Bending-neutral deformations of minimal surfaces

André M. Sonnet[1] and Epifanio G. Virga[2]

[1]Department of Mathematics and Statistics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, UK
[2]Department of Mathematics, University of Pavia, via Ferrata 5, 27100 Pavia, Italy

AMS, 0000-0003-2583-7897; EGV, 0000-0002-2295-8055

Minimal surfaces are ubiquitous in nature. Here, they are considered as geometric objects that bear a deformation content. By refining the resolution of the surface deformation gradient afforded by the polar decomposition theorem, we identify a *bending* content and a class of deformations that leave it unchanged. These are the *bending-neutral* deformations, fully characterized by an integrability condition; they preserve normals. We prove that: (i) every minimal surface is transformed into a minimal surface by a bending-neutral deformation; (ii) given two minimal surfaces with the same system of normals, there is a bending-neutral deformation that maps one into the other; and (iii) *all* minimal surfaces have indeed a *universal* bending content.

1. Introduction

The *Universe is parsimonious*: so says the title of a celebrated book [1], which among many other things contains a fascinating account of minimal surfaces (disguised as soap bubbles). The theory of plates and shells should make no exception to this universal rule. However, researchers are still debating the most appropriate kinematic measures of deformation that would enter the elastic energy functional in an *intrinsic* (direct) theory of these bodies. In the language of the book [2], an intrinsic theory represents plates and shells as truly two-dimensional bodies, with mass distributed on a surface $\mathscr{S}$ in three-dimensional Euclidean space $\mathscr{E}$. The following works provide but a partial sample of the current debate [3–10].

Here, we build on an earlier proposal for measures of *pure* bending of a surface $\mathscr{S}$ [7], which was formulated as an invariance requirement under the class of

bending-neutral deformations, which preserve normals. In this paper, we identify as *bending content* of $\mathscr{S}$ one geometric object (a vector field) left unchanged by these deformations.

We prove that *all* minimal surfaces share *one* and the same bending content and those that share the same system of normals can be seen as the image of one another under a bending-neutral deformation. Thus, making it impossible to properly *bend* one minimal surface into another, if the system of normals is to be *preserved*.

The paper is organized as follows. In §2, to make our development self-contained, we collect a few results about surface tensor calculus that are used in the rest of the paper. In §3, we lay out our kinematic analysis: we recall the definition of bending-neutral deformations and identify the bending content of a surface. In §4, we show how minimal surfaces are tightly connected by bending-neutral deformations; we accordingly introduce the notion of bending-neutral *associates*, which extends Bonnet's classical one. Section 5 collects the conclusions of this work and a few comments on their possible bearing on the theory of plates and shells, which was our original motivation and remains our primary intent.

The paper is closed by two appendices, where we give details about proofs left out of the main text to ease the flow of our presentation. Two animations and a Maple script are also provided as electronic supplementary material for the reader. The animations show motions traversing bending-neutral associates of a minimal surface: they visibly convey an impression of *gliding* instead of bending, in accord with the theory. The Maple script was used to generate the animations and can be modified by the reader to explore further minimal surfaces and their deformations.

2. Glimpses of surface differential calculus

To make our paper self-contained, we recall some basic notions of differential calculus on smooth surfaces in three-dimensional space, especially those playing a role in the development that follows.

Let $\mathscr{S}$ be a smooth, orientable surface in three-dimensional Euclidean space $\mathscr{E}$; it can be represented (at least locally) by a mapping of class C^2 defined on a domain $S \subset \mathbb{R}^2$ described by a system of coordinates. Here, however, we shall keep our discussion independent of any particular choice of coordinates.

A scalar field $\varphi : \mathscr{S} \to \mathbb{R}$ is differentiable at a point $p \in \mathscr{S}$, if for every curve $c : t \mapsto c(t) \in \mathscr{S}$ such that $c(t_0) = p$ there is a vector $\nabla_s \varphi$ on the tangent plane $\mathscr{T}_p$ to $\mathscr{S}$ at p such that:

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi(c(t)) \Big|_{t=t_0} = \nabla_s \varphi \cdot \boldsymbol{t}, \tag{2.1}$$

where $\boldsymbol{t}$ is the unit tangent vector to c at p. We call $\nabla_s \varphi$ the *surface gradient* of φ.[1]

Equivalently, $\nabla_s \varphi$ can be introduced by extending φ in a three-dimensional neighbourhood that contains $\mathscr{S}$ to a smooth function $\tilde{\varphi}$ such that $\tilde{\varphi}|_{\mathscr{S}} = \varphi$. Letting $\nabla \tilde{\varphi}$ denote the ordinary, three-dimensional gradient of $\tilde{\varphi}$, we can uniquely decompose it in its tangential and normal components relative to $\mathscr{S}$:

$$\nabla \tilde{\varphi} = \mathbf{P}(\boldsymbol{\nu}) \nabla \tilde{\varphi} + (\nabla \tilde{\varphi} \cdot \boldsymbol{\nu}) \boldsymbol{\nu}, \tag{2.2}$$

where $\boldsymbol{\nu}$ is a selected unit normal to $\mathscr{S}$ at $p \in \mathscr{S}$ and $\mathbf{P}(\boldsymbol{\nu}) := \mathbf{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}$ is the projection onto the tangent plane $\mathscr{T}_p$. We can then set

$$\nabla_s \varphi := \mathbf{P}(\boldsymbol{\nu}) \nabla \tilde{\varphi} \quad \text{and} \quad \nabla_\nu \tilde{\varphi} := \nabla \tilde{\varphi} \cdot \boldsymbol{\nu}, \tag{2.3}$$

the former depending only on φ and the latter on its extension $\tilde{\varphi}$. One can devise $\tilde{\varphi}$ so that $\nabla_\nu \tilde{\varphi} = 0$, as in Šilhavý [12]: this is called a *normal* extension of φ. Here, we shall not make use of such a restricted class of extensions.

The advantage of defining $\nabla_s \varphi$ as in (2.3) is that this definition is easily extended to higher derivatives. Denoting by the same symbol also the extension of φ (no confusion can arise, as only the normal derivative depends on the extension), we have that:

$$\nabla_s^2 \varphi := \nabla[\nabla\varphi - (\nabla\varphi \cdot \boldsymbol{\nu})\boldsymbol{\nu}]\mathbf{P}(\boldsymbol{\nu}), \tag{2.4}$$

where ∇ denotes the standard gradient. By expanding (2.4) and setting $\nabla_s\boldsymbol{\nu} := (\nabla\boldsymbol{\nu})\mathbf{P}(\boldsymbol{\nu})$, having also extended $\boldsymbol{\nu}$ out of $\mathscr{S}$, by use of (2.3), we arrive at:

$$\nabla_s^2 \varphi = \mathbf{P}(\boldsymbol{\nu})(\nabla^2\varphi)\mathbf{P}(\boldsymbol{\nu}) - (\nabla_\nu\varphi)\nabla_s\boldsymbol{\nu} - \boldsymbol{\nu} \otimes (\nabla_s\boldsymbol{\nu})(\nabla_s\varphi). \tag{2.5}$$

Two consequences follow immediately from (2.5). First, $\nabla_s^2\varphi$ clearly depends on the extension of φ outside $\mathscr{S}$; an intrinsic definition can be achieved by resorting to normal extensions of φ. Second, since both $\nabla^2\varphi$ and the *curvature tensor* $\nabla_s\boldsymbol{\nu}$, which at each point $p \in \mathscr{S}$ maps $\mathscr{T}_p$ into itself, are symmetric, the skew-symmetric part of $\nabla_s^2\varphi$ does not generally vanish, but it is independent of the extension of φ,

$$\mathrm{skw}(\nabla_s^2\varphi) = \mathrm{skw}[(\nabla_s\boldsymbol{\nu})(\nabla_s\varphi) \otimes \boldsymbol{\nu}], \tag{2.6}$$

which is both intrinsic and fully determined by $\nabla_s\varphi$ and the curvature of $\mathscr{S}$ (see also [13,14]). Letting the axial vector associated with the skew-symmetric part of $\nabla_s^2\varphi$ be the *surface* curl of $\nabla_s\varphi$, we can rewrite (2.6) as:

$$\mathrm{curl}_s \nabla_s\varphi = \boldsymbol{\nu} \times (\nabla_s\boldsymbol{\nu})(\nabla_s\varphi). \tag{2.7}$$

More generally, for $\boldsymbol{h}$ a given *tangential* vector field of class C^1 on $\mathscr{S}$, (2.6) is replaced by:

$$\mathrm{skw}(\nabla_s\boldsymbol{h}) = \mathrm{skw}((\nabla_s\boldsymbol{\nu})\boldsymbol{h} \otimes \boldsymbol{\nu}). \tag{2.8}$$

Similarly, for a given surface second-rank tensor field $\mathbf{F}$ of class C^1 on $\mathscr{S}$ such that $\mathbf{F}\boldsymbol{\nu} = \boldsymbol{0}$,

$$\mathrm{skw}(\nabla_s\mathbf{F}) = \mathrm{skw}(\mathbf{F}(\nabla_s\boldsymbol{\nu}) \otimes \boldsymbol{\nu}), \tag{2.9}$$

where the skew-symmetric part is taken on the last two legs of the third-rank tensors involved.

Consider now a closed curve c over $\mathscr{S}$ and denote by $s \subset \mathscr{S}$ the portion of $\mathscr{S}$ that it encloses. The *circulation theorem* (e.g. [11, p. 243]) says that for a (not necessarily tangential) vector field $\boldsymbol{h}$ defined on $\mathscr{S}$ the following identity holds:

$$\oint_c \boldsymbol{h} \cdot \boldsymbol{t}\,\mathrm{d}s = \int_s \mathrm{curl}_s \boldsymbol{h} \cdot \boldsymbol{\nu}\,\mathrm{d}A, \tag{2.10}$$

where $\boldsymbol{t}$ is the unit tangent to c (oriented anticlockwise relative to $\boldsymbol{\nu}$), s is the arc-length measure along c and A is the area measure on $\mathscr{S}$.

It readily follows from (2.10) and (2.7) that a sufficient and necessary condition for a surface vector field $\boldsymbol{h}$ to be the surface gradient of a scalar field φ is that both $\boldsymbol{h}$ and $\mathrm{curl}_s\boldsymbol{h}$ are tangential to $\mathscr{S}$ [15, p. 244]. Said differently, for a tangential surface vector field $\boldsymbol{h}$, (2.8) is also a sufficient *integrability* condition, which guarantees the existence (at least locally) of a scalar field φ such that $\boldsymbol{h} = \nabla_s\varphi$. Similarly, (2.9) is also a sufficient integrability condition for a surface second-rank tensor field $\mathbf{F}$ that annihilates the normal: it guarantees the existence of a surface vector field $\boldsymbol{h}$ such that $\mathbf{F} = \nabla_s\boldsymbol{h}$.

Finally, we define the *surface* Laplacian as $\triangle_s \varphi := \mathrm{div}_s\nabla_s\varphi$ and we shall say that a scalar function φ of class C^2 on $\mathscr{S}$ such that $\triangle_s \varphi = 0$ is *surface harmonic*.[2]

3. Bending-neutral deformations

In this section, we are concerned with the kinematics of plates. We think of $\mathscr{S}$ as the image of the mid-surface of a plate $S \subset \mathbb{R}^2$ in the (x_1, x_2) plane of a Cartesian frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ under a deformation $\boldsymbol{f} : S \to \mathscr{E}$, which is assumed to be of class C^2 (see figure 1). In turn, $\mathscr{S}$ can be regarded as the reference configuration of a *shell*, which can be further deformed, as it will be at a later stage.

(a) Kinematics

Letting $\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2$ denote the position vector in S and extending to the present geometric setting results well-known from three-dimensional kinematics (e.g. chapter 6 of [17]), the deformation gradient $\nabla \boldsymbol{f}$ can be represented as:

$$\nabla \boldsymbol{f} = \lambda_1 \boldsymbol{b}_1 \otimes \boldsymbol{c}_1 + \lambda_2 \boldsymbol{b}_2 \otimes \boldsymbol{c}_2, \tag{3.1}$$

where ∇ denotes the gradient in $\boldsymbol{x}$, the positive scalars λ_1, λ_2 are the principal stretches and the unit vectors $\boldsymbol{c}_i$, $\boldsymbol{b}_i$ are the corresponding *right* and *left* principal directions of stretching. While $\boldsymbol{c}_1(\boldsymbol{x})$ and $\boldsymbol{c}_2(\boldsymbol{x})$ live on the $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ plane for all $\boldsymbol{x} \in S$, $\boldsymbol{b}_1(\boldsymbol{x})$ and $\boldsymbol{b}_2(\boldsymbol{x})$ live on the tangent plane $\mathscr{T}_{\boldsymbol{f}(\boldsymbol{x})}$ to $\mathscr{S}$ at $\boldsymbol{f}(\boldsymbol{x})$. The right and left Cauchy–Green tensors are correspondingly given by

$$\mathbf{C}_{\boldsymbol{f}} := (\nabla \boldsymbol{f})^{\top}(\nabla \boldsymbol{f}) = \lambda_1^2 \boldsymbol{c}_1 \otimes \boldsymbol{c}_1 + \lambda_2^2 \boldsymbol{c}_2 \otimes \boldsymbol{c}_2, \tag{3.2a}$$

$$\mathbf{B}_{\boldsymbol{f}} := (\nabla \boldsymbol{f})(\nabla \boldsymbol{f})^{\top} = \lambda_1^2 \boldsymbol{b}_1 \otimes \boldsymbol{b}_1 + \lambda_2^2 \boldsymbol{b}_2 \otimes \boldsymbol{b}_2. \tag{3.2b}$$

By the polar decomposition theorem for the deformation of surfaces established in Man & Cohen [18] within the general coordinate-free theory introduced in Gurtin & Murdoch [19,20] (see also [21]), the deformation gradient $\nabla \boldsymbol{f}$ can be written as:

$$\nabla \boldsymbol{f} = \mathbf{R}_{\boldsymbol{f}} \mathbf{U}_{\boldsymbol{f}} = \mathbf{V}_{\boldsymbol{f}} \mathbf{R}_{\boldsymbol{f}}, \tag{3.3}$$

where $\mathbf{R}_{\boldsymbol{f}}$ is a rotation of the special orthogonal group in three dimensions SO(3) and,

$$\mathbf{U}_{\boldsymbol{f}} = \lambda_1 \boldsymbol{c}_1 \otimes \boldsymbol{c}_1 + \lambda_2 \boldsymbol{c}_2 \otimes \boldsymbol{c}_2 \quad \text{and} \quad \mathbf{V}_{\boldsymbol{f}} = \lambda_1 \boldsymbol{b}_1 \otimes \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2 \otimes \boldsymbol{b}_2 \tag{3.4}$$

are the *stretching* tensors. It readily follows from (3.3) that $\boldsymbol{b}_i = \mathbf{R}_{\boldsymbol{f}} \boldsymbol{c}_i$.

(b) Bending and drilling rotations

In (3.3), the rotation $\mathbf{R}_{\boldsymbol{f}}$, which can be different at different places $\boldsymbol{x}$, represents the non-metric component of the deformation $\boldsymbol{f}$. It follows from Euler's theorem on rotations (e.g. [22], for a modern discussion) that for every $\mathbf{R} \in$ SO(3) there is an *axis*, designated by a unit vector $\boldsymbol{e} \in \mathbb{S}^2$, and a scalar $\alpha \in [0, \pi]$ such that $\mathbf{R}$ is a (conventionally anticlockwise) rotation about $\boldsymbol{e}$ by angle α. More precisely, the following representation applies,

$$\mathbf{R} = \mathbf{I} + \sin \alpha \mathbf{W}(\boldsymbol{e}) + (1 - \cos \alpha) \mathbf{W}(\boldsymbol{e})^2, \tag{3.5}$$

where $\mathbf{I}$ is the identity tensor, $\mathbf{W}(\boldsymbol{e})$ is the skew-symmetric tensor associated with $\boldsymbol{e}$ (so that $\mathbf{W}(\boldsymbol{e})\boldsymbol{v} = \boldsymbol{e} \times \boldsymbol{v}$ for all vectors $\boldsymbol{v}$). SO(3) is set by (3.5) into a one-to-one correspondence with the ball $\mathbb{B}_{\pi}^3 \subset \mathscr{E}$ of radius π centred at the origin. This latter corresponds to the identity $\mathbf{I}$, whereas the points on $\partial \mathbb{B}_{\pi}^3$ are π-*turns*, that is rotations by an angle π, which are the only symmetric members of SO(3) other than $\mathbf{I}$.
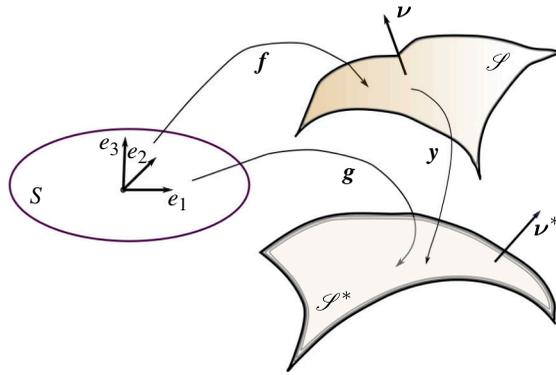
Figure 1. The surface $\mathscr{S}$ is obtained from the reference configuration S in the (x_1, x_2) plane of a Cartesian frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ via the deformation $\boldsymbol{f}$. $\mathscr{S}^*$ is similarly obtained via the deformation $\boldsymbol{g}$. $\mathscr{S}$ is mapped onto $\mathscr{S}^*$ by the deformation $\boldsymbol{y}$; $\boldsymbol{\nu}$ and $\boldsymbol{\nu}^*$ are the unit vector fields that orient $\mathscr{S}$ and $\mathscr{S}^*$, respectively, while $\boldsymbol{e}_3$ is the unit normal orienting S.

As shown in appendix A, $\mathbf{R}_f$ can be uniquely decomposed into two rotations, $\mathbf{R}_f^{(d)}$ with axis along the normal $\boldsymbol{e}_3$ to S and $\mathbf{R}_f^{(b)}$ with axis in the $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ plane. We shall interpret $\mathbf{R}_f^{(b)}$ as the *bending* component of $\mathbf{R}_f$ and $\mathbf{R}_f^{(d)}$ as the *drilling* component.[3] Formally, we write:

$$\mathbf{R}_f = \mathbf{R}_f^{(b)}\mathbf{R}_f^{(d)}, \tag{3.6}$$

where the order of composition matters, as bending and drilling rotations are about different axes. A pure drilling rotation, for which $\mathbf{R}_f^{(b)}$ is uniform but $\mathbf{R}_f^{(d)}$ is not, would pertain to a deformation $\boldsymbol{f}$ that strains S but leaves it planar. A pure bending rotation, for which $\mathbf{R}_f^{(d)}$ is uniform but $\mathbf{R}_f^{(b)}$ is not, would instead pertain to a deformation $\boldsymbol{f}$ that brings S out of a plane.

To identify both $\mathbf{R}_f^{(d)}$ and $\mathbf{R}_f^{(b)}$ for a given $\mathbf{R}_f$, it is expedient to represent a member $\mathbf{R} \in \mathsf{SO}(3)$ via Rodrigues' formula,[4]

$$\mathbf{R}(\boldsymbol{a}) = \frac{1}{1 + a^2}\left\{(1 - a^2)\mathbf{I} + 2\boldsymbol{a} \otimes \boldsymbol{a} + 2\mathbf{W}(\boldsymbol{a})\right\}, \tag{3.7}$$

where $\boldsymbol{a}$ is a vector of arbitrary length $a := \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$. This is the *vector* representation of a rotation, where $\mathsf{SO}(3)$ is set into a one-to-one correspondence with the whole translation space $\mathscr{V}$ of $\mathscr{E}$: the identity is still represented by the origin, but here π-turns are points at infinity.

Equation (3.7) follows from (3.5) by setting $\boldsymbol{a} = \tan(\alpha/2)\boldsymbol{e}$ and remarking that $\mathbf{W}(\boldsymbol{e})^2 = -\mathbf{I} + \boldsymbol{e} \otimes \boldsymbol{e}$. It can also be easily inverted: for given $\mathbf{R} \in \mathsf{SO}(3)$, the representing vector $\boldsymbol{a}$ is identified by:

$$\mathbf{W}(\boldsymbol{a}) = \frac{\mathbf{R} - \mathbf{R}^{\mathsf{T}}}{1 + \operatorname{tr} \mathbf{R}}, \tag{3.8}$$

whence

$$a^2 = \frac{3 - \operatorname{tr} \mathbf{R}}{1 + \operatorname{tr} \mathbf{R}}, \tag{3.9}$$

which shows that a π-turn is characterized by having $\operatorname{tr} \mathbf{R} = -1$.

Rodrigues [27] proved a composition formula, which in modern terms can be phrased as follows (e.g. also [28]): for $\mathbf{R}(\boldsymbol{a}_1)$ and $\mathbf{R}(\boldsymbol{a}_2)$ in $\mathsf{SO}(3)$ represented as in (3.7),

$$\mathbf{R}(\boldsymbol{a}_2)\mathbf{R}(\boldsymbol{a}_1) = \mathbf{R}(\boldsymbol{a}), \tag{3.10}$$

where

$$\boldsymbol{a} = \frac{\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_2 \times \boldsymbol{a}_1}{1 - \boldsymbol{a}_1 \cdot \boldsymbol{a}_2}. \tag{3.11}$$

In particular, (3.11) shows that the composed rotation $\mathbf{R}(\boldsymbol{a})$ is a π-turn whenever $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ belong to the cone defined by $\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = 1$.

Let $\mathbf{R}(\boldsymbol{a})$ be given in SO(3); we prove in appendix A that it can uniquely be decomposed as in (3.10) into a rotation $\mathbf{R}(\boldsymbol{a}_1)$ with $\boldsymbol{a}_1$ parallel to a given unit vector $\boldsymbol{e}$ and a rotation $\mathbf{R}(\boldsymbol{a}_2)$ with $\boldsymbol{a}_2$ orthogonal to $\boldsymbol{e}$. The vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are explicitly determined as:

$$\boldsymbol{a}_1 = (\boldsymbol{a} \cdot \boldsymbol{e})\boldsymbol{e}, \quad \boldsymbol{a}_2 = \frac{1}{1 + (\boldsymbol{a} \cdot \boldsymbol{e})^2}\{\mathbf{I} + (\boldsymbol{a} \cdot \boldsymbol{e})\mathbf{W}(\boldsymbol{e})\}\mathbf{P}(\boldsymbol{e})\boldsymbol{a}, \tag{3.12}$$

where $\mathbf{P}(\boldsymbol{e}) := \mathbf{I} - \boldsymbol{e} \otimes \boldsymbol{e}$ is the projector on the plane orthogonal to $\boldsymbol{e}$. It easily follows from (3.12) that:

$$a_1^2 = (\boldsymbol{a} \cdot \boldsymbol{e})^2 \quad \text{and} \quad a_2^2 = \frac{a^2 - a_1^2}{1 + a_1^2}. \tag{3.13}$$

It is interesting to note that the rotation $\mathbf{R}(\boldsymbol{a}_1)$ just determined can also be characterized as the member in the subgroup $\mathrm{SO}(\boldsymbol{e}) \subset \mathrm{SO}(3)$ of all rotations with axis $\boldsymbol{e}$ that is the *closest* to $\mathbf{R}(\boldsymbol{a})$ in the sense made precise in appendix A, where this result is proved in detail.[5]

For $\mathbf{R}_f \in \mathrm{SO}(3)$, we set $\mathbf{R}_f^{(\mathrm{d})} := \mathbf{R}(\boldsymbol{d}_f)$ and $\mathbf{R}_f^{(\mathrm{b})} := \mathbf{R}(\boldsymbol{b}_f)$, where, repeating (3.12) verbatim,

$$\boldsymbol{d}_f = (\boldsymbol{a}_f \cdot \boldsymbol{e}_3)\boldsymbol{e}_3, \quad \boldsymbol{b}_f = \frac{1}{1 + (\boldsymbol{a}_f \cdot \boldsymbol{e}_3)^2}\{\mathbf{I} + (\boldsymbol{a}_f \cdot \boldsymbol{e}_3)\mathbf{W}(\boldsymbol{e}_3)\}\mathbf{P}(\boldsymbol{e}_3)\boldsymbol{a}_f, \tag{3.14}$$

and $\boldsymbol{a}_f$ is the vector associated with $\mathbf{R}_f$ via (3.7), so that (3.6) applies. We shall call $\boldsymbol{d}_f$ and $\boldsymbol{b}_f$ the *drilling* and *bending* contents of $\mathbf{R}_f$, respectively. It should be noted that, as typical of any kinematic quantities, both $\boldsymbol{d}_f$ and $\boldsymbol{b}_f$ inherit from $\mathbf{R}_f$ its frame-dependent character; neither should be mistaken for *measures* of drilling and bending, for which we need frame-indifferent quantities.

(c) Bending neutrality

We wish to identify a class of deformations that do not alter the bending content of a surface $\mathscr{S}$ (relative to its flat reference configuration S). Let $\mathscr{S}$ be obtained as above from S via the deformation $\boldsymbol{f}$ and let $\boldsymbol{y} : \mathscr{S} \to \mathscr{E}$ be a deformation of $\mathscr{S}$ that transforms it into $\mathscr{S}^*$ (see figure 1). As for $\nabla \boldsymbol{f}$ in (3.3), we can decompose the surface gradient $\nabla_{\mathrm{s}}\boldsymbol{y}$ of the incremental deformation $\boldsymbol{y}$ as:

$$\nabla_{\mathrm{s}}\boldsymbol{y} = \mathbf{R}_y\mathbf{U}_y, \tag{3.15}$$

where $\mathbf{U}_y$ is the *surface* stretching tensor that maps the tangent plane $\mathscr{T}_{\boldsymbol{\xi}}$ onto itself for all $\boldsymbol{\xi} \in \mathscr{S}$ and $\mathbf{R}_y$ is a rotation of SO(3). We say that $\boldsymbol{y}$ is a *bending-neutral* deformation if:

$$\mathbf{R}_y = \mathbf{R}_f^{(\mathrm{d})} = \mathbf{R}_\nu \in \mathrm{SO}(\nu) \quad \text{on } \mathscr{S}, \tag{3.16}$$

where $\boldsymbol{\nu}$ is the unit normal to $\mathscr{S}$ oriented coherently with the orientation of S. This amounts to require that the bending component $\mathbf{R}_y^{(\mathrm{b})}$ of $\boldsymbol{y}$ be any *uniform* rotation $\mathbf{R}_0$, which can always be chosen to be the identity $\mathbf{I}$ by an appropriate change of frame.

The surface $\mathscr{S}^*$ can also be regarded as the image of S under the deformation $\boldsymbol{g} := \boldsymbol{y} \circ \boldsymbol{f}$. We now show that for every bending-neutral deformation $\boldsymbol{y}$ of $\mathscr{S}$, the deformations $\boldsymbol{g}$ and $\boldsymbol{f}$ have

one and the same bending content. To this end, we first remark that the unit normal $\boldsymbol{\nu}$ at $\boldsymbol{f}(\boldsymbol{x})$ is given by:

$$\boldsymbol{\nu} = \frac{(\nabla \boldsymbol{f})\boldsymbol{e}_1 \times (\nabla \boldsymbol{f})\boldsymbol{e}_2}{|(\nabla \boldsymbol{f})\boldsymbol{e}_1 \times (\nabla \boldsymbol{f})\boldsymbol{e}_2|} = \frac{\mathrm{co}(\nabla \boldsymbol{f})\boldsymbol{e}_3}{|\mathrm{co}(\nabla \boldsymbol{f})\boldsymbol{e}_3|}, \tag{3.17}$$

where $\mathrm{co}(\cdot)$ denotes the cofactor tensor (a definition is given, for example in section 2.11 of [17]). Since, for any two tensors $\mathbf{L}$ and $\mathbf{M}$, $\mathrm{co}(\mathbf{LM}) = \mathrm{co}(\mathbf{L})\,\mathrm{co}(\mathbf{M})$ and $\mathrm{co}(\mathbf{R}) = \mathbf{R}$ for all $\mathbf{R} \in \mathrm{SO}(3)$, it follows from (3.3) and (3.17) that

$$\boldsymbol{\nu}(\boldsymbol{f}(\boldsymbol{x})) = \mathbf{R}_f \boldsymbol{e}_3 \quad \forall \boldsymbol{x} \in S, \tag{3.18a}$$

as, by (3.4), $\mathrm{co}(\mathbf{U}_f) = (\det \mathbf{U}_f)\boldsymbol{e}_3 \otimes \boldsymbol{e}_3$. Similarly, denoting by $\boldsymbol{\nu}^*$ the unit normal to $\mathscr{S}^*$, we also have that,

$$\boldsymbol{\nu}^*(\boldsymbol{g}(\boldsymbol{x})) = \mathbf{R}_g \boldsymbol{e}_3 \quad \forall \boldsymbol{x} \in S. \tag{3.18b}$$

On the other hand, since $\mathscr{S}^* = \boldsymbol{y}(\mathscr{S})$,

$$\boldsymbol{\nu}^*(\boldsymbol{\xi}) = \frac{\mathrm{co}(\nabla_{\mathrm{s}} \boldsymbol{y})\boldsymbol{\nu}}{|\mathrm{co}(\nabla_{\mathrm{s}} \boldsymbol{y})\boldsymbol{\nu}|} \quad \forall \boldsymbol{\xi} \in \mathscr{S}. \tag{3.19}$$

Moreover, since $\nabla_{\mathrm{s}} \boldsymbol{y}$ can be decomposed as in (3.15), and $\mathbf{U}_y$ can be written as

$$\mathbf{U}_y(\boldsymbol{\xi}) = \mu_1 \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 + \mu_2 \boldsymbol{u}_2 \otimes \boldsymbol{u}_2 \quad \text{with} \quad \mu_1, \mu_2 > 0, \tag{3.20}$$

in its eigenframe $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ on the tangent plane $\mathscr{T}_{\boldsymbol{\xi}}$ to $\mathscr{S}$, we see that $\boldsymbol{\nu}^* = \mathbf{R}_y \boldsymbol{\nu}$, as $\mathrm{co}(\mathbf{U}_y) = (\det \mathbf{U}_y)\boldsymbol{\nu} \otimes \boldsymbol{\nu}$. Then (3.16) implies that:

$$\boldsymbol{\nu}^*(\boldsymbol{g}(\boldsymbol{x})) = \boldsymbol{\nu}(\boldsymbol{f}(\boldsymbol{x})) \quad \text{for all} \quad \boldsymbol{x} \in S, \tag{3.21}$$

which by (3.18) and (3.7) amounts to the equation:

$$(\boldsymbol{b}_f - \boldsymbol{b}_g) \times \boldsymbol{e}_3 = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{b}_f = \boldsymbol{b}_g = \boldsymbol{b} \quad \text{since} \quad \boldsymbol{b}_f \cdot \boldsymbol{e}_3 = \boldsymbol{b}_g \cdot \boldsymbol{e}_3 = 0. \tag{3.22}$$

This is the desired result, which justifies our identifying the vector $\boldsymbol{b}$ as the *universal* bending content, as it is common to all surfaces related via a bending-neutral deformation.

While, as already remarked, both drilling and bending contents are frame-dependent, the notion of bending-neutrality is frame-indifferent: it says that whatever bending content is assigned to a surface $\mathscr{S}$ in a frame, it will be the same as the one assigned to $\mathscr{S}^*$, if $\boldsymbol{y}$ is bending-neutral.

The definition of bending-neutral deformations was introduced in [7] in an attempt to provide a solid foundation to the notion of *pure measures* of bending in plate theory. These were defined as deformation measures (either scalars or tensors) invariant under the action of all bending-neutral deformations. For example, it was shown in Virga [7] that the tensor:

$$\mathbf{A} := (\nabla \boldsymbol{\nu})^{\mathsf{T}}(\nabla \boldsymbol{\nu}) \tag{3.23}$$

is a pure measure of bending, whereas the tensor $\mathbf{K} := (\nabla \boldsymbol{\nu})(\nabla \boldsymbol{\nu})^{\mathsf{T}}$ is not, even if it shares with $\mathbf{A}$ all scalar invariants.

(d) Compatibility condition

The existence of a bending-neutral deformation is subject to a compatibility condition that involves the curvature of $\mathscr{S}$: as shown in Virga [7], the rotation $\mathbf{R}_{\boldsymbol{\nu}}$ and the surface stretching $\mathbf{U}_y$ that decompose $\nabla_{\mathrm{s}} \boldsymbol{y}$ must be such that:

$$\nabla_{\mathrm{s}} \boldsymbol{\nu} = \mathbf{R}_{\boldsymbol{\nu}} \mathbf{U}_y^{-1}(\nabla_{\mathrm{s}} \boldsymbol{\nu})\mathbf{R}_{\boldsymbol{\nu}} \mathbf{U}_y. \tag{3.24}$$

This condition ensures that the tensor $\nabla_s \boldsymbol{\nu}^*$ be symmetric, as it should. We represent $\nabla_s \boldsymbol{\nu}$ as:

$$\nabla_s \boldsymbol{\nu} = \kappa_1 \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \kappa_2 \boldsymbol{n}_2 \otimes \boldsymbol{n}_2, \tag{3.25}$$

where (κ_1, κ_2) are the principal curvatures of $\mathscr{S}$ and $(\boldsymbol{n}_1, \boldsymbol{n}_2)$ the corresponding principal directions of curvature, and we let β be the angle the eigenframe $(\boldsymbol{n}_1, \boldsymbol{n}_2)$ of $\nabla_s \boldsymbol{\nu}$ makes with the eigenframe $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ of $\mathbf{U}_y$ in (3.20). By writing $\mathbf{R}_\nu$ as in (3.5),

$$\mathbf{R}_\nu(\alpha) = \mathbf{I} + \sin \alpha \mathbf{W}(\boldsymbol{\nu}) - (1 - \cos \alpha)\mathbf{P}(\boldsymbol{\nu}) \quad \text{with} \quad \alpha \in [-\pi, \pi], \tag{3.26}$$

equation (3.24) can then be reduced to a single scalar equation, see [7],

$$\frac{\mu_2}{\mu_1} = 1 - \frac{(\kappa_1 + \kappa_2)\sin \alpha}{\kappa_2 \sin \alpha + (\kappa_1 - \kappa_2)\cos \beta \sin(\alpha + \beta)}, \tag{3.27}$$

which is thus the general compatibility condition for the existence of a bending-neutral deformation of $\mathscr{S}$. Since only the ratio between the principal surface stretches is prescribed by (3.27), $\mathbf{U}_y$ can only be determined to within a multiplicative surface dilation $\lambda\mathbf{P}(\boldsymbol{\nu})$, with λ an arbitrary *positive* scalar field on $\mathscr{S}$.

A special class of solutions of (3.27) is obtained for $\beta = 0$ and $\alpha \notin \{0, \pi\}$. This requires $\mathscr{S}$ to be a *hyperbolic* surface, that is such that the Gaussian curvature $K := \kappa_1 \kappa_2 < 0$, as (3.27) reduces to:

$$\frac{\mu_2}{\mu_1} = -\frac{\kappa_2}{\kappa_1} > 0. \tag{3.28}$$

In this class of solutions of (3.27), $\mathbf{U}_y$ can be represented as:

$$\mathbf{U}_y = \lambda(\kappa_1 \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 - \kappa_2 \boldsymbol{n}_2 \otimes \boldsymbol{n}_2). \tag{3.29}$$

It is not restrictive to assume that both $\kappa_1 > 0$ and $\lambda > 0$. As proved in Virga [7], for $\mathbf{U}_y$ as in (3.29), the curvature tensor of $\mathscr{S}^*$ can be written as:[6]

$$\nabla_s \boldsymbol{\nu}^* = \frac{\kappa_1}{\mu_1} \boldsymbol{n}_1 \otimes \mathbf{R}_\nu \boldsymbol{n}_1 + \frac{\kappa_2}{\mu_2} \boldsymbol{n}_2 \otimes \mathbf{R}_\nu \boldsymbol{n}_2, \tag{3.30}$$

which, by (3.26) and (3.28), implies that the total curvature of $\mathscr{S}^*$ is

$$2H^* := \text{tr}(\nabla_s \boldsymbol{\nu}^*) = \cos \alpha \left(\frac{\kappa_1}{\mu_1} + \frac{\kappa_2}{\mu_2} \right) = 0. \tag{3.31}$$

Thus, if there exists a bending-neutral deformation $\boldsymbol{y}$ of a hyperbolic surface $\mathscr{S}$ with stretching tensor $\mathbf{U}_y$ as in (3.29), it necessarily maps $\mathscr{S}$ into a *minimal* surface $\mathscr{S}^*$.

(e) Integrability condition

Having solved (in a special case) the compatibility condition (3.24), we found a class of possible bending-neutral deformations $\boldsymbol{y}$ of a hyperbolic surface $\mathscr{S}$ characterized by:

$$\nabla_s \boldsymbol{y} = \mathbf{R}_\nu(\alpha)\mathbf{U}_y, \tag{3.32}$$

where $\mathbf{U}_y$ is given by (3.29) and both λ and α are surface fields to be determined. However, there is no guarantee that (3.32) actually represents a surface gradient. For this to be the case, the tensor field $\mathbf{F} := \mathbf{R}_\nu(\alpha)\mathbf{U}_y$ must satisfy the integrability condition (2.9). It is shown in appendix B that (2.9) is satisfied if and only if λ and α obey the following system of equations:

$$\begin{cases} \nabla_s(\lambda\kappa_1\cos\alpha) \cdot \boldsymbol{n}_2 - \nabla_s(\lambda\kappa_2\sin\alpha) \cdot \boldsymbol{n}_1 = \lambda(\kappa_1 + \kappa_2)(\cos\alpha\, \boldsymbol{c} \cdot \boldsymbol{n}_1 + \sin\alpha\, \boldsymbol{c} \cdot \boldsymbol{n}_2), \\ \nabla_s(\lambda\kappa_1\sin\alpha) \cdot \boldsymbol{n}_2 + \nabla_s(\lambda\kappa_2\cos\alpha) \cdot \boldsymbol{n}_1 = \lambda(\kappa_1 + \kappa_2)(\sin\alpha\, \boldsymbol{c} \cdot \boldsymbol{n}_1 - \cos\alpha\, \boldsymbol{c} \cdot \boldsymbol{n}_2), \end{cases} \tag{3.33}$$

where $\boldsymbol{c}$ is the *Cartesian connector* field defined as:[7]

$$c := (\nabla_s \boldsymbol{n}_1)^\mathsf{T} \boldsymbol{n}_2 = -(\nabla_s \boldsymbol{n}_2)^\mathsf{T} \boldsymbol{n}_1\,. \tag{3.34}$$

The system (3.33) acquires a more compact form when $\mathscr{S}$ itself is a *minimal* surface,

$$\nabla_s(\mu \sin \alpha) = \boldsymbol{\nu} \times \nabla_s(\mu \cos \alpha), \tag{3.35}$$

where $\mu := \lambda \kappa > 0$ and $\kappa := \kappa_1 = -\kappa_2$, so that (3.32) reduces to:

$$\nabla_s \boldsymbol{y} = \mu \mathbf{R}_{\boldsymbol{\nu}}(\alpha) \mathbf{P}(\boldsymbol{\nu})\,. \tag{3.36}$$

It follows from (3.35) that:

$$\begin{cases} \nabla_s(\mu \sin \alpha) \cdot \nabla_s(\mu \cos \alpha) = 0, \\ |\nabla_s(\mu \sin \alpha)| = |\nabla_s(\mu \cos \alpha)|\,. \end{cases} \tag{3.37}$$

Expanding this system, we arrive at:

$$\begin{cases} \mu \nabla_s \mu \cdot \nabla_s \alpha = 0, \\ |\nabla_s \mu|^2 = \mu^2 |\nabla_s \alpha|^2, \end{cases} \tag{3.38}$$

which amounts to the alternative $\nabla_s \mu = \pm \mu \boldsymbol{\nu} \times \nabla_s \alpha$. Insertion of either forms in (3.35) shows that the latter is only solved by $\nabla_s \mu = -\mu \boldsymbol{\nu} \times \nabla_s \alpha$, which can equivalently be written as:

$$\nabla_s \alpha = \boldsymbol{\nu} \times \nabla_s \varphi, \tag{3.39}$$

where we have set $\varphi := \ln \mu$.

As recalled in §2, for (3.39) to be integrable, the field $\boldsymbol{h} = \boldsymbol{\nu} \times \nabla_s \varphi$, which is tangential to $\mathscr{S}$, must also be such that $\mathrm{curl}_s \boldsymbol{h} \cdot \boldsymbol{\nu} = 0$. For φ of class C^2, we easily see that:

$$\mathrm{curl}_s \boldsymbol{h} = (\nabla_s \boldsymbol{\nu}) \nabla_s \varphi - (\nabla_s^2 \varphi) \boldsymbol{\nu} + (\triangle_s \varphi) \boldsymbol{\nu} - (\mathrm{div}_s \boldsymbol{\nu}) \nabla_s \varphi, \tag{3.40}$$

where $\mathrm{div}_s \boldsymbol{\nu} := \mathrm{tr}(\nabla_s \boldsymbol{\nu}) = 2H$ is the total curvature of $\mathscr{S}$, which vanishes for a minimal surface. By use of (2.5) in (3.40), we conclude that $\mathrm{curl}_s \boldsymbol{h} \cdot \boldsymbol{\nu} = 0$ whenever

$$\triangle_s \varphi = 0. \tag{3.41}$$

Thus, for any surface harmonic function φ on $\mathscr{S}$, we can set $\mu = \exp(\varphi)$ in (3.36). Correspondingly, (3.39) is integrable and it too delivers a surface harmonic function α, as it implies that, for α of class C^2,

$$\triangle_s \alpha = \mathrm{curl}_s \boldsymbol{\nu} \cdot \nabla_s \varphi - \boldsymbol{\nu} \cdot \mathrm{curl}_s \nabla_s \varphi = 0 \tag{3.42}$$

because $\mathrm{curl}_s \boldsymbol{\nu} = \boldsymbol{0}$, by the symmetry of the curvature tensor. This identifies (at least locally) a drilling rotation $\mathbf{R}_{\boldsymbol{\nu}}$ that makes equation (3.36) integrable, so as to determine locally (to within a translation) a bending-neutral deformation of $\mathscr{S}$. We have actually found a whole class of bending-neutral deformations of $\mathscr{S}$, each represented by a surface harmonic function φ on $\mathscr{S}$.

We shall see in the following section that this bond between bending-neutral deformations of minimal surfaces and surface harmonic functions is indeed much stronger.

4. Minimal surfaces

Here, we show that every two minimal surfaces of class C^2 with the *same* spherical image[8] are one the image of the other under a bending-neutral deformation in the family described in §3. To this end, it is particularly instrumental to resort to the celebrated Enneper–Weierstrass representation of minimal surfaces (e.g. section 3.3 of [35]), which requires the use of *isothermal* parameters (u, v) to describe a surface $\mathscr{S}$.

In an isothermal parameterization, the mapping $\boldsymbol{r} : \Omega \to \mathscr{E}$ that represents $\mathscr{S}$ as image of a simply connected set Ω of the (u, v) plane enjoys the following properties:

$$|\boldsymbol{r}_{,u}| = |\boldsymbol{r}_{,v}| \quad \text{and} \quad \boldsymbol{r}_{,u} \cdot \boldsymbol{r}_{,v} = 0, \tag{4.1}$$

where a comma denotes differentiation.[9] Moreover, since also

$$\triangle \boldsymbol{r} = \boldsymbol{r}_{,uu} + \boldsymbol{r}_{,vv} = \boldsymbol{0} \tag{4.2}$$

(e.g. [38, p.268]), for any scalar-valued function φ defined on $\mathscr{S}$ in terms of the (u, v) parameters,

$$\triangle_s \varphi = \frac{1}{|\boldsymbol{r}_{,u}|^2} \triangle \varphi = \frac{1}{|\boldsymbol{r}_{,u}|^2}(\varphi_{,uu} + \varphi_{,vv}), \tag{4.3}$$

so that φ is surface harmonic on $\mathscr{S}$ whenever it is harmonic in Ω.

(a) Weierstrass representation

Let $w := u + iv \in \mathbb{C}$ be a complex representation of the (u, v) plane. A minimal surface $\mathscr{S}$ such that the normal $\boldsymbol{\nu}$ regarded as a map from Ω into the unit sphere $\mathbb{S}^2$ is injective, but does not cover the whole of $\mathbb{S}^2$, can be represented (to within a translation) by a *holomorphic* function $F : \Omega \to \mathbb{C}$ as

$$\boldsymbol{r}(w) = \Re\left(\frac{1}{2}\int (1 - w^2)F(w)\mathrm{d}w\,\boldsymbol{e}_1 + \frac{i}{2}\int (1 + w^2)F(w)\mathrm{d}w\,\boldsymbol{e}_2 + \int wF(w)\mathrm{d}w\,\boldsymbol{e}_3\right), \tag{4.4}$$

where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a Cartesian frame in $\mathscr{V}$ and $\Re$ denotes the real part (e.g. [35, p. 117]). F is also called the *Weierstrass function* that represents $\mathscr{S}$ (see also [39,40]).

A number of interesting consequences follow from (4.4), which can easily be interpreted geometrically.

We first write F as:

$$F = \mathrm{e}^{\Phi + i\chi} = \mathrm{e}^{\Phi}(\cos \chi + i \sin \chi). \tag{4.5}$$

Since F is holomorphic, the Cauchy–Riemann relations imply that both Φ and χ are harmonic in Ω. Explicit computations give:

$$\boldsymbol{r}_{,u} = \mathrm{e}^{\Phi}\left\{\left[uv \sin \chi + \frac{1}{2}(1 - u^2 + v^2) \cos \chi\right]\boldsymbol{e}_1 - \left[uv \cos \chi + \frac{1}{2}(1 + u^2 - v^2) \sin \chi\right]\boldsymbol{e}_2\right.$$
$$\left. + (u \cos \chi - v \sin \chi)\boldsymbol{e}_3\right\}, \tag{4.6a}$$

$$\boldsymbol{r}_{,v} = \mathrm{e}^{\Phi}\left\{\left[uv \cos \chi - \frac{1}{2}(1 - u^2 + v^2) \sin \chi\right]\boldsymbol{e}_1 + \left[uv \sin \chi - \frac{1}{2}(1 + u^2 - v^2) \cos \chi\right]\boldsymbol{e}_2\right.$$
$$\left. - (u \sin \chi + v \cos \chi)\boldsymbol{e}_3\right\}, \tag{4.6b}$$

which satisfy

$$|\boldsymbol{r}_{,u}| = |\boldsymbol{r}_{,v}| = \frac{1}{2}\mathrm{e}^{\Phi}(u^2 + v^2 + 1) \quad \text{and} \quad \boldsymbol{r}_{,u} \cdot \boldsymbol{r}_{,v} = 0. \tag{4.6c}$$

Letting $\boldsymbol{e}_u$ and $\boldsymbol{e}_v$ denote the unit vectors of $\boldsymbol{r}_{,u}$ and $\boldsymbol{r}_{,v}$, respectively, we form an orthonormal frame $(\boldsymbol{e}_u, \boldsymbol{e}_v)$ on the tangent plane $\mathscr{T}_{\boldsymbol{r}(w)}$ on $\mathscr{S}$; the unit normal $\boldsymbol{\nu}$ is then obtained from:

$$\boldsymbol{\nu} = \boldsymbol{e}_u \times \boldsymbol{e}_v = \frac{1}{u^2 + v^2 + 1}\{2u\boldsymbol{e}_1 + 2v\boldsymbol{e}_2 + (u^2 + v^2 - 1)\boldsymbol{e}_3\}. \tag{4.7}$$

This equation shows clearly the geometrical meaning of the (u, v) coordinates in Weierstrass' representation: w is the stereographic projection of $\boldsymbol{\nu}$ (on the equatorial plane of $\mathbb{S}^2$ with the

North pole identified with e_3). Thus, all functions F defined on the *same* domain Ω represent minimal surfaces with the *same* spherical image.

Consider now a trajectory in Ω parameterized by $(u(t), v(t))$; r maps it into a trajectory on $\mathscr{S}$ for which:

$$\dot{r} = |r_{,u}|(\dot{u}e_u + \dot{v}e_v), \tag{4.8}$$

where a superimposed dot denotes differentiation with respect to t. Correspondingly,

$$\dot{\nu} = \dot{u}\nu_{,u} + \dot{v}\nu_{,v} = (\nabla_s\nu)\dot{r}. \tag{4.9}$$

By differentiating with respect to u and v both sides of (4.7) and projecting both $\nu_{,u}$ and $\nu_{,v}$ in the frame (e_u, e_v), we obtain that:

$$\nu_{,u} = \frac{2}{u^2 + v^2 + 1}(\cos\chi e_u - \sin\chi e_v), \tag{4.10a}$$

$$\nu_{,v} = -\frac{2}{u^2 + v^2 + 1}(\sin\chi e_u + \cos\chi e_v). \tag{4.10b}$$

Inserting (4.10) and (4.7) into (4.9), with the aid of (4.6c), since $\dot{u}$ and $\dot{v}$ are arbitrary, we arrive at the following representation of the curvature tensor in terms of the functions Φ and χ,

$$\nabla_s\nu = \frac{4}{e^\Phi(u^2 + v^2 + 1)^2}\mathbf{R}_\nu(-\chi)(e_u \otimes e_u - e_v \otimes e_v), \tag{4.11}$$

from which we obtain at once that:

$$2H = \text{tr}(\nabla_s\nu) = 0 \quad \text{and} \quad K = \det(\nabla_s\nu) = -\frac{16}{e^{2\Phi}(u^2 + v^2 + 1)^4}. \tag{4.12}$$

Comparing (4.11) and (3.25), we easily see that:

$$\kappa_1 = -\kappa_2 = \frac{4}{e^\Phi(u^2 + v^2 + 1)^2}, \quad n_1 = \cos\frac{\chi}{2}e_u - \sin\frac{\chi}{2}e_v, \quad n_2 = \sin\frac{\chi}{2}e_u + \cos\frac{\chi}{2}e_v. \tag{4.13}$$

(b) Bending-neutral associates

Let $\mathscr{S}^*$ be a minimal surface, different from $\mathscr{S}$ but with the same spherical image, represented by the holomorphic function:

$$F^* = e^{\Phi^* + i\chi^*} = e^{\Phi^*}(\cos\chi^* + i\sin\chi^*), \tag{4.14}$$

where both Φ^* and χ^* are harmonic functions in Ω. Precisely as in (4.4), F^* induces the mapping $r^* : \Omega \to \mathscr{E}$ that represents $\mathscr{S}^*$. The vectors $r^*_{,u}$ and $r^*_{,v}$ are still given by (4.6) with Φ replaced by Φ^* and χ by χ^*. The unit vectors (e^*_u, e^*_v) that identify a frame on the tangent plane $\mathscr{T}_{r^*(w)}$ to $\mathscr{S}^*$ are defined accordingly and the unit normal $\nu^* = e^*_u \times e^*_v$ is still given by (4.7) in the Cartesian frame (e_1, e_2, e_3), so that $\nu^* \equiv \nu$, although $r^* \neq r$. It is not difficult to show by direct computation that:

$$e^*_u \cdot e_u = e^*_v \cdot e_v = \cos(\chi^* - \chi). \tag{4.15}$$

Let y be a deformation that maps $\mathscr{S}$ onto $\mathscr{S}^*$. For any trajectory $(u(t), v(t))$ in Ω, we have that:

$$\dot{r}^* = (\nabla y)\dot{r}, \tag{4.16}$$

which by (4.6c) and (4.8), along with the corresponding equations for r^*, can also be written as:

11

royalsocietypublishing.org/journal/rspa   *Proc. R. Soc. A* **480:** 20240394

12

royalsocietypublishing.org/journal/rspa   Proc. R. Soc. A **480**: 20240394

$$e^{\Phi^*}(\dot{u}e_u^* + \dot{v}e_v^*) = e^{\Phi}(\dot{u}e_u + \dot{v}e_v). \tag{4.17}$$

Since the frames (e_u^*, e_v^*) and (e_u, e_v) are on parallel planes (both orthogonal to $\boldsymbol{\nu}$) and both $\dot{u}$ and $\dot{v}$ are arbitrary, (4.17) and (4.15) imply that:

$$\nabla_s y = e^{\Phi^* - \Phi} R_{\boldsymbol{\nu}}(\chi^* - \chi)P(\boldsymbol{\nu}). \tag{4.18}$$

By comparing (4.18) and (3.36), we readily conclude that y is a bending-neutral deformation with:

$$\mu = e^{\varphi} = e^{\Phi^* - \Phi} \quad \text{and} \quad \alpha = \chi^* - \chi \tag{4.19}$$

directly derived from the Weierstrass functions representing the surfaces $\mathscr{S}^*$ and $\mathscr{S}$. This proves our claim that *any two minimal surfaces with the same spherical image are associated through a bending-neutral deformation.*

In the classical theory of minimal surfaces, a special role is played by the families of *associate surfaces*. Given a minimal surface $\mathscr{S}$, the classical family of its associates $\mathscr{S}_\theta = y_\theta(\mathscr{S})$, described by a real parameter θ, consists of minimal surfaces in isometric correspondence with one another and such that normals are preserved for all θ. The mapping that changes $\mathscr{S}$ into $\mathscr{S}_\theta$ is also called *Bonnet's transformation* (e.g. section 3.1 of [35]); in the Weierstrass representation, it has the following simple form [35, p. 119]:

$$F(w) \mapsto F_\theta(w) := e^{i\theta}F(w) \tag{4.20}$$

with θ a constant.

Perhaps the best known of these families is that transforming catenoids into helicoids, for which,

$$F_\theta = \frac{c}{w^2}(\cos\theta + i\sin\theta), \tag{4.21}$$

where $c \in \mathbb{R}$ is a constant and $\theta \in [0, \frac{\pi}{2}]$. For $\theta = 0$ or $\theta = \frac{\pi}{2}$, F_θ represents a catenoid or a helicoid, respectively; for $0 < \theta < \frac{\pi}{2}$, the intermediate associate surface $\mathscr{S}_\theta$ is *Scherk's second surface* (e.g. [35, p. 148]).

Bending-neutral deformations generate far more general families of associate minimal surfaces through the transformation:

$$F(w) \mapsto F^*(w) := e^{\varphi + i\vartheta}F(w), \tag{4.22}$$

where both φ and ϑ are harmonic functions. Furthermore, in this broad sense, any minimal surface is the bending-neutral associate of another with the same spherical image. Like Bonnet's, the transformation (4.22) preserves normals, and it is isometric for $\varphi \equiv 0$. In particular, by (4.19), we may say that for any $\theta \in (0, \frac{\pi}{2})$ Scherk's second surface $\mathscr{S}_\theta$ is the image of a catenoid (or a helicoid) under an isometric bending-neutral deformation with $\mu \equiv 1$ and $\alpha \equiv \theta$.

(c) Universal bending content

We learned in §3 that in general two surfaces related by a bending-neutral deformation have one and the same bending content. Here, we prove that the bending content $\boldsymbol{b}$ of a minimal surface is as universal as the normal $\boldsymbol{\nu}$ in Weierstrass' representation.

Interpreting $\boldsymbol{r}$ as a deformation from a reference configuration Ω, by (3.3), we can write $\nabla\boldsymbol{r} = R_r U_r$, where

$$U_r = \frac{1}{2}e^{\Phi}(u^2 + v^2 + 1)P(e_3) \quad \text{and} \quad R_r = e_u \otimes e_1 + e_v \otimes e_2 + \boldsymbol{\nu} \otimes e_3, \tag{4.23}$$

having chosen $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ as the frame of the (u, v) coordinates in Ω with $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$. By use of (3.8), we identify the vector $\boldsymbol{a_r}$ that represents $\mathbf{R_r}$ in (3.7),

$$\boldsymbol{a_r} = \frac{1}{u \sin \chi + v(1 + \cos \chi)}\{-(1 + \cos \chi)\boldsymbol{e}_1 + \sin \chi \boldsymbol{e}_2 + [v \sin \chi - u(1 + \cos \chi)]\boldsymbol{e}_3\}. \tag{4.24a}$$

Then, we obtain from (3.14) the vectors $\boldsymbol{d_r}$ and $\boldsymbol{b_r}$ that represent the drilling and bending components $\mathbf{R}_r^{(b)}$ and $\mathbf{R}_r^{(d)}$ of $\mathbf{R_r}$,

$$\boldsymbol{d_r} = \frac{v \sin \chi - u(1 + \cos \chi)}{u \sin \chi + v(1 + \cos \chi)}\boldsymbol{e}_3 = -\cot\left(\frac{\chi}{2} + \phi\right)\boldsymbol{e}_3, \tag{4.24b}$$

$$\boldsymbol{b_r} = \frac{1}{u^2 + v^2}(-v\boldsymbol{e}_1 + u\boldsymbol{e}_2) = \frac{1}{\rho}\boldsymbol{e}_\phi, \tag{4.24c}$$

where we have expressed (u, v) in polar form,

$$u = \rho \cos \phi, \quad v = \rho \sin \phi, \tag{4.25}$$

and set $\boldsymbol{e}_\phi := -\sin \phi \, \boldsymbol{e}_1 + \cos \phi \, \boldsymbol{e}_2$. It is clear from (4.24c) that $\boldsymbol{b_r} = \boldsymbol{b}$ is the same for all minimal surfaces, as it is independent of both Φ and χ, precisely like $\boldsymbol{\nu}$ in (4.7).

In a similar way, we can prove directly that the pure measure of bending $\mathbf{A}$ in (3.23) is the same for all minimal surfaces. Since

$$\nabla \boldsymbol{\nu} = \boldsymbol{\nu}_{,u} \otimes \boldsymbol{e}_1 + \boldsymbol{\nu}_{,v} \otimes \boldsymbol{e}_2, \tag{4.26}$$

by use of (4.10), we give $\mathbf{A}$ the following explicit form:

$$\mathbf{A} = \frac{4}{(u^2 + v^2 + 1)^2}\mathbf{P}(\boldsymbol{e}_3). \tag{4.27}$$

(d) Illustration by Bour's surfaces

To illustrate graphically our findings, we consider an exemplary family $\mathscr{S}_t$ of bending-neutral associates of two minimal surfaces $\mathscr{S}$ and $\mathscr{S}^*$, both with historical pedigree. The parameter t ranges in the interval $[0,1]$, so that $\mathscr{S}_t$ is $\mathscr{S}$ for $t = 0$, and $\mathscr{S}^*$ for $t = 1$. The Weierstrass function representing $\mathscr{S}_t$ is

$$F_t(w) = w^t = \rho^t e^{it\phi}, \tag{4.28}$$

where we have set $w = \rho \exp(i\phi)$. It is immediately seen that $\mathscr{S}_t$ is the image of $\mathscr{S}$ under the bending-neutral deformation characterized by $\mu = \rho^t$ and $\alpha = t\phi$.

Here, $\mathscr{S}$ is Enneper's surface, while $\mathscr{S}^*$ is Bour's surface of index $m = 3$. The Weierstrass function of the former is $F(w) \equiv 1$, while that of the latter is obtained by setting $m = 3$ into the general formula:

$$F_B(w) := cw^{m-2}, \tag{4.29}$$

where $c \neq 0$ is a complex constant and $m \in \mathbb{R}$ (see [35, p. 156]).[10] Actually, $\mathscr{S}_t$ is a Bour's surface for all $t \in [0,1]$, with $m = t + 2$.

Figure 2 shows a gallery of surfaces extracted from the family $\mathscr{S}_t$ represented by (4.29): these are images of a disk $\mathbb{B}_R$ centred at $w = 0$ (described in polar coordinates by $0 < \rho < R$ and $-\pi \leqslant \phi \leqslant \pi$); the colour coding is such that each individual radius of $\mathbb{B}_R$ conveys on $\mathscr{S}_t$ a specific colour.
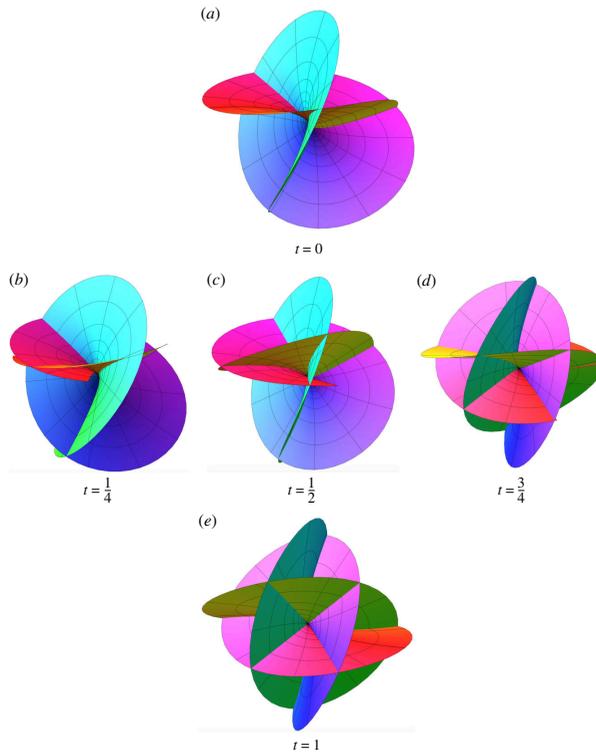
Figure 2. Surfaces $\mathscr{S}_t$ extracted from the family described by the Weierstrass function F_t in (4.28). They are images via the mapping $\boldsymbol{r}_t(w)$ induced by $F_t(w)$ through (4.4) of the disk $\mathbb{B}_R$ of radius R centred at $w = 0$; the scaling factor of each image is $|\boldsymbol{r}_t(R)|$. The selected values of the parameter t are shown in panels (a)-(e).

As a supplement to this paper, we also provide an animation showing the whole family $\mathscr{S}_t$. We think that this makes it apparent how the moving surface progressively unfolds *gliding* over itself, with *no* sign of *bending*. In a companion animation, we also capture the motion that illustrates the classical association between catenoid and helicoid described by (4.21): it clearly conveys the same gliding impression.

We finally remark that by the change of variable $w \mapsto 1/w$ in (4.4) for $F = F_B$ as in (4.29) Bour's surfaces are shown to be invariant (to within a reflection) under the transformation $m \mapsto -m$. This makes catenoids and helicoids somehow special members of Bour's family, as they are obtained from (4.29) for $m = 0$.

5. Conclusions

Understanding how surfaces deform is the proper foundation for a sound, intrinsic theory of plates and shells (as abundantly discussed, for example in section XIV.13 of [2]). Expressing the appropriate energy contents involved in the elasticity of these bodies requires a clear identification of the local independent modes for changing their shape.

Intuitively, we may say that there are only two such modes, that is, *stretching* and *bending*. Rigorously, we may associate unambiguously the former mode with the stretching component $\mathbf{U}$ of the surface deformation gradient $\mathbf{F}$, as delivered by the polar decomposition theorem. But associating the bending mode with the whole orthogonal component $\mathbf{R}$ of $\mathbf{F}$ cannot be right, as this deformation descriptor would in general fail to be intrinsic to the surface.

By decomposing uniquely $\mathbf{R}$ into a *drilling* rotation $\mathbf{R}^{(d)}$ about the surface normal $\boldsymbol{\nu}$ and a complementary rotation $\mathbf{R}^{(b)}$ about an axis orthogonal to $\boldsymbol{\nu}$, we identified the latter as the correct *bending* descriptor. *Bending-neutral* deformations were then naturally defined as those deformations that would leave the bending content unchanged, while affecting the drilling one. These deformations preserve normals. In a variational elastic theory for shells, these deformations, if existent, would not affect the bending energy, while still prompting effective changes in shape, at the expenses of a drilling energy.

Whether bending-neutral deformations do actually exist is not a trivial issue. In this paper, we gave a necessary and sufficient integrability condition that guarantees their existence, at least locally. A large class of surfaces can indeed be subjected to bending-neutral deformations: *all* minimal surfaces. Actually, there is more to it: (i) every minimal surface is transformed into a minimal surface by a bending-neutral deformation; (ii) given two minimal surfaces with the same system of normals, there is a bending-neutral deformation that maps one into the other; and (iii) *all* minimal surfaces have indeed a *universal* bending content.

All conceivable normal preserving motions driving one minimal surface into another can only consist of drilling and stretching, as is apparent from the gliding motions shown in the animations accompanying this paper.

In the light of our finer decomposition of surface deformations, speaking of 'a bending procedure which, at every stage, passes through a minimal surface' [35, p. 102], as done in almost every textbook, would be inaccurate whenever normals are preserved: there is indeed *no* way to *bend* a minimal surface, other than by altering the system of its normals.

Although we deem our kinematic analysis of surfaces possibly also relevant to the energetics of elastic plates and shells, we have not attempted any systematic treatment of this issue. A preliminary step toward this aim would be to extract from both $\boldsymbol{d}_f$ and $\boldsymbol{b}_f$ objective measures of drilling and bending that would feature in a stored-energy functional.

We are also guilty of another omission. Our deformation study was local in nature: we left out all global considerations. This came with the price of allowing self-intersecting surfaces as deformed images, which is forbidden in continuum mechanics by the principle of identification of material body points.

Appendix A: Rotation decomposition

Here, we use equation (3.10) for the decomposition of a rotation $\mathbf{R}(\boldsymbol{a}) \in \mathsf{SO}(3)$ to prove that the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ can be identified uniquely by requiring the former to be parallel to a given unit vector $\boldsymbol{e}$ and the latter orthogonal to it.

15

royalsocietypublishing.org/journal/rspa *Proc. R. Soc. A* **480:** 20240394

Let $\boldsymbol{a} \in \mathscr{V}$ and $\boldsymbol{e} \in \mathbb{S}^2$ be given so that $\boldsymbol{a} \times \boldsymbol{e} \neq \boldsymbol{0}$.[11] We solve (3.11) with $\boldsymbol{a}_1 = u\boldsymbol{e}$ and $\boldsymbol{a}_2 \cdot \boldsymbol{e} = 0$. Projecting both sides of (3.11) along $\boldsymbol{e}$ and on the plane orthogonal to it, we find that:

$$u = \boldsymbol{a} \cdot \boldsymbol{e}, \tag{A 1a}$$

$$\mathbf{P}(\boldsymbol{e})\boldsymbol{a} = \boldsymbol{a}_2 - u\boldsymbol{e} \times \boldsymbol{a}_2. \tag{A 1b}$$

Representing $\boldsymbol{a}_2$ as:

$$\boldsymbol{a}_2 = b_1\mathbf{P}(\boldsymbol{e})\boldsymbol{a} + b_2\boldsymbol{e} \times \mathbf{P}(\boldsymbol{e})\boldsymbol{a}, \tag{A 2}$$

where b_1 and b_2 are scalar parameters to be determined, we easily see that the only solution of (A 1b) is obtained for:

$$b_1 = \frac{1}{1 + u^2} \quad \text{and} \quad b_2 = \frac{u}{1 + u^2}. \tag{A 3}$$

Use of (A 3) and (A 1a) in (A 2) readily leads us to (3.12) in the main text.

To prove that $\mathbf{R}(\boldsymbol{a}_1)$ with $\boldsymbol{a}_1$ as in (3.12) is indeed the rotation of $\mathsf{SO}(\boldsymbol{e})$ closest to $\mathbf{R}(\boldsymbol{a})$, we introduce the (squared) distance:

$$d(u) := |\mathbf{R}(\boldsymbol{a}) - \mathbf{R}(u\boldsymbol{e})|^2 := \mathrm{tr}\{[\mathbf{R}(\boldsymbol{a}) - \mathbf{R}(u\boldsymbol{e})]^\mathsf{T}[\mathbf{R}(\boldsymbol{a}) - \mathbf{R}(u\boldsymbol{e})]\}. \tag{A 4}$$

A direct computation shows that

$$d(u) = 6 + \frac{2}{1 + a^2}\frac{1}{1 + u^2}\{[1 + a^2 - 4(\boldsymbol{a} \cdot \boldsymbol{e})^2]u^2 - 8(\boldsymbol{a} \cdot \boldsymbol{e})u + a^2 - 3\}, \tag{A 5}$$

and it is then a simple matter to conclude that d attains its minimum for $u = \boldsymbol{a} \cdot \boldsymbol{e}$ and its maximum for $u = -1/\boldsymbol{a} \cdot \boldsymbol{e}$.

Appendix B: Gradient integrability

In this appendix, we gather a number of details needed to derive the integrability conditions in (3.33). The objective is to enforce (2.9) for the surface tensor field $\mathbf{F} = \mathbf{R}_\nu(\alpha)\mathbf{U}_y$ where $\mathbf{U}_y$ is given by (3.29) and $\mathbf{R}_\nu(\alpha)$ by (3.26).

We begin by writing $\mathbf{F}$ explicitly in the eigenframe $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{\nu})$ of the curvature tensor,

$$\mathbf{F} = \lambda\{\kappa_1\cos\,\alpha\boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \kappa_2\sin\,\alpha\boldsymbol{n}_1 \otimes \boldsymbol{n}_2 + \kappa_1\sin\,\alpha\boldsymbol{n}_2 \otimes \boldsymbol{n}_1 - \kappa_2\cos\,\alpha\boldsymbol{n}_2 \otimes \boldsymbol{n}_2\}. \tag{B 1}$$

To compute $\nabla_s\mathbf{F}$, we also need to express both $\nabla_s\boldsymbol{n}_1$ and $\nabla_s\boldsymbol{n}_2$ in the frame $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{\nu})$,

$$\begin{cases} \nabla_s\boldsymbol{n}_1 = \boldsymbol{n}_2 \otimes \boldsymbol{c} - \kappa_1\boldsymbol{\nu} \otimes \boldsymbol{n}_1, \\ \nabla_s\boldsymbol{n}_2 = -\boldsymbol{n}_1 \otimes \boldsymbol{c} - \kappa_2\boldsymbol{\nu} \otimes \boldsymbol{n}_2, \end{cases} \tag{B 2}$$

where $\boldsymbol{c}$ is the tangential field defined by (3.34).

A lengthy but simple calculation shows that equation (2.9) acquires the following structure:

$$\boldsymbol{n}_1 \otimes \mathbf{W}_1 + \boldsymbol{n}_2 \otimes \mathbf{W}_2 = \boldsymbol{0}, \tag{B 3}$$

where both $\mathbf{W}_1$ and $\mathbf{W}_2$ are skew-symmetric second-rank tensors, which by (B 1), (B 2) and (3.25) are associated with the following axial vectors

$$\begin{cases} \boldsymbol{w}_1 = \nabla_s(\lambda\kappa_1\cos\alpha) \times \boldsymbol{n}_1 + \nabla_s(\lambda\kappa_2\sin\alpha) \times \boldsymbol{n}_2 - \lambda(\kappa_1 + \kappa_2)\mathbf{R}_\nu(\alpha)\boldsymbol{n}_2 \times \boldsymbol{c}, \\ \boldsymbol{w}_2 = \nabla_s(\lambda\kappa_1\sin\alpha) \times \boldsymbol{n}_1 - \nabla_s(\lambda\kappa_2\cos\alpha) \times \boldsymbol{n}_2 - \lambda(\kappa_1 + \kappa_2)\mathbf{R}_\nu(\alpha)\boldsymbol{n}_1 \times \boldsymbol{c}, \end{cases} \tag{B 4}$$

which are both parallel to $\boldsymbol{\nu}$. The system of scalar equations (3.33) in the main text follows from requiring that $\boldsymbol{w}_1 = \boldsymbol{w}_2 \equiv \boldsymbol{0}$ on $\mathscr{S}$.

Endnotes

[1]This definition agrees with that given by Weatherburn [11, p. 220], who identifies $\nabla_s\varphi$ with the 'vector quantity whose direction is that direction on the surface p at which gives the maximum arc-rate of increase of φ, and whose magnitude is this maximum rate of increase', which is perhaps more descriptive.

[2]As remarked in Weatherburn [15 p. 5], $\triangle_s\varphi$ is just the same as the *differential parameter of the second order* introduced by Beltrami (see pp. 148, 321 of [16]).

[3]We have retraced the first occurrence of 'drilling degrees of freedom' to [23]; it was soon picked up [24]; see also [25] for an instance of more recent usage.

[4]The reader is referred to Altmann [26] for a witty account on Rodrigues' representation of rotations and its connections with Hamilton's quaternions.

[5]Much in the spirit of the variational characterization of the polar decomposition itself [29,30] initiated by a kinematic note of Grioli [31] recently revived and commented in Neff *et al*. [32].

[6]It would be a simple matter for the reader to check directly that use of (3.29) in (3.24) makes the latter equation identically satisfied for any $\mathbf{R}_\nu$ as in (3.26) and that use of (3.28) in (3.30) makes $\nabla_s\boldsymbol{\nu}^*$ correspondingly symmetric for any $\mathbf{R}_\nu$.

[7]The definition of Cartesian connectors is further expounded in Ozenda *et al*. and Ozenda & Virga [33,34].

[8]The spherical image of a surface is the image of its normal map (or Gauss map) on $\mathbb{S}^2$ (e.g. [35 , p. 9]). Saying that two surfaces share the same spherical image amounts to saying that they have collectively the same system of normals, which is perhaps a more colloquial expression.

[9]The existence of isothermal parameters for a generic surface is not obvious, and for surfaces of class C^1 it is even not generally true. For surfaces of class C^2, they always exist (at least locally) by a classical theorem (see also [36]) whose proof is indeed much easier for minimal surfaces (e.g. [37 p. 31]).

[10]Bour [41] proved that the surfaces represented by (4.29) constitute the complete set of minimal surfaces that can be mapped isometrically onto a surface of revolution. The paper [41] is the published version of the manuscript that won Bour the 1859 *Grand Prix des Mathématiques* granted by the French Academy of Sciences in Paris. A terse account of Bour's manuscript and two others competing with it for the same prize can be found in [42].

[11]Were $\boldsymbol{a} \times \boldsymbol{e} = \mathbf{0}$, our claim would be trivially valid with $\boldsymbol{a}_1 = \boldsymbol{a}$ and $\boldsymbol{a}_2 = \mathbf{0}$.

References

1. Hildebrandt S, Tromba A. 1996 *The Parsimonious Universe. Shape and form in the natural world*. New York, NY: Springer-Verlag.
2. Antman SS. 1995 Nonlinear problems of elasticity. In *Applied mathematical sciences*, vol. **107**. New York, NY: Springer.
3. Ghiba ID, Birsan M, Lewintan P, Neff P. 2020 The isotropic Cosserat shell model including terms up to O(h⁵): part I: derivation in matrix notation. *J. Elast.* **142**, 201–262. (doi:10.1007/s10659-020-09796-3)
4. Ghiba ID, Bîrsan M, Lewintan P, Neff P. 2021 A constrained Cosserat shell model up to order O(h⁵): modelling, existence of minimizers, relations to classical shell models and scaling invariance of the bending tensor. *J. Elast.* **146**, 83–141. (doi:10.1007/s10659-021-09851-7)
5. Vitral E, Hanna JA. 2023 Dilation-invariant bending of elastic plates, and broken symmetry in shells. *J. Elast.* **153**, 571–579. (doi:10.1007/s10659-022-09894-4)
6. Vitral E, Hanna JA. 2023 Energies for elastic plates and shells from quadratic-stretch elasticity. *J. Elast.* **153**, 581–598. (doi:10.1007/s10659-022-09895-3)
7. Virga EG. 2023 Pure measures of bending for soft plates. *Soft Matter* **20**, 144–151. (doi:10.1039/D3SM01123B)
8. Acharya A. 2024 Mid-surface scaling invariance of some bending strain measures. *J. Elast.* **156**, 619–621. (doi:10.1007/s10659-024-10066-9)
9. Ghiba ID, Lewintan P, Sky A, Neff P. 2023 An essay on deformation measures in isotropic thin shell theories. Bending versus curvature. *arXiv preprint*.(doi:10.48550/arXiv.2312.10928)
10. Vitral E, Hanna JA. 2024 Assorted remarks on bending measures and energies for plates and shells and their invariance properties. *arXiv Preprint.* (doi:10.48550/arXiv.2405.06638)

11. Weatherburn CE. 2016 *Differential geometry of three dimensions*. vol. **I**. Cambridge, UK: Cambridge University Press.

12. Šilhavý M. 2021 A new approach to curvature measures in linear shell theories. *Math. Mech. Solids* **26**, 1241–1263. (doi:10.1177/1081286520972752)

13. Kralj S, Rosso R, Virga EG. 2011 Curvature control of valence on nematic shells. *Soft Matter* **7**, 670–683. (doi:10.1039/C0SM00378F)

14. Rosso R, Virga EG, Kralj S. 2012 Parallel transport and defects on nematic shells. *Continuum Mech. Thermodyn.* **24**, 643–664. (doi:10.1007/s00161-012-0259-4)

15. Weatherburn CE. 2016 *Differential geometry of three dimensions*. vol. **II**. Cambridge, UK: Cambridge University Press.

16. Beltrami E. 1902 Opere Matematiche vol. 1. Milan, Italy: Hoepli. See https://gallica.bnf.fr/ark:/12148/bpt6k99432q/f1.item.

17. Gurtin ME, Fried E, Anand L. 2010 *The mechanics and thermodynamics of continua*. Cambridge, UK: Cambridge University Press.

18. Man CS, Cohen H. 1986 A coordinate-free approach to the kinematics of membranes. *J. Elast.* **16**, 97–104. (doi:.10.1007/BF00041068)

19. Gurtin ME, Murdoch AI. 1975 A continuum theory of elastic material surfaces. *Arch. Ration. Mech. Anal.* **57**, 291–323. (doi:10.1007/BF00261375)

20. Gurtin ME, Murdoch AI. 1975 Addenda to our paper a continuum theory of elastic material surfaces. *Arch. Ration. Mech. Anal.* **59**, 389–390. (doi:10.1007/BF00250426)

21. Pietraszkiewicz W, Szwabowicz ML, Vallée C. 2008 Determination of the midsurface of a deformed shell from prescribed surface strains and bendings via the polar decomposition. *Int. J. Non Linear Mech.* **43**, 579–587. (doi:10.1016/j.ijnonlinmec.2008.02.003)

22. Palais B, Palais R, Rodi S. 2009 A disorienting look at Euler's theorem on the axis of a rotation. *Am. Math. Mon.* **116**, 892–909. (doi:10.4169/000298909X477014)

23. Hughes TJR, Brezzi F. 1989 On drilling degrees of freedom. *Comput. Methods Appl. Mech. Eng.* **72**, 105–121. (doi:10.1016/0045-7825(89)90124-2)

24. Fox DD, Simo JC. 1992 A drill rotation formulation for geometrically exact shells. *Comput. Methods Appl. Mech. Eng.* **98**, 329–343. (doi:10.1016/0045-7825(92)90002-2)

25. Mohammadi Saem M, Lewintan P, Neff P. 2021 On in-plane drill rotations for Cosserat surfaces. *Proc. R. Soc. A* **477**. (doi:10.1098/rspa.2021.0158)

26. Altmann SL. 1989 Hamilton, Rodrigues, and the quaternion scandal. *Math. Mag.* **62**, 291–308. (doi:10.1080/0025570X.1989.11977459)

27. Rodrigues O. 1840 Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendamment des causes qui peuvent les produire. *J. Math. Pures Appl.* **5**, 380–440.

28. Mladenova CD, Mladenov IM. 2011 Vector decomposition of finite rotations. *Rep. Math. Phys.* **68**, 107–117. (doi:10.1016/S0034-4877(11)60030-X)

29. Martins LC, Podio-Guidugli P. 1979 A variational approach to the polar decomposition theorem. *Atti. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. Rend.* **66**, 487–493.

30. Martins LC, Podio-Guidugli P. 1980 An elementary proof of the polar decomposition theorem. *Am. Math. Mon.* **87**, 288–290. (doi:10.1080/00029890.1980.11995017)

31. Grioli G. 1940 Una proprieta di minimo nella cinematica delle deformazioni finite. *Boll. Un. Math. Ital.* **2**, 252–255.

32. Neff P, Lankeit J, Madeo A. 2014 On Grioli's minimum property and its relation to Cauchy's polar decomposition. *Int. J. Eng. Sci.* **80**, 209–217. (doi:10.1016/j.ijengsci.2014.02.026)

33. Ozenda O, Sonnet AM, Virga EG. 2020 A blend of stretching and bending in nematic polymer networks. *Soft Matter* **16**, 8877–8892. (doi:10.1039/d0sm00642d)

34. Ozenda O, Virga EG. 2021 On the Kirchhoff-Love hypothesis (revised and vindicated). *J. Elast.* **143**, 359–384. (doi:(10.1007/s10659-021-09819-7)

35. Dierkes U, Hildebrandt S, Sauvigny F. 2010 Minimal surfaces. In *Series of comprehensive studies in mathematics*, vol. **339A**, 2nd edn. Berlin, Germany: Springer.

36. Chern SS. 1955 An elementary proof of the existence of isothermal parameters on a surface. *Proc. Amer. Math. Soc.* **6**, 771–782. (doi:10.1090/S0002-9939-1955-0074856-1)

37. Osserman R 1986 A Survey of Minimal Surfaces, Mineola, NY: Dover.

38. Spivak M. 1999 A comprehensive introduction to differential geometry. vol. **4**, 3rd edn. Houston, TX: Publish or Perish.

39. Weierstrass K. 1903 Untersuchungen über die Flächen, deren mittlere Krümmung überall gleich null ist. In *Mathematische Werke: Herausgegeben unter Mitwirkung einer von der königlich preussischen Akademie der Wissenschaften eingesetzten Commission* (ed. J Knoblauch), pp. 39–52. Berlin, Germany: Mayer & Muller. See https://archive.org/details/mathematischewer03weieuoft/page/38/mode/2up.

40. Weierstrass K. 2013 Untersuchungen über die Flächen, deren mittlere Krümmung überall gleich Null ist. In *Mathematische werke: herausgegeben unter mitwirkung einer von der königlich preussischen akademie der wissenschaften eingesetzten commission, Cambridge Library collection - mathematics* (ed. J Knoblauch), pp. 39–52. Cambridge, UK: Cambridge University Press.

41. Bour E. 1862 Théorie de la déformation des surfaces. *J. Éc. Imp. Polytech.* **39**, 1–148.

42. Cogliati A, Rivis R. 2022 The origins of the fundamental theorem of surface theory. *Hist. Math.* **61**, 45–79. (doi:10.1016/j.hm.2022.09.001)

43. Sonnet AM, Virga E. 2024 Data from: Bending-neutral deformations of minimal surfaces. Figshare. (doi:10.6084/m9.figshare.c.7477933)