A New Criterion on Stability in Distribution for a Hybrid Stochastic Delay Differential Equation

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Abstract

A new sufficient condition for stability in distribution of a hybrid stochastic delay differential equation (SDDE) has been proposed. In the new criterion leading to stability for an SDDE, its main component only depends on the coefficients of a corresponding SDE without delay. The Lyapunov method is applied to find an upper bound, so that the SDDE is stable in distribution if the delay is less than the upper bound. Also, the criterion shows that delay terms can be impetuses toward the stability in distribution.

Keywords: hybrid stochastic delay differential equations, Brownian motion, delay-dependent, stability in distribution

1. Introduction

Hybrid stochastic differential delay equations (SDDEs) have been widely applied to model stochastic delay systems, when they experience abrupt changes in their structures and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. Mao et al. [1] is the first book on theories and applications of hybrid SDDEs. One of the important issues in automatic control focuses on the stability of a hybrid SDDE. On stability analysis of SDDEs, most articles have focused on the stability of equilibrium states in the sense of moment, almost sure and so on (see, e.g.[2–9]). When the system has no equilibrium states, it will be meaningless to study its asymptotic stability. In this case, we often focus on the stability in distribution. A well-known example is the Ornstein-Uhlenbeck process in financial model (see, e.g.[3]): $dx(t) = a(r - x(t))dt + bdB_t$; where a, b and r are all positive numbers and B_t is a scalar Brownian motion. It can be verified as on [10, p.306], the probability distribution of its solution $x(t)$ will converge to normal distribution $N(r, b^2/2a)$ for any the initial value $x(0)$.

Now we have seen many references dealing with the stability in distribution for hybrid SDDEs. Basak et al. [11] considered the stability in distributions for semilinear SDEs with Markov switching. Yuan and Mao [12] studied the same stability for a nonlinear SDE with Markov switching with the help of Lyapunov functions. It can be considered as an improvement on the result given by [11]. After that, Yuan, Zhou and Mao [13] gave a sufficient condition on stability in distribution for an SDDE with Markov switching. Hu and Wang [14] proposed sufficient conditions for general NSFDEs with Markov switching. Du, Dang and Dieu [15] provided a new sufficient condition on stability in distribution for an SDDE with Markovian switching, improving the result given by [13]. Li, Wang and Suo [16] studied the stability in distribution for a class of NSFDEs by means of weak convergence methods. We also have seen many work by Bao and his coauthors ([17–19]) about the existence and uniqueness of invariant measures for different classes of SFDEs. Wang,

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Wu and Mao [20] studied the stability in distribution for a class of SFDEs, where the equation are highly nonlinear.

Here are two common points in above cited references. One point is that the term without delay is the impetus toward stability of the equation. For example, for the stability in distribution of equation

$$dX(t) = f(X(t), X(t-\tau), r(t))dt + g(X(t), X(t-\tau), r(t))dB(t) \tag{1.1}$$

where $r(t)$ is a Markovian process independent of the Brownian motion $B(t)$. Two auxiliary positive functions $V(x,i)$ and $w(x)$ are applied to give following sufficient conditions:

$$\begin{aligned} & c_1|x|^2 \le V(x,i) \le c_2 w(x) \\ & LV(x,y,i) \le -\lambda_1 w(x) + \lambda_2 w(y) + \beta \end{aligned} \tag{1.2}$$

for suitable positive constants $c_1, c_2, \lambda_1, \lambda_2, \beta$, where $LV(x,y,i)$ is the diffusion operator of V, and $\lambda_1 > \lambda_2$.

Now consider following scalar linear equation without Markovian switching

$$dx(t) =(0.1 + 0.1x(t) - 0.3x(t-\tau))dt + (0.2 + 0.1x(t) - 0.2x(t-\tau))dB(t). \tag{1.3}$$

Using $V(x) = x^2$ for discussion, we have

$$\begin{aligned} LV &= 2x(0.1 + 0.1x - 0.3y) + (0.2 + 0.1x - 0.2y)^2 \\ &= 0.21x^2 + 0.04y^2 - 0.14xy + 0.24x - 0.08y + 0.04, \end{aligned}$$

which can't be reduced to the inequality in (1.2) for any $w(x)$.

Roughly speaking, in this special kind of equations as (1.3), the term with delay is the dominated one. Can the delay term be the driving force for stability in distribution? We will have a positive answer to this question. In section 4, by applying our new criterion, it can verified that (1.3) is stable in distribution as $\tau < 0.0643$. Also, the criterion (1.2) can't be applied to those equations with coefficients only depending on delay terms. Our new criterion can solve such problems.

In recent years, we have seen many results on stabilization by delay feedback controls, such as in [21–28]. You et al. [29] has proposed a procedure to stabilize an unstable SDE in distribution with a delay feedback control. These references show that delay term can be the impetus to stability. In this article, we will propose a new sufficient condition for the stability in distribution, which reflects the impetuses of delay terms.

The other common point is that all existing criteria are delay-independent. As well known, the delay size will also affect the stability of an equation. Here we mention some delay-dependent criteria on moment or almost sure stability. In Mao and Leonid [5] and Dong and Han [8], delay-dependent criteria had been given for asymptotic stability of equilibrium states in the sense of moment and also sure for hybrid SDDEs. Guo et. al [30] had given a bound for delay size to get almost sure exponential stability when the corresponding SDE without delay was almost surely exponential stable. Fei, Hu, Mao et al. [31] established delay-dependent criteria for highly nonlinear hybrid SDDEs. Fei, Fei, Mao et al. [32], explored the delay-dependent criteria on the asymptotic stability for a class of highly nonlinear SDDEs driven by G-Brownian motion by using the nonlinear expectation theories. To our knowledge, there exist no delay-dependent criteria on stability in distribution for hybrid SDDEs. We will fix this gap because the new criteria proposed in this article depends on the delay size.

The main contributions of this article are:

(1) proposing a new delay-dependent criterion on stability in distribution for an SDDE.

(2) showing that delay terms can be impetuses toward stability in distribution.

This article is arranged as follows. In section 2, some fundamental concepts and notations are listed. In section 3, main theorems are stated. Three examples are given in section 4 to verify results obtained and conclusions are made in the last section.

2. Preliminaries and Notations

Throughout this article, following notations will be used. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathcal{B}(\mathbb{R}^n)$ be the family of all Borel measurable sets in $\mathbb{R}^n$. For $\tau > 0$, $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ ($\mathcal{C}_\tau$ in short) denotes the family of continuous functions $\xi : [-\tau, 0] \to \mathbb{R}^n$ with norm $\|\xi\|_\tau = \sup_{-\tau \le u \le 0} |\xi(u)|$. Denote $|x|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$. For a matrix A, $\|A\| = \max\{|Ax| : |x| = 1\}$ means its operator norm. For a symmetric matrix $A(A = A^T)$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are its smallest and largest eigenvalues, respectively. $A > 0$ ($A < 0$) means that A is a positively definite (negatively definite) matrix. If both a and b are real numbers, we denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual condition, and $B(t) = (B_1(t), B_2(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on this space. Also, there is a right continuous irreducible Markov chain $r(t), t > 0$, taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & i = j, \end{cases}$$

where $\Delta > 0$, and $\gamma_{ij} > 0 (i \ne j)$ is the transition rate from state i to j with $\gamma_{ii} = -\sum_{j \ne i} \gamma_{ij}$. Assume that $r(t)$ and $B(t)$ are independent.

Consider the n-dimensional SDDE with Markovian switching (1.1) on $t \ge 0$ with the initial condition

$$\{X(\theta) : -\tau \le \theta \le 0\} = \xi \in \mathcal{C}([-\tau, 0]; \mathbb{R}^n), r(0) = i_0, \tag{2.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$, and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n \times m}$ satisfy following global Lipschitz condition.

Assumption 2.1. *There exists a positive constant H_1, such that for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ and $i \in \mathbb{S}$,*

$$|f(x, y, i) - f(\bar{x}, \bar{y}, i)|^2 \vee |g(x, y, i) - g(\bar{x}, \bar{y}, i)|^2 \le H_1(|x - \bar{x}|^2 + |y - \bar{y}|^2). \tag{2.2}$$

It is easy to see from Assumption 2.1 that for any $x, y \in \mathbb{R}^n$

$$|f(x, y, i)|^2 \vee |g(x, y, i)|^2 \le 2H_1(|x|^2 + |y|^2) + a_0 \tag{2.3}$$

with $a_0 = 2 \max_{i \in \mathbb{S}} \{f(0, 0, i) \vee g(0, 0, i)\}$.

It is known that under Assumption 2.1, the SDDE (1.1) with the initial data (2.1) has a unique global solution on $t \ge -\tau$. Moreover, define $X_t = \{X(t + u) : -\tau \le u \le 0\}$ for $t \ge 0$, which is a $\mathcal{C}_\tau$-valued process. To emphasize the role of the initial data (2.1), we will write the solution as $X^{\xi, i_0}(t)$, and the Markov chain starting from i_0 at time 0 as $r^{i_0}(t)$.

Denote $Y^{\xi, i_0}(t) = (X_t^{\xi, i_0}, r(t))$ be an $\mathcal{C}_\tau \times \mathbb{S}$ valued process. Then $Y^{\xi, i_0}(t)$ is a time homogeneous Markov process and denote $p(t, \xi, i_0, d\zeta \times \{j\})$ its transition probability. Equation (1.1) is said to be stable in distribution, if there exists a probability measure $\pi(\cdot)$ on $\mathcal{C}_\tau \times \mathbb{S}$ such that $p(t, \xi, i_0, d\zeta \times \{j\})$ converges to $\pi(d\zeta \times \{j\})$ weakly as $t \to \infty$ for any initial data $(\xi, i_0) \in \mathcal{C}_\tau \times \mathbb{S}$. Let $\mathcal{P}(\mathcal{C}_\tau)$ be the space of all probability measures on $\mathcal{C}_\tau$. Because we assume that $r(t)$ is irreducible, we only need to show that the probability measure $\mathcal{L}(X_t^{\xi, i_0})$, generated by $X_t^{\xi, i_0} \in \mathcal{C}_\tau$, converges to a probability measure $\mu_\tau \in \mathcal{P}(\mathcal{C}_\tau)$.

For two probability measures $P_1, P_2 \in \mathcal{P}(\mathcal{C}_\tau)$, define the distance between P_1 and P_2 as

$$d(P_1, P_2) = \sup_{\psi \in L} \left| \int \psi(\xi) P_1(d\xi) - \int \psi(\xi) P_2(d\xi) \right|$$

where

$$L = \{\psi : \mathcal{C}_\tau \to \mathbb{R} \big| |\psi(\xi) - \psi(\eta)| \le \|\xi - \eta\| \text{ and } |\psi(\cdot)| \le 1 \text{ for any } (\xi, \eta) \in \mathcal{C}_\tau \times \mathcal{C}_\tau\}.$$

Definition 2.2. *The equation* (1.1) *is said to be stable in distribution, if there exists a probability measure* $\mu_\tau \in \mathcal{P}(\mathcal{C}_\tau)$ *such that*

$$\lim_{t\to\infty} d(\mathcal{L}(X_t^{\xi,i_0}), \mu_\tau) = 0$$

holds for any initial data $(\xi, i_0) \in \mathcal{C}_\tau \times \mathbb{S}$.

Assumption 2.3. *There exist two positive numbers* λ_1, δ *and* N *symmetric positive definite matrices* $W_i, 1 \le i \le N$ *such that*

$$\begin{aligned}\Psi(z_1, z_2, i) :=& 2(z_1 - z_2)^T W_i\big(f(z_1, z_1, i) - f(z_2, z_2, i)\big)\\ &+ (1+\delta)\mathrm{trace}\Big(\big(g(z_1, z_1, i) - g(z_2, z_2, i)\big)^T W_i\big(g(z_1, z_1, i) - g(z_2, z_2, i)\big)\Big)\\ &+ \sum_{j=1}^N \gamma_{ij}(z_1 - z_2)^T W_j (z_1 - z_2)\\ \le& -\lambda_1|z_1 - z_2|^2 \end{aligned} \tag{2.4}$$

holds for all $(z_1, z_2, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$.

The advantage of condition (2.4) lies that we can make use of delay terms to produce better effect. Taking equation (1.3) as an example, we will have $f(x, x) = 0.1 - 0.2x$ and $g(x, x) = 0.2 - 0.1x$. Now if we use $V(x) = x^2$ for discussion, we see

$$2xf(x, x) + g^2(x, x) = 0.04 + 0.16x - 0.39x^2 \le -0.385x^2 + 1.32 \tag{2.5}$$

which is just the condition what we want for further discussion. Also, as shown in [12], under Assumptions 2.1 and 2.3, when $\delta = 0$, following equation without delay

$$dX(t) = f(X(t), X(t), r(t))dt + g(X(t), X(t), r(t))dB(t) \tag{2.6}$$

will be stable in distribution. Our assumption is a little more stronger than that in [13], because we need $\delta > 0$ to balance the difference $X(t) - X(t-\tau)$ in forthcoming discussion. Now there arises a question: can we give a bound τ^* for τ such that the delay equation (1.1) is also stable in distribution when $\tau < \tau^*$?

It is straightforward to show from Assumptions 2.1 and 2.3, there exist positive numbers λ_0 and λ_2 such that for all $(z, i) \in \mathbb{R}^n \times \mathbb{S}$

$$\begin{aligned}&\Phi(z, i) := 2z^T W_i f(z, z, i) + (1+\delta)\mathrm{trace}\big(g(z, z, i)^T W_i g(z, z, i)\big) + \sum_{j=1}^N \gamma_{ij} z^T W_j z\\ &\le -\lambda_1|z|^2 + \lambda_2|z| + \lambda_0.\end{aligned} \tag{2.7}$$

3. Main Results

Arguments to prove the stability in distribution for an SDDE (1.1) are rather standard, as proposed and proved in [13], and then widely used such as in [14–16, 28, 29]. We conclude as a lemma.

Lemma 3.1 (Theorem 3.2, [13]). *Denote* $B_R = \{\xi \in \mathcal{C}_\tau | \|\xi\| \le R\}$. *If for any given* $R > 0$, *following two assertions are verified:*

(A) for any $(\xi, i_0) \in B_R \times \mathbb{S}$, *the solution* $X^{\xi,i_0}(t)$ *of* (1.1) *satisfies*

$$\sup_{\xi\in B_R}\left(\sup_{0\le t<\infty} \mathbb{E}\|X_t^{\xi,i_0}\|^2\right) < \infty,$$

(B) *for any* $(\xi, \eta, i_0) \in B_R \times B_R \times \mathbb{S}$*, two solutions* $X^{\xi,i_0}(t), X^{\eta,i_0}(t)$ *from different initial data* (ξ, i_0) *and* (η, i_0) *of* (1.1) *satisfy*

$$\lim_{t\to\infty} \mathbb{E}\|X_t^{\xi,i_0} - X_t^{\eta,i_0}\|^2 = 0$$

uniformly in ξ *and* η*,*

then equation (1.1) *is stable in distribution.*

As explained in [13], assertion (A) guarantees that for any $(\xi, i_0) \in \mathcal{C}_\tau \times \mathbb{S}$, the family $\{p(t, \xi, i_0, d\zeta \times \{j\})|t \geq 0\}$ is tight, while assertion (B) guarantees that solutions from different initial data will have the same asymptotic tendency. Now we give sufficient conditions as in next two lemmas such that equation (1.1) satisfies two assertions.

Lemma 3.2. *Under Assumption 2.1 and Assumption 2.3, there exists* $\tau_1^* > 0$ *as defined in* (3.15)*, such that as* $\tau < \tau_1^*$*, we will have*

$$\mathbb{E}\|X_t^{\xi,i_0}\|^2 \leq \gamma_1(1 + \|\xi\|^2) \tag{3.1}$$

for any $t > 0$*, where* γ_1 *is independent of* (ξ, i_0)*.*

Proof. We will divide the proof into three steps.

Step 1: Introduce the auxiliary Lyapunov functional.

Denote $\hat{X}_t = \{X(t+s) : -2\tau \leq s \leq 0\}$. We will apply following Lyapunov functional for subsequent analysis

$$\begin{aligned} V(\hat{X}_t, r(t)) :=& X^T(t)W(r(t))X(t) \\ &+ \theta \int_{-\tau}^{0} \int_{t+s}^{t} \big(\tau|f(X(v), X(v-\tau), r(v))|^2 + |g(X(v), X(v-\tau), r(v))|^2\big) dv ds. \end{aligned} \tag{3.2}$$

For this functional, it is easily to obtain that

$$\begin{aligned} \hat{\lambda}_m|X(t)|^2 \leq& V(\hat{X}_t, r(t)) \\ \leq& \hat{\lambda}_M|X(t)|^2 + \theta\tau \int_{t-\tau}^{t} \big(\tau|f(X(v), X(v-\tau), r(v))|^2 + |g(X(v), X(v-\tau), r(v))|^2\big) dv, \end{aligned} \tag{3.3}$$

and

$$dV(\hat{X}_t, r(t)) = LV(X(t), X(t-\tau), r(t))dt + 2X^T(t)W(r(t))g(X(t), X(t-\tau), r(t))dB(t), \tag{3.4}$$

where $\hat{\lambda}_m = \min_{i\in\mathbb{S}} \lambda_{\min}(W_i)$, $\hat{\lambda}_M = \max_{i\in\mathbb{S}} \lambda_{\max}(W_i)$, and for any $i \in \mathbb{S}$,

$$\begin{aligned} LV(x, y, i) =& 2x^T W_i f(x, y, i) + \text{trace}\big(g(x, y, i)^T W_i g(x, y, i)\big) \\ &+ \sum_{j=1}^{N} \gamma_{ij} x^T W_j x + \theta\tau\big(\tau|f(x, y, i)|^2 + |g(x, y, i)|^2\big) \\ &- \theta \int_{t-\tau}^{t} \big(\tau|f(x, y, i)|^2 + |g(x, y, i)|^2\big) dv. \end{aligned} \tag{3.5}$$

Step 2: Evaluate LV in (3.5).

It can be directly derived that

$$\begin{aligned} \text{trace}\big(g(x, y, i)^T W_i g(x, y, i)\big) \leq& (1+\delta)\text{trace}\big(g(x, x, i)^T W_i g(x, x, i)\big) \\ &+ (1 + \frac{1}{\delta})\text{trace}\Big(\big(g(x, x, i) - g(x, y, i)\big)^T W_i \big(g(x, x, i) - g(x, y, i)\big)\Big). \end{aligned} \tag{3.6}$$

By Assumption 2.3, We can see that

$$LV(x,y,i) \le \Phi(x,i) + J_1(x,y,i) + J_2(x,y,i) + J_3(x,y,i) + J_4(x,y,i), \tag{3.7}$$

where

$$J_1(x,y,i) = 2x^T W_i(f(x,y,i) - f(x,x,i)) \le \hat{\lambda}_M \left(\theta_1|x|^2 + \frac{H_1}{\theta_1}|x-y|^2\right), \tag{3.8}$$

$$\begin{aligned} J_2(x,y,i) &= (1+\frac{1}{\delta})\mathrm{trace}\Big(\big(g(x,x,i)-g(x,y,i)\big)^T W_i\big(g(x,x,i)-g(x,y,i)\big)\Big) \\ &\le \hat{\lambda}_M H_1(1+\frac{1}{\delta})|x-y|^2, \end{aligned} \tag{3.9}$$

$$J_3(x,y,i) = \theta\tau\big(\tau|f(x,y,i)|^2 + |g(x,y,i)|^2\big) \le \theta\tau(\tau+1)\left(2H_1(|x|^2+|y|^2)+a_0\right), \tag{3.10}$$

$$J_4(x,y,i) = -\theta\int_{t-\tau}^{t}\big(\tau|f(x,y,i)|^2 + |g(x,y,i)|^2\big)dv. \tag{3.11}$$

Setting $\theta_1 = \frac{\lambda_1}{2\hat{\lambda}_M}$ in J_1, and rearranging terms, we finally have

$$\begin{aligned} LV(x,y,i) \le &-\frac{\lambda_1}{2}|x|^2 + \lambda_2|x| + \lambda_0 + \hat{\lambda}_M H_1\left(\frac{2\hat{\lambda}_M}{\lambda_1} + (1+\frac{1}{\delta})\right)|x-y|^2 \\ &+ \theta\tau(\tau+1)\left(2H_1(|x|^2+|y|^2)+a_0\right) - \theta\int_{t-\tau}^{t}\big(\tau|f(x,y,i)|^2 + |g(x,y,i)|^2\big)dv. \end{aligned} \tag{3.12}$$

Step 3: Find a bound τ_1^* such that the assertion (3.1) holds as $\tau < \tau_1^*$.
Applying the Hölder inequality and martingale inequality to

$$X(t) - X(t-\tau) = \int_{t-\tau}^{t} f(X(v), X(v-\tau), r(v))dv + \int_{t-\tau}^{t} g(X(v), X(v-\tau), r(v))dB_v,$$

it is easily to get

$$\mathbb{E}|X(t)-X(t-\tau)|^2 \le 2\mathbb{E}\int_{t-\tau}^{t}\left(\tau|f(X(v),X(v-\tau),r(v))|^2 + |g(X(v),X(v-\tau),r(v))|^2\right)dv. \tag{3.13}$$

For any $\epsilon > 0$, it follows that

$$\begin{aligned} \hat{\lambda}_m e^{\epsilon t}\mathbb{E}|X(t)|^2 \le& \mathbb{E}(e^{\epsilon t}V(\hat{X}_t, r(t))) \\ \le& \mathbb{E}V(\xi, i_0) + \mathbb{E}\int_0^t e^{\epsilon s}\epsilon V(\hat{X}_s, r(s))ds + \mathbb{E}\int_0^t e^{\epsilon s} LV(X(s), X(s-\tau), r(s))ds \\ \le& \mathbb{E}V(\xi, i_0) + 2H_1\theta\tau(\tau+1)\mathbb{E}\int_0^t e^{\epsilon s}|X(s-\tau)|^2 ds \\ &+ \left(-\frac{\lambda_1}{2} + \epsilon\hat{\lambda}_M + 2H_1\theta\tau(\tau+1)\right)\mathbb{E}\int_0^t e^{\epsilon s}|X(s)|^2 ds \\ &+ \lambda_2\mathbb{E}\int_0^t e^{\epsilon s}|X(s)|ds + (\lambda_0 + a_0\theta\tau(\tau+1))\int_0^t e^{\epsilon s}ds \\ &+ \left(\epsilon\theta\tau - \theta + 2\hat{\lambda}_M H_1(\frac{2\hat{\lambda}_M}{\lambda_1} + 1 + \frac{1}{\delta})\right) \\ &\times \mathbb{E}\int_0^t e^{\epsilon s}\int_{s-\tau}^{s}\Big(\tau|f(X(v),X(v-\tau),r(v))|^2 + |g(X(v),X(v-\tau),r(v))|^2\Big)dvds. \end{aligned} \tag{3.14}$$

Fix a $\theta > 2\hat{\lambda}_M H_1 \left(\frac{2\hat{\lambda}_M}{\lambda_1} + 1 + \frac{1}{\delta}\right)$, there exists a unique positive solution τ_1^* such that

$$8H_1\theta\tau_1^*(\tau_1^* + 1) - \lambda_1 = 0. \tag{3.15}$$

For any fixed $\tau \in (0, \tau_1^*)$, $8H_1\theta\tau(\tau+1) < \lambda_1$ holds, and there then exists a positive number ϵ_0, such that

$$-\frac{\lambda_1}{2} + \epsilon_0\hat{\lambda}_M + 2H_1\theta\tau(\tau+1)(1+e^{\epsilon_0\tau}) < 0$$

and

$$\epsilon_0\theta\tau - \theta + 2\hat{\lambda}_M H_1(\frac{2\hat{\lambda}_M}{\lambda_1} + 1 + \frac{1}{\delta}) < 0$$

hold simultaneously. Setting

$$\beta_0 = -\left(-\frac{\lambda_1}{2} + \epsilon_0\hat{\lambda}_M + 2H_1\theta\tau(\tau+1)(1+e^{\epsilon_0\tau})\right) > 0,$$

(3.14) can be further evaluated as

$$\begin{aligned}\hat{\lambda}_m e^{\epsilon_0 t}\mathbb{E}|X(t)|^2 \leq & \mathbb{E}V(\xi, i_0) + 2H_1\theta\tau(\tau+1)e^{\epsilon_0\tau}\mathbb{E}\int_{-\tau}^0 e^{\epsilon_0 s}|X(s)|^2 ds \\ & + \mathbb{E}\int_0^t e^{\epsilon_0 s}(-\beta_0|X(s)|^2 + \lambda_2|X(s)|)ds + \frac{1}{\epsilon_0}\left(\lambda_0 + a_0\theta\tau(\tau+1)\right)(e^{\epsilon_0 t} - 1).\end{aligned} \tag{3.16}$$

Obviously, for any s, $-\beta_0|X(s)|^2 + \lambda_2|X(s)| \leq \frac{\lambda_2^2}{4\beta_0}$, and consequently, we obtain

$$\hat{\lambda}_m e^{\epsilon_0 t}\mathbb{E}|X(t)|^2 \leq K_1 + K_2(e^{\epsilon_0 t} - 1)$$

with $K_1 = \mathbb{E}V(\xi, i_0) + 2H_1\theta\tau(\tau+1)e^{\epsilon_0\tau}\mathbb{E}\int_{-\tau}^0 e^{\epsilon_0 s}|X(s)|^2 ds$ and $K_2 = \frac{1}{\epsilon_0}\left(\lambda_0 + a_0\theta\tau(\tau+1) + \frac{\lambda_2^2}{4\beta_0}\right)$, and then

$$\mathbb{E}|X(t)|^2 \leq \frac{1}{\hat{\lambda}_m}\left(K_2 + (K_1 - K_2)e^{-\epsilon_0 t}\right) \leq K_0(1 + \|\xi\|^2),$$

where K_0 is a constant independent on (ξ, i_0).

By the well-known BDG inequality, we can derive following bound for $\mathbb{E}\|X_t\|^2$ as $t \geq \tau$, which is

$$\begin{aligned}\mathbb{E}\|X_t\|^2 \leq & 3\mathbb{E}|X(t-\tau)|^2 + 3\mathbb{E}\left(\sup_{t-\tau\leq s\leq t}\Big|\int_{s-\tau}^s f(X(v), X(v-\tau), r(v))dv\Big|\right)^2 \\ & + 3\mathbb{E}\left(\sup_{t-\tau\leq s\leq t}\Big|\int_{s-\tau}^s g(X(v), X(v-\tau), r(v))dB_v\Big|\right)^2 \\ \leq & 3K_0(1 + \|\xi\|^2) + 12(\tau+1)\mathbb{E}\int_{t-\tau}^t 2H_1(|X(v)|^2 + |X(v-\tau)|^2)dv \\ \leq & \gamma_1(1 + \|\xi\|^2)\end{aligned}$$

with γ_1 independent on the initial data. Now (3.1) is verified. □

By Lemma 3.2, it is obvious that assertion (A) in Lemma 3.1 is satisfied. Now we check assertion (B). For that, denote $H^{\xi,\eta,i_0}(t) = X^{\xi,i_0}(t) - X^{\eta,i_0}(t)$ and $\hat{H}_t^{\xi,\eta,i_0} = \{H^{\xi,\eta,i_0}(t+s) | -2\tau \leq s \leq 0\}$. We have following differential rule for $H^{\xi,\eta,i_0}(t)$,

$$\begin{aligned}dH^{\xi,\eta,i_0}(t) = & \Big(f(X^{\xi,i_0}(t), X^{\xi,i_0}(t-\tau), r(t)) - f(X^{\eta,i_0}(t), X^{\eta,i_0}(t-\tau), r(t))\Big)dt \\ & + \Big(g(X^{\xi,i_0}(t), X^{\xi,i_0}(t-\tau), r(t)) - g(X^{\eta,i_0}(t), X^{\eta,i_0}(t-\tau), r(t))\Big)dB(t).\end{aligned} \tag{3.17}$$

with the initial condition

$$H^{\xi,\eta,i_0}(0)=\xi-\eta, r(0)=i_0.$$

We will use another Lyapunov functional $\hat{V}$ for analysis:

$$\begin{aligned}\hat{V}(\hat{H}_t^{\xi,\eta,i_0},r(t)) :=&(X^{\xi,i_0}(t)-X^{\eta,i_0}(t))^T W(r(t))(X^{\xi,i_0}(t)-X^{\eta,i_0}(t))\\ &+\bar{\theta}\int_{-\tau}^0\int_{t+s}^t\Big(\tau\Big|f(X^{\xi,i_0}(v),X^{\xi,i_0}(v-\tau),r(v))-f(X^{\eta,i_0}(v),X^{\xi,i_0}(v-\eta),r(v))\Big|^2\\ &+\Big|g(X^{\xi,i_0}(v),X^{\xi,i_0}(v-\tau),r(v))-g(X^{\eta,i_0}(v),X^{\eta,i_0}(v-\tau),r(v))\Big|^2\Big)dvds, \end{aligned} \tag{3.18}$$

where $\bar{\theta}$ is a free parameter.

When we evaluate square moment of the difference $|H^{\xi,\eta,i_0}(t)-H^{\xi,\eta,i_0}(t-\tau)|$, it is inevitable to relate $f(X^{\xi,i_0}(t),X^{\xi,i_0}(t-\tau),r(t))-f(X^{\eta,i_0}(t),X^{\eta,i_0}(t-\tau),r(t))$ and $g(X^{\xi,i_0}(t),X^{\xi,i_0}(t-\tau),r(t))-g(X^{\eta,i_0}(t),X^{\eta,i_0}(t-\tau),r(t))$ with $H^{\xi,\eta,i_0}(t)-H^{\xi,\eta,i_0}(t-\tau)$. Equivalently, we should relate $f(x,y,i)-f(\bar{x},\bar{y},i)$ and $g(x,y,i)-g(\bar{x},\bar{y},i)$ with $(x-\bar{x})-(y-\bar{y})$. Because our main criterion (2.4) only involves $f(x,x,i)$ and $g(x,x,i)$, we make following technical assumption to meet the requirement.

Assumption 3.3. *There exist three positive constants σ_1, σ_2 and σ_3, such that for any $x,y,\bar{x},\bar{y}\in\mathbb{R}^n$ and $i\in\mathbb{S}$,*

$$\begin{aligned}&|f(x,y,i)-f(x,x,i)+f(\bar{x},\bar{x},i)-f(\bar{x},\bar{y},i)|^2\vee\\ &|g(x,y,i)-g(x,x,i)+g(\bar{x},\bar{x},i)-g(\bar{x},\bar{y},i)|^2\\ \le&\sigma_1|(x-\bar{x})-(y-\bar{y})|^2+\sigma_2|x-\bar{x}|^2+\sigma_3|y-\bar{y}|^2.\end{aligned} \tag{3.19}$$

Remark: Actually, assumption 3.3 can be derived from assumption 2.1. For example, we can see that under assumption 2.1,

$$\begin{aligned}|f(x,y,i)-f(x,x,i)+f(\bar{x},\bar{x},i)-f(\bar{x},\bar{y},i)|^2\le&2H_1(|x-y|^2+|\bar{x}-\bar{y}|^2)\\ \le&4H_1(|x-y-\bar{x}+\bar{y}|^2)+6H_1|\bar{x}-\bar{y}|^2.\end{aligned} \tag{3.20}$$

The reason why we make such assumption is that we need σ_1,σ_2 and σ_3 to determine the control delay size. Smaller σ_1,σ_2 and σ_3 will produce better results. There are some types of functions f and g that can produce $|x-y-\bar{x}+\bar{y}|^2$ directly without deriving from the global Lipschitz condition. In such cases, good estimations for $\sigma_1,\sigma_2,\sigma_3$ will be achieved. Take $f(x,y,i)=f_1(x,i)+A_iy+f_2(y,i)$ as an example, where f_1 and f_2 satisfy the global Lipschitz condition. For such f, it can be directly derived that

$$\begin{aligned}|f(x,y,i)-f(\bar{x},\bar{y},i)|^2&\le 3|f_1(x,i)-f_1(\bar{x},i)|^2+3\|A_i\|^2|y-\bar{y}|^2+3|f_2(y,i)-f_2(\bar{y},i)|^2\\ &\le\tilde{H}\left(|x-\bar{x}|^2+|y-\bar{y}|^2\right)\end{aligned} \tag{3.21}$$

with $\tilde{H}=3\max\left(\tilde{H}_1,\max_{i\in\mathbb{S}}\|A_i\|^2\vee\tilde{H}_2\right)$, where $\tilde{H}_1$ and $\tilde{H}_2$ are Lipschitz constants of f_1 and f_2, respectively. And now we can get $\sigma_1,\sigma_2,\sigma_3$ as in (3.20). But if we estimate directly, we can calculate as

$$\begin{aligned}&|(f(x,y,i)-f(x,x,i))-(f(\bar{x},\bar{y},i)-f(\bar{x},\bar{x},i))|^2\\ =&|A_i(y-x-\bar{y}+\bar{x})+f_2(y,i)-f_2(\bar{y},i)+f_2(\bar{x},i)-f_2(x,i)|^2\\ \le&3\max_{i\in\mathbb{S}}\|A_i\|^2|y-x-\bar{y}+\bar{x}|^2+3\tilde{H}_2\left(|x-\bar{x}|^2+|y-\bar{y}|^2\right).\end{aligned}$$

Then we have $\sigma_1=3\max_{i\in\mathbb{S}}\|A_i\|^2$ and $\sigma_2+\sigma_3=6\tilde{H}_2$, substantially smaller than those derived from the Lipschitz condition.

Lemma 3.4. *Under Assumption 2.1, Assumption 2.3 and Assumption 3.3, if*

$$\sigma_2 + \sigma_3 < \frac{\lambda_1}{2(1 + \frac{2\hat{\lambda}_M}{\lambda_1} + \delta^{-1})\hat{\lambda}_M}$$

there exists a $\tau_2^* > 0$, such that as $0 < \tau < \tau_2^*$,

$$\lim_{t\to\infty} \mathbb{E}\|X_t^{\xi,i_0} - X_t^{\eta,i_0}\|^2 = 0 \tag{3.22}$$

holds uniformly for any initial data $(\xi, \eta, i_0) \in B_R \times B_R \times \mathbb{S}$.

Proof. For the Lyapunov functional (3.18), we have

$$\begin{aligned}
L\hat{V}(x,y,\bar{x},\bar{y},i) =& 2(x-\bar{x})^T W_i\big(f(x,y,i) - f(\bar{x},\bar{y},i)\big) \\
&+ \text{trace}\Big(\big(g(x,y,i) - g(\bar{x},\bar{y},i)\big)^T W_i\big(g(x,y,i) - g(\bar{x},\bar{y},i)\big)\Big) \\
&+ \sum_{j=1}^{N} \gamma_{ij}(x-\bar{x})^T W_j(x-\bar{x}) \\
&+ \bar{\theta}\tau\big(\tau|f(x,y,i) - f(\bar{x},\bar{y},i)|^2 + |g(x,y,i) - g(\bar{x},\bar{y},i)|^2\big) \\
&- \bar{\theta}\int_{t-\tau}^{t} \Big(\tau|f(x,y,i) - f(\bar{x},\bar{y},i)|^2 + |g(x,y,i) - g(\bar{x},\bar{y},i)|^2\Big)dv \\
\le& \Psi(x,\bar{x},i) + J_1' + J_2' + J_3' + J_4',
\end{aligned} \tag{3.23}$$

where

$$J_1' = 2(x-\bar{x})W_i\big(f(x,y,i) - f(x,x,i) + f(\bar{x},\bar{x},i) - f(\bar{x},\bar{y},i)\big), \tag{3.24}$$

$$\begin{aligned}
J_2' =& (1 + \frac{1}{\delta})\text{trace}\Big(\big(g(x,y,i) - g(x,x,i) + g(\bar{x},\bar{x},i) - g(\bar{x},\bar{y},i)\big)^T W_i \\
&\big(g(x,y,i) - g(x,x,i) + g(\bar{x},\bar{x},i) - g(\bar{x},\bar{y},i)\big)\Big),
\end{aligned} \tag{3.25}$$

$$J_3' = \bar{\theta}\tau\big(\tau|f(x,y,i) - f(\bar{x},\bar{y},i)|^2 + |g(x,y,i) - g(\bar{x},\bar{y},i)|^2\big), \tag{3.26}$$

$$J_4' = -\bar{\theta}\int_{t-\tau}^{t}\Big(\tau|f(x,y,i) - f(\bar{x},\bar{y},i)|^2 + |g(x,y,i) - g(\bar{x},\bar{y},i)|^2\Big)dv. \tag{3.27}$$

Applying similar arguments as those used for evaluating J_1, J_2 and J_3 in Lemma 3.2,we can get

$$\begin{aligned}
&L\hat{V}(x,y,\bar{x},\bar{y},i) \\
\le& \left(-\frac{1}{2}\lambda_1 + \bar{\theta}\tau(\tau+1)H_1 + (\frac{2\hat{\lambda}_M}{\lambda_1} + (1+\frac{1}{\delta}))\hat{\lambda}_M\sigma_2\right)|x-\bar{x}|^2 \\
&+ \left(\bar{\theta}\tau(\tau+1)H_1 + (\frac{2\hat{\lambda}_M}{\lambda_1} + (1+\frac{1}{\delta}))\hat{\lambda}_M\sigma_3\right)|y-\bar{y}|^2 \\
&+ \left(\frac{2\hat{\lambda}_M}{\lambda_1} + (1+\frac{1}{\delta})\right)\hat{\lambda}_M\sigma_1|(x-\bar{x}) - (y-\bar{y})|^2 \\
&- \bar{\theta}\int_{t-\tau}^{t}\Big(\tau|f(x,y,i) - f(\bar{x},\bar{y},i)|^2 + |g(x,y,i) - g(\bar{x},\bar{y},i)|^2\Big)dv.
\end{aligned} \tag{3.28}$$

From

$$\begin{aligned}
H^{\xi,\eta,i_0}(t) - H^{\xi,\eta,i_0}(t-\tau) =& \int_{t-\tau}^{t}\Big(f(X^{\xi,i_0}(v), X^{\xi,i_0}(v-\tau), r(v)) - f(X^{\eta,i_0}(v), X^{\eta,i_0}(v-\tau), r(v))\Big)dv \\
&+ \int_{t-\tau}^{t}\Big(g(X^{\xi,i_0}(v), X^{\xi,i_0}(v-\tau), r(v)) - g(X^{\eta,i_0}(v), X^{\eta,i_0}(v-\tau), r(v))\Big)dB_v,
\end{aligned}$$

we have $\mathbb{E}|H^{\xi,\eta,i_0}(t)-H^{\xi,\eta,i_0}(t-\tau)|^2 \le 2\mathbb{E}(F(t))$, where

$$F(t)=\int_{t-\tau}^{t}\Big(\tau\Big|f(X^{\xi,i_0}(v),X^{\xi,i_0}(v-\tau),r(v))-f(X^{\eta,i_0}(v),X^{\eta,i_0}(v-\tau),r(v))\Big|^2 \\ +\Big|g(X^{\xi,i_0}(v),X^{\xi,i_0}(v-\tau),r(v))-g(X^{\eta,i_0}(v),X^{\eta,i_0}(v-\tau),r(v))\Big|^2\Big)dv.$$

By the definition of $\hat{V}$ and Itô's lemma, it can be derived directly that for any $\epsilon>0$,

$$\begin{aligned}
\hat{\lambda}_m e^{\epsilon t}\mathbb{E}|H(t)|^2 \le&\mathbb{E}(e^{\epsilon t}\hat{V}(\hat{H}_t,r(t)))\\
\le&\mathbb{E}\hat{V}(\xi-\eta,i_0)\\
&+\mathbb{E}\int_0^t e^{\epsilon s}\Big(\epsilon V(\hat{H}_s,r(s))+LV(X^{\xi,i_0}(s),X^{\xi,i_0}(s-\tau),X^{\eta,i_0}(s),X^{\eta,i_0}(s-\tau),r(s))\Big)ds\\
\le&\mathbb{E}\hat{V}(\xi-\eta,i_0)+\left(\bar{\theta}\tau(\tau+1)H_1+(\frac{2\hat{\lambda}_M}{\lambda_1}+1+\frac{1}{\delta})\hat{\lambda}_M\sigma_3\right)\mathbb{E}\int_0^t e^{\epsilon s}|H(s-\tau)|^2ds\\
&+\left(-\frac{1}{2}\lambda_1+\bar{\theta}\tau(\tau+1)H_1+(\frac{2\hat{\lambda}_M}{\lambda_1}+1+\frac{1}{\delta})\hat{\lambda}_M\sigma_2+\epsilon\hat{\lambda}_M\right)\mathbb{E}\int_0^t e^{\epsilon s}|H(s)|^2ds\\
&+\left(\epsilon\bar{\theta}\tau-\bar{\theta}+2(\frac{2\hat{\lambda}_M}{\lambda_1}+1+\frac{1}{\delta})\hat{\lambda}_M\sigma_1\right)\mathbb{E}\int_0^t e^{\epsilon s}F(s)ds\\
\le&\mathbb{E}\hat{V}(\xi-\eta,i_0)+\left(\bar{\theta}\tau(\tau+1)H_1+(\frac{2\hat{\lambda}_M}{\lambda_1}+1+\frac{1}{\delta})\hat{\lambda}_M\sigma_3\right)e^{\epsilon\tau}\mathbb{E}\int_{-\tau}^0 e^{\epsilon s}|H(s)|^2ds\\
&+\gamma_1(\epsilon,\tau)\mathbb{E}\int_0^t e^{\epsilon s}|H(s)|^2ds+\gamma_2(\epsilon,\tau)\mathbb{E}\int_0^t e^{\epsilon s}F(s)|ds
\end{aligned}\tag{3.29}$$

where

$$\begin{aligned}
\gamma_1(\epsilon,\tau)=&-\frac{1}{2}\lambda_1+\bar{\theta}\tau(\tau+1)H_1(1+e^{\bar{\epsilon}\tau})+\left(\frac{2\hat{\lambda}_M}{\lambda_1}+1+\frac{1}{\delta}\right)\hat{\lambda}_M(\sigma_2+\sigma_3e^{\bar{\epsilon}\tau})+\bar{\epsilon}\hat{\lambda}_M,\\
\gamma_2(\epsilon,\tau)=&\bar{\epsilon}\bar{\theta}\tau-\bar{\theta}+2\hat{\lambda}_M\sigma_1\left(\frac{2\hat{\lambda}_M}{\lambda_1}+1+\frac{1}{\delta}\right).
\end{aligned}$$

Fix a $\bar{\theta}>2(\frac{2\hat{\lambda}_M}{\lambda_1}+1+\frac{1}{\delta})\hat{\lambda}_M\sigma_1$, we can have a unique positive solution τ_2^* for the equation

$$-\lambda_1+4\bar{\theta}\tau(\tau+1)H_1+2\left(\frac{2\hat{\lambda}_M}{\lambda_1}+1+\frac{1}{\delta}\right)\hat{\lambda}_M(\sigma_2+\sigma_3)=0.\tag{3.30}$$

Obviously, for any $\tau<\tau_2^*$, there exists a positive number $\bar{\epsilon}_0$, such that

$$\gamma_1(\bar{\epsilon}_0,\tau)<0,\gamma_2(\bar{\epsilon}_0,\tau)<0$$

hold simultaneously.

(3.29) can then be further reduced to

$$\hat{\lambda}_m e^{\bar{\epsilon}_0 t}\mathbb{E}|H(t)|^2\le\mathbb{E}\hat{V}(\xi-\eta,i_0)+\left(\bar{\theta}\tau(\tau+1)H_1+\hat{\lambda}_M\sigma_3\left(\frac{2\hat{\lambda}_M}{\lambda_1}+1+\frac{1}{\delta}\right)\right)e^{\bar{\epsilon}_0\tau}\mathbb{E}\int_{-\tau}^0 e^{\bar{\epsilon}_0 s}|H(s)|^2ds.\tag{3.31}$$

We then obtain

$$\mathbb{E}|H(t)|^2 \leq \bar{K}_0 e^{-\bar{\epsilon}_0 t}\hat{\lambda}_m^{-1}\|\xi-\eta\|^2$$

for some positive $\bar{K}_0$ independent on ξ and η.

Now, applying the well-known BDG inequality and (2.2), we can derive following bound for $\mathbb{E}\|H_t\|^2$ as $t \geq \tau_2^*$, which is

$$\begin{aligned}
\mathbb{E}\|H_t\|^2 \leq & 3\mathbb{E}|H(t-\tau)|^2 \\
& + 3\mathbb{E}\left(\sup_{t-\tau\leq s\leq t}\left|\int_{s-\tau}^{s}\Big(f(X^{\xi,i_0}(v),X^{\xi,i_0}(v-\tau),r(v)) - f(X^{\eta,i_0}(v),X^{\eta,i_0}(v-\tau),r(v))\Big)dv\right|\right)^2 \\
& + 3\mathbb{E}\left(\sup_{t-\tau\leq s\leq t}\left|\int_{s-\tau}^{s}\Big(g(X^{\xi,i_0}(v),X^{\xi,i_0}(v-\tau),r(v)) - g(X^{\eta,i_0}(v),X^{\eta,i_0}(v-\tau),r(v))\Big)dB_v\right|\right)^2 \\
\leq & 3\bar{K}_0(\|\xi-\eta\|^2) + 12(\tau+1)\mathbb{E}\left(\int_{t-\tau}^{t} 2H_1(|H(v)|^2 + |H(v-\tau)|^2)dv\right) \\
\leq & \gamma_2 e^{-\bar{\epsilon}_0 t}(\|\xi-\eta\|^2)
\end{aligned} \tag{3.32}$$

with γ_2 independent on the initial data. And consequently, (3.22) is verified. $\square$

On the base of Lemmas 3.2 and 3.4, we can state our final theorem on the stability in distribution of equation (1.1) by using Lemma 3.1.

Theorem 3.5. *If $\tau < \tau^* = \tau_1^* \wedge \tau_2^*$ and $\sigma_2 + \sigma_3 < \frac{\lambda_1}{2\hat{\lambda}_M\left(1+\frac{2\hat{\lambda}_M}{\lambda_1}+\delta^{-1}\right)}$, there exists a unique probability measure $\mu_\tau \in \mathcal{P}(\mathcal{C}_\tau)$ such that*

$$\lim_{t\to\infty} d(\mathcal{L}(X_t^{\xi,i}),\mu_\tau) = 0.$$

for any $(\xi,i_0) \in \mathcal{C}_\tau \times \mathbb{S}$.

4. Illustrative Examples

In this section, three illustrative examples will be applied to validate the new criterion for some SDDEs. The first scale SDDE without Markovian switching is used to verify the assertion made in Introduction section. In the second example, a general hybrid linear SDDE is analyzed. It will be shown that for a linear SDDE, we can have a sufficient condition with a form like LMIs, which can be checked directly. Also, in such special equations, the upper bounds can be expressed explicitly. In the third example, we want to show that the criterion can also be applied to an SDDE with nonlinear coefficients.

Example 1. Consider the scalar equation (1.3) in the introduction section

$$dx(t) = [0.1 + 0.1x(t) - 0.3x(t-\tau)]dt + [0.2 + 0.1x(t) - 0.2x(t-\tau)]dB(t). \tag{4.1}$$

Obviously, the equation satisfies the Assumption 2.1 with $H_1 = 0.18$. Assumption 3.3 is also satisfied with $\sigma_1 = 0.09$ and $\sigma_2 = \sigma_3 = 0$. We still use $V(x(t)) = x^2(t)$ for discussion. Setting $\delta = 0.5$, it can be directly verified that $\Psi(z_1,z_2)$ in Assumption 2.3 will be $\Psi(z_1,z_2) = 2(z_1-z_2)\big(-0.2(z_1-z_2)\big) + (1+\delta)\big(-0.1(z_1-z_2)\big)^2 = -\lambda_1|z_1-z_2|^2$, with $\lambda_1 = 0.385$.

Following the procedures as in Lemmas 3.2 and 3.4, we can choose $\theta = 2.3823$ and $\bar{\theta} = 1.0712$ to calculate $\tau_1^* = 0.3689$ and $\tau_2^* = 0.8220$, respectively. By Theorem 3.5, (1.3) is stable in distribution as long as $\tau < \tau_1^* \wedge \tau_2^* = 0.3689$.

Setting $X(0) = 2$ and $\tau = 0.35$ for simulation, one thousand sample paths are simulated on $0 \leq t \leq 100$ with step size $h = 0.01$. As shown in Fig. 1, the shapes of empirical density functions at $t = 1, t = 5$ and $t = 10$ are rather different. But those shapes at $t = 20, t = 50$ and $t = 100$ are similar, which indicates the existence of the limit probability measure.

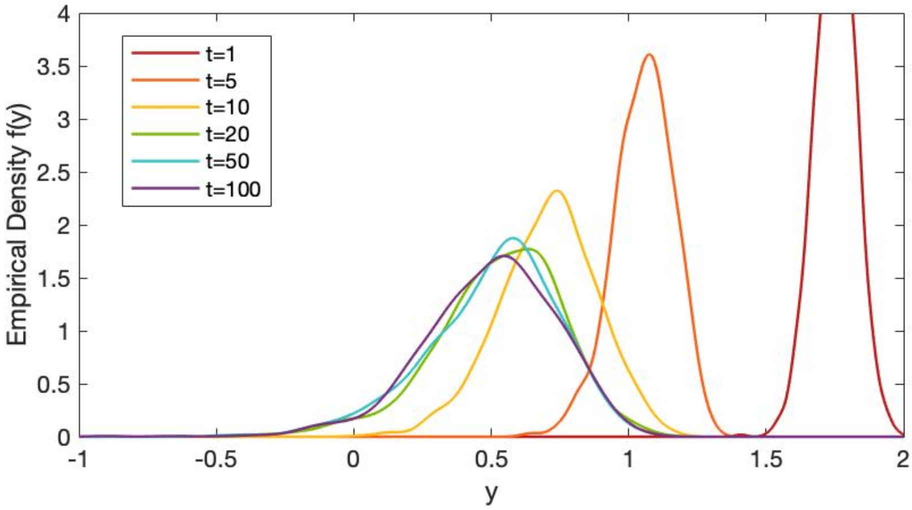

Fig. 1. Empirical density functions at different time.

To measure the difference of density functions at consecutive time points $t_k = kh$ and $t_{k+1} = (k+1)h$ $fork = 0, 1, \cdots$, we employ Kolmogorov-Smirnov test (K-S test) to test following hypotheses:

H_0 : Two samples at t_k and t_{k+1} are from the same distribution

H_1 : Two samples at t_k and t_{k+1} are from different distributions

It can be observed from the left subgraph in Fig. 2 that the differences of empirical density functions at consecutive time points tend to zero as time gets large. Also the right subgragh shows that p values are close to 1 as time advances, which confirms the conclusion too.

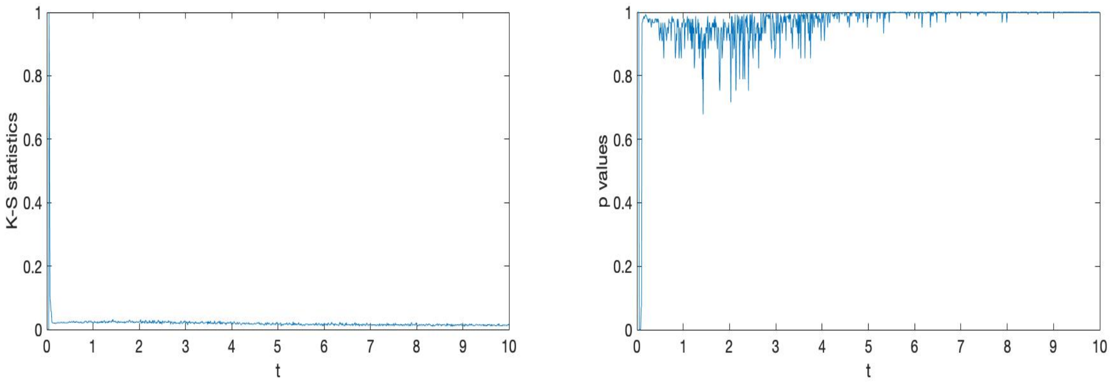

Fig. 2. K-S test and p-values for samples at consecutive time points.

Example 2. In the second example, we consider a general hybrid linear equation (1.1) with

$$\begin{aligned} f(X(t), X(t-\tau), i) &= K_{1,i} + K_{2,i}X(t) + K_{3,i}X(t-\tau) \\ g(X(t), X(t-\tau), i) &= L_{1,i} + L_{2,i}X(t) + L_{3,i}X(t-\tau) \end{aligned} \tag{4.2}$$

where $K_{1,i}, L_{1,i} \in \mathbb{R}^n$ and $K_{2,i}, K_{3,i}, L_{2,i}, L_{3,i} \in \mathbb{R}^{n\times n}$. To this equation, we look forward to form a criterion composed of its matrices coefficients and checked directly.

Obviously, Assumption 2.1 is satisfied with

$$H_1 = 2 \max_{i\in\mathbb{S}} \left(\|K_{2,i}\|^2 \vee \|K_{3,i}\|^2 \vee \|L_{2,i}\|^2 \vee \|L_{3,i}\|^2 \right).$$

$\Psi(z_1, z_2, i)$ in Assumption 2.3 will be $\Psi(z_1, z_2, i) = (z_1 - z_2)^T \Lambda_i (z_1 - z_2)$ with

$$\Lambda_i = (K_{2,i} + K_{3,i})^T W_i + W_i (K_{2,i} + K_{3,i}) + (1+\delta)(L_{2,i} + L_{3,i})^T W_i (L_{2,i} + L_{3,i}) + \sum_{j=1}^{N} \gamma_{ij} W_j.$$

If there exist N positively definite matrices $W_i, i \in \mathbb{S}$ and $\delta > 0$ such that Λ_i is negatively definite for any $i \in \mathbb{S}$, Assumption 2.3 will be satisfied with $\lambda_1 = -\max_{i\in\mathbb{S}} \lambda_M(\Lambda_i)$.

Toward Assumption 3.2, we can easily deduce $\sigma_1 = \max_{i\in\mathbb{S}} \left(\|K_{3,i}\|^2 \vee \|L_{3,i}\|^2\right)$ and $\sigma_2 = \sigma_3 = 0$. Now following the argument in Lemma 3.2, and setting $\theta = 2\hat{\lambda}_M H_1 \left(\frac{2\hat{\lambda}_M}{\lambda_1} + 1 + \frac{1}{\delta}\right)$, we can calculate $\tau_1^* = -\frac{1}{2} + \frac{1}{2}(1 + \frac{\lambda_1}{2H_1\theta})^{1/2}$. Similarly, letting $\bar{\theta} = 2\hat{\lambda}_M \sigma_1 \left(\frac{2\hat{\lambda}_M}{\lambda_1} + 1 + \frac{1}{\delta}\right)$ in Lemma 3.4, we also have $\tau_2^* = -\frac{1}{2} + \frac{1}{2}\left(1 + \frac{\lambda_1}{H_1\theta}\right)^{1/2}$. Obviously, we have $\tau_1^* < \tau_2^*$ and the equation will be stable in distribution as long as $\tau < \tau_1^*$.

Specifically, let us consider an equation defined in $\mathbb{R}^2$ with $\mathbb{S} = \{1, 2\}$ and $\Gamma = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$. The matrices are given by

$$K_{1,1} = \begin{pmatrix} 0.1 \\ 0.08 \end{pmatrix}, \quad L_{1,1} = \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix}, \quad K_{2,1} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad L_{2,1} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},$$
$$K_{3,1} = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \end{pmatrix}, \quad L_{3,1} = \begin{pmatrix} -0.2 & 0 \\ 0 & -0.2 \end{pmatrix}, \quad K_{1,2} = \begin{pmatrix} 0.12 \\ 0.1 \end{pmatrix}, \quad L_{1,2} = \begin{pmatrix} 0.21 \\ 0.09 \end{pmatrix},$$
$$K_{2,2} = \begin{pmatrix} 0.15 & 0 \\ 0 & 0.15 \end{pmatrix}, \quad L_{2,2} = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.12 \end{pmatrix}, \quad K_{3,2} = \begin{pmatrix} -0.32 & 0 \\ 0 & -0.3 \end{pmatrix}, \quad L_{3,2} = \begin{pmatrix} -0.15 & 0 \\ 0 & -0.2 \end{pmatrix}.$$

Choose $\delta = 2$, $W_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1.8 \end{pmatrix}$ and $W_2 = \begin{pmatrix} 2.2 & 0 \\ 0 & 2 \end{pmatrix}$ for analysis. It can be verified that Assumption 2.3 is satisfied with $\lambda_1 = 0.266$. Now we then figure out $\tau^* = 0.0099$.

For simulation, we select $X_1(0) = X_2(0) = 1$ and $\tau = 0.009$. One thousand sample paths are simulated on $0 \le t \le 30$ with time step size $h = 0.003$. As in the first example, we can draw empirical density functions for $X_1(t)$ and $X_2(t)$ at different time, as shown in Fig. 3. It can be seen that shapes the density functions of $X_1(t)$ at $t = 15, 20, 30$ are very similar, and those of $X_2(t)$ are also similar at $t = 20, 30$, which shows that both $X_1(t)$ and $X_2(t)$ have their limit probability measures as $t \to \infty$. Also, K-S tests and their p-values are depicted in Fig. 4, which confirm above assertions.

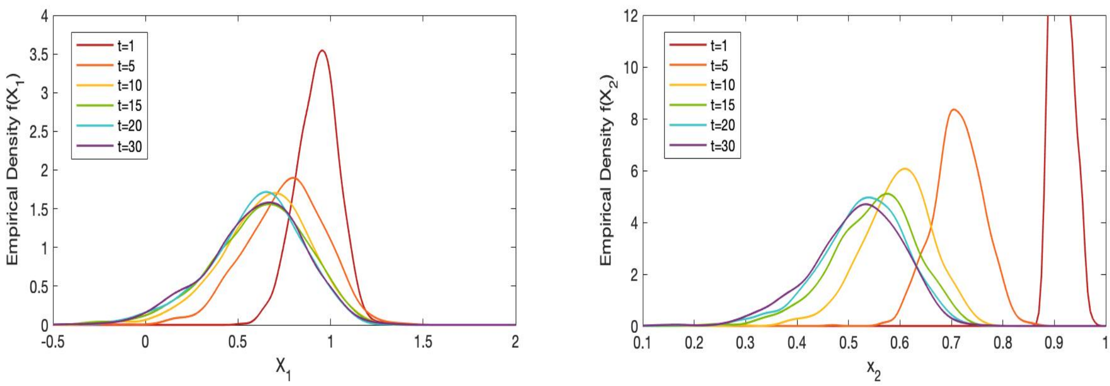

Fig. 3. Empirical density functions at different time for $X_1(t)$ and $X_2(t)$.

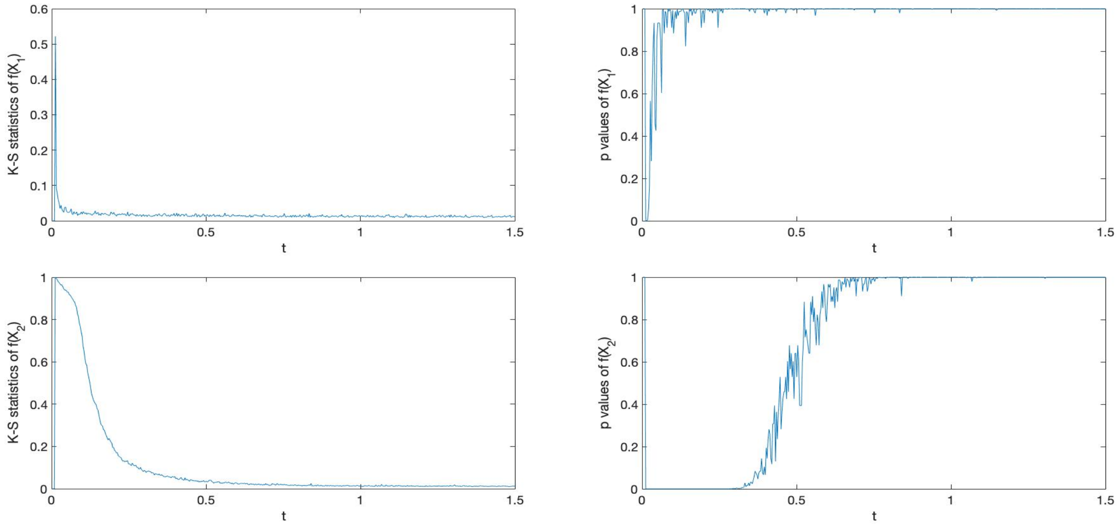

Fig. 4. K-S test and p-values for samples at consecutive time points for $X_1(t)$ and $X_2(t)$.

Example 3. Consider a nonlinear hybrid scalar equation (1.1) with $\mathbb{S} = \{1, 2\}$ and $\Gamma = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$, where

$$f(x, y, 1) = 0.1 - 0.2y + 0.01(\sin(x) + \sin(y)), f(x, y, 2) = 0.11 - 0.22y + 0.013(\sin(x) - \sin(y)),$$

$$g(x, y, 1) = 0.2 - 0.1y + 0.005(\sin(x) + \sin(y)), g(x, y, 2) = 0.22 - 0.12y + 0.006(\sin(x) - \sin(y)).$$

For this equation, Assumption 2.1 is satisfied with $H_1 = 0.1452$. Take $V(x, i) = \begin{cases} 2x^2 & i = 1 \\ 2.1x^2 & i = 2 \end{cases}$ for discussion. We see that Assumption 2.3 is satisfied with $\lambda_1 = 0.3$. Toward Assumption 3.3, we can take $\sigma_1 = 0.1452$, $\sigma_2 = 0.0004$ and $\sigma_3 = 0.0004$ to meet conditions in Lemma 3.4 simultaneously. Choosing $\theta = 11.2869$ and $\bar{\theta} = 11.2869$, we will have $\tau_1^* = 0.0224, \tau_2^* = 0.0359$, respectively. Now by our theories, the equation will be stable in distribution as long as $\tau < \tau^* = 0.0224$.

We let $\tau = 0.02$ and $X(0) = 2$ for simulation on $0 \le t \le 100$ with step size $h = 0.01$. The empirical density functions at $t = 1, 5, 10, 20, 50, 100$ are drawn in Fig.5, with K-S test and p-values shown in Fig.6. All three graphs show that the equation is stable in distribution.

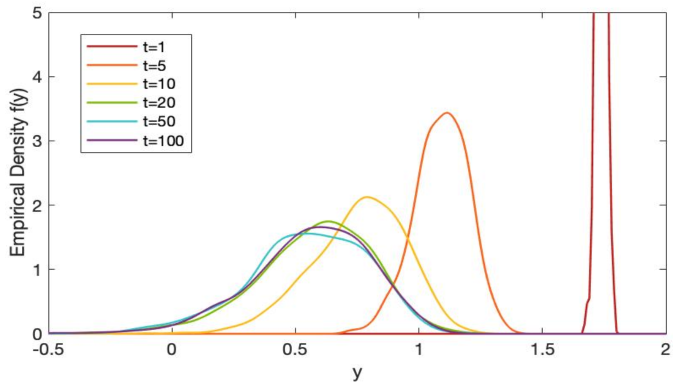

Fig. 5. Empirical density functions at different time.

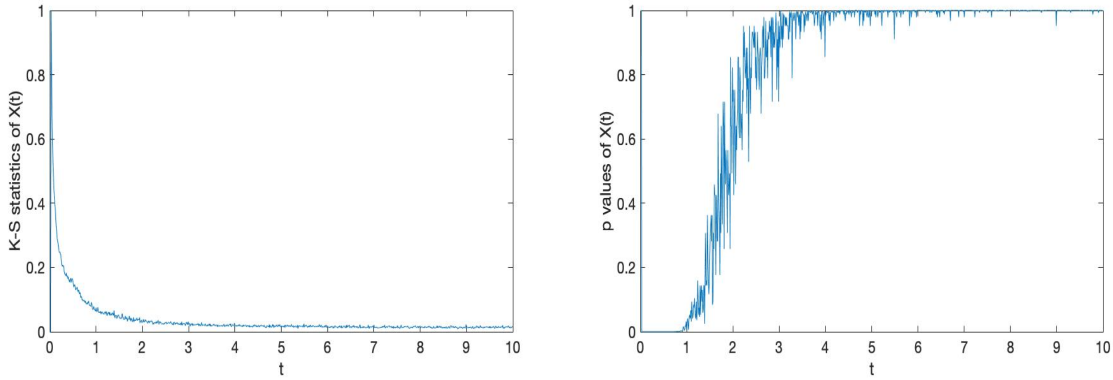

Fig. 6. K-S test and p-values for samples at consecutive time points.

5. Conclusion

A new criterion has been proposed to guarantee asymptotic stability in distribution for a SDDE. Compared to other related results, this criterion is delay-dependent. An upper bound for the delay size can be calculated from some equations. Also, this criterion shows the positive role of the delay term. It means that we can use delay terms as impetuses toward asymptotic stability for SDDEs.

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