu})}(x)$. In order to obtain the inequality $W_n(\boldsymbol{s};\boldsymbol{\sigma}) \geq_{hr} V_n(\boldsymbol{s};\boldsymbol{\mu})$, it suffices to show that

$$\frac{\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)}{\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\mu})}(x)} = \begin{cases} \frac{0}{m_1^*(x)} = \frac{0}{1-s_nF\left(\frac{x-\mu_n}{\lambda}\right)} = 0 & ;\text{if} \quad \mu_n < x \leq \mu_{n-1} \\ \frac{0}{m_2^*(x)} = \frac{0}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+s_{n-1}F\left(\frac{x-\mu_{n-1}}{\lambda}\right)\right]} = 0 & ;\text{if} \quad \mu_{n-1} < x \leq \mu_{n-2} \\ \frac{0}{m_3^*(x)} = \frac{0}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+s_{n-2}F\left(\frac{x-\mu_{n-2}}{\lambda}\right)\right]} = 0 & ;\text{if} \quad \mu_{n-2} < x \leq \mu_{n-3} \\ \qquad\vdots & \qquad\vdots \\ \frac{0}{m_{n-2}^*(x)} = \frac{0}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+s_3F\left(\frac{x-\mu_3}{\lambda}\right)\right]} = 0 & ;\text{if} \quad \mu_3 < x \leq \mu_2 \\ \frac{0}{m_{n-1}^*(x)} = \frac{0}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+\ldots+s_2F\left(\frac{x-\mu_2}{\lambda}\right)\right]} = 0 & ;\text{if} \quad \mu_2 < x \leq \mu_1 \\ \frac{0}{m_n^*(x)} = \frac{0}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+\ldots+s_1F\left(\frac{x-\mu_1}{\lambda}\right)\right]} = 0 & ;\text{if} \quad \mu_1 < x \leq \sigma_n \\ \frac{m_1(x)}{m_1^*(x)} = \frac{1-s_nF\left(\frac{x-\sigma_n}{\lambda}\right)}{1-s_nF\left(\frac{x-\mu_n}{\lambda}\right)} & ;\text{if} \quad \sigma_n < x \leq \sigma_{n-1} \\ \frac{m_2(x)}{m_2^*(x)} = \frac{1-\left[s_nF\left(\frac{x-\sigma_n}{\lambda}\right)+s_{n-1}F\left(\frac{x-\sigma_{n-1}}{\lambda}\right)\right]}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+s_{n-1}F\left(\frac{x-\mu_{n-1}}{\lambda}\right)\right]} & ;\text{if} \quad \sigma_{n-1} < x \leq \sigma_{n-2} \\ \frac{m_3(x)}{m_3^*(x)} = \frac{1-\left[s_nF\left(\frac{x-\sigma_n}{\lambda}\right)+s_{n-2}F\left(\frac{x-\sigma_{n-2}}{\lambda}\right)\right]}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+s_{n-2}F\left(\frac{x-\mu_{n-2}}{\lambda}\right)\right]} & ;\text{if} \quad \sigma_{n-2} < x \leq \sigma_{n-3} \\ \qquad\vdots & \qquad\vdots \\ \frac{m_{n-2}(x)}{m_{n-2}^*(x)} = \frac{1-\left[s_nF\left(\frac{x-\sigma_n}{\lambda}\right)+\ldots+s_3F\left(\frac{x-\sigma_3}{\lambda}\right)\right]}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+\ldots+s_3F\left(\frac{x-\mu_3}{\lambda}\right)\right]} & ;\text{if} \quad \sigma_3 < x \leq \sigma_2 \\ \frac{m_{n-1}(x)}{m_{n-1}^*(x)} = \frac{1-\left[s_nF\left(\frac{x-\sigma_n}{\lambda}\right)+\ldots+s_2F\left(\frac{x-\sigma_2}{\lambda}\right)\right]}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+\ldots+s_2F\left(\frac{x-\mu_2}{\lambda}\right)\right]} & ;\text{if} \quad \sigma_2 < x \leq \sigma_1 \\ \frac{m_n(x)}{m_n^*(x)} = \frac{1-\left[s_nF\left(\frac{x-\sigma_n}{\lambda}\right)+\ldots+s_1F\left(\frac{x-\sigma_1}{\lambda}\right)\right]}{1-\left[s_nF\left(\frac{x-\mu_n}{\lambda}\right)+\ldots+s_1F\left(\frac{x-\mu_1}{\lambda}\right)\right]} & ;\text{if} \quad \sigma_1 < x < \infty, \end{cases} \tag{50}$$

is increasing in $x \in (\sigma_n, \infty) \cup (\mu_n, \infty)$ i.e., $x \in (\mu_n, \infty)$, where $\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)$ and $\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\mu})}(x)$ are given by

$$\bar{F}_{W_n(\boldsymbol{s};\boldsymbol{\sigma})}(x)=\begin{cases} m_1(x) = 1 - s_n F\left(\frac{x-\sigma_n}{\lambda}\right) & ;\text{if} \quad \sigma_n < x \le \sigma_{n-1} \\ m_2(x)=1-\left[s_n F\left(\frac{x-\sigma_n}{\lambda}\right)+s_{n-1} F\left(\frac{x-\sigma_{n-1}}{\lambda}\right)\right] & ;\text{if} \quad \sigma_{n-1} < x \le \sigma_{n-2} \\ \vdots & \vdots \\ m_{n-1}(x) = 1 - \left[s_n F\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + s_2 F\left(\frac{x-\sigma_2}{\lambda}\right)\right] & ;\text{if} \quad \sigma_2 < x \le \sigma_1 \\ m_n(x) = 1 - \left[s_n F\left(\frac{x-\sigma_n}{\lambda}\right) + \ldots + s_1 F\left(\frac{x-\sigma_1}{\lambda}\right)\right] & ;\text{if} \quad \sigma_1 < x < \infty \end{cases} \tag{51}$$

and

$$\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\mu})}(x)=\begin{cases} m_1^*(x) = 1 - s_n F\left(\frac{x-\mu_n}{\lambda}\right) & ;\text{if} \quad \mu_n < x \le \mu_{n-1} \\ m_2^*(x)=1-\left[s_n F\left(\frac{x-\mu_n}{\lambda}\right)+s_{n-1} F\left(\frac{x-\mu_{n-1}}{\lambda}\right)\right] & ;\text{if} \quad \mu_{n-1} < x \le \mu_{n-2} \\ \vdots & \vdots \\ m_{n-1}^*(x) = 1 - \left[s_n F\left(\frac{x-\mu_n}{\lambda}\right) + \ldots + s_2 F\left(\frac{x-\mu_2}{\lambda}\right)\right] & ;\text{if} \quad \mu_2 < x \le \mu_1 \\ m_n^*(x) = 1 - \left[s_n F\left(\frac{x-\mu_n}{\lambda}\right) + \ldots + s_1 F\left(\frac{x-\mu_1}{\lambda}\right)\right] & ;\text{if} \quad \mu_1 < x < \infty, \end{cases} \tag{52}$$

respectively. First we have to establish that

$$\frac{m_n(x)}{m_n^*(x)} = \frac{1 - \sum_{i=1}^{n} s_i F\left(\frac{x-\sigma_i}{\lambda}\right)}{1 - \sum_{i=1}^{n} s_i F\left(\frac{x-\mu_i}{\lambda}\right)} = \chi_1(x), \ (say), \tag{53}$$

is increasing with respect to $x \in (\sigma_1, \infty)$. For this, differentiating $\chi_1(x)$ with respect to x, we obtain

$$\begin{aligned} \chi_1'(x) &\overset{sign}{=} \left[1 - \sum_{i=1}^{n} s_i F\left(\frac{x-\mu_i}{\lambda}\right)\right]\left[-\sum_{i=1}^{n} s_i f\left(\frac{x-\sigma_i}{\lambda}\right)\frac{1}{\lambda}\right] \\ &\quad - \left[1 - \sum_{i=1}^{n} s_i F\left(\frac{x-\sigma_i}{\lambda}\right)\right]\left[-\sum_{i=1}^{n} s_i f\left(\frac{x-\mu_i}{\lambda}\right)\frac{1}{\lambda}\right] \\ &\overset{sign}{=} \left[1 - \sum_{i=1}^{n} s_i F\left(\frac{x-\sigma_i}{\lambda}\right)\right]\left[\sum_{i=1}^{n} s_i f\left(\frac{x-\mu_i}{\lambda}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&-\left[1-\sum_{i=1}^{n} s_i F\left(\frac{x-\mu_i}{\lambda}\right)\right]\left[\sum_{i=1}^{n} s_i f\left(\frac{x-\sigma_i}{\lambda}\right)\right] \\
=&\left[\sum_{i=1}^{n} s_i f\left(\frac{x-\mu_i}{\lambda}\right)-\sum_{i=1}^{n} s_i f\left(\frac{x-\sigma_i}{\lambda}\right)\right] \\
&+\left[\sum_{i=1}^{n} s_i F\left(\frac{x-\mu_i}{\lambda}\right)\right]\left[\sum_{i=1}^{n} s_i f\left(\frac{x-\sigma_i}{\lambda}\right)\right]-\left[\sum_{i=1}^{n} s_i F\left(\frac{x-\sigma_i}{\lambda}\right)\right]\left[\sum_{i=1}^{n} s_i f\left(\frac{x-\mu_i}{\lambda}\right)\right].
\end{aligned}
$$

Simplifying further, we get from the preceding equation as

$$
\begin{aligned}
\chi_1'(x) =& \sum_{i=1}^{n} s_i\left[f\left(\frac{x-\mu_i}{\lambda}\right)-f\left(\frac{x-\sigma_i}{\lambda}\right)\right] \\
&+\sum_{i=1}^{n}\sum_{j=1}^{n} s_i s_j\left[F\left(\frac{x-\mu_i}{\lambda}\right) f\left(\frac{x-\sigma_j}{\lambda}\right)-F\left(\frac{x-\sigma_j}{\lambda}\right) f\left(\frac{x-\mu_i}{\lambda}\right)\right] \\
=& \sum_{i=1}^{n} s_i\left[f\left(\frac{x-\mu_i}{\lambda}\right)-f\left(\frac{x-\sigma_i}{\lambda}\right)\right] \\
&+\sum_{i=1}^{n}\sum_{j=1}^{n} s_i s_j F\left(\frac{x-\mu_i}{\lambda}\right) F\left(\frac{x-\sigma_j}{\lambda}\right)\left[\frac{f\left(\frac{x-\sigma_j}{\lambda}\right)}{F\left(\frac{x-\sigma_j}{\lambda}\right)}-\frac{f\left(\frac{x-\mu_i}{\lambda}\right)}{F\left(\frac{x-\mu_i}{\lambda}\right)}\right] \\
=& \sum_{i=1}^{n} s_i\left[f\left(\frac{x-\mu_i}{\lambda}\right)-f\left(\frac{x-\sigma_i}{\lambda}\right)\right] \\
&+\sum_{i=1}^{n}\sum_{j=1}^{n} s_i s_j F\left(\frac{x-\mu_i}{\lambda}\right) F\left(\frac{x-\sigma_j}{\lambda}\right)\left[\tilde{h}\left(\frac{x-\sigma_j}{\lambda}\right)-\tilde{h}\left(\frac{x-\mu_i}{\lambda}\right)\right]. \quad (54)
\end{aligned}
$$

Under the assumptions made, $\max\{\mu_1,\ldots,\mu_n\} \le \min\{\sigma_1,\ldots,\sigma_n\}$, implies that $\mu_i \le \sigma_j$, $i, j = 1,\ldots,n$ and also $\mu_i \le \sigma_i$, $i = 1,\ldots,n$. Now, for $\mu_i \le \sigma_i$, we obtain $\frac{x-\mu_i}{\lambda} \ge \frac{x-\sigma_i}{\lambda}$, implies that

$$
f\left(\frac{x-\mu_i}{\lambda}\right) \ge f\left(\frac{x-\sigma_i}{\lambda}\right). \quad (55)
$$

Again, for $\mu_i \le \sigma_j$, $i, j = 1,\ldots,n$, the decreasing property of $\tilde{h}(.)$ implies that

$$
\tilde{h}\left(\frac{x-\mu_i}{\lambda}\right) \le \tilde{h}\left(\frac{x-\sigma_j}{\lambda}\right). \quad (56)
$$

Using the inequalities given by Eqs. 55 and 56 in Eq. 54, we obtain $\chi_1'(x) \ge 0$, which implies that $\chi_1(x)$ is increasing with respect to $x \in (\sigma_1, \infty)$. Then, we have to establish that

$$
\frac{m_{n-1}(x)}{m_{n-1}^*(x)} = \frac{1-\sum_{i=2}^{n} s_i F\left(\frac{x-\sigma_i}{\lambda}\right)}{1-\sum_{i=2}^{n} s_i F\left(\frac{x-\mu_i}{\lambda}\right)} = \chi_2(x),\ (say), \quad (57)
$$

is increasing with respect to $x \in (\sigma_2, \sigma_1]$. Now, differentiating $\chi_2(x)$ with respect to x and after some calculations, we obtain

$$\chi_2'(x) \overset{sign}{=} \sum_{i=2}^{n} s_i \left[f\left(\frac{x-\mu_i}{\lambda}\right) - f\left(\frac{x-\sigma_i}{\lambda}\right)\right] + \sum_{i=2}^{n}\sum_{j=2}^{n} s_i s_j F\left(\frac{x-\mu_i}{\lambda}\right) F\left(\frac{x-\sigma_j}{\lambda}\right)\left[\tilde{h}\left(\frac{x-\sigma_j}{\lambda}\right) - \tilde{h}\left(\frac{x-\mu_i}{\lambda}\right)\right]. \tag{58}$$

Under the assumptions made, $\max\{\mu_2, \ldots, \mu_n\} \leq \min\{\sigma_2, \ldots, \sigma_n\}$, implies that $\mu_i \leq \sigma_j$, $i, j = 2, \ldots, n$ and also $\mu_i \leq \sigma_i, i = 2, \ldots, n$. After some calculations, it can be established that

$$f\left(\frac{x-\mu_i}{\lambda}\right) \geq f\left(\frac{x-\sigma_i}{\lambda}\right) \tag{59}$$

and

$$\tilde{h}\left(\frac{x-\sigma_j}{\lambda}\right) \geq \tilde{h}\left(\frac{x-\mu_i}{\lambda}\right). \tag{60}$$

Now, substituting Eqs. 59 and 60 in Eq. 58, we get $\chi_2'(x) \geq 0$, implies that $\chi_2(x)$ is increasing with respect to $x \in (\sigma_2, \sigma_1]$. Using similar arguments, it can be established that $\chi_3(x) = m_{n-2}(x)/m_{n-2}^*(x), \ldots, \chi_{n-1}(x) = m_2(x)/m_2^*(x), \chi_n(x) = m_1(x)/m_1^*(x)$ are also increasing with respect to x belongs to the other subintervals, say $(\sigma_3, \sigma_2], \ldots, (\sigma_{n-1}, \sigma_{n-2}], (\sigma_n, \sigma_{n-1}]$, respectively. Also, $0/m_n^*(x), \ldots, 0/m_2^*(x), 0/m_1^*(x)$ are increasing with respect to x belongs to the another subintervals, say $(\mu_1, \sigma_n], \ldots, (\mu_{n-1}, \mu_{n-2}], (\mu_n, \mu_{n-1}]$, respectively, which are obvious. Hence, the theorem is proved. □

Remark 11 If we consider $\boldsymbol{r} \in \mathcal{E}_n^+$ and $\boldsymbol{\sigma} \in \mathcal{E}_n^+$ instead of $\boldsymbol{r} \in \mathcal{D}_n^+$ and $\boldsymbol{\sigma} \in \mathcal{D}_n^+$, then the established result in Theorem 10 also holds.

To illustrate the result stated in Theorem 10, we consider the following example.

Example 5 Assume that $\boldsymbol{r} = (r_1, r_2, r_3) = (0.5, 0.3, 0.2) \in \mathcal{D}_3^+$, $\boldsymbol{s} = (s_1, s_2, s_3) = (0.6, 0.3, 0.1) \in \mathcal{D}_3^+$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (0.8, 0.7, 0.4) \in \mathcal{D}_3^+$, and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) = (0.4, 0.3, 0.2) \in \mathcal{D}_3^+$. Further, take $\lambda = 4$, $c = 3$, and $l = 2$. Clearly, $\boldsymbol{r} \overset{m}{\succcurlyeq} \boldsymbol{s}$ and $\max\{\mu_1, \mu_2, \mu_3\} \leq \min\{\sigma_1, \sigma_2, \sigma_3\}$. Now, consider power distribution with cdf $F(t) = (\frac{t}{l})^c$, $0 < t < l$, $c > 0$ and pdf $f(t) = \frac{c}{l^c} t^{c-1}$, $0 < t < l$, $c > 0$ as the baseline distribution. Clearly, the pdf is increasing with respect to t in its domain when $c > 1$ and $\tilde{h}(t)$ is decreasing with respect to t in its domain. Based on these information, the difference between the hazard rate functions of $U_3(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_3(\boldsymbol{s}; \boldsymbol{\mu})$ is plotted in Fig. 4(b), from which we can readily observe that $U_3(\boldsymbol{r}; \boldsymbol{\sigma}) \geq_{hr} V_3(\boldsymbol{s}; \boldsymbol{\mu})$. Here, we have considered $x = -\ln y$, $0 < y < 1$ to capture the whole positive real axis.

In Remarks 5-11, it is written that similar results with other sufficient conditions hold if we change the ordering of some parameters. Here, we consider examples for the illustration of Remarks 7 and 9. The illustrative examples for other remarks can be constructed; however they are omitted for brevity.

Example 6

(i) Assume that $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (16, 18, 22) \in \mathcal{E}_3^+$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) = (2, 8, 12) \in \mathcal{E}_3^+$. Then, under the similar setup in Example 3(ii), the ratio of the pdfs of $U_3(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_3(\boldsymbol{r}; \boldsymbol{\mu})$ is displayed in Fig. 5(a), which clearly justifies the likelihood ratio order between $U_3(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_3(\boldsymbol{r}; \boldsymbol{\mu})$ in Remark 7.

(ii) Set $\boldsymbol{r} = (r_1, r_2, r_3) = (0.5, 0.3, 0.2) \in \mathcal{D}_3^+$, $\boldsymbol{s} = (s_1, s_2, s_3) = (0.6, 0.3, 0.1) \in \mathcal{D}_3^+$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (5.2, 8.3, 10.2) \in \mathcal{E}_3^+$, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) = (4.3, 9.8, 12.5) \in \mathcal{E}_3^+$. Under the similar settings in Example 4, the sfs of $U_3(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_3(\boldsymbol{s}; \boldsymbol{\theta})$ are plotted in Fig. 5(b), from which the result in Remark 9 can be easily justified.

Next, we consider heterogeneity in mixing proportion parameter vectors and scale parameter vectors. The concept of T-transform matrix is required in the following results. For the definition of T-transform matrix, please refer to Hazra et al. (2017); Barmalzan et al. (2022), and Panja et al. (2022). First, we consider mixtures having two components and establish sufficient conditions, under which the usual stochastic order between two MRVs holds.

Theorem 11 *Let* $\bar{F}_{U_2(\boldsymbol{r};\boldsymbol{\lambda})}(x) = \sum_{i=1}^{2} r_i[1 - F(\frac{x-\sigma}{\lambda_i})]$ *and* $\bar{F}_{V_2(\boldsymbol{s};\boldsymbol{\theta})}(x) = \sum_{i=1}^{2} s_i[1 - F(\frac{x-\sigma}{\theta_i})]$ *be the sfs of the MRVs* $U_2(\boldsymbol{r}; \boldsymbol{\lambda})$ *and* $V_2(\boldsymbol{s}; \boldsymbol{\theta})$*, respectively. Further, suppose* $t^2 f(t)$ *is increasing in* $t > 0$*. Then, for* $\boldsymbol{r} \in \mathcal{D}_2^+$ *(or* $\mathcal{E}_2^+$*),* $\boldsymbol{\lambda} \in \mathcal{E}_2^+$ *(or* $\mathcal{D}_2^+$*), and for fixed* $\sigma > 0$*, we have*

$$\begin{pmatrix} r_1 & r_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \gg \begin{pmatrix} s_1 & s_2 \\ \theta_1 & \theta_2 \end{pmatrix} \Rightarrow U_2(\boldsymbol{r}; \boldsymbol{\lambda}) \leq_{st} V_2(\boldsymbol{s}; \boldsymbol{\theta}).$$

Proof To establish the desired result, we have to check the Conditions (i) and (ii) of Lemma 7. Clearly, $\bar{F}_{U_2(\boldsymbol{r};\boldsymbol{\lambda})}(x)$ is permutation invariant on $(\boldsymbol{r}; \boldsymbol{\lambda})$, which confirms Condition (i). Now, for fixed $x > 0$ and $i \neq j$, consider the function

$$\begin{aligned} \xi(x; \boldsymbol{r}, \boldsymbol{\lambda}) &= (r_1 - r_2)\left(\frac{\partial \bar{F}_{U_2(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_1} - \frac{\partial \bar{F}_{U_2(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial r_2}\right) \\ &\quad +(\lambda_1 - \lambda_2)\left(\frac{\partial \bar{F}_{U_2(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial \lambda_1} - \frac{\partial \bar{F}_{U_2(\boldsymbol{r};\boldsymbol{\lambda})}(x)}{\partial \lambda_2}\right) \\ &= \xi_1(x; \boldsymbol{r}, \boldsymbol{\lambda}) + \xi_2(x; \boldsymbol{r}, \boldsymbol{\lambda}), \text{ (say)}, \end{aligned} \tag{61}$$

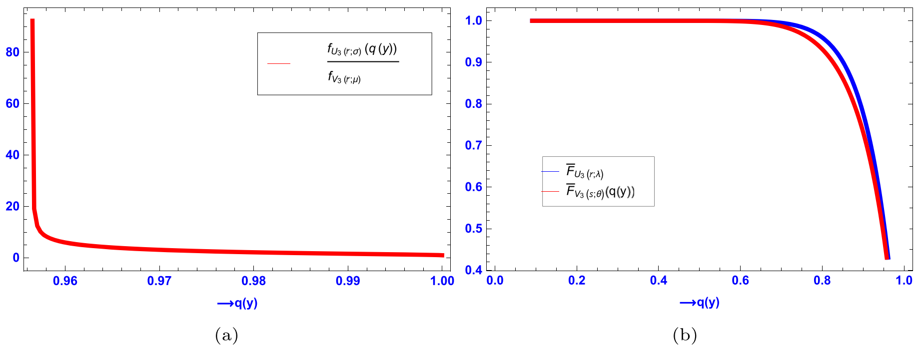


Fig. 5 (a) Plot of the ratio of the pdfs of $U_3(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_3(\boldsymbol{r}; \boldsymbol{\mu})$ in Example 6(i). (b) Graphs of the sfs of $U_3(\boldsymbol{r}; \boldsymbol{\lambda})$ (in blue colour) and $V_3(\boldsymbol{s}; \boldsymbol{\theta})$ (in red colour) in Example 6(ii)

where

$$\begin{aligned}\xi_1(x;\boldsymbol{r},\boldsymbol{\lambda}) &= (r_1-r_2)\left[F\left(\frac{x-\sigma}{\lambda_2}\right)-F\left(\frac{x-\sigma}{\lambda_1}\right)\right] \text{ and}\\ \xi_2(x;\boldsymbol{r},\boldsymbol{\lambda}) &= (\lambda_1-\lambda_2)\left[(x-\sigma)\frac{r_1}{\lambda_1^2}f\left(\frac{x-\sigma}{\lambda_1}\right)-(x-\sigma)\frac{r_2}{\lambda_2^2}f\left(\frac{x-\sigma}{\lambda_2}\right)\right]\\ &\overset{sign}{=} (\lambda_1-\lambda_2)\left[r_1\left(\frac{x-\sigma}{\lambda_1}\right)^2 f\left(\frac{x-\sigma}{\lambda_1}\right)-r_2\left(\frac{x-\sigma}{\lambda_2}\right)^2 f\left(\frac{x-\sigma}{\lambda_2}\right)\right].\end{aligned}$$

Let $\boldsymbol{r}\in\mathcal{D}_2^+(or\ \mathcal{E}_2^+)$ and $\boldsymbol{\lambda}\in\mathcal{E}_2^+(or\ \mathcal{D}_2^+)$. That is, $r_1\geq (or \leq)\ r_2$ and $\lambda_1\leq (or\geq)\ \lambda_2$. We present the proof only for the case when $r_1\geq r_2$ and $\lambda_1\leq\lambda_2$, since the other case is quite similar. For $r_1\geq r_2$ and $\lambda_1\leq\lambda_2$, we obtain $\frac{1}{\lambda_1}\geq\frac{1}{\lambda_2}$ and $\frac{x-\sigma}{\lambda_2}\leq\frac{x-\sigma}{\lambda_1}$, implies $F(\frac{x-\sigma}{\lambda_2})\leq F(\frac{x-\sigma}{\lambda_1})$, that is, $\xi_1\leq 0$. Further, $t^2f(t)$ is increasing with respect to $t>0$, implies $(\frac{x-\sigma}{\lambda_1})^2f(\frac{x-\sigma}{\lambda_1})\geq(\frac{x-\sigma}{\lambda_2})^2f(\frac{x-\sigma}{\lambda_2})$, that is, $\xi_2\leq 0$. Thus, from Eq. 61, we see that $\xi(x;\boldsymbol{r},\boldsymbol{\lambda})\leq 0$. Therefore, from Lemma 7, we get the required result. This completes the proof of the theorem. □

To illustrate the aforementioned theorem, we consider the following example.

Example 7 Consider inverted exponential distribution as the baseline distribution with cdf as in Example 4. The condition on pdf is clearly satisfied. Assume $\boldsymbol{r}=(r_1,r_2)=(0.6,0.4)\in\mathcal{D}_2^+$, $\boldsymbol{s}=(s_1,s_2)=(0.56,0.44)\in\mathcal{D}_2^+$, $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)=(0.2,0.7)\in\mathcal{E}_2^+$, and $\boldsymbol{\theta}=(\theta_1,\theta_2)=(0.30,0.60)\in\mathcal{E}_2^+$. Take $\sigma=0.1$. Next, let us consider the T-transform matrix $T_{0.2}=\begin{pmatrix}0.2 & 0.8\\ 0.8 & 0.2\end{pmatrix}$. It can then be seen that

$$\begin{pmatrix}0.56 & 0.44\\ 0.30 & 0.60\end{pmatrix}=\begin{pmatrix}0.6 & 0.4\\ 0.2 & 0.7\end{pmatrix}\times\begin{pmatrix}0.2 & 0.8\\ 0.8 & 0.2\end{pmatrix},$$

which implies that

$$\begin{pmatrix}0.6 & 0.4\\ 0.2 & 0.7\end{pmatrix}\gg\begin{pmatrix}0.56 & 0.44\\ 0.30 & 0.60\end{pmatrix}.$$

Therefore, from Theorem 11, we get $U_2(\boldsymbol{r};\boldsymbol{\lambda})\leq_{st} V_2(\boldsymbol{s};\boldsymbol{\theta})$, which can be verified from Fig. 1(b). Here, $q(y)=\frac{y}{1-y}$, $0<y<1$ is used in order to capture the positive real axis.

The following counterexample describes that the result stated in Theorem 11 may not be true if $\boldsymbol{r}\notin\mathcal{D}_2^+$ and $\boldsymbol{\lambda}\in\mathcal{E}_2^+$.

Counterexample 7 Consider Burr type-III distribution as the baseline distribution for the location-scale model with cdf $F(t)=\frac{t}{1+t}$, $t>0$ and pdf $f(t)=\frac{1}{(1+t)^2}$, $t>0$. The condition on pdf is satisfied. Let us take

$$\begin{pmatrix}r_1 & r_2\\ \lambda_1 & \lambda_2\end{pmatrix}=\begin{pmatrix}0.2 & 0.8\\ 0.4 & 0.6\end{pmatrix}\ and \begin{pmatrix}s_1 & s_2\\ \theta_1 & \theta_2\end{pmatrix}=\begin{pmatrix}0.62 & 0.38\\ 0.54 & 0.46\end{pmatrix}.$$

It is then easy to check that

$$\begin{pmatrix}0.62 & 0.38\\ 0.54 & 0.46\end{pmatrix}=\begin{pmatrix}0.2 & 0.8\\ 0.4 & 0.6\end{pmatrix}\times\begin{pmatrix}0.3 & 0.7\\ 0.7 & 0.3\end{pmatrix},$$

where $T_{0.3} = \begin{pmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{pmatrix}$ is a T-transform matrix, which implies that

$$\begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix} \gg \begin{pmatrix} 0.62 & 0.38 \\ 0.54 & 0.46 \end{pmatrix}.$$

The difference between the sfs of the MRVs $U_2(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_2(\boldsymbol{s}; \boldsymbol{\theta})$ is plotted in Fig. 6(a). Clearly, the difference is negative as well as positive, which means that $U_2(\boldsymbol{r}; \boldsymbol{\lambda}) \nleq_{st} V_2(\boldsymbol{s}; \boldsymbol{\theta})$. Here, $q(y) = \frac{y}{1-y}$, $0 < y < 1$ is considered for capturing the positive real axis.

Next, we present a generalization of Theorem 11 in terms of the number of mixing components. The proof of the following theorem follows using similar arguments as in the proof of Theorem 21 of Balakrishnan et al. (2018), and thus it is omitted. In the following theorem, T_ω is used to denote the T-transform matrix of the form $T_\omega = \omega I_n + (1-\omega)\Pi_n$, where I_n and Π_n are the identity and permutation matrices of order $n \times n$, respectively.

Theorem 12 *Let $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) = \sum_{i=1}^n r_i[1 - F(\frac{x-\sigma}{\lambda_i})]$ and $\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\theta})}(x) = \sum_{i=1}^n s_i[1 - F(\frac{x-\sigma}{\theta_i})]$ be the sfs of the MRVs $U_n(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_n(\boldsymbol{s}; \boldsymbol{\theta})$, respectively. Further, suppose $t^2 f(t)$ is increasing in $t > 0$. Then, for $\boldsymbol{r} \in \mathcal{D}_n^+$ (or $\mathcal{E}_n^+$), $\boldsymbol{\lambda} \in \mathcal{E}_n^+$ (or $\mathcal{D}_n^+$), and for fixed $\sigma > 0$, we have*

$$\begin{pmatrix} s_1 \dots s_n \\ \theta_1 \dots \theta_n \end{pmatrix} = \begin{pmatrix} r_1 \dots r_n \\ \lambda_1 \dots \lambda_n \end{pmatrix} T_w \Rightarrow U_n(\boldsymbol{r}; \boldsymbol{\lambda}) \leq_{st} V_n(\boldsymbol{s}; \boldsymbol{\theta}).$$

The following corollary is a direct consequence of Theorem 11 due to the fact that a finite product of T-transform matrices with same structure is also a T-transform matrix. We note that the T-transform matrices $T_{\omega_1}, \dots, T_{\omega_k}$ have the same structure if they have all zero and non-zero elements at the same positions which means that

$$T_{\omega_i} = \omega_i I_n + (1-\omega_i)\Pi_{i,n}, \ i = 1, \dots, k,$$

for the same $\Pi_{i,n}$ and possibly different ω_i, $i = 1, \dots, k$. On the other hand, $T_{\omega_1}, \dots, T_{\omega_k}$ have different structures if at least one of $\Pi_{i,n}$ differs from the others (Table 1).

Corollary 2 *Let $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) = \sum_{i=1}^n r_i[1 - F(\frac{x-\sigma}{\lambda_i})]$ and $\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\theta})}(x) = \sum_{i=1}^n s_i[1 - F(\frac{x-\sigma}{\theta_i})]$ be the sfs of the MRVs $U_n(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_n(\boldsymbol{s}; \boldsymbol{\theta})$, respectively. Further, suppose $t^2 f(t)$*

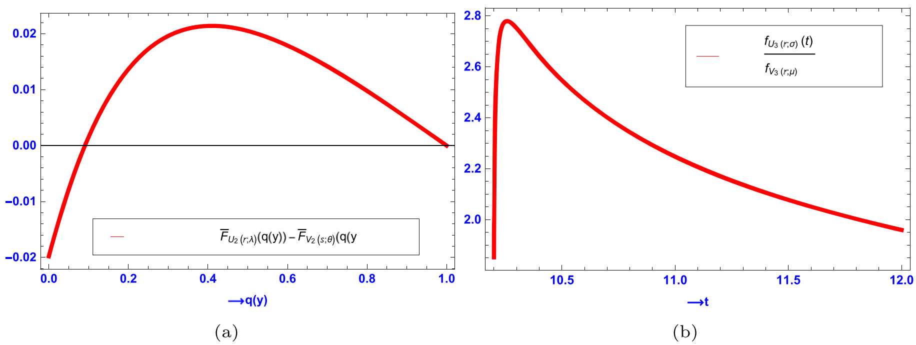


Fig. 6 (a) Plot of the difference between the sfs of MRVs $U_2(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_2(\boldsymbol{s}; \boldsymbol{\theta})$ as in Counterexample 7. (b) Plot of the ratio of the pdfs of MRVs $U_3(\boldsymbol{r}; \boldsymbol{\sigma})$ and $V_3(\boldsymbol{r}; \boldsymbol{\mu})$ as in Counterexample 6

Table 1 List of (baseline) lifetime distributions satisfying the conditions in the established results

Distribution	pdf ($f(t) =$)	Condition	Useful for
Pareto	$\frac{1}{t^2},\ t \geq 1$	$tf(t)$ is decreasing $[f(t)$ is decreasing$]$ $\{f(t)$ is log-convex$\}$	Theorems 3.1 and 3.9, Corollary 3.1 [Theorem 3.6] {Theorem 3.7}
IE	$\frac{\lambda}{t^2} e^{-\frac{\lambda}{t}},\ t, \lambda > 0$	$t^2 f(t)$ is increasing	Theorems 3.2,3.8,3.11,3.12,3.13 and Corollary 3.2
Burr-III	$\frac{1}{(1+t)^2},\ t > 0$	$t^2 f(t)$ is increasing	Theorems 3.2,3.8,3.11,3.12,3.13 and Corollary 3.2
GG	$\frac{pt^{q-1}}{\Gamma(\frac{q}{p}) e^{t^p}},\ t, p, q > 0$	$f(t)$ is decreasing	Theorem 3.6
Exponential	$\lambda e^{-\lambda t},\ t, \lambda > 0$	$f(t)$ is decreasing	Theorem 3.6
Weibull	$\frac{ct^{c-1}}{e^{t^c}}, t > 0, c > 0$	$f(t)$ is decreasing for $c \in (0, 1]$ $[f(t)$ is log-convex for $0 < c \leq 1$ $\{f(t)$ is log-concave for $c > 1\}]$ $(h(t)$ is increasing for $c > 1)$	Theorem 3.6 [Theorem 3.7] (Theorem 3.5)
Gamma	$\frac{t^{q-1}}{\Gamma(q) e^t},\ t, q > 0$	$f(t)$ is log-convex (log-concave) for $0 < q \leq 1$ $(q > 1)$	Theorem 3.7
Rayleigh	$\frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}},\ t, \sigma > 0$	$h(t)$ is increasing	Theorem 3.5
G-M	$(\alpha e^{\beta t} + \lambda) e^{-\lambda t - \frac{\alpha}{\beta} e^{\beta t}}$, $t, \alpha, \beta, \lambda > 0$	$h(t)$ is increasing	Theorem 3.5
Power	$\frac{c}{l^c} t^{c-1},\ 0 < t < l, c > 0$	$f(t)$ is increasing for $c > 1$ and $\tilde{h}(t)$ is decreasing	Theorems 3.3, 3.4 and 3.10
	$\frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}},\ t, \sigma > 0$	$h(t)$ is increasing	Theorem 3.5

Here, IE, GG, G-M denote for the inverted exponential, generalized gamma, and Gompertz-Makeham distributions, respectively. $\Gamma(.)$ denotes the Euler gamma function

is increasing in $t > 0$. If the T-transform matrices $T_{w_1}, \ldots, T_{w_k}$ have the same structure, then for $\boldsymbol{r} \in \mathcal{D}_n^+$ (or $\mathcal{E}_n^+$), $\boldsymbol{\lambda} \in \mathcal{E}_n^+$ (or $\mathcal{D}_n^+$), and for fixed $\sigma > 0$, we have

$$\begin{pmatrix} s_1 \ldots s_n \\ \theta_1 \ldots \theta_n \end{pmatrix} = \begin{pmatrix} r_1 \ldots r_n \\ \lambda_1 \ldots \lambda_n \end{pmatrix} T_{w_1} \ldots T_{w_k} \Rightarrow U_n(\boldsymbol{r}; \boldsymbol{\lambda}) \leq_{st} V_n(\boldsymbol{s}; \boldsymbol{\theta}).$$

The following result gives an illustration if the T-transform matrices T_{w_i}, $i = 1, \ldots, k$, $k \geq 2$ have the different structure. The proof of theorem follows using similar arguments as in Theorem 22 of Balakrishnan et al. (2018), and therefore it is omitted for the sake of conciseness.

Theorem 13 *Let $\bar{F}_{U_n(\boldsymbol{r};\boldsymbol{\lambda})}(x) = \sum_{i=1}^n r_i[1 - F(\frac{x-\sigma}{\lambda_i})]$ and $\bar{F}_{V_n(\boldsymbol{s};\boldsymbol{\theta})}(x) = \sum_{i=1}^n s_i[1 - F(\frac{x-\sigma}{\theta_i})]$ be the sfs of the MRVs $U_n(\boldsymbol{r}; \boldsymbol{\lambda})$ and $V_n(\boldsymbol{s}; \boldsymbol{\theta})$, respectively. Further, suppose*

$t^2 f(t)$ is increasing in $t > 0$. If the T-transform matrices $T_{w_1}, \ldots, T_{w_k}$ have the different structures, then for $\boldsymbol{r} \in \mathcal{D}_n^+$(or $\mathcal{E}_n^+$), $\boldsymbol{\lambda} \in \mathcal{E}_n^+$(or $\mathcal{D}_n^+$), $\begin{pmatrix} r_1 \ldots r_n \\ \lambda_1 \ldots \lambda_n \end{pmatrix} \in \mathcal{M}_n$, and $\begin{pmatrix} r_1 \ldots r_n \\ \lambda_1 \ldots \lambda_n \end{pmatrix} T_{w_1} \ldots T_{w_i} \in \mathcal{M}_n$, $i = 1, \ldots, (k-1)$, where $k \geq 2$, we have

$$\begin{pmatrix} s_1 \ldots s_n \\ \theta_1 \ldots \theta_n \end{pmatrix} = \begin{pmatrix} r_1 \ldots r_n \\ \lambda_1 \ldots \lambda_n \end{pmatrix} T_{w_1} \ldots T_{w_k} \Rightarrow U_n(\boldsymbol{r}; \boldsymbol{\lambda}) \leq_{st} V_n(\boldsymbol{s}; \boldsymbol{\theta}).$$

4 Concluding Remarks

In this article, we have obtained several stochastic comparison results, comparing two FMMs with respect to various stochastic senses. The usual stochastic, hazard rate, reversed hazard rate, and likelihood ratio orders have been proved for MMs when heterogeneity is considered in one parameter. When heterogeneity in two parameters are considered, the usual stochastic order and hazard rate order have been established. To illustrate the established results, various examples and counterexamples have been provided.

We note that when considering heterogeneity in three parameters of the location-scale model, then the sf of the MRV with two components, say $U_2(\boldsymbol{r}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ is given by

$$\begin{aligned}\bar{F}_{U_2(\boldsymbol{r},\boldsymbol{\sigma},\boldsymbol{\lambda})}(x) &= \begin{cases} 1 - r_2 F\left(\frac{x-\sigma_2}{\lambda_2}\right) & ; \text{if} \quad \sigma_2 < x \leq \sigma_1 \\ \\ 1 - \left[r_1 F\left(\frac{x-\sigma_1}{\lambda_1}\right) + r_2 F\left(\frac{x-\sigma_2}{\lambda_2}\right)\right] & ; \text{if} \quad \sigma_1 < x < \infty \end{cases} \\ &= \varphi(P), \ (say),\end{aligned}$$

where $P = \begin{pmatrix} r_1 & r_2 \\ \sigma_1 & \sigma_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}$, $\boldsymbol{r} = (r_1, r_2)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. Now, to check Condition (i) of Lemma 7, we need to consider all the permutation matrices of order 2×2, which are given by $\Pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Pi_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Further, $P\Pi_1 = P$ and

$$\begin{aligned} P\Pi_2 &= \begin{pmatrix} r_1 & r_2 \\ \sigma_1 & \sigma_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} r_2 & r_1 \\ \sigma_2 & \sigma_1 \\ \lambda_2 & \lambda_1 \end{pmatrix}. \end{aligned}$$

Then,

$$\begin{aligned}\varphi(P\Pi_2) &= \begin{cases} 1 - r_1 F\left(\frac{x-\sigma_1}{\lambda_1}\right) & ; \text{if} \quad \sigma_1 < x \leq \sigma_2 \\ \\ 1 - \left[r_1 F\left(\frac{x-\sigma_1}{\lambda_1}\right) + r_2 F\left(\frac{x-\sigma_2}{\lambda_2}\right)\right] & ; \text{if} \quad \sigma_2 < x < \infty \end{cases} \\ &\neq \varphi(P).\end{aligned}$$

Thus, the sf is not permutation invariant. So, one can not apply Lemma 7 to obtain result based on matrix chain majorization when heterogeneity is taken in three parameters. Thus, further research/development is required in this direction. This can be considered as open problem.

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References

Asadi M, Ebrahimi N, Kharazmi O, Soofi ES (2018) Mixture models, Bayes Fisher information, and divergence measures. IEEE Trans Inf Theory 65(4):2316–2321

Balakrishnan N, Haidari A, Masoumifard K (2015) Stochastic comparisons of series and parallel systems with generalized exponential components. IEEE Trans Reliab 64(1):333–348

Balakrishnan N, Nanda P, Kayal S (2018) Ordering of series and parallel systems comprising heterogeneous generalized modified Weibull components. Appl Stoch Models Bus Ind 34(6):816–834

Barmalzan G, Kosari S, Zhang Y (2021) On stochastic comparisons of finite α-mixture models. Stat Probab Lett 173:109083

Barmalzan G, Kosari S, Balakrishnan N (2022) Orderings of finite mixture models with location-scale distributed components. Probab Eng Inf Sci 36(2):461–481

Dedecius K, Ettler P (2013) Overview of bounded support distributions and methods for Bayesian treatment of industrial data. In: ICINCO vol 1 pp 380–387

Finkelstein M (2008) Failure rate modeling for reliability and risk. Springer, London

Haidari A, Najafabadi ATP, Balakrishnan N (2019) Comparisons between parallel systems with exponentiated generalized gamma components. Commun Stat - Theory Methods 48(6):1316–1332

Hazra NK, Finkelstein M (2018) On stochastic comparisons of finite mixtures for some semiparametric families of distributions. Test 27(4):988–1006

Hazra NK, Kuiti MR, Finkelstein M, Nanda AK (2017) On stochastic comparisons of maximum order statistics from the location-scale family of distributions. J Multivar Anal 160:31–41

Hazra NK, Finkelstein M, Cha JH (2017) On optimal grouping and stochastic comparisons for heterogeneous items. J Multivar Anal 160:146–156

Jiang R (2013) A new bathtub curve model with a finite support. Reliab Eng Syst Saf 119:44–51

Kayal S, Bhakta R, Balakrishnan N (2023) Some results on stochastic comparisons of two finite mixture models with general components. Stoch Models 39(2):363–382

Lindsay BG (1995) Mixture models: theory, geometry, and applications. In NSF-CBMS regional conference series in probability and statistics. Hayward, CA: Institute of Math- ematical Statistics. Springer

Marshall AW, Olkin I, Arnold BC (2011) Inequalities: theory of majorization and its applications, volume 143. Springer (Second Edition)

McLachlan GJ, Peel D (2000) Finite mixture models. Wiley, New York

Nadeb H, Torabi H (2022) New results on stochastic comparisons of finite mixtures for some families of distributions. Commun Stat - Theory Methods 51(10):3104–3119

Panja A, Kundu P, Pradhan B (2022) On stochastic comparisons of finite mixture models. Stoch Models 38(2):190–213

Sattari M, Barmalzan G, Balakrishnan N (2021) Stochastic comparisons of finite mixture models with generalized Lehmann distributed components. Commun Stat - Theory Methods 51(22):7767–7782

Shaked M, Shanthikumar JG (2007) Stochastic orders. Springer, New York