Discrete Applied Mathematics 359 (2024) 83–88

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On semi-transitivity of (extended) Mycielski graphs

Humaira Hameed

Department of Mathematics and Statistics, University of Strathclyde, 26 Richmond Street, Glasgow G1, 1XH, United Kingdom

ARTICLE INFO

Article history:
Received 21 September 2023
Received in revised form 11 July 2024
Accepted 16 July 2024
Available online xxxx

Keywords:
Semi-transitive graph
Semi-transitive orientation
Word-representable graph
Mycielski graph
Extended mycielski graph

ABSTRACT

An orientation of a graph is semi-transitive if it is acyclic and shortcut-free. An undirected graph is semi-transitive if it admits a semi-transitive orientation. Semi-transitive graphs generalise several important classes of graphs and they are precisely the class of word-representable graphs studied extensively in the literature.

The Mycielski graph of an undirected graph is a larger graph, constructed in a certain way, that maintains the property of being triangle-free but enlarges the chromatic number. These graphs are important as they allow to prove the existence of triangle-free graphs with arbitrarily large chromatic number. An extended Mycielski graph is a certain natural extension of the notion of a Mycielski graph that we introduce in this paper.

In this paper we characterise completely semi-transitive extended Mycielski graphs and Mycielski graphs of comparability graphs. We also conjecture a complete characterisation of semi-transitive Mycielski graphs. Our studies are a far-reaching extension of the result of Kitaev and Pyatkin on non-semi-transitive orientability of the Mycielski graph $\mu(C_5)$ of the cycle graph C_5. Using a recent result of Kitaev and Sun, we shorten the length of the original proof of non-semi-transitive orientability of $\mu(C_5)$ from 2 pages to a few lines.

1. Introduction

There is a long line of research dedicated to the theory of semi-transitive graphs (equivalently, word-representable graphs, e.g. see [2–5] and references therein). The motivation to study these graphs is their relevance to algebra, graph theory, computer science, combinatorics on words, and scheduling [4]. In particular, semi-transitive graphs generalise several fundamental classes of graphs (e.g. *circle graphs*, 3-*colourable graphs* and *comparability graphs*).

This paper is a contribution to the theory of semi-transitive graphs, where we characterise completely semi-transitive extended Mycielski graphs and Mycielski graphs of comparability graphs to be introduced next.

1.1. Mycielski graphs

The *Mycielski graph* of an undirected graph is a larger graph that preserves the property of being triangle-free but enlarges the chromatic number. These graphs were introduced by Mycielski in 1955 (see [8]) to prove the existence of triangle-free graphs with arbitrarily large chromatic number. Since its introduction, Mycielski graphs attracted much attention in the literature from various points of view; see, for example, [1] and references therein.

Let the n vertices of a given graph G be $v_1, v_2, \dots, v_n$. The *Mycielski* graph $\mu(G)$ contains G itself as a subgraph, together with $n+1$ additional vertices: a vertex u_i corresponding to each vertex v_i of G and an extra vertex x. Each vertex u_i is

E-mail address: humaira.hameed@strath.ac.uk.

https://doi.org/10.1016/j.dam.2024.07.028
0166-218X/

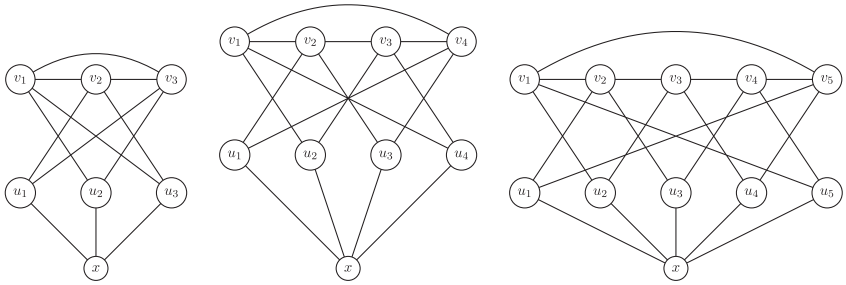

Fig. 1. The graphs $\mu(C_3)$, $\mu(C_4)$, and $\mu(C_5)$.

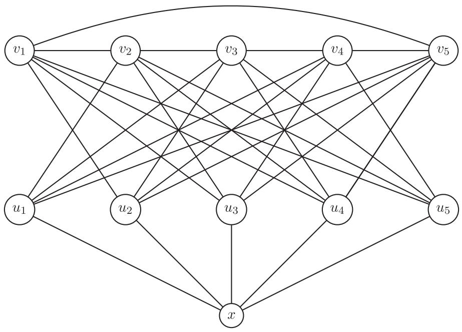

Fig. 2. The graph $\mu'(C_5)$.

connected by an edge to x so that these vertices form a subgraph in the form of a star $K_{1,n}$. In addition, for each edge v_iv_j of G, the *Mycielski* graph includes two edges, u_iv_j and v_iu_j.

Thus, if G has n vertices and m edges, $\mu(G)$ has $2n+1$ vertices and $3m+n$ edges. The only new triangles in $\mu(G)$ are of the form $v_iv_ju_k$, where $v_iv_jv_k$ is a triangle in G. Thus, if G is triangle-free, so is $\mu(G)$. The graphs $\mu(C_3)$, $\mu(C_4)$, and $\mu(C_5)$ are in Fig. 1, where C_n is the cycle graph on n vertices.

1.2. Extended Mycielski graphs

The *extended Mycielski graph* $\mu'(G)$ is obtained from a Mycielski graph $\mu(G)$ by connecting every vertex u_i to every vertex v_j except for when $i = j$. For example, $\mu'(C_5)$ can be found in Fig. 2. Note that for a complete graph K_n, $\mu'(K_n) = \mu(K_n)$.

It is easy to see that if G has n vertices and m edges then $\mu'(G)$ has $2n+1$ vertices and n^2+m edges. For $n \geq 3$, $\mu'(G)$ contains triangles even if G is triangle-free.

1.3. Semi-transitive graphs

An orientation of a graph is *semi-transitive* if it is acyclic (there are no directed cycles), and for any directed path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k$ either there is no edge between v_0 and v_k, or $v_i \rightarrow v_j$ is an edge for all $0 \leq i < j \leq k$. An induced subgraph on vertices $\{v_0, v_1, \ldots, v_k\}$ of an oriented graph is a *shortcut* if its orientation is acyclic and non-transitive, and there is the directed path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k$ and the edge $v_0 \rightarrow v_k$ called the *shortcutting edge*. A semi-transitive orientation can then be alternatively defined as an acyclic shortcut-free orientation. A non-oriented graph is *semi-transitive* if it admits a semi-transitive orientation.

The following results are important for us.

Lemma 1 ([6]). *Suppose that an undirected graph G has a cycle $C = x_1x_2\cdots x_mx_1$, where $m \geq 4$ and the vertices in $\{x_1, x_2, \ldots, x_m\}$ do not induce a clique in G. If G is oriented semi-transitively, and $m-2$ edges of C are oriented in the same direction (i.e. from x_i to x_{i+1} or vice versa, where the index $m+1 := 1$) then the remaining two edges of C are oriented in the opposite direction.*

A *source* (resp., *sink*) in a directed graph is a vertex with no incident to it edges oriented towards (resp., from) it.

Theorem 2 ([7]). *Suppose that a graph G is semi-transitive, and v is a vertex in G. Then, there exists a semi-transitive orientation of G where v is a source (or a sink).*

Relevant to our paper, Kitaev and Pyatkin [6] proved that $\mu(C_5)$ is non-semi-transitive. In Section 3.1 we use Theorem 2 to shorten the length of the original proof of non-semi-transitive orientability of $\mu(C_5)$ from 2 pages to a few lines.

1.4. The format of a proof of non-semi-transitivity

Proving that a given graph is not semi-transitive often involves going through all possible extensions of partial orientations of the graph and showing that none of them results in a semi-transitive orientation. Lemma 1 and Theorem 2 are of great importance here because they allow us to reduce dramatically the number of orientations to be considered. We refer to [7] for more details about the approach.

By a "line" of a proof we mean a sequence of instructions that directs us in orienting a partially oriented graph and necessarily ends with detecting a shortcut showing that this particular orientation branch will not produce a semi-transitive orientation. The idea is that if no branch produces a semi-transitive orientation then the graph is non-semi-transitively orientable.

Each proof begins with assumptions on orientations of certain edges, and there are four types of instructions:

- "B$a \to b$ (Copy x)" means "Branch on edge ab, orient the edge as $a \to b$, create a copy of the current version of the graph except orient the edge ab there as $b \to a$, and call the new copy x; leave Copy x aside and continue to follow the instructions". The instruction B occurs when no application of Lemma 1 is possible in the partially oriented graph.
- "MCx" means "Move to Copy x", where Copy x of the graph in question is a partially oriented version of the graph that was created at some point in the branching process. This instruction is always followed by an oriented edge $a \to b$ reminding on the directed edge obtained after application of the branching process.
- "O$a \to b$(Cabc)" means orient the edge ab as $a \to b$ in the cycle abc to avoid a directed cycle. If instead of a triangle we see a longer cycle, then we deal with an application of Lemma 1 to a cycle where all but two edges are oriented in one direction, and one of the remaining two edges is oriented in the opposite direction.
- "O$a \to b$ O$c \to d$ (C$xyz\cdots$)" means that Lemma 1 is applied to cycle $xyz\cdots$ to create new directed edges, $a \to b$ and $c \to d$.

Each line ends with "S: $xy\cdots z$" indicating a shortcut with the shortcutting edge $x \to z$ is obtained.

1.5. Organisation of this paper

In Section 2 we discuss semi-transitivity of extended Mycielski graph $\mu'(G)$ and characterise it completely in Theorem 4, our main result in this paper. In Section 3 we consider semi-transitivity of Mycielski graph $\mu(G)$ and characterise it completely for comparability graphs in Theorem 6. Section 3 also provides short proofs of non-semi-transitive orientability of the graphs $\mu(C_3)$, $\mu(C_5)$ and $\mu(C_7)$; the longer proofs of non-semi-transitivity of $\mu(C_9)$ and $\mu(C_{11})$ can be obtained (and verified) using the software [9]. In Section 4 we provide concluding remarks and conjecture a complete characterisation of semi-transitive Mycielski graphs.

2. Semi-transitivity of $\mu'(G)$

The following theorem is important in proving our main result.

Theorem 3. *The graph $\mu'(C_{2k+1})$ is non-semi-transitive for all $k \geq 1$.*

Proof. Suppose a semi-transitive orientation of $\mu'(C_{2k+1})$ exists. By Theorem 2 we can assume that $1'$ is a source. We distinguish four cases.

Case 1. $1\to2\to3$; see Fig. 3 and its caption for an argument leading to a contradiction.
Case 2. $1\leftarrow2\to3$; see Fig. 4 and its caption for an argument leading to a contradiction.
Case 3. $1\leftarrow2\leftarrow3$; see Fig. 5 and its caption for an argument leading to a contradiction.
Case 4. $1\to2\leftarrow3$; see Fig. 6 and its caption for an argument leading to a contradiction.
Hence, we conclude that no semi-transitive orientation for $\mu'(C_{2n+1})$ exists. □

The main result in this paper is the following theorem.

Theorem 4. *The graph $\mu'(G)$ is semi-transitive if and only if G is a bipartite graph.*

Proof. Suppose that G is *not* a bipartite graph. Then G must contain an odd cycle. A minimal odd cycle in G is an induced odd cycle, namely C_{2k+1} for some $k \geq 1$. Then, $\mu'(G)$ contains $\mu'(C_{2k+1})$ as an induced subgraph, so by Theorem 3, $\mu'(G)$ is not semi-transitive.

Suppose that G is a bipartite graph with n vertices. Orient G transitively from one part to the other so that the longest directed path in such an orientation will be of length 1. Extend this transitive orientation of G to a semi-transitive orientation of $\mu'(G)$ by letting x be a source and orienting edges $z \to y'$ for all $z, y \in \{1, 2, \ldots, n\}$ (the longest directed path in such an orientation is of length 2, so no shortcut is possible and clearly the orientation is acyclic). Hence, $\mu'(G)$ is semi-transitive in this case. □

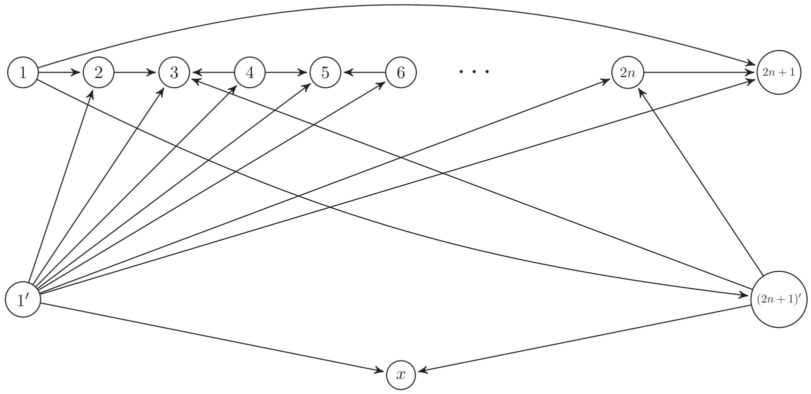

Fig. 3. A subgraph of the graph $\mu'(C_{2n+1})$ in Case 1, for which the proof goes as follows: O4→3 (C1′234) O1→(2n+1)′ O(2n+1)′→3 (C123(2n+1)′) O1→(2n+1) (C1′(2n+1)12) O(2n+1)′→ x (C1′3(2n+1)′x) O(2n+1)′→(2n) (C1′(2n)(2n+1)′x) S: 1(2n+1)′(2n)(2n+1).

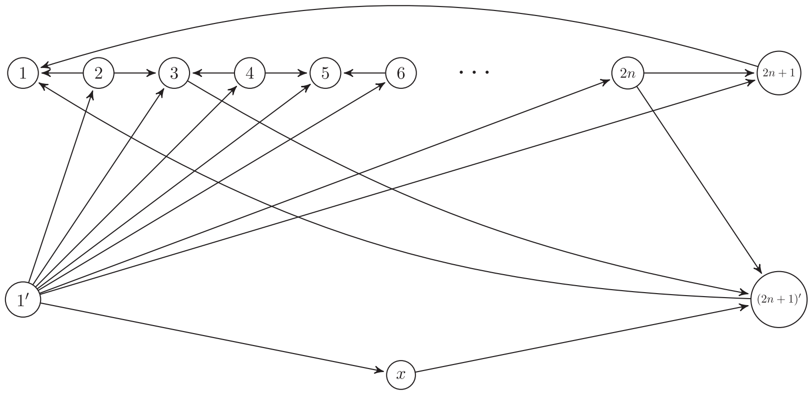

Fig. 4. A subgraph of the graph $\mu'(C_{2n+1})$ in Case 2, for which the proof goes as follows: O(2n + 1)→1 (C1′21(2n + 1)) O4→3 (C1′234) O4→5 (C1′345) O6→5 (C1′456)... O(2n) → (2n + 1) (C1′(2n − 1)(2n)(2n + 1)) O(2n)→(2n + 1)′ O(2n + 1)′→1 (C1(2n + 1)(2n)(2n + 1)′) Ox→(2n + 1)′ (C1′(2n)(2n + 1)′x) Oi→(2n + 1)′ (C1′i(2n + 1)′x) ∀i∈{2, 3, . . . , 2n − 1} S: 23(2n + 1)′1.

A direct corollary to Theorem 4 is the following statement, where P_n is the path graph on n vertices.

Corollary 5. *$\mu'(P_n)$ and $\mu'(C_{2k})$ are semi-transitive for all $k, n \geq 1$ and $\mu'(K_n)$ is not semi-transitive for $n \geq 3$.*

Proof. Clearly, P_n and C_{2k} are bipartite graphs, while K_n is not (it contains a triangle), and we obtain the desired result by Theorem 4. □

3. Semi-transitivity of $\mu(G)$

Unlike the case of extended Mycielski graphs, we cannot classify completely semi-transitivity of $\mu(G)$, even though we conjecture such a classification in Section 4. However, we can characterise semi-transitive $\mu(G)$ in the case when G is a comparability graph.

Theorem 6. *Let G be a comparability graph. Then $\mu(G)$ is semi-transitive if and only if G is bipartite.*

Proof. If G is not bipartite, it contains a triangle, so $\mu(G)$ contains $\mu(C_3) = \mu'(C_3)$ as an induced subgraph, and hence $\mu(G)$ is not semi-transitive by Theorem 3.

If G is bipartite then orient G transitively (from one part to the other). The longest directed path in such an orientation is of length 1. Extend this transitive orientation of G to a semi-transitive orientation of $\mu(G)$ by letting x be a source and

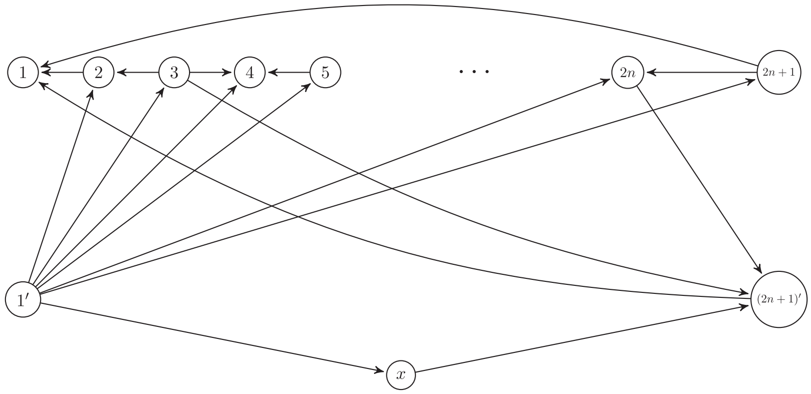

Fig. 5. A subgraph of the graph $\mu'(C_{2n+1})$ in Case 3, for which the proof goes as follows: O3→4 (C1′234) O5→4 (C1′345)... O(2n + 1)→(2n) (C1′(2n − 1)(2n)(2n + 1)) O(2n + 1)→1 (C121′(2n + 1)) O3→(2n + 1)′ O(2n + 1)′→1 (C123(2n + 1)′) Ox→(2n + 1)′ (C1′3(2n + 1)′x) O(2n)→(2n + 1)′ (C1′(2n)(2n + 1)′x) S: (2n + 1)(2n)(2n + 1)′1.

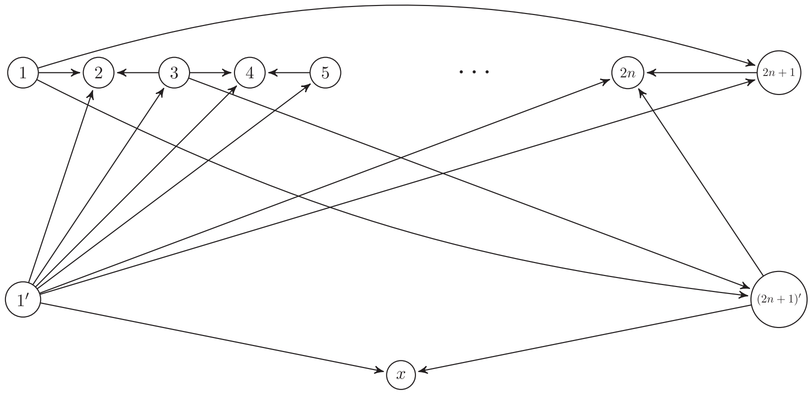

Fig. 6. A subgraph of the graph $\mu'(C_{2n+1})$ in Case 4, for which the proof goes as follows: O3→4 (C1′234) O5→4 (C1′345) O5→6 (C1′456)... O(2n + 1)→(2n) (C1′(2n − 1)(2n)(2n + 1)) O1→(2n + 1) (C1′(2n + 1)12) O1→(2n + 1)′ O(2n + 1)′→(2n) (C1(2n + 1)′(2n)(2n + 1)) O3→(2n + 1)′ (C123(2n + 1)′) O(2n + 1)′→x (C1′(2n)(2n + 1)′x) S: 1′3(2n + 1)′x.

orienting edges $z \rightarrow y'$ for all $z, y \in \{1, 2, \ldots, n\}$ (assuming G has n vertices). Hence, $\mu(G)$ is semi-transitive in this case. □

As an immediate corollary to Theorem 6, we get the following result.

Corollary 7. *$\mu(P_n)$ and $\mu(C_{2k})$ are semi-transitive for all $k, n \geq 1$ and $\mu(K_n)$ is not semi-transitive for $n \geq 3$.*

Proof. The same proof as that of Corollary 5, with Theorem 4 replaced by Theorem 6. □

3.1. Semi-transitivity of $\mu(C_n)$

By Corollary 7, $\mu(C_{2k})$ is semi-transitive for all $k \geq 1$. Unlike the case of $\mu'(C_{2k+1})$, we cannot completely describe semi-transitivity of $\mu(C_{2k+1})$, even though we conjecture non-semi-transitivity for all $k \geq 1$. Using the software [9], for the graphs $\mu(C_3)$, $\mu(C_5)$, $\mu(C_7)$, $\mu(C_9)$ and $\mu(C_{11})$ we have 1, 2, 4, 12 and 30 line proofs, respectively. Next, we present the proofs for $\mu(C_3)$, $\mu(C_5)$ and $\mu(C_7)$, and refer to [9] for producing the longer proofs for $\mu(C_9)$ and $\mu(C_{11})$ that can be verified within reasonable time. In all these proofs, for C_{2k+1}, we label vertex i' by $i + 2k + 1$, for $1 \leq i \leq 2k + 1$, and x by $4k + 3$, and also we assume, without loss of generality, that

(i) $x = 4k + 3$ is a source (by Theorem 2);
(ii) the edges 12 and 23 are oriented as $1 \rightarrow 2 \rightarrow 3$ (because in an acyclically oriented odd cycle, we must have a directed path of length 2; the rest is given by symmetry).

The non-semi-transitivity of the graphs $\mu(C_3)$, $\mu(C_5)$ and $\mu(C_7)$ is proved as follows (of course, $\mu'(C_3) = \mu(C_3)$ and Theorem 3 can be applied).

- **Graph** $\mu(C_3)$:
O1→3 (C132) O1→5 O5→3 (C1532) O1→6 (C1675) O2→6 (C1623) O2→4 (C2674) O3→4 (C1342) S: 7534
- **Graph** $\mu(C_5)$ (the original, much longer proof can be found in [6]):
1. O1→7 O7→3 (C1732) O1→10 (C1(10)(11)7) O9→3 (C39(11)7) B2→8 (Copy 2) O2→6 (C28(11)6) O1→5 O5→6 (C1562) O5→9 (C59(11)6) O5→4 O4→3 (C3954) O(10)→4 (C1(10)45) O4→8 (C2843) S: (11)(10)48
2. MC2 8→2 O8→4 O4→3 (C2843) O6→2 (C28(11)6) O(10)→4 (C4(10)(11)8) O5→4 O1→5 (C1(10)45) O6→5 (C1562) O5→9 (C3954) S: (11)659
- **Graph** $\mu(C_7)$:
1. O1→9 O9→3 (C1932) O1→(14) (C1(14)(15)9) O(11)→3 (C3(11)(15)9) B2→(10) (Copy 2) O2→8 (C2(10)(15)8) O1→7 O7→8 (C1782) O7→(13) (C7(13)(15)8) B5→(13) (Copy 3) O5→(11) (C5(13)(15)(11)) O5→4 O4→3 (C3(11)54) O4→(10) (C2(10)43) O4→(12) (C4(12)(15)(10)) O6→(12) O5→6 (C4(12)65) O7→6 (C5(13)76) O(14)→6 (C1(14)67) S:(15)(14)6(12)
2. MC3 (13)→5 O7→6 O6→5 (C5(13)76) O(14)→6 (C1(14)67) O(11)→5 (C5(13)(15)(11)) O(12)→6 (C6(14)(15)(12)) O(12)→4 O4→5 (C4(12)65) O4→3 (C3(11)54) O4→(10) (C2(10)43) S: (15)(12)4(10)
3. MC2 (10)→2 O(10)→4 O4→3 (C2(10)43) O8→2 (C2(10)(15)8) O(12)→4 (C4(12)(15)(10)) B6→(14) (Copy 4) O6→(12) (C6(14)(15)(12)) O6→5 O5→4 (C4(12)65) O5→(11) (C3(11)54) O5→13 (C5(13)(15)(11)) O7→(13) O6→7 (C5(13)76) O1→7 (C1(14)67) O8→7 (C1782) S: (15)87(13)
4. MC4 (14)→6 O7→6 O1→7 (C1(14)67) O8→7 (C1782) O(12)→6 (C6(14)(15)(12)) O(13)→7 (C7(13)(15)8) O(13)→5 O5→6 (C5(13)76) O5→4 (C4(12)65) O5→(11) (C3(11)54) S:(15)(13)5(11)

4. Conclusion

In this paper we completely characterised semi-transitive extended Mycielski graphs (in Theorem 4) and Mycielski graphs obtained from comparability graphs (in Theorem 6). We believe that a complete characterisation of Mycielski graphs is identical to that of extended Mycielski graphs, which we state as the following conjecture.

Conjecture 1. *The graph $\mu(G)$ is semi-transitive if and only if G is a bipartite graph.*

Because of Theorem 6, only the case of non-comparability graphs need to be considered, and Conjecture 1 will be true if the following conjecture is true.

Conjecture 2. *The graph $\mu(C_{2k+1})$ is non-semi-transitive for all $k \geq 1$.*

In this paper, we verified Conjecture 2 for $k \leq 3$ (and omitted the longer proofs for $k = 4, 5$ produced by the software [9]).

Acknowledgements

We are grateful to the anonymous referees for their helpful comments, which have improved the presentation of this paper.

References

[1] E.Z. Bidine, T. Gadi, M. Kchikech, Independence number and packing coloring of generalized Mycielski graphs, Discuss. Math. Graph Theory 41 (3) (2021) 725–747.
[2] M.M. Halldórsson, S. Kitaev, A. Pyatkin, Semi-transitive orientations and word-representable graphs, Discrete Appl. Math. 201 (2016) 164–171.
[3] S. Kitaev, A comprehensive introduction to the theory of word-representable graphs, Lect. Not. Comput. Sci. 10396 (2017) 36–67.
[4] S. Kitaev, V. Lozin, Words and Graphs, Springer, 2015.
[5] S. Kitaev, A. Pyatkin, On representable graphs, J. Autom. Lang. Combin. 13 (1) (2008) 45–54.
[6] S. Kitaev, A. Pyatkin, On semi-transitive orientability of triangle-free graphs, Discuss. Math. Graph Theory 43 (2) (2023) 533–547.
[7] S. Kitaev, H. Sun, Human-verifiable proofs in the theory of word-representable graphs, RAIRO – Theor. Inform. Appl. 58 (2024) 10, Special Issue: Randomness and Combinatorics - Edited by Luca Ferrari & Paolo Massazza. 10pp.
[8] J. Mycielski, Sur le coloriage des graphes, Colloq. Math. 3 (1955) 161–162.
[9] H. Sun, Software available at personal.strath.ac.uk/sergey.kitaev/human-verifiable-proofs.html.