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ORIGINAL PAPER

New Tools to Study 1-11-Representation of Graphs

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Abstract

The notion of a k-11-representable graph was introduced by Jeff Remmel in 2017 and studied by Cheon et al. in 2019 as a natural extension of the extensively studied notion of word-representable graphs, which are precisely 0-11-representable graphs. A graph G is k-11-representable if it can be represented by a word w such that for any edge (resp., non-edge) xy in G the subsequence of w formed by x and y contains at most k (resp., at least $k+1$) pairs of consecutive equal letters. A remarkable result of Cheon at al. is that *any* graph is 2-11-representable, while it is unknown whether every graph is 1-11-representable. Cheon et al. showed that the class of 1-11-representable graphs is strictly larger than that of word-representable graphs, and they introduced a useful toolbox to study 1-11-representable graphs. In this paper, we introduce new tools for studying 1-11-representation of graphs. We apply them for establishing 1-11-representation of Chvátal graph, Mycielski graph, split graphs, and graphs whose vertices can be partitioned into a comparability graph and an independent set.

Keywords 1-11-representable graph · Word-representable graph · Chvátal graph · Split graph · Mycielski graph · Comparability graph

1 Introduction

Various ways to represent graphs have evolved into a field of study, interesting from both mathematical and computer science perspectives [22]. Of more relevance to

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us is the theory of word-representable graphs [14], admitting a myriad of various generalizations. The basic idea here is to encode a given graph by a word using specified rules for defining edges/non-edges. For example, in the word-representable graphs alternations of letters in words define edges/non-edges, whilst this idea has been generalized by utilizing other patterns [10]. A given graph may, or may not admit representation under a given set of rules, so the main concern in the area of interest to us is whether a given graph is representable. Other research questions may include studying algorithmic aspects of representations, its minimal lengths, connections to other structures like graph orientations, applications, etc.

A particular way to represent graphs is k-11-representation introduced by Jeff Remmel in 2017 and studied by Cheon et al. in [4]. This way to represent graphs, formally defined in Sect. 2.2, is a natural way to generalize the notion of a word-representable graph that are precisely 0-11-representable graphs. Remarkably, *any* graph is 2-11-representable and the class of 1-11-representable graphs is strictly larger than that of 0-11-representable graphs (i.e. word-representable graphs); see [4]. It is still unknown whether there exist graphs that are not 1-11-representable. Clearly, such graphs (if they exist) must be non-word-representable. Hence, proving that various classes of non-word-representable graphs are 1-11-representable is a worthwhile direction of research.

1.1 Our Results and Organization of the Paper

In this paper, we observe the need of introducing new tools to study 1-11-representable graphs as the known set of tools does not allow to establish 1-11-representation of some known non-word-representable graphs. In particular, we introduce a new tool for establishing 1-11-representation of the Chvátal graph and another tool for proving that every split graph is 1-11-representable. We also generalize these tools to prove 1-11-representability for certain more general classes of graphs. Finally, we revisit the proof in [4] that every graph on at most 7 vertices is 1-11-representable to fill in the gap in the proof caused by usage of an incomplete list of small non-word-representable graphs, where two graphs were missing.

The paper is organized as follows. In Sect. 2 we introduce all (classes of) graphs considered in this paper highlighting in separate subsections more important word-representable graphs and related to them semi-transitive orientations (Sect. 2.1) and k-11-representable graphs (Sect. 2.2). Also, in Sect. 2.3 we provide a comprehensive list of known results about 1-11-representable graphs that provide a powerful base to study 1-11-representation of graphs. In Sect. 3 we introduce new tools to study 1-11-representable graphs and discuss its applications for the Chvátal graph in Sect. 3.1 and for split graphs and for graphs whose vertices can be partitioned into a comparability graph and an independent set in Sect. 3.2. Also, we complete justification of the fact that all graphs on at most 7 vertices are 1-11-representable in Sect. 3.3 and provide concluding remarks in Sect. 4.

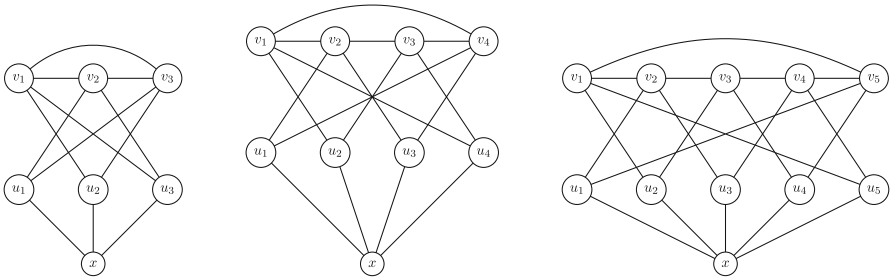

Fig. 1 The graphs $\mu(C_3)$, $\mu(C_4)$, and $\mu(C_5)$

2 Preliminaries

We begin with defining (classes of) graphs appearing in this paper under various contexts. Throughout this paper, we denote by $G \setminus v$ the graph obtained from a graph G by deleting a vertex $v \in V(G)$ and all edges adjacent to it. Also, for any $A \subseteq V$ and $v \in V$ let $N_A(v) := \{u \in A \mid uv \in E\}$, that is, $N_A(v)$ is the set of *neighbours* of v in A. If $A = V$ we write simply $N(v)$. We use the notation $G[A]$ for the subgraph of G induced by the subset A.

A *circle graph* is the intersection graph of a set of chords of a circle, i.e. it is an undirected graph whose vertices can be associated with chords of a circle such that two vertices are adjacent if and only if the corresponding chords cross each other [21]. An *interval graph* has one vertex for each interval in a family of intervals on a line, and an edge between every pair of vertices corresponds to intervals that intersect [19]. A *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set [8, 12]. For an arbitrary graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$, define the $Mycielski$ graph $\mu(G) = (V \cup U \cup \{x\}, E \cup E')$ where $U = \{u_1, \dots, u_n\}$ and

$$E' = \cup_{i=1}^{n} (\{xu_i\} \cup \{yu_i \text{ for all } y \in N_V(v_i)\}).$$

In other words, $\mu(G)$ contains G itself as a subgraph, the independent set consisting of a copy of each its vertex, and a vertex x adjacent to all these copies. For example, the graphs $\mu(C_3)$, $\mu(C_4)$, and $\mu(C_5)$ are in Fig. 1. The importance of Mycielski graphs follows from the well-known fact [20] that this construction allows to increase the chromatic number of a triangle-free graph without adding new triangles (i.e if G is a triangle-free k-chromatic graph then $\mu(G)$ is a triangle-free $(k+1)$-chromatic graph). The Chvátal graph is presented to the left in Fig. 2.

An orientation of a graph is *transitive*, if the presence of the edges $u \to v$ and $v \to z$ implies the presence of the edge $u \to z$. An undirected graph G is a *comparability graph* if G admits a transitive orientation.

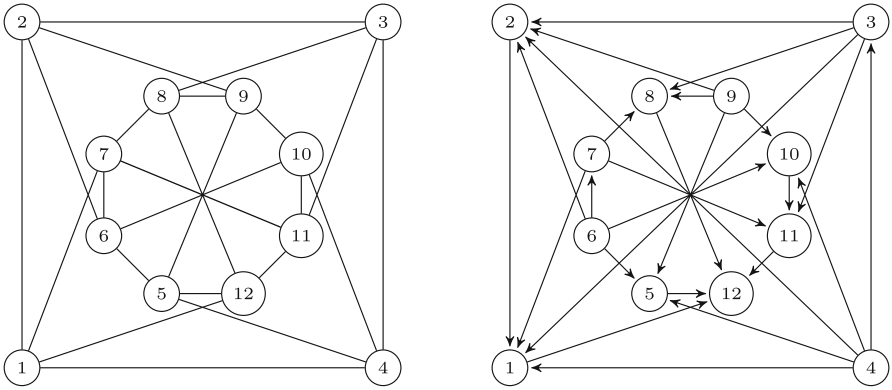

Fig. 2 The Chvátal graph (to the left) and a semi-transitive orientation of the Chvátal graph extended by the edges 31 and 42 (to the right)

2.1 Word-Representable Graphs and Semi-transitive Orientations

Two letters x and y alternate in a word w if after deleting in w all letters but the copies of x and y we either obtain a word $xyxy\cdots$ or a word $yxyx\cdots$ (of even or odd length). A graph $G = (V, E)$ is *word-representable* if and only if there exists a word w over the alphabet V such that letters x and y, $x \neq y$, alternate in w if and only if $xy \in E$. The unique minimum (by the number of vertices) non-word-representable graph on 6 vertices is the wheel graph W_5, while there are 25 non-word-representable graphs on 7 vertices. We note that the original list of 25 non-word-representable graphs on 7 vertices presented, for example, in [14] contains two incorrect graphs, so we refer to [18] for the corrected catalog of the 25 graphs.

A graph is *permutationally representable* if it can be represented by concatenation of permutations of (all) vertices. Thus, the class of permutationally representable graphs is a subclass of word-representable graphs. The following theorem classifies these graphs.

Theorem 1 ([14]) *A graph is permutationally representable if and only if it is a comparability graph.*

An orientation of a graph is *semi-transitive* if it is acyclic, and for any directed path $v_0 \to v_1 \to \cdots \to v_k$ either there is no arc from v_0 to v_k, or $v_i \to v_j$ is an arc for all $0 \leq i < j \leq k$. An induced subgraph on at least four vertices $\{v_0, v_1, \ldots, v_k\}$ of an oriented graph is a *shortcut* if it is acyclic, non-transitive, and contains both the directed path $v_0 \to v_1 \to \cdots \to v_k$ and the arc $v_0 \to v_k$, that is called the *shortcutting edge*. A semi-transitive orientation can then be alternatively defined as an acyclic shortcut-free orientation. A fundamental result in the area of word-representable graphs is the following theorem.

Theorem 2 ([11]) *A graph is word-representable if and only if it admits a semi-transitive orientation.*

For instance, it follows from Theorem 2 that each 3-colorable graph is word-representable (just direct each edge from a lesser color to a larger one).

2.2 *k*-11-Representable Graphs

A *factor* in a word $w_1w_2\dots w_n$ is a word $w_iw_{i+1}\dots w_j$ for $1 \leq i \leq j \leq n$. For any word w, let $\pi(w)$ be the *initial permutation* of w obtained by reading w from left to right and recording the leftmost occurrences of the letters in w. Denote by $r(w)$ the *reverse* of w, that is, w written in the reverse order. Finally, for a pair of letters x and y in a word w, let $w|_{\{x,y\}}$ be the subword induced by the letters x and y. For example, if $w = 42535214421$ then $\pi(w) = 42531$, $r(w) = 12441253524$, and $w|_{\{4,5\}} = 45544$.

Let $k \geq 0$. A graph $G = (V, E)$ is k-11-*representable* if there exists a word w over the alphabet V such that the word $w|_{\{x,y\}}$ contains in total at most k occurrences of the factors in $\{xx, yy\}$ if and only if xy is an edge in E. Such a word w is called G's k-11-*representant*. Note that 0-11-representable graphs are precisely word-representable graphs, and that 0-11-representants are precisely word-representants. A graph $G = (V, E)$ is *permutationally* k-11-*representable* if it has a k-11-representant that is a concatenation of permutations of V. The "11" in "k-11-representable" refers to counting occurrences of the *consecutive pattern* 11 in the word induced by a pair of letters $\{x, y\}$, which is exactly the total number of occurrences of the factors in $\{xx, yy\}$.

A *uniform* (resp., *t-uniform*) representant of a graph G is a word, satisfying the required properties, in which each letter occurs the same (resp., t) number of times. It is known that each word-representable graph has a uniform representant [15], the class of 2-uniformly representable graphs is exactly the class of circle graphs [14], while the class of 2-uniformly 1-11-representable graphs is the class of interval graphs [4]. Interestingly, 2-uniformly representable graphs appear in the literature under the name of "*alternance graph*", and other names, in [1, 2, 6–8] well before the introduction of word-representable graphs; see [2] for a discussion and more references on alternance graphs. The main result in [4] is the following theorem.

Theorem 3 ([4]) *Every graph G is permutationally* 2-11-*representable.*

So, when understanding whether each graph is k-11-representable for a fixed k, the only open case to study is $k = 1$.

2.3 Known Tools to Study 1-11-Representable Graphs

Clearly, each word-representable graph is 1-11-representable. Indeed, if w is a word-representant of G then, for instance, ww or $r(\pi(w))w$ are its 1-11-representants. There are three types of tools for finding 1-11-representable graphs suggested in [4]:

- Modifying known 1-11-representable graphs;
- Removing edges from word-representable graphs;
- Adding vertices to certain classes of graphs.

Below we unify all known tools from [4] into three statements according to their type.

Lemma 1 ([4])

(a) *Let G_1 and G_2 be 1-11-representable graphs. Then their disjoint union, glueing them in a vertex or connecting them by an edge results in a 1-11-representable graph.*
(b) *If G is 1-11-representable then for any edge xy adding a new vertex adjacent to x and y only, gives a 1-11-representable graph.*

Lemma 2 ([4]) *Let G be a word-representable graph, $A \subseteq V$ and $v \in V$. Then*

(a) $G \setminus \{xy \in E(G) \mid x, y \in A\}$ *is a 1-11-representable graph;*
(b) $G \setminus \{uv \in E(G) \mid u \in N_A(v)\}$ *is a 1-11-representable graph.*

Lemma 3 ([4]) *Let G be a graph with a vertex v. G is 1-11-representable if at least one of the following conditions holds:*

(a) $G \setminus v$ *is a comparability graph;*
(b) $G \setminus v$ *is a circle graph.*

Note that the tool in Lemma 3(b) (that is a partial case of Theorem 2.7 in [4] for $k = 2$) looks to be the strongest one. For instance, it allows to establish 1-11-representability of such known non-word-representable graphs as odd wheels. In the next statement we use it to prove a new result on 1-11-representability of $\mu(C_n)$. Note that $\mu(C_n)$ is conjectured to be non-word-representable for all odd $n \geq 3$, and it is known that the conjecture is true for $\mu(C_5)$ [16].

Proposition 1 *The Mycielski graphs $\mu(C_n)$ are 1-11-representable for all $n \geq 3$.*

Proof By Lemma 3(b) it is sufficient to show that the graph $\mu(C_n) \setminus x$ is a circle graph, i.e. that it is 2-uniformly representable. It is easy to check that the following 2-uniform word represents $\mu(C_n) \setminus x$:

$$v_2 u_1 u_2 v_1 v_3 u_2 u_3 v_2 v_4 \dots v_i u_{i-1} u_i v_{i-1} v_{i+1} u_i u_{i+1} v_i \dots v_n u_{n-1} u_n v_{n-1} v_1 u_n u_1 v_n.$$

Indeed, it is easy to see that the 2-uniform word $v_2 v_1 v_3 v_2 \dots v_n v_{n-1} v_1 v_n$ represents the cycle C_n. The u's are inserted into this word in such a way that between two copies of u_i one finds only v_{i-1} and v_{i+1} for every i (including the cyclical shifts of the word with the indices $0 = n$ and $n + 1 = 1$). So, $N(u_i) = \{v_{i-1}, v_{i+1}\} = N_{C_n}(v_i)$, as required. □

3 New Tools to Study 1-11-Representation of Graphs and Their Applications

Our first tool (Theorem 4 below) is a far-reaching generalization of Lemma 2. We begin with the following easy observation.

Proposition 2 *Let Π_1, Π_2, Π_3 be three permutations over $[n] = \{1, \ldots, n\}$. Then the word $w = \Pi_1\Pi_2\Pi_3$ permutationally 1-11-represents the graph with the vertex set $[n]$ in which x and y are not connected by an edge if and only if in Π_1 and Π_3, x and y are in the same relative order, while in Π_2 they are in the opposite order.*

Proof We may assume that $x < y$ in Π_1. Then the word $w|_{\{x,y\}}$ is either one of $xyxyxy$, $xyxyyx$, $xyyxyx$ (then xy is an edge) or $xyyxxy$ (then x and y are not adjacent). □

In the proof of the next theorem, and in other places in the rest of the paper, for convenience, we slightly abuse the notation by denoting a set A and a certain permutation of elements in A by the same letter. This will not cause any confusion.

Theorem 4 *Let $V_1, \ldots, V_k$ be pairwise disjoint subsets of $[n]$, the set of vertices of a word-representable graph G. We denote by $E(V_i)$ the set of all edges of G having both end-points in V_i. Then, the graph $H = G\backslash(\cup_{1\leq i\leq k}E(V_i))$, obtained by removing all edges belonging to $E(V_i)$ for all $1 \leq i \leq k$, is 1-11-representable.*

Proof Let w be a word representing G and recall that $\pi(w)$ denotes the initial permutation of w. By [15], we can assume that w is uniform. Also, we let $R := [n]\backslash(\cup_{1\leq i\leq k}V_i)$ and we define the permutation $\Pi_1 := V_1V_2\ldots V_kR$, where all letters in each subset follow the same order as they have in $\pi(w)$. Let $\Pi_2 := r(V_1)r(V_2)\ldots r(V_k)R$. We will next prove that the word $W = \Pi_1\Pi_2\pi(w)ww$ 1-11-represents[1] the graph H.

Note that the word $\pi(w)ww$ 1-11-represents G and since w is uniform, each edge of G is represented in w by strict alternation of letters (avoiding occurrences of the pattern 11). Clearly, all non-edges in G remain non-edges in H.

If xy is an edge in G that belongs to $E(V_i)$ for some i, then by Proposition 2, $(\Pi_1\Pi_2\pi(w))|_{\{x,y\}}$ contains at least two occurrences of the patterns 11, and hence x and y are not connected by an edge in H.

Suppose that xy is an edge in both G and H. Hence, x and y cannot belong to any V_i. But then in the permutations Π_1 and Π_2 the letters x and y are in the same order. By Proposition 2, the word $(\Pi_1\Pi_2\pi(w))|_{\{x,y\}}$ contains at most one occurrence of the pattern 11. As it was shown above, the word $(\pi(w)ww)|_{\{x,y\}}$ has no such occurrences. So, $W|_{\{x,y\}}$ has at most one occurrence of the pattern 11, which is consistent with xy being an edge in H. □

A particular case of Theorem 4, when each V_i is of size 2, is useful from an applications point of view and hence is stated as a separate result.

Corollary 1 *Let the graph G be obtained from a graph H by adding a matching (that is, by adding new edges no pair of which shares a vertex). If G is word-representable then H is 1-11-representable.*

[1] In fact, the shorter word $\Pi_1\Pi_2ww$ also represents the graph H, but we inserted $\pi(w)$ for the convenience of the reader, making it easier to follow our arguments.

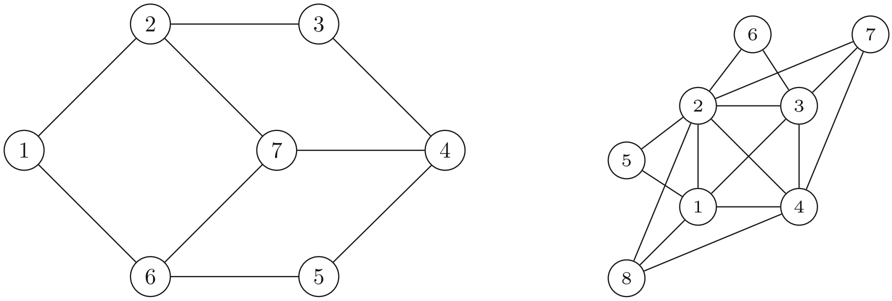

Fig. 3 The graph BW_3 and a minimal non-word-representable split graph

3.1 The Chvátal Graph is 1-11-Representable

The Chvátal graph, given to the left in Fig. 2, is the smallest triangle-free 4-chromatic 4-regular graph on 12 vertices [5]. This graph is non-word-representable [16]. Firstly, we show that no known tool from [4] can be applied for proving its 1-11-representability.

Proposition 3 1-11*-representability of the Chvátal graph does not follow from Lemmas 1, 2, and 3.*

Proof It is evident that Lemma 1 cannot be applied.

Assume that Lemma 2 can be applied, i.e. that there is a word-representable graph G, its vertex subset A and a vertex v such that $G \setminus E'$ is the Chvátal graph where either $E' = \{xy \in E(G) \mid x, y \in A\}$ or $E' = \{uv \in E(G) \mid v \in N_A(v)\}$. Consider a semi-transitively oriented copy of G (that exists by Theorem 2) and remove from it the edges in E'. The obtained oriented graph must contain a shortcut S since the Chvátal graph is not word-representable [16]. Since the Chvátal graph is triangle-free, S must contain a directed path $u_1 \to u_2 \to u_3 \to u_4$ with edges u_1u_3 and u_2u_4 missing. However, none of the variants of E' can simultaneously contain the edges u_1u_3 and u_2u_4 and miss the edges u_1u_2, u_2u_3, and u_3u_4. Hence, S must be a shortcut in G, a contradiction.

Finally, let us show that Lemma 3 cannot be applied. Since the Chvátal graph contains two non-intersecting cycles of length 5 induced by the sets $\{1, 2, 3, 7, 8\}$ and $\{5, 6, 10, 11, 12\}$ (see the left graph in Fig. 2 for the notations), removing any vertex in the graph cannot produce a comparability graph. Moreover, it is known [2] that a circle graph cannot contain a graph BW_3 (the left one in Fig. 3) as an induced subgraph. It is straightforward to verify that each of the subsets $V_1 = \{2, 3, 5, 6, 8, 9, 12\}$, $V_2 = \{3, 4, 6, 7, 8, 10, 11\}$, $V_3 = \{1, 4, 5, 8, 9, 10, 12\}$, and $V_4 = \{1, 2, 6, 7, 10, 11, 12\}$ induces a copy of BW_3. Since $V_1 \cap V_2 \cap V_3 \cap V_4 = \emptyset$, the Chvátal graph cannot be turned into a circle graph by removing one vertex. □

Remark 1 Note that the same arguments as those in Proposition 3 for non-applicability of Lemma 2 work not only for the Chvátal graph, but for any triangle-free graph.

However, the new tool from Theorem 4 works well for the Chvátal graph.

Theorem 5 *The Chvátal graph is* 1-11*-representable.*

Proof Add to the Chvátal graph the edges 13 and 24 and consider the orientation of the obtained graph G presented to the right in Fig. 2. It is easy to verify that this orientation is acyclic. Assume that it has a shortcut. Note that a shortcut must contain a path of length at least 3. There are exactly seven such paths in G, namely,

$$9 \to 10 \to 11 \to 12, \quad 6 \to 10 \to 11 \to 12, \quad 6 \to 7 \to 11 \to 12,$$
$$6 \to 7 \to 8 \to 12, \quad 4 \to 10 \to 11 \to 12, \quad 4 \to 3 \to 11 \to 12,$$
$$4 \to 3 \to 2 \to 1.$$

First six of them are not shortcuts since the vertex 12 is not adjacent to 4, 6, or 9. The last one is not a shortcut since the subgraph induced by the vertices 1, 2, 3, 4 is transitive. So, the orientation of G is semi-transitive and by Corollary 1 the Chvátal graph is 1-11-representable. □

3.2 1-11-Representability of Split Graphs and Their Generalizations

Our second tool is a new technique of finding permutational 1-11-representants for certain graphs. We first present the technique for split graphs and then generalize it to a class of graphs that can be partitioned into an independent set and a comparability graph. However, we believe that the new technique could be applicable in proving 1-11-representability of other classes of graphs.

Studying word-representation of split graphs is a hard problem, and it has been the subject of interest in [3, 9, 13, 17]. It is remarkable that each split graph is 1-11-representable as is shown in the following theorem.

Theorem 6 *Any split graph is permutationally* 1-11*-representable.*

Proof Let $A = \{a_1, \ldots, a_k\}$ be a clique and $B = \{b_1, \ldots, b_\ell\}$ be an independent set in a split graph S, so that $A \cup B$ is the set of all vertices in S. For a vertex $a_i \in A$ let $N_i = N_B(a_i)$ (resp., $O_i = B \backslash N_i$) be the set of neighbours (resp., non-neighbours) of a_i in B. We put

$$w_0 := a_1a_2 \ldots a_kb_1b_2 \ldots b_\ell \; a_1a_2 \ldots a_kb_\ell b_{\ell-1} \ldots b_1 \; a_1a_2 \ldots a_kb_1b_2 \ldots b_\ell$$

and define the permutations

$$\Pi_k := a_1a_2 \ldots a_{k-1}O_ka_kN_k;$$
$$\Pi_j := a_ka_{k-1} \ldots a_{j+1}a_1a_2 \ldots a_{j-1}O_ja_jN_j, \text{ for } 0 < j < k;$$
$$\Pi_0 := a_ka_{k-1} \ldots a_1b_1b_2 \ldots b_\ell.$$

Then the word $w = w_0\Pi_k\Pi_{k-1} \ldots \Pi_0$ permutationally 1-11-represents the graph S.

Indeed, the factor w_0 of w ensures independence of the set B. Moreover, for each pair $a_i, a_j \in A$ where $i < j$ in $w|_{a_i,a_j}$ we have a subsequence $a_ia_ja_ia_j \ldots a_ia_j$ to the left of the permutation Π_j (including Π_j itself), and a subsequence $a_ja_ia_ja_i \ldots a_ja_i$

to the right of Π_j. So, there is exactly one occurrence of the pattern 11 in $w|_{a_i,a_j}$ ensuring that a_i and a_j are connected. Next, suppose that $a_i \in A$ and $b \in B$. If a_i is adjacent to b, then $w|_{\{a_i,b\}} = a_i b a_i b \ldots a_i b$, which has no pattern 11. Finally, if a_i is not adjacent to b then $(w \backslash \Pi_i)|_{\{a_i,b\}} = a_i b a_i b \ldots a_i b$ but $\Pi_i|_{\{a_i,b\}} = b a_i$, so $w|_{\{a_i,b\}}$ has two occurrences of the pattern 11 that is consistent with a_i being not adjacent to b.

Thus, w 1-11-represents G. Since w_0 is a concatenation of three permutations, w is also a concatenation of permutations. □

To illustrate the construction in the proof of Theorem 6, we give a permutational 1-11-representation of the split graph given in Fig. 3 to the right that is observed in [13] to be minimal non-word-representable (removing any of its vertices results in a word-representable graph). We have $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8\}$, $k = \ell = 4$, $N_1 = \{5, 8\}$, $O_1 = \{6, 7\}$, $N_2 = \{5, 6, 7, 8\}$, $O_2 = \emptyset$, $N_3 = \{6, 7\}$, $O_3 = \{5, 8\}$, $N_4 = \{7, 8\}$ and $O_4 = \{5, 6\}$. Separating permutations by space for more convenient visual representation, we have:

$$
\begin{aligned}
w_0 &= 12345678\ 12348765\ 12345678\\
\Pi_4 &= 123O_44N_4 = 12356478\\
\Pi_3 &= 412O_33N_3 = 41258367\\
\Pi_2 &= 431O_22N_2 = 43125678\\
\Pi_1 &= 432O_11N_1 = 43267158\\
\Pi_0 &= 43215678
\end{aligned}
$$

and so a permutational 1-11-representation of the graph to the right in Fig. 3 is

$$12345678\ 12348765\ 12345678\ 12356478\ 41258367\ 43125678\ 43267158\ 43215678.$$

The following theorem is a far-reaching generalization of Theorem 6. However, we do keep Theorem 6 as a separate result as we need the construction in its proof in what follows.

Theorem 7 *Suppose that the vertices of a graph G can be partitioned into a comparability graph formed by vertices in $A = \{a_1, \ldots, a_k\}$ and an independent set formed by vertices in $B = \{b_1, \ldots, b_\ell\}$. Then G is permutationally* 1-11*-representable.*

Proof Denote by G' the split graph obtained from G by substitution of A by a clique A'. By Theorem 6 G' can be permutationally 1-11-represented by the word $w = w_0 \Pi_k \Pi_{k-1} \ldots \Pi_1 \Pi_0$. Moreover, for each $a_i, a_j \in A'$ the subword $w|_{a_i,a_j}$ contains exactly one occurrence of the pattern 11.

By Theorem 1, the subgraph $G[A]$ is permutationally representable. So, let $Q_1 Q_2 \ldots Q_t$ be its representation by permutations Q_i over the set A. Let $\Pi'_i = Q_i b_1 b_2 \ldots b_\ell$ for all $i \in \{1, 2, \ldots, t\}$ and rename, if necessary, the vertices in A so that $Q_1 = a_k a_{k-1} \ldots a_1$ (i.e. so that $\Pi'_1 = \Pi_0$ in the word w). We put

$$W = w_0 \Pi_k \Pi_{k-1} \ldots \Pi_1 \Pi'_1 \Pi'_2 \ldots \Pi'_t$$

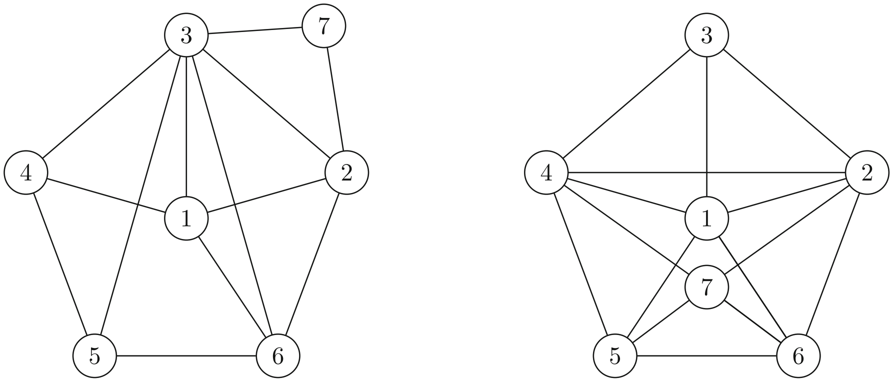

Fig. 4 Non-word-representable Graph 12 (to the left) and Graph 17 (to the right)

and show that it permutationally 1-11-represents G.

Indeed, the factor $w_0\Pi_k\Pi_{k-1}\dots\Pi_1\Pi'_1$ of W defines the split graph with a clique formed by the vertices in A and an independent set B. Also, any edge a_ib_j of the split graph remains an edge in G since the order of these vertices is a_ib_j in all permutations of W.

Let $i < j$ and consider vertices $a_i, a_j \in A$. By construction, in the word $w = w_0\Pi_k\Pi_{k-1}\dots\Pi_1\Pi'_1$ each edge a_ia_j of the clique A' is defined by the subsequence

$$a_ia_ja_ia_j\dots a_ia_ja_ja_ia_ja_i\dots a_ja_i$$

containing exactly one occurrence of the pattern 11. If a_ia_j is an edge of the comparability graph $G[A]$, then in all permutations Π'_s vertices a_i and a_j are in the same order a_ja_i, and so $\Pi'_1\Pi'_2\dots\Pi'_t|_{\{a_i,a_j\}}$ avoids the pattern 11 and hence a_ia_j remains an edge in G. Finally, if a_ia_j is not an edge of the comparability graph $G[A]$, then in $\Pi'_1\Pi'_2\dots\Pi'_t|_{\{a_i,a_j\}}$ we have at least one occurrence of the pattern 11, and hence $w|_{\{a_i,a_j\}}$ has at least two occurrences of the pattern 11, so in G a_i and a_j are not connected by an edge. □

3.3 1-11-Representability of all Graphs on at most 7 Vertices

1-11-representation of all graphs on at most 7 vertices is established in [4]. However, the arguments in [4] are based on the incorrect list of 25 non-word-representable graphs published in several places in the literature, in particular, in [14]. The problem with the list was spotted in [18], and the two incorrect graphs, Graphs 12 and 17, were replaced in [18] by the correct graphs given in Fig. 4. Hence, technically, 1-11-representation of all graphs on at most 7 vertices, but Graph 12 and Graph 17, is known, and next we complete the classification by confirming 1-11-representability of the graphs in Fig. 4.

Proposition 4 *Graphs* 12 *and* 17 *are permutationally* 1-11*-representable.*

Proof Note that removing the independent set $\{1, 5, 7\}$ from Graph 12 results in a triangle with a pending edge, that is a comparability graph. Similarly, removing the independent set $\{1, 7\}$ from Graph 17 results in a 5-cycle with a chord, that is also a comparability graph. So, by Theorem 7 both graphs are permutationally 1-11-representable. □

Note that there exist shorter non-permutational 1-11-representants for these graphs found using software:

$$w_{12} = 4573275465142631256 \qquad w_{17} = 23474625731436251645.$$

4 Concluding Remarks

In this paper we introduce new tools to study 1-11-representable graphs, which allows to confirm 1-11-representability of Chvátal graph, Mycielski graph, split graphs and graphs whose vertices can be partitioned into a comparability graph and an independent set. Finally, we confirm a claim in [4] that all graphs on at most 7 vertices are 1-11-representable.

It is still an open problem whether each graph is 1-11-representable. Moreover, it is still unknown whether each graph is permutationally 1-11-representable, and towards constructing potential counterexamples, one should look for a graph for which none of the known existing tools is applicable. Note that even if all graphs are (permutationally) 1-11-representable, the constructions of 1-11-representations presented in this paper can still be useful for finding explicit representations of graphs, with an aim towards potential applications.

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Declarations

Conflict of Interest The authors declare no Conflict of interest.

References

1. Bouchet, A.: Caractérisation des symboles croisés de genre nul. C. R. Acad. Sci. Paris **274**, 724–727 (1972)
2. Bouchet, A.: Circle graph obstructions. J. Combin. Theory, Ser. B **60**(1), 107–144 (1994)

3. Chen, H.Z.Q., Kitaev, S., Saito, A.: Representing split graphs by words. Discuss. Math. Graph Theory **42**, 1263 (2022). (**Article ID: 4314**)
4. Cheon, G.-S., Kim, J., Kim, M., Kitaev, S., Pyatkin, A.: On k-11-representable graphs. J. Combin. **10**(3), 491–513 (2019)
5. Chvátal, V.: The smallest triangle-free 4-chromatic 4-regular graph. J. Combin. Theory **9**(1), 93–94 (1970)
6. Even, S., Itai, A.: Queues, stacks and graphs. In: Theory of Machines and Computations (Proc. Internat. Sympos., Technion, Haifa, 1971), pp. 71–86, Academic Press, New York, (1971)
7. Fournier, J.C.: Une caractérisation des graphes de cordes. C. R. Acad. Sci. Paris **286A**, 811–813 (1978)
8. Golumbic, M.C.: Algorithm Graph Theory. Academic Press, New York (1980)
9. Iamthong, K.: Word-representability of split graphs generated by morphisms. Discrete Appl. Math. **314**, 284–303 (2022)
10. Jones, M., Kitaev, S., Pyatkin, A., Remmel, J.: Representing graphs via pattern avoiding words. Electron. J. Combin. **22**(2), 20 (2015). (**#P2.53**)
11. Halldórsson, M.M., Kitaev, S., Pyatkin, A.: Semi-transitive orientations and word-representable graphs. Discr. Appl. Math. **201**, 164–171 (2016)
12. Hammer, P.L., Simeone, B.: The Splittance of a Graph. Combinatorica. **1**, 275–284 (1981)
13. Kitaev, S., Long, Y., Ma, J., Wu, H.: Word-representability of split graphs. J. Combin. **12**(4), 725–746 (2021)
14. Kitaev, S., Lozin, V.: Words and Graphs. Springer (2015)
15. Kitaev, S., Pyatkin, A.: On representable graphs. J. Autom. Lang. Combin. **13**(1), 45–54 (2008)
16. Kitaev, S., Pyatkin, A.: On semi-transitive orientability of triangle-free graphs. Discuss. Math.Graph Theory **43**, 533 (2023). (**ID: 4621**)
17. Kitaev, S., Pyatkin, A.: On semi-transitive orientability of split graphs. Inform. Process. Lett. **184**, 106435 (2024)
18. Kitaev, S., Sun, H.: Human-verifiable proofs in the theory of word-representable graph, RAIRO – Theor. Inform. and Appl. 58 (2024), Special Issue: Randomness and Combinatorics -Edited by Luca Ferrari & Paolo Massazza, Art. 10, 10pp.
19. Lekkerkerker, C.G., Boland, J.C.: Representation of a finite graph by a set of intervals on the real line. Fund. Math. **51**, 45–64 (1962)
20. Mycielski, J.: Sur le coloriage des graphes. Colloq. Math. **3**(2), 161–162 (1955)
21. Spinrad, J.P.: Recognition of circle graphs. J. Algorithms **16**(2), 264–282 (1994)
22. Spinrad, J.P.: Efficient graph representations. Fields Inst. Monogr. **19**, 342 (2003). (**ISBN 978-1-4704-3146-4**)