Concurrent Games over Relational Structures: The Origin of Game Comonads

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Abstract

Spoiler-Duplicator games are used in finite model theory to examine the expressive power of logics. Their strategies have recently been reformulated as coKleisli maps of game comonads over relational structures, providing new results in finite model theory via categorical techniques. We present a novel framework for studying Spoiler-Duplicator games by viewing them as event structures. We introduce a first systematic method for constructing comonads for all one-sided Spoiler-Duplicator games: game comonads are now realised by adjunctions to a category of games, generically constructed from a comonad in a bicategory of game schema (called signature games). Maps of the constructed categories of games are strategies and generalise coKleisli maps of game comonads; in the case of one-sided games they are shown to coincide with suitably generalised homomorphisms. Finally, we provide characterisations of strategies on two-sided Spoiler-Duplicator games; in a common special case they coincide with spans of event structures.

1 Introduction

Spoiler-Duplicator games are used in finite model theory to study the expressive power of logical languages. These games make use of resources (*e.g.* number of rounds) which correspond to different restrictions on fragments of first-order logic (*e.g.* quantifier rank). One may venture beyond Spoiler-Duplicator games to a framework that supports both resources and structure. In the semantic world, concurrent games and strategies based on event structures have been demonstrated to achieve that goal: they can be composed and also support quantitative extensions, *e.g.* to probabilistic and quantum computation [7–9, 20, 32], as well as resource usage [5]. As a result, concurrent games and strategies provide a rich arena in which structure meets power—a line of research that became prominent with the paper of Abramsky, Dawar and Wang [1] on the pebbling comonad in finite model theory.

That seminal work has been followed up by many more examples of "game comonads," for example [1, 3, 12, 19]. Though obtaining the comonads has been something of a mystery, largely because algebraic structures do not in themselves express the operational features of games. To some extent this has been addressed through *arboreal categories* [2, 4] which attempt to axiomatise the Eilenberg-Moore coalgebras of the known examples; as their name suggests arboreal categories and their "covers" impose a treelike behaviour on relational structures so making them amenable to operational concerns. But, as Bertrand Russell joked, postulation has "the advantages of theft over honest toil" [23]. And, the coalgebras of newer comonads such as the all-in-one pebble-relation comonad [19] are not arboreal and consequently the original definition is being readdressed in *linear arboreal categories* [2].

Our contribution is to provide a sweeping definition of Spoiler-Duplicator games out of which game comonads, both old and new, their coalgebras and characterisations are derived as general theorems. To do so, we introduce concurrent games and strategies over relational structures. Through their foundation in event structures, concurrent games and strategies represent closely the operational nature of games, their interactivity, dependence, independence, and conflict of moves. This makes the usual, largely informal and implicit description of Spoiler-Duplicator games formal and explicit. In particular, the concurrency expressible in event structures plays an essential role in enforcing the independence of moves.

A game will be represented by an event structure in which each event stands for a move occurrence of Player (Duplicator) or Opponent (Spoiler); a move is associated with a constant or variable. Positions of the game are represented by configurations of the event structure. With respect to a many-sorted relational structure $\mathcal{A}$, a strategy (for Player) assigns values in $\mathcal{A}$ to Player variables (those associated with Player moves) in response to *challenges* or assignments of values in $\mathcal{A}$ to variables by Opponent. A winning condition specifies those configurations, with latest assignments, which represent a win for Player. A strategy is winning if it results in a winning configuration of the game regardless of Opponent's strategy.

We exploit that such games form a bicategory to exhibit a deconstruction of traditional Spoiler-Duplicator games; the choice of Spoiler-Duplicator game is parameterised by a comonad δ in the bicategory— δ specifies the allowed pattern of interaction between Spoiler and Duplicator. Special cases follow from particular choices of δ. They include Ehrenfeucht-Fraïssé games, pebbling games, such as the k-pebble game [1] and the all-in-one k-pebble game [19]; and their versions on transition systems, *i.e.* simulation and trace inclusion.

With respect to a choice of δ, we obtain a category of strategies on a Spoiler-Duplicator game $\mathbf{SD}_\delta$; and, for suitable one-sided games, a comonad $\mathrm{Rel}_\delta(\_)$ on the category of appropriate relational structures. For general δ we provide characterisations of the strategies $\mathbf{SD}_\delta$. In the one-sided case, the strategies $\mathbf{SD}_\delta$ correspond to δ-homomorphisms—strictly more general than coKleisli maps of the comonad $\mathrm{Rel}_\delta(\_)$; we delineate when $\mathbf{SD}_\delta$ coincides with the coKleisli category. We characterise the Eilenberg-Moore coalgebras of $\mathrm{Rel}_\delta(\_)$ as relational structures which also possess a certain event-structure shape—the coalgebras are not always (linear) arboreal categories. The derived comonads on relational structures coincide with those in the literature we know, particularly for the examples mentioned. This provides a systematic method for constructing comonads for one-sided Spoiler-Duplicator games, one in which game comonads are now realised by adjunctions to a category of

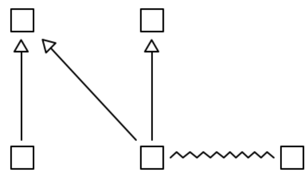


Figure 1: An event structure

games; this reconciles the composition of strategies as coKleisli maps, prevalent in the category-theoretic approach to finite model theory, with the more standard composition of strategies following the paradigm of Conway and Joyal [11, 16].

2 Preliminaries

We cover the basic definitions of games as event structures [22].

2.1 Event structures. An *event structure* is a triple $E = (|E|, \leq, \#)$, where $|E|$ is a set of *events*, $\leq$ is a partial order of *causal dependency* on E, and $\#$ is a binary irreflexive symmetric relation, called the *conflict relation*, such that $\{e' \mid e' \leq e\}$ is finite and $e \# e' \leq e''$ implies $e \# e''$. Event structures have an accompanying notion of a state or history: a *configuration* is a (possibly infinite) subset $x \subseteq |E|$ which is *consistent*, *i.e.* $\forall e, e' \in x.\ (e, e') \notin \#$, and *down-closed*, *i.e.* $e' \leq e \in x$ implies $e' \in x$. We denote by $C(E)$ the set of configurations of E and by $C(E)^o$ the subset of finite configurations.

Two events e, e' are called *concurrent* when they are neither in conflict nor causally related; we denote this by $e\ co\ e'$. In games, the relation $e \rightarrowtriangle e'$ of *immediate dependency*, meaning that e and e' are distinct with $e \leq e'$ and no events are in between, plays an important role. We write $[X]$ for the down-closure of a subset of events X; when X is a singleton $\{e\}$, we denote its down-closure by $[e]$. Note that $[e]$ is necessarily a configuration. Two events e and e' are in *immediate conflict* whenever $e \# e'$ and if $e \geq e_1 \# e_1' \leq e'$ then $e = e_1$ and $e' = e_1'$. To avoid ambiguity, we sometimes distinguish the precise event structure with which a relation is associated and write, for instance, $\leq_E$, $\rightarrowtriangle_E$, $\#_E$ and co_E.

In diagrams, events are depicted as squares, immediate causal dependencies by arrows and immediate conflicts by squiggly lines. For example, the diagram in Figure 1 represents an event structure with five events. The event to the far-right is in immediate conflict with one event—as shown, but in non-immediate conflict with all events besides the one on the lower far-left, with which it is concurrent.

Let E and E' be event structures. A *map of event structures* $f : E \to E'$ is a partial function $f : |E| \rightharpoonup |E'|$ on events such that, for all $x \in C(E)$, we have $fx \in C(E')$, and if $e, e' \in x$ and $f(e) = f(e')$, with both defined, then $e = e'$. Maps of event structures compose as partial functions. Notice that for a total map f, the condition on maps says that f is *locally injective*, in the sense that w.r.t. every configuration x of the domain, the restriction of f to a function from x is injective; the restriction to a function from x to fx is thus bijective.

Although a map $f : E \to E'$ of event structures does not generally preserve causal dependency, it does reflect causal dependency locally: whenever $e, e' \in x \in C(E)$ and $f(e') \leq f(e)$, with $f(e)$ and $f(e')$ both defined, then $e' \leq e$. Consequently, f preserves the concurrency relation: if $e\ co\ e'$ then $f(e)\ co\ f(e')$, when defined. A total map of event structures is called *rigid* when it preserves causal dependency.

Proposition 2.1 ([29]). *A total map $f : E \to E'$ of event structures is rigid iff for all $x \in C(E)$ and $y \in C(E')$, if $y \subseteq fx$, then there exists $z \in C(E)$ such that $z \subseteq x$ and $fz = y$. Moreover, z is necessarily unique.*

Open maps give a general treatment of bisimulation; they are defined by a path-lifting property [17]. Here a characterisation of open maps of event structures will suffice.

Proposition 2.2 (open map [17]). *A total map $f : E \to E'$ of event structures is* open *iff it is rigid and for all $x \in C(E)$ and $y \in C(E')$, $fx \subseteq y$ implies that there exists $z \in C(E)$ such that $x \subseteq z$ and $fz = y$.*

Event structures possess a hiding operation. Let E be an event structure and V a subset of its events. The *projection* of E to V, written $E{\downarrow}V$, is defined to be the event structure $(V, \leq_V, \#_V)$ in which the relations of causal dependency and conflict are simply restrictions of those in E [appendix A.1].

2.2 Concurrent games and strategies. The driving idea of concurrent games is to replace the traditional role of game trees by that of event structures [22]. Both games and strategies will be represented by an *event structure with polarity (esp)*, which comprises (A, pol_A), where A is an event structure and $pol_A : A \to \{+, -, 0\}$ is a polarity function ascribing a polarity + to Player, polarity − to opponent, and polarity 0 to neutral events. The events of A correspond to (occurrences of) moves. Events of neutral polarity arise in a play between a strategy and a counter-strategy. Maps between esps are those of event structures that preserve polarity. A *game* is represented by an event structure with polarities restricted to + or −, with no neutral events.

In an event structure with polarity, for configurations x and y, we write $x \subseteq^- y$ (resp. $x \subseteq^+ y$) to mean inclusion in which all the intervening events $(y \setminus x)$ are Opponent (resp. Player) moves. For a subset of events X we write X^+ and X^- for its restriction to Player and Opponent moves, respectively. There are two fundamentally important operations on games: Given a game A, the *dual game* $A^\perp$ is the same as A but with the polarities reversed. The other operation, the *simple parallel composition* $A \| B$, is achieved by simply juxtaposing A and B; ensuring that two events are in conflict only if they are in conflict in a component. Any configuration x of $A \| B$ decomposes into $x_A \| x_B$, where x_A and x_B are configurations of A and B, respectively.

Definition 2.3 (strategy). A *strategy* in a game A is an esp S together with a total map $\sigma : S \to A$ of esps, where

- if $\sigma x \subseteq^- y$, for $x \in C(S)$ and $y \in C(A)$, then there exists a unique $x' \in C(S)$ such that $x \subseteq x'$ and $\sigma x' = y$;
- if $s \rightarrowtriangle_S s'$ & $(pol(s) = +$ or $pol(s') = -)$, then $\sigma(s) \rightarrowtriangle_A \sigma(s')$.

The strategy is *deterministic* iff all immediate conflict in S is between Opponent events. We say the strategy is *rigid/open* according as the map σ is rigid/open.

The first condition is called *receptivity*. It ensures that the strategy is open to all moves of Opponent permitted by the game. The second condition, called *innocence* in [13], ensures that the only additional immediate causal dependencies a strategy can enforce beyond those of the game are those in which a Player move causally depends on Opponent moves. An important feature of strategies is that they admit composition to form a bicategory [22] [appendix A.1]. A *map of strategies* $f : \sigma \Rightarrow \sigma'$, where $\sigma : S \to A$ and $\sigma' : S' \to A$, is a map $f : S \to S'$ such that $\sigma = \sigma' f$; this determines when strategies are isomorphic. Note that such a map f of strategies is itself receptive and innocent and so a strategy in S'.

Following Conway and Joyal [11, 16], we define a *strategy from a game A to a game B* as a strategy in the game $A^\perp \| B$. The conditions of receptivity and innocence precisely ensure that copycat strategies as follow behave as identities w.r.t. composition [22]. Given a game A, the *copycat strategy* $\text{æ}_A : \mathbb{C}_A \to A^\perp \| A$ is an instance of a strategy from A to A. The event structure $\mathbb{C}_A$ is based on the idea that Player moves in one component of the game $A^\perp \| A$ always copy corresponding moves of Opponent in the other component. For $c \in A^\perp \| A$ we use $\bar{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component. The event structure $\mathbb{C}_A$ comprises $A^\perp \| A$ and its causal dependencies with extra causal dependencies $\bar{c} \le c$ for all events c such that $pol_{A^\perp \| A}(c) = +$; this generates a partial order. Two events in $\mathbb{C}_A$ are in conflict if they now causally depend on events originally in conflict in $A^\perp \| A$. Figure 2 illustrates $\mathbb{C}_A$ when A is $\boxminus \rightarrowtriangle \boxplus$. The map æ_A acts as the identity function on events. The copycat strategy on a game A is deterministic iff A is *race-free*, *i.e.* there is no immediate conflict between a Player and an Opponent move [30].

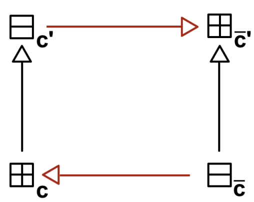


Figure 2: A copycat strategy

A *winning condition* of a game A is a subset W_A of configurations $\mathcal{C}(A)$. Informally, a strategy (for Player) is winning if it always prescribes moves for Player to end up in a winning configuration regardless of what opponent does. Formally, $\sigma : S \to A$ is a *winning strategy* in (A, W_A), for a concurrent game A with winning condition $W_A \subseteq \mathcal{C}(A)$, if σx is in W_A for all +-maximal configurations x of S; a configuration is +-maximal if the only additional moves enabled at it are those of Opponent. That σ is winning can be shown equivalent to all plays of σ against every counter-strategy of Opponent result in a win for Player [10, 34].

For the dual of a game with winning condition (A, W_A), we again reverse the roles of Player and Opponent, and take its winning condition to be the set-complement of W_A, *i.e.* $(A, W_A)^\perp = (A^\perp, \mathcal{C}(A) \setminus W_A)$. In a parallel composition of two games with winning conditions, we deem a configuration winning if its component in either game is winning: $(A, W_A) \| (B, W_B) := (A \| B, W)$, where $W = \{x \in \mathcal{C}(A \| B) \mid x_A \in W_A \text{ or } x_B \in W_B\}$. With these extensions, we take a winning strategy from a game (A, W_A) to a game (B, W_B) to be a winning strategy in the game $A^\perp \| B$, *i.e.* a strategy for which a win in A implies a win in B. Whenever games are race-free, copycat is a winning strategy; moreover, the composition of winning strategies is a winning strategy [10, 34].

3 Games over relational structures

We are concerned with games in which Player or Opponent moves instantiate variables (in a set V) with elements of a relational structure, subject to moves including challenges (in a set C) by either player. By restricting the size of the set of variables V, we can bound the memory intrinsic to a game, in the manner of pebble games [14].

We will work with *many-sorted relational structures* because it will be useful to consider different relational structures, say $\mathcal{A}$ and $\mathcal{B}$, in parallel, via their sum $\mathcal{A} + \mathcal{B}$; the sum has sorts given by the disjoint union of the sorts of $\mathcal{A}$ and $\mathcal{B}$, with relations and arities lifted from those of the components [appendix B.1]. A *game signature* (Σ, C, V) comprises Σ, a *many-sorted* relational signature including equality for each sort; a set C of *constants*; and a set $V = \{\alpha, \beta, \gamma, \dots\}$ of sorted *variables*, disjoint from C, with sorts in Σ – we write $sort(\alpha)$ for the sort of $\alpha \in V$. We assume each sort in a Σ-relational structure is associated with a nonempty set.

Definition 3.1 (signature game). A *signature game* with signature (Σ, C, V), alternatively a (Σ, C, V)*-game* G, comprises a race-free esp $G = (|G|, \le_G, \#_G, pol_G)$ and

- a *labelling* $\mathrm{vc} : |G| \to V \cup C$ such that $g \; co \; g'$ implies $\mathrm{vc}(g) \ne \mathrm{vc}(g')$; and $\mathrm{vc}(g) = \mathrm{vc}(g')$ implies $pol_G(g) = pol_G(g')$;
- a *winning condition* W_G, which is an assertion in the free logic over (Σ, C, V). (Free logic is explained in detail in the Addendum, Section 9.)

Events in G are either V*-moves* or C*-moves*. Note the elements of C and V appearing in a (Σ, C, V)*-game* G are associated with a unique polarity. We shall specify the set of moves in G labelled in $V_0 \subseteq V$ by $|G|_{V_0}$. We will consider G to be played over a Σ-structure $\mathcal{A}$; the pair $(G, \mathcal{A})$ is called *a game over a* Σ*-structure* $\mathcal{A}$.

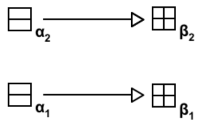
Example 3.2 (a multigraph homomorphism game). The signature game in Figure 3 is the multigraph homomomorphism game for two graphs $\mathcal{A}$ and $\mathcal{B}$ with red and green edge relations; so its signature has two sorts, one for $\mathcal{A}$ and one for $\mathcal{B}$, each with their equality, together with red and green edge relations. The game comprises four events assigned variables, α_1, α_2 with sort that of $\mathcal{A}$ and β_1, β_2 with sort that of $\mathcal{B}$. Its winning condition W is written under the illustration of the game. Looking ahead to Definition 4.3, a rigid deterministic winning strategy in this game over $\mathcal{A} + \mathcal{B}$ is necessarily open and corresponds to a homomorphism $\mathcal{A}$ to $\mathcal{B}$.

Proposition 3.3. *Given a* (Σ, C, V)*-game, if for two events* g, g' *in a configuration* $\mathrm{vc}(g) = \mathrm{vc}(g')$*, then they are causally dependent, i.e.* $g \le_G g'$ *or* $g' \le_G g$.

Consequently, the set of events in a configuration of G which is labelled with the same variable or constant is totally ordered w.r.t. causal dependency. This fact is reflected in an esp S via a total map $\sigma : S \to G$. For $x \in \mathcal{C}(S)$, it is sensible to define

$$\mathrm{last}_S(x) := \big\{ s \in x \mid \forall s' \in x.\ \big(s \le_S s' \text{ and } \mathrm{vc}(\sigma(s)) = \mathrm{vc}(\sigma(s')) \big) \implies s = s' \big\}.$$

We shall write last_G when σ is the identity on G.



$$W \equiv (\alpha_1 =_{\mathcal{A}} \alpha_2 \to \beta_1 =_{\mathcal{B}} \beta_2) \land$$
$$(red_{\mathcal{A}}(\alpha_1, \alpha_2) \to red_{\mathcal{B}}(\beta_1, \beta_2)) \land$$
$$(green_{\mathcal{A}}(\alpha_1, \alpha_2) \to green_{\mathcal{B}}(\beta_1, \beta_2))$$

Figure 3: The two-edge multigraph homomorphism game

Let $(G, \mathcal{A})$ be a (Σ, C, V)-game over a Σ-structure $\mathcal{A}$. A strategy in G will assign values in $\mathcal{A}$ to V-moves, providing a dynamically changing environment for the variables—their values being the latest assigned. The dynamic environment, which can both extend and update the set of labelled events, is captured by the notion of an *instantiation*.

Definition 3.4 (instantiation). An *instantiation* of G in $\mathcal{A}$ consists of $\sigma : S \to G$, where S is an esp and σ is a total esp map; and a function $\rho : |S|_V \to |A|$, where $|S|_V := \{s \in |S| \mid \mathrm{vc}(\sigma(s)) \in V\}$ and $sort(\rho(s)) = sort(\mathrm{vc}(\sigma(s)))$.

A map from instantiation (σ, ρ), where $\sigma : S \to G$ and $\rho : |S|_V \to |\mathcal{A}|$, to instantiation (σ', ρ'), where $\sigma' : S' \to G$ and $\rho' : |S'|_V \to |\mathcal{A}|$, is a total esp map $f : S \to S'$ such that the following diagrams commute:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "\sigma"'] & S' \arrow[d, "\sigma'"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} |S|_V \arrow[r, "f"] \arrow[dr, "\rho"'] & |S'|_V \arrow[d, "\rho'"] \\ & |\mathcal{A}| \end{tikzcd}$$

We will shortly construct a universal instantiation for a (Σ, C, V)-game over a relational structure. Since total maps of event structures reflect causal dependency locally, the notion of last moves associated with variables or constants remains the same, whether according to the game or the instantiation, and indeed across maps of instantiations.

Proposition 3.5. *Let S be an esp and let $\sigma : S \to G$ be a total esp map. Then, $\sigma \mathrm{last}_S(x) = \mathrm{last}_G(\sigma x)$. Moreover, if f is a map of instantiations from $(\sigma : S \to G, \rho)$ to $(\sigma' : S' \to G, \rho')$, then $f \mathrm{last}_S(x) = \mathrm{last}_{S'}(fx)$.*

An instantiation (σ, ρ), with $\sigma : S \to G$ and $\rho : |S|_V \to |A|$, forms a model of the free logic. It specifies through $x \models_{\sigma,\rho} \varphi$ those configurations x of S which satisfy φ. A novelty is that a variable's value at x is defined by its latest move in $\mathrm{last}_S(x)$, if there is such. The naturalness of free logic stems from there being undefined terms. A variable may label an event in one configuration and not do so in another; without free logic, allowing undefined terms, we would have to cope with undefinedness in a more ad-hoc way. Moreover, traditional predicate logic assumes a non-empty universe, which would make dealing with the empty configurations an issue.

Our semantics of the free logic ensures:

Proposition 3.6. *Let f be a map of instantiations of G in $\mathcal{A}$, from (σ, ρ) to (σ', ρ'). Let φ be an assertion in free logic. Then, $x \models_{\sigma,\rho} \varphi$ iff $fx \models_{\sigma',\rho'} \varphi$, for every configuration x.*

4 Strategies over structures

A signature game over a Σ-structure $(G, \mathcal{A})$ expands to a (traditional) concurrent game $expn(G, \mathcal{A})$ (Section 2.2), from which we can derive strategies in games over relational structures. The idea is that each move associated with a variable $\alpha \in V$, denoted $\Box_\alpha$, is expanded to all its conflicting instances $\Box_\alpha^{a_1} \sim\!\sim\!\sim \Box_\alpha^{a_2} \sim\!\sim\!\sim \ldots$, where a_i are elements of $\mathcal{A}$ with matching sort. Precisely, paying more careful attention to causal dependencies—that such expansions can depend on earlier expansions, we arrive at the following definition.

Definition 4.1 (expansion). The *expansion* of a (Σ, C, V)-game over a Σ-structure $(G, \mathcal{A})$ is the esp $expn(G, \mathcal{A})$, with

- events defined as pairs (g, γ), where $\gamma : [g]_V \to |\mathcal{A}|$ assigns an element of $\mathcal{A}$ of the correct sort to each V-move on which $g \in |G|$ causally depends;
- causal dependency $\leq$ defined as $(g', \gamma') \leq (g, \gamma)$ iff $g' \leq_G g$ and $\gamma' = \gamma \upharpoonright [g']_V$;
- conflict relation # defined as $(g, \gamma) \# (g', \gamma')$ iff $g \#_G g'$ or $\exists g'' \leq_G g, g' .\, \gamma(g'') \neq \gamma'(g'')$;
- polarity map $pol(g, \gamma) = pol_G(g)$ inherited from G.

There is an obvious generalisation to an expansion w.r.t. $V_0 \subseteq V$; the role of variables V above is simply replaced by the subset V_0. In Section 6, we shall use a *partial expansion* w.r.t. Opponent variables.

Let $\epsilon_{G,\mathcal{A}} : |expn(G, \mathcal{A})| \to |G|$ be the map such that $\epsilon_{G,\mathcal{A}} : (g, \gamma) \mapsto g$; it is an open map of event structures since each sort of $\mathcal{A}$ is nonempty. Let $\rho_{G,\mathcal{A}} : |expn(G, \mathcal{A})|_V \to |\mathcal{A}|$ be the map such that $\rho_{G,\mathcal{A}}(g, \gamma) = \gamma(g)$ for $g \in |G|$ with $\mathrm{vc}(g) \in V$. The *winning condition* of the expansion is

$$W := \{x \in C(expn(G, \mathcal{A})) \mid x \models_{\epsilon_{G,\mathcal{A}}, \rho_{G,\mathcal{A}}} W_G\}.$$

The expansion provides a universal instantiation of $(G, \mathcal{A})$.

Lemma 4.2 (universal instantiation)**.** *The pair $(\epsilon_{G,\mathcal{A}}, \rho_{G,\mathcal{A}})$ forms an* instantiation *of G in $\mathcal{A}$. For each instantiation $(\sigma : S \to G, \rho)$ of G in $\mathcal{A}$, there is a unique map σ_0 of instantiations from (σ, ρ) to $(\epsilon_{G,\mathcal{A}}, \rho_{G,\mathcal{A}})$ such that the following diagrams commute:*

$$\begin{tikzcd} S \arrow[r, dashed, "\sigma_0"] \arrow[dr, "\sigma"'] & expn(G, \mathcal{A}) \arrow[d, "\epsilon_{G,\mathcal{A}}"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} |S|_V \arrow[r, dashed, "\sigma_0"] \arrow[dr, "\rho"'] & |expn(G, \mathcal{A})|_V \arrow[d, "\rho_{G,\mathcal{A}}"] \\ & |\mathcal{A}| \end{tikzcd}$$

Moreover, $\epsilon_{G,\mathcal{A}}$ is open; σ is rigid iff σ_0 is rigid; and σ is open iff σ_0 is open.

From the universality of Lemma 4.2 and Proposition 3.6, the truth of an assertion w.r.t. an instantiation reduces to its truth w.r.t. the universal instantiation $(\epsilon_{G,\mathcal{A}}, \rho_{G,\mathcal{A}})$ of the expansion. We shall write $x \models \varphi$ for $x \models_{\epsilon_{G,\mathcal{A}}, \rho_{G,\mathcal{A}}} \varphi$. Given $x \in C(G)$ and a sort-respecting $\rho : |x|_V \to |\mathcal{A}|$, write $x[\rho] \models \varphi$ when $x \models_{j,\rho} \varphi$ w.r.t. the instantiation (j, ρ) based on the inclusion map $j : x \hookrightarrow G$; we shall identify $x[\rho]$ with the configuration of $expn(G, \mathcal{A})$ obtained via universality as an image of x.

We can now give a central definition of this paper:

Definition 4.3 (strategy in a game over a Σ-structure). Let $(G, \mathcal{A})$ be a game over a Σ-structure. A *strategy in* $(G, \mathcal{A})$ is an instantiation (σ, ρ) of G in $\mathcal{A}$ for which σ_0, the unique map of instantiations from Lemma 4.2, is a concurrent strategy in $expn(G, \mathcal{A})$.

Say (σ, ρ) is *rigid/open* according as σ_0 is rigid/open. Say (σ, ρ) is *winning* if σ_0 is winning in $expn(G, \mathcal{A})$ and *deterministic* if σ_0 is deterministic.

An explicit characterisation of strategies in $(G, \mathcal{A})$ can be given [appendix C.1]. As expected, a *strategy* in a game over $\mathcal{A}$ assigns values in $\mathcal{A}$ to Player moves of the game G in answer to assignments of Opponent. It satisfies the same condition of innocence as concurrent strategies while the original condition of receptivity is generalised so that Opponent is free to make arbitrary assignments to accessible variables of negative polarity [15].

Strategies in $(G, \mathcal{A})$, a game over a Σ-structure, are related by maps of instantiations. Lemma 4.2 provides an isomorphism between the ensuing category of strategies in $(G, \mathcal{A})$ and the category of strategies in $expn(G, \mathcal{A})$.

4.1 The bicategories of signature games. Signature games are forms of game schema, useful for describing possible patterns of play without committing to particular instantiations. Let G be a (Σ, C, V)-game. Its *dual* $G^\perp$ is the (Σ, C, V)-game obtained by reversing polarities and negating the winning condition. If H is a (Σ', C', V')-game, the *parallel composition* $G \| H$ is the obvious $(\Sigma + \Sigma', C + C', V + V')$-game comprising the parallel juxtaposition of event structures with winning condition the disjunction of those of G and H (strictly, after a renaming of constants and variables associated with the sums $C + C'$ and $V + V'$). Consequently $G^\perp \| H$ has winning condition the implication $W_G \to W_H$. The bicategory of signature games **Sig** has as arrows from G to H concurrent strategies $\delta : D \to G^\perp \| H$ —strictly speaking in the esp of $G^\perp \| H$ —with 2-cells and composition inherited from that of concurrent strategies. Notice the esp D inherits the form of a signature game from $G^\perp \| H$ and itself describes a pattern of strategical play, now from G to H. The bicategory **Sig** describes schematic patterns of play between signature games.

Let $(G, \mathcal{A})$, $(H, \mathcal{B})$ be games over relational structures. A *winning strategy from* $(G, \mathcal{A})$ *to* $(H, \mathcal{B})$, written $(\sigma, \rho) : (G, \mathcal{A}) \mathrel{\longrightarrow\!\!\!\!\!\!\!+} (H, \mathcal{B})$, is a winning strategy in the game $(G^\perp \| H, \mathcal{A} + \mathcal{B})$. Lemma 4.2 provides an isomorphism between strategies over structures and concurrent strategies, so providing a bicategory **Red** of signature games and strategies instantiated over relational structures [appendix C.2]. The identity, copycat strategy of $(G, \mathcal{A})$, is the instantiation $(\text{æ}_{G,A}, \gamma_{G,A})$ which corresponds to the copycat strategy of $expn(G, \mathcal{A})$. Because winning strategies compose, a winning strategy (σ, ρ) in **Red** from $(G, \mathcal{A})$ to $(H, \mathcal{B})$ is a form of *reduction.* It reduces the problem of finding a winning strategy in $(H, \mathcal{B})$ to finding a winning strategy in $(G, \mathcal{A})$: a winning strategy in $(G, \mathcal{A})$ is a winning strategy from the empty game and structure $(\emptyset, \emptyset)$ to $(G, \mathcal{A})$; its composition with σ is a winning strategy in $(H, \mathcal{B})$.

4.2 Examples of games. As remarked above, w.r.t. a strategy in **Sig** such as æ_G, the copycat strategy on a signature game G, we obtain a signature game, in this case based on the esp $\mathbb{CC}_G$, describing a pattern of play of strategies from G to G. All the following examples illustrate such copycat patterns of play.

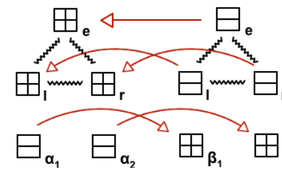


Figure 4: The multigraph game as a $\mathbb{CC}_G$ game

Example 4.4 (multigraph game revisited). Example 3.2 can be recast as a copycat signature game $\mathbb{CC}_G$ w.r.t. a signature game G with winning condition W. This decomposes the original multigraph game to a game *from G to G*, so a form of function space. The resulting game has winning condition $W_A \to W_B$, where in W_A the variables of W have been replaced by α_1, α_2 and in W_B by β_1, β_2. Here we are adopting a convention maintained throughout our examples. If G has variables V then $G^\perp \| G$ and $\mathbb{CC}_G$ have variables the disjoint union $V + V$: we shall write $\alpha, \alpha_1, \alpha_2, \cdots$ for the variables in the left component, and $\beta, \beta_1, \beta_2, \cdots$ for variables in that on the right. To ensure that the winning condition $(W_A \to W_B)$ implies the required homomorphism property, that each edge relation is preserved individually, we adjoin *challenge events* associated with each edge relation—a small adjustment needed for compositionality.

In detail, take G to comprise challenge events, three constant-labelled Opponent moves e, r and g, pairwise in conflict with each other, in parallel with two variable-labelled concurrent Player moves $\boxplus_{\beta_1}$ and $\boxplus_{\beta_2}$. Its winning condition is

$$W_B \equiv (\mathbf{E}(e) \to \beta_1 = \beta_2) \wedge \\ (\mathbf{E}(r) \to red(\beta_1, \beta_2)) \;\wedge\; (\mathbf{E}(g) \to green(\beta_1, \beta_2)).$$

The game $\mathbb{CC}_G$ is illustrated in Figure 4. An open winning deterministic strategy of $\mathbb{CC}_G$ over $\mathcal{A} + \mathcal{B}$, for multigraphs $\mathcal{A}$ and $\mathcal{B}$, has the form (σ, ρ) with $\sigma : S \to \mathbb{CC}_G$. For any +-maximal configuration x of S, we have $x \models_{\sigma,\rho} W_A \to W_B$. If x contains a challenge $\boxminus_e$, $\boxminus_r$ or $\boxminus_g$, by +-maximality it must also contain the corresponding $\boxplus_e$, $\boxplus_r$ or $\boxplus_g$. Hence if also $x \models_{\sigma,\rho} W_A$, *i.e.* it establishes the equality or edge between α_1 and α_2 corresponding to the challenge, it will establish the corresponding equality or edge between β_1 and β_2. More completely, open winning deterministic strategies of $\mathbb{CC}_G$ over $\mathcal{A} + \mathcal{B}$ correspond to homomorphisms from $\mathcal{A}$ to $\mathcal{B}$. The key observation here is that because (σ, ρ) is deterministic and open, each assignment $\boxminus^a_{\alpha_i}$ in any +-maximal configuration of S always determines the same assignment $\boxplus^b_{\beta_i}$, and moreover, because equality is preserved, this assignment is the same whether $i = 1$ or $i = 2$; thus (σ, ρ) determines a function from $|\mathcal{A}|$ to $|\mathcal{B}|$ which, as argued above also preserves red and green edges.

Example 4.5 (k-pebble game). The (one-sided) k-pebble game [18] on two relational structures $\mathcal{A}$ and $\mathcal{B}$ is prominent in the field of finite model theory and the study of constraint satisfaction problems. It can be explained as a copycat signature game $\mathbb{CC}_G$ w.r.t. a signature game G. Thus $\mathbb{CC}_G$ describes a pattern of strategy between a copy $G^\perp$ associated with the structure $\mathcal{A}$ —accordingly its variables are α_j, for $j \in \{1, \ldots, k\}$ —and a copy G associated with $\mathcal{B}$ —its variables are β_j, for $j \in \{1, \ldots, k\}$ (Figure 5). The right copy G describes sequences of potential assignments by Duplicator (Player) to variables β_j as well as *challenge events* of Spoiler (Opponent),

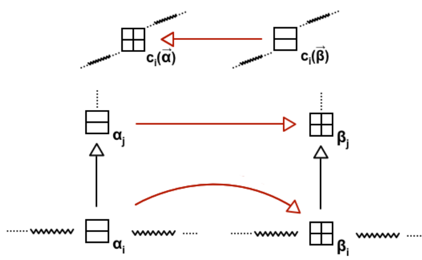


Figure 5: The k-pebble game

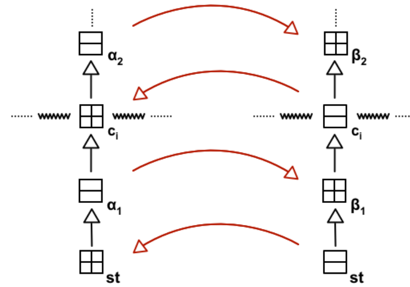


Figure 6: The simulation game

labelled with a constant $c_i(\vec{\beta})$, associated with realising a relation $R_i(\vec{\beta})$. Challenge events are all in conflict with each other and concurrent with all other events; again, the concurrency supported by event structures is essential. The winning condition of G is

$$W_B \equiv \bigwedge_{i,\vec{\beta}} \mathbf{E}(c_i(\vec{\beta})) \rightarrow R_i(\vec{\beta}).$$

The conjunction above is over all pairs $i, \vec{\beta}$ such that $R_i(\vec{\beta})$ is well-sorted. Its existence predicate $\mathbf{E}(c_i(\vec{\beta}))$ is true only at configurations in which an event labelled $c_i(\vec{\beta})$ has occurred. So W_B is true precisely at a configuration where any occurrence of a challenge $c_i(\vec{\beta})$ is accompanied by latest instantiations which make $R_i(\vec{\beta})$ true in $\mathcal{B}$. The dual, left copy of $G^\perp$ has variables α_i, awaiting an assignment in $\mathcal{A}$ by Spoiler (Opponent), and challenge events labelled $c_i(\vec{\alpha})$ which are now moves of Duplicator (Player). Its winning condition is $\neg W_A$ where $W_A \equiv \bigwedge_{i,\vec{\alpha}} \mathbf{E}(c_i(\vec{\alpha})) \rightarrow R_i(\vec{\alpha})$ —a conjunction over pairs $i, \vec{\alpha}$ with $R_i(\vec{\alpha})$ well-sorted. The signature game $\mathbb{C}_G$ comprises the parallel composition $G^\perp \| G$ with the additional red arrows, further constraining the play so Duplicator awaits the corresponding move of Spoiler. Its winning condition is $W_A \rightarrow W_B$. Its open deterministic winning strategies coincide with those of the traditional k-pebbling game. The arbitrary and independent nature of Spoiler's challenges, events labelled $c_i(\vec{\alpha})$, ensures that in any +-maximal configuration, if $R_i(\vec{\alpha})$ in $\mathcal{A}$ then $R_i(\vec{\beta})$ in $\mathcal{B}$. Its rigid deterministic winning strategies are similar but need not force a Player assignment to $\vec{\beta}$ in response to an assignment of Opponent to $\vec{\alpha}$ for which $R_i(\vec{\alpha})$ fails.

Example 4.6 (simulation game). The simulation game expresses the one-sided version of bisimulation [21]. Its open winning deterministic strategies correspond to simulations. We consider the simulation game (Figure 6) on two transition systems $(\mathcal{A}, a)$ and $(\mathcal{B}, b)$ with start states a and b respectively. The game has two variables for each player. A player makes assignments (in $\mathcal{A}$ for Spoiler and $\mathcal{B}$ for Duplicator) to infinite alternating sequences of their variables intertwined with challenges from the other player; the challenges c_i specify transitions R_i. Distinct sequences are in conflict with each other. The red arrows represent the copycat strategy. The two initial events labelled by constants st are challenges associated with the *Start* predicate used to identify the start states, which we assume have no transitions into them. The simulation game can be constructed as a copycat signature game $\mathbb{C}_G$, this time w.r.t. a signature game G with winning condition W_1 below; the order constraints ensure that only the transitions in the direction of the transition system need be preserved:

$$\begin{aligned}(\mathbf{E}(st) \rightarrow Start(\beta_1)) \ \wedge \bigwedge_i c_i < \beta_2 \rightarrow R_i(\beta_1, \beta_2) \\ \wedge \bigwedge_i c_i < \beta_1 \rightarrow R_i(\beta_2, \beta_1).\end{aligned}$$

Example 4.7 (Ehrenfeucht-Fraïssé game). As an example of a two-sided game, we present an Ehrenfeucht-Fraïssé game for checking the isomorphism of two Σ-structures. In such a game the aim of Player is to establish a partial bijection in which the relations of the structures are both preserved and reflected. For this reason we extend the "challenge" events we have seen earlier in the k-pebble game with "negative challenges" associated with relations failing to hold; in addition to constants $c_i(\vec{\beta})$, we have constants $nc_i(\vec{\beta})$ in a winning condition

$$W_B \equiv \bigwedge_{i,\vec{\beta}} [(\mathbf{E}(c_i(\vec{\beta})) \rightarrow R_i(\vec{\beta})) \wedge (\mathbf{E}(nc_i(\vec{\beta})) \rightarrow \neg R_i(\vec{\beta})]$$

—the conjunction above is over all pairs $i, \vec{\beta}$ s.t. $R_i(\vec{\beta})$ is well-sorted.

The Ehrenfeucht-Fraïssé game is defined as a copycat signature game $\mathbb{C}_G$, where the (Σ, C, V)-game $G = G_0 \| Ch$ comprises a subgame associated with challenge events Ch in parallel with a subgame G_0. The subgame Ch has events comprising all the constants $c_i(\vec{\beta})$ and $nc_i(\vec{\beta})$ in pairwise conflict with each other—they are all Opponent moves. It is convenient to provide a recursive definition of G_0, borrowing from early ideas on defining event structures recursively in [28], where also the (nondeterministic) sum and prefix operation are introduced. From there, recall the relation $\trianglelefteq$ between event structures: $E' \trianglelefteq E$ iff $|E'| \subseteq |E|$ and the inclusion $E' \hookrightarrow E$ is a rigid map; then $\trianglelefteq$ forms a (large) cpo w.r.t. which operations such as sum, prefix and dual are continuous. Using these techniques, we construct G_0 as the $\trianglelefteq$-least solution to

$$G_0 = \boxminus_l . \Sigma_{\beta \in V} \boxplus_\beta . G_0 + \boxminus_r . \Sigma_{\beta \in V} \boxminus_\beta . G_0^\perp .$$

In G_0, initially Spoiler chooses to "leave"—the move $\boxminus_l$ —or "remain"—the move $\boxminus_r$ —at the current structure. If they remain, they assign a value in the current structure before the game resumes as the dual $G_0^\perp$; whereas if they leave, Duplicator assigns a value in the current structure and the game resumes as G_0.

Through this definition of G, we achieve the following behaviour in $\mathbb{C}_G$ which, recalling $\alpha_G : \mathbb{C}_G \rightarrow G^\perp \| G$, relates assignments in

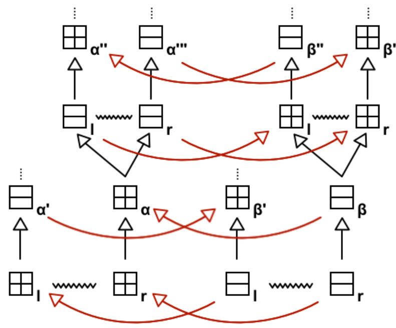


Figure 7: The Ehrenfeucht-Fraïssé game

Σ-structure $\mathcal{A}$ to variables α over $G^\perp$ on the left and assignments in Σ-structure $\mathcal{B}$ to corresponding variables β over G on the right. (For simplicity, in Figure 7 we only draw that part of $\mathbb{C}_G$ involving G_0.) Initially Spoiler chooses to leave $\boxminus_l$ or remain $\boxminus_r$ at the current structure $\mathcal{B}$. If Spoiler chooses to remain, they follow this by assigning a value in the structure $\mathcal{B}$ to a variable, a move $\boxminus_\beta$ over G; through the causal constraints of copycat, these moves are answered by Duplicator moves, the first of which acknowledges the choice of side while the second answers Spoiler's choice in $\mathcal{B}$ with a choice of element in $\mathcal{A}$, a move $\boxplus_\alpha$. This is followed by a choice of Spoiler to leave or remain at $\mathcal{A}$ on the left. If alternatively Spoiler chooses to leave and play on the left, they will make an assignment in $\mathcal{A}$ and the next move on the right will be an assignment by Duplicator in $\mathcal{B}$. This accomplished, Spoiler resumes by again choosing a side on which to play. Open winning strategies of $\mathbb{C}_G$ maintain partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ whatever the moves of Spoiler.

5 Spoiler-Duplicator games

We provide a general method for generating Spoiler-Duplicator games between relational structures; it goes far beyond the examples of the last section, all of which were based on copycat signature games. A category of Spoiler-Duplicator games $\mathbf{SD}_\delta$ is determined by a deterministic, idempotent comonad δ in the bicategory of signature games [25], so a deterministic strategy $\delta \in \mathbf{Sig}(G, G)$ forming an idempotent comonad. The comonad δ describes a pattern of play: its counit ensures that this pattern respects copycat; its idempotence that the pattern is preserved under composition. To characterise the strategies in $\mathbf{SD}_\delta$ we shall introduce the partial expansion of a signature game.

5.1 Spoiler-Duplicator games deconstructed. A deterministic, idempotent comonad δ in **Sig** provides us with a deterministic strategy $\delta : D \to G^\perp \| G$, on a game G with signature (Σ, C, V), with counit $c : \delta \Rightarrow \text{æ}_G$ and invertible comultiplication $d : \delta \Rightarrow \delta \odot \delta$ satisfying the usual comonad laws.

The objects of $\mathbf{SD}_\delta$ are Σ-sorted relational structures $\mathcal{A}$, $\mathcal{B}$, etc. Its maps $\mathbf{SD}_\delta(\mathcal{A}, \mathcal{B})$, written $(\sigma, \rho) : \mathcal{A} \mathrel{-\!\!+\!\!\twoheadrightarrow}_\delta \mathcal{B}$, are deterministic winning strategies $(\sigma, \rho) : (G, \mathcal{A}) \mathrel{-\!\!+\!\!\twoheadrightarrow} (G, \mathcal{B})$, composing as in **Red**, which factor *openly* through δ.[1] To explain this in detail, note that σ is a map $\sigma : S \to G^\perp \| G$. By σ factoring openly through δ, we mean that for some open map σ_1 the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, dashed, "\sigma_1"] \arrow[dr, "\sigma"'] & D \arrow[d, "\delta"] \\ & G^\perp \| G \end{tikzcd}$$

As δ is deterministic, it is mono [30, 34], ensuring a bijective correspondence between strategies $(\sigma, \rho) : \mathcal{A} \mathrel{-\!\!+\!\!\twoheadrightarrow}_\delta \mathcal{B}$ and open deterministic strategies (σ_1, ρ) in $(D, \mathcal{A} + \mathcal{B})$; the correspondence respects the composition of strategies. However, identities aren't in general copycat strategies and have to be constructed as pullbacks of δ along $\text{æ}_{G,B}$, where $(\text{æ}_{G,B}, \gamma_{G,B})$ is the copycat strategy of $(G, \mathcal{B})$:

$$\begin{tikzcd} I_B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\delta"] \\ \mathbb{C}_{G,B} \arrow[r, "\text{æ}_{G,B}"] & G^\perp \| G \end{tikzcd}$$

Define $\iota_B := \delta \circ \pi_2 = \text{æ}_{G,B} \circ \pi_1$ and $\rho_B := \gamma_{G,B} \circ \pi_1$. Provided (ι_B, ρ_B) is winning, it will constitute an identity in the category $\mathbf{SD}_\delta$ [see appendix D.1].

By choosing a suitable δ we can design Spoiler-Duplicator games with different patterns of play and constraints on the way resources, *e.g.* number of variables, are used. For the k-pebble game (Example 4.5) and the simulation game (Example 4.6), a simple choice of comonad δ being a copycat strategy æ_G suffices. Besides δ acting as copycat other behaviours of games can be found in the literature: the all-in-one k-pebble game [19] is a version of the k-pebble game where Spoiler plays their entire sequence at once; then Duplicator answers with a similar sequence, preserving partial homomorphism in each prefix of the sequence. Similarly, a game for trace inclusion [26] is obtained from that for simulation by insisting Spoiler present an entire trace before Duplicator provides a matching trace. In such games δ expresses that Duplicator awaits Spoiler's complete play before making their moves in a matching complete play.

Example 5.1 (Trace-inclusion game). The trace-inclusion game in Figure 8 is another example of an "all-in-one" game. It is an adaptation of the earlier simulation game, Example 4.6, in two ways: so that Duplicator awaits all of Spoiler's choices of transitions before making theirs; so that Duplicator answers correctly to every subsequence of Spoiler's choices, not only their latest pebble placements. The former we ensure by using "stopper" events \$ to mark the explicit termination of Spoiler's assignments (primed); in any configuration the start move on the right causally depends on a stopper \$ on the left. For the latter, we make copies (unprimed) of the sequences of assignments. In Figure 8 we have illustrated the comonad $\delta : D \to G^\perp \| G$ in $\mathbf{Sig}(G, G)$ for the trace inclusion game. The esp D comprises the signature game $\mathbb{C}_G$ with extra causal dependencies; for simplicity, we show only one causal dependency from a stopper and omit copycat arrows. The winning condition of G extends the winning condition W_1 of the simulation game:

$$W_1 \wedge \mathbf{E}(\$) \wedge (\mathbf{E}(\beta_1) \to \beta_1' \asymp \beta_1) \wedge (\mathbf{E}(\beta_2) \to \beta_2' \asymp \beta_2).$$

The two equality constraints ensure that the assignments to original sequences and their copies agree. There is a winning strategy $(\mathcal{A}, a) \mathrel{-\!\!+\!\!\twoheadrightarrow}_\delta (\mathcal{B}, b)$ between two transition systems iff the traces

[1] Openness, rather than just rigidity, is essential to the existence of identities in $\mathbf{SD}_\delta$.

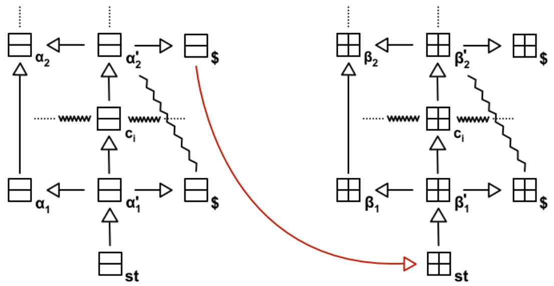


Figure 8: The trace-inclusion game

of $(\mathcal{A}, a)$ are included in the traces of $(\mathcal{B}, b)$. A similar δ can be constructed for the pebble-relation comonad [19]—with $\mathbf{SD}_\delta$ isomorphic to its coKleisi category.

5.2 Partial expansion. The expansion of Section 4 generalises to an expansion w.r.t. a subset of variables. A key to understanding the strategies of $\mathbf{SD}_\delta$ is the *partial expansion* of D w.r.t. its Opponent moves [appendix E].

Recall that δ is a map of esps $\delta : D \to G^\perp \| G$ where G has signature (Σ, C, V) and winning condition W. The signature of $G^\perp \| G$, inherited by D, is $(\Sigma + \Sigma, V + V, C + C)$. In particular, the variables are either $V_1 := \{1\} \times V$ of the left component, $G^\perp$, or $V_2 := \{2\} \times V$ of the right component, G. Accordingly the moves of D associated with variables are either moves in the set $|D|_{V_1}$ or moves in $|D|_{V_2}$. We form the *partial expansion* of $(D, \mathcal{A} + \mathcal{B})$ w.r.t. Σ-structures $\mathcal{A}$ and $\mathcal{B}$; it expands Opponent moves in $|D|_{V_1}$ to instances in $\mathcal{A}$ and Opponent moves in $|D|_{V_2}$ to instances in $\mathcal{B}$.

The partial expansion of $(D, \mathcal{A} + \mathcal{B})$, written $D(\mathcal{A}, \mathcal{B})$, is characterised as a pullback:

$$\begin{tikzcd} D(\mathcal{A},\mathcal{B}) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\delta"] \\ G^\perp(\mathcal{A}) \| G(\mathcal{B}) \arrow[r, "\epsilon"'] & G^\perp \| G \end{tikzcd}$$

Above, $G^\perp(\mathcal{A})$ is the partial expansion of $(G^\perp, \mathcal{A})$, while $G(\mathcal{B})$ the partial expansion of $(G, \mathcal{B})$, and ϵ the map projecting expansions to the original moves in $G^\perp \| G$. Write $\rho(\mathcal{A})$ for the sort-respecting function $\rho(\mathcal{A}) : |D(\mathcal{A}, \mathcal{B})|_{V_1^-} \to |\mathcal{A}|$ sending $s \in |D(\mathcal{A}, \mathcal{B})|_{V_1^-}$ to $\rho_{G^\perp, \mathcal{A}}(e)$ when $\pi_2(s) = (1, e)$; similarly, $\rho(\mathcal{B})$ is the sort-respecting function $\rho(\mathcal{B}) : |D(\mathcal{A}, \mathcal{B})|_{V_2^-} \to |\mathcal{B}|$ sending $s \in |D(\mathcal{A}, \mathcal{B})|_{V_2^-}$ to $\rho_{G, \mathcal{B}}(e)$ when $\pi_2(s) = (2, e)$. Write $\sigma(\mathcal{A}, \mathcal{B})$ for the composite map $\delta \circ \pi_1 = \epsilon \circ \pi_2$.

Through the partial expansion $D(\mathcal{A}, \mathcal{B})$ we can show the sense in which every strategy from $\mathcal{A}$ to $\mathcal{B}$ in $\mathbf{SD}_\delta$ follows essentially the same behavioural pattern: as we shall see, such a strategy corresponds to a function assigning values in $\mathcal{A} + \mathcal{B}$ to $(V + V)$-moves of Player in the partial expansion $D(\mathcal{A}, \mathcal{B})$. Let $(\sigma, \rho) : \mathcal{A} \twoheadrightarrow_\delta \mathcal{B}$ be a strategy in $\mathbf{SD}_\delta$. Then σ factors through the expansion of $G^\perp \| G$ w.r.t. $\mathcal{A}$ and $\mathcal{B}$, as

$$\sigma : S \xrightarrow{\sigma_0} expn(G^\perp, \mathcal{A}) \| expn(G, \mathcal{B}) \to G^\perp \| G.$$

Moreover, the expansion w.r.t. $\mathcal{A}$ and $\mathcal{B}$ factors into the Opponent-expansion followed by the expansion w.r.t. Player moves:

$$expn(G^\perp, \mathcal{A}) \| expn(G, \mathcal{B}) \to G^\perp(\mathcal{A}) \| G(\mathcal{B}) \to G^\perp \| G.$$

Consequently, we obtain the commuting diagram

$$\begin{tikzcd} S \arrow[rrr] \arrow[d, "\sigma_0"'] \arrow[drr, dotted, "\theta"] & & & D \arrow[d, "\delta"] \\ expn(G^\perp, \mathcal{A}) \| expn(G, \mathcal{B}) \arrow[drr] & & D(\mathcal{A}, \mathcal{B}) \arrow[ur, "\pi_1"] \arrow[d, "\pi_2"'] & G^\perp \| G \\ & & G^\perp(\mathcal{A}) \| G(\mathcal{B}) \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

where the dotted arrow represents the unique mediating map θ to the pullback.

Lemma 5.2. *The map θ in the diagram above is an isomorphism.*

Via the isomorphism of Lemma 5.2, we can reformulate the strategies of $\mathbf{SD}_\delta(\mathcal{A}, \mathcal{B})$.

Theorem 5.3. *Strategies $(\sigma_0, \rho_0) \in \mathbf{SD}_\delta(\mathcal{A}, \mathcal{B})$ are isomorphic to strategies*

$$(\sigma(\mathcal{A}, \mathcal{B}), (\rho(\mathcal{A}) \cup k) \cup (\rho(\mathcal{B}) \cup h)),$$

where

$$k : |D(\mathcal{A}, \mathcal{B})|_{V_1^+} \to |\mathcal{A}| \text{ and } h : |D(\mathcal{A}, \mathcal{B})|_{V_2^+} \to |\mathcal{B}|$$

are sort-respecting functions from Player moves in V_1 and V_2 respectively, such that for all $+$-maximal $x \in \mathcal{C}(D(\mathcal{A}, \mathcal{B}))$,

$$x \models_{\sigma(\mathcal{A}, \mathcal{B}), (\rho(\mathcal{A}) \cup k) \cup (\rho(\mathcal{B}) \cup h)} W_1 \to W_2.$$

(Above, W_1 and W_2 are copies of the winning condition W of G, renamed with variables and constants of the left, respectively right, parts of the signature $(\Sigma + \Sigma, C + C, V + V)$.)

The theorem makes precise how strategies of $\mathbf{SD}_\delta$ are determined by the extra part of the instantiation to variable-moves of Player (given by k and h above).

Of course, we would wish to understand composition in terms of the above reformulation of strategies. This is most easily done in the case of one-sided games, when game comonads and composition of strategies via coKleisli composition appear almost automatically out of the partial expansion.

6 One-sided games

As above, let G be a game with signature (Σ, C, V) and winning condition W. Assume $\delta : D \to G^\perp \| G$ is deterministic and forms an idempotent comonad $\delta \in \mathbf{Sig}(G, G)$. The game G with signature (Σ, V, C) is called *one-sided* if its every V-move is a Player move. Then a strategy (σ, ρ) in $\mathbf{SD}_\delta$ from $\mathcal{A}$ to $\mathcal{B}$ provides assignments of variable moves in $\mathcal{B}$ by Player on the right in response to assignments of variable moves in $\mathcal{A}$ by Opponent on the left; in this sense the strategies of $\mathbf{SD}_\delta$ are *one-sided* strategies. In most examples above, the games are one-sided. However, there are many well-known two-sided games, when the V-moves of G are of mixed polarity, *e.g.* for bisimulation, trace equivalence and in Ehrenfeucht–Fraïssé games. Throughout this section we focus on understanding $\mathbf{SD}_\delta$ where δ is a deterministic idempotent comonad and G one-sided.

The signature of $G^\perp \| G$ is $(\Sigma+\Sigma, V+V, C+C)$. As earlier, the variables are either $V_1 := \{1\}\times V$ of the left component, $G^\perp$, or $V_2 := \{2\}\times V$ of the right component, G. Accordingly the moves of D associated with variables are either Opponent moves in the set $|D|_{V_1}$ or Player moves in $|D|_{V_2}$. Now the partial expansion of D is w.r.t. just a single Σ-structure $\mathcal{A}$ and written $D(\mathcal{A})$. A typical event of $D(\mathcal{A})$ takes the form (e,γ) where $e \in |D|$ and γ is a sort-respecting function from $[e]_{V_1}$ to $|\mathcal{A}|$, reflecting that the expansion just instantiates V_1-moves. Specialising the earlier characterisation of the partial expansion as a pullback we observe:

$$\begin{tikzcd} D(\mathcal{A}) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\delta"] \\ expn(G^\perp,\mathcal{A})\|G \arrow[r, "\epsilon_{G,\mathcal{A}}\|G"'] & G^\perp\|G \end{tikzcd}$$

Write $\sigma(\mathcal{A})$ for the map $\delta \circ \pi_1$. The function $\rho(\mathcal{A})$ sends $(e,\gamma) \in |D(\mathcal{A})|_{V_1}$ to $\gamma(e) \in |\mathcal{A}|$.

6.1 Obtaining game comonads from δ. We rely on the notion of *companion events* w.r.t a given map $\sigma : S \to \mathbb{C}_G$. Recall that $\mathbb{C}_G$ comprises $G^\perp\|G$ with additional causal dependencies $\bar{g} \twoheadrightarrow g$ or $g \twoheadrightarrow \bar{g}$, across the components of the parallel composition, according to whether g has positive or negative polarity. Say $\bar{s}, s \in S$ are *companions* iff $\bar{s}, s \in S$ are not in conflict and $\sigma(\bar{s}) = \overline{\sigma(s)}$ in $\mathbb{C}_G$. Note that since σ reflects causal dependency locally, we have $\bar{s} \leq_S s$ or $s \leq_S \bar{s}$ according to whether s is positive or negative.

The counit of δ is a map $c : D \to \mathbb{C}_G$; so every V_2-move of Player over G in D causally depends on a companion over $G^\perp$ in D. Composing the counit with the map $\pi_1 : D(\mathcal{A}) \to D$, we obtain a map $c \circ \pi_1 : D(\mathcal{A}) \to \mathbb{C}_G$. Whence, every V_2-move e of Player over G in $D(\mathcal{A})$ causally depends on a companion $\bar{e}$ over $G^\perp$ in $D(\mathcal{A})$. The event e is not yet instantiated to a value in a Σ-structure $\mathcal{B}$ whereas, in this case, its companion $\bar{e}$ is instantiated to a value $\rho(\mathcal{A})(\bar{e}) \in |\mathcal{A}|$ by the Opponent expansion. We use their companions to equip the V_2-moves of Player over G in $D(\mathcal{A})$ with a relational structure.

Endow $|D(\mathcal{A})|_{V_2}$ with the structure of a Σ-structure $\mathrm{Rel}_\delta(\mathcal{A})$ as follows: for a relation symbol R of Σ and $e_1,\dots,e_k \in |D(\mathcal{A})|_{V_2}$ such that $R(\mathrm{vc}(e_1),\dots,\mathrm{vc}(e_k))$ is well-sorted, we say $R(e_1,\dots,e_k)$ holds in $\mathrm{Rel}_\delta(\mathcal{A})$ iff

$$\exists x \in C(D(\mathcal{A})).\ x \text{ is } +\text{-maximal } \ \&\ e_1,\dots,e_k \in \mathrm{last}(x)\ \& \\ R_{\mathcal{A}}(\rho(\mathcal{A})(\bar{e}_1),\dots,\rho(\mathcal{A})(\bar{e}_k))\ \&\ x_{G^\perp} \models W$$

– $x_{G^\perp}$ is the projection of x to a configuration of $C(expn(G^\perp,\mathcal{A}))$.

Generally, we can lift a sort-respecting function $h : |D(\mathcal{A})|_{V_2} \to |\mathcal{B}|$ to a function $h^\dagger : |D(\mathcal{A})|_{V_2} \to |D(\mathcal{B})|_{V_2}$. To do so we cannot immediately use the universality of the Opponent expansion as this concerns the variables V_1. Instead we have to rely on companion events in D and the idempotence of δ. Write D_1 for the set of events in D over $G^\perp$, *i.e.* such that $\delta(e) = (1,g)$ for some $g \in G^\perp$. Similarly, write D_2 for those events in D over G. Each Player event $e \in D_2$ has a unique companion $\bar{e} \in D_1$. The converse need not hold. However, from the idempotence of δ it follows that if $e_1 \in D_1$, $e \in D_2$ and $e_1 \leq_D e$, then there is a unique companion $\bar{e}_1 \in D_2$ of e_1; moreover, $[\bar{e}_1]_{V_1} \subseteq [e]_{V_1}$ [appendix, Lemma F.1].

Let $h : |D(\mathcal{A})|_{V_2} \to |\mathcal{B}|$ be a sort-respecting function to a Σ-structure $\mathcal{B}$. We use the concrete partial expansion w.r.t. Opponent moves (see Definition 4.1) in defining the *coextension* of h to a function $h^\dagger : |D(\mathcal{A})|_{V_2} \to |D(\mathcal{B})|_{V_2}$. Let (e,γ) be an event of $|D(\mathcal{A})|_{V_2}$; so $e \in D_2$ is a V_2-move with sort-respecting function $\gamma : [e]_{V_1} \to |\mathcal{A}|$. We define $h^\dagger$ as

$$h^\dagger((e,\gamma)) := (e,\gamma'),$$

where $\gamma'(e_1) = h((\bar{e}_1, \gamma \upharpoonright [\bar{e}_1]_{V_1}))$ for all $e_1 \in [e]_{V_1}$.

Definition 6.1. Say G has *homomorphic winning condition* iff for all instantiations (σ,ρ) in G, with $\rho : |G|_V \to |\mathcal{A}|$ and homomorphisms $h : \mathcal{A} \to \mathcal{B}$, whenever $x \models_{(\sigma,\rho)} W$ then $x \models_{(\sigma, h\circ\rho)} W$.

Under the assumption that G has homomorphic winning condition, if h is a homomorphism $h : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathcal{B}$, then its coextension $h^\dagger : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathrm{Rel}_\delta(\mathcal{B})$ is a homomorphism. In fact, $\mathrm{Rel}_\delta(\_)$ with counit $\mathrm{Rel}_\delta(\mathcal{A}) \to \mathcal{A}$ acting as $e \mapsto \rho(\mathcal{A})(\bar{e})$ and coextension $(\_)^\dagger$ forms a comonad on $\mathfrak{R}(\Sigma)$, the category of Σ-structures:

Theorem 6.2 (comonadic characterisation)**.** *Assume G has homomorphic winning condition. The operation $\mathrm{Rel}_\delta(\_)$ extends to a unique comonad on $\mathfrak{R}(\Sigma)$, which*

- *maps every Σ-structure $\mathcal{A}$ to $\mathrm{Rel}_\delta(\mathcal{A})$;*
- *has counit $\rho_{\mathcal{A}} : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathcal{A}$ acting as $e \mapsto \rho(\mathcal{A})(\bar{e})$;*
- *has coextension mapping a homomorphism $h : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathcal{B}$ to a homomorphism $h^\dagger : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathrm{Rel}_\delta(\mathcal{B})$.*

We can simplify the definition of the comonad in the common case where δ is copycat; it is then based on the expansion of G.

Corollary 6.3. *Assume G has homomorphic winning condition. When $\delta = \mathrm{c\!c}_G$, under $e \mapsto \bar{e}$ the Σ-structure $\mathrm{Rel}_\delta(\mathcal{A})$ is isomorphic to $|expn(G,\mathcal{A})|_V$ with Σ-relations*

$$R(e_1,\cdots,e_k) \textit{ in } |\exp(G,\mathcal{A})|_V \textit{ iff } \exists x \in C(\exp(G,\mathcal{A})). \\ e_1,\cdots,e_k \in \mathrm{last}(x)\ \&\ R_{\mathcal{A}}(\rho_{G,\mathcal{A}}(e_1),\cdots,\rho_{G,\mathcal{A}}(e_k))\ \&\ x \models W.$$

Under this isomorphism the counit acts as $\rho_{G,\mathcal{A}}$ and coextension takes $h : |expn(G,\mathcal{A})|_V \to |\mathcal{B}|$ to $h^\dagger : |expn(G,\mathcal{A})|_V \to |expn(G,\mathcal{B})|_V$, where $h^\dagger(g,\gamma) = (g,\gamma')$ with $\gamma'(g') = h(g', \gamma \upharpoonright [g']_G)$, for all $g' \leq_G g$.

In particular, from Corollary 6.3 we obtain that the pebbling comonad $\mathbb{P}_k$ from [1] arises as a special case: $\mathbb{P}_k(\mathcal{A}) \cong \mathrm{Rel}_{\mathrm{c\!c}_G}(\mathcal{A})$, where k is the size of the set of variables in the k-pebble game G of Example 4.5. Similarly, from Theorem 6.2 and its corollary, we can obtain other well-known game comonads such as the modal comonad [3], pebble-relation comonad [19], and the Ehrenfeucht–Fraïssé commonad [3]. This raises the question of when a coKleisli category is isomorphic to $\mathbf{SD}_\delta$, and what its relationship with $\mathbf{SD}_\delta$ is. More broadly, we seek a characterisation of one-sided strategies. In general, they are associated with a more refined preservation property than that of homomorphism, *viz.* that winning conditions are preserved. This will be captured in Section 6.3 by the concept of *δ-homomorphism.* It specialises to usual homomorphisms w.r.t. appropriate winning conditions.

6.2 Eilenberg-Moore coalgebras. Arboreal categories axiomatise many of those categories of Eilenberg-Moore coalgebras that arise from game comonads [4]. In this section we explore the Eilenberg-Moore coalgebras of $\mathrm{Rel}_\delta(\_)$ for a deterministic idempotent comonad $\delta : D \to G^\perp \| G$ on a one-sided game G; and see that they correspond to certain event algebras. Categories of event algebras specialise to arboreal categories when the V-moves form a tree-like event structure and to linear arboreal categories [2] when they form a path-like event structure.

Assume G has homomorphic winning condition. As might be expected, the coalgebras of $\mathrm{Rel}_\delta(\_)$ can be represented by Σ-structures which also take an event-structure shape, a characterisation we now make precise. First note that $\mathrm{Rel}_\delta(\mathcal{A})$ inherits the form of an event structure, $D(\mathcal{A}) \downarrow |\mathrm{Rel}_\delta(\mathcal{A})|$ from the partial expansion $D(\mathcal{A})$ w.r.t. a Σ-structure $\mathcal{A}$; the event structure, also denoted by $\mathrm{Rel}_\delta(\mathcal{A})$, is the projection of $D(\mathcal{A})$ to its +ve assignments, all in V_2. The construction is functorial: given a Σ-homomorphism $h : \mathcal{A} \to \mathcal{B}$, the function $\mathrm{Rel}_\delta(h)$ is both a homomorphism and a rigid map of event structures.

The coalgebras of $\mathrm{Rel}_\delta(\_)$ correspond to δ-event algebras, *viz.* pairs $(\mathcal{A}, f)$ consisting of the Σ-structure $\mathcal{A}$ and a rigid map of event structures,

$$f : (|\mathcal{A}|, \leq_{\mathcal{A}}, \#_{\mathcal{A}}) \to \mathrm{Rel}_\delta(\mathcal{A}),$$

whose domain is an event structure with events the elements of $\mathcal{A}$, such that whenever $f(a) = (e, \gamma)$,

$$\begin{aligned} &\gamma(\bar{e}) = a \text{ (so } f \text{ is injective) and} \\ &e_1 \in [e]_{V_1} \implies f(\gamma(e_1)) = (\bar{e}_1, \gamma \upharpoonright [\bar{e}_1]_{V_1}), \end{aligned} \qquad (i)$$

and whenever $R_{\mathcal{A}}(a_1, \cdots, a_k)$, for R in the signature,

$$\begin{aligned} &\exists x \in C(D(\mathcal{A})).\ x \text{ is +-maximal } \& \\ &f(a_1), \cdots, f(a_k) \in \mathrm{last}_{D(\mathcal{A})}(x) \ \&\ x_{G^\perp} \models W. \end{aligned} \qquad (ii)$$

Suppose

$$f : (|\mathcal{A}|, \leq_{\mathcal{A}}, \#_{\mathcal{A}}) \to \mathrm{Rel}_\delta(\mathcal{A}) \text{ and } g : (|\mathcal{B}|, \leq_{\mathcal{B}}, \#_{\mathcal{B}}) \to \mathrm{Rel}_\delta(\mathcal{B})$$

are δ-event algebras. A map from f to g is a map of event structures $h : (|\mathcal{A}|, \leq_{\mathcal{A}}, \#_{\mathcal{A}}) \to (|\mathcal{B}|, \leq_{\mathcal{B}}, \#_{\mathcal{B}})$, which is also a homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that $\mathrm{Rel}_\delta(h) f = gh$. Such maps compose (and are necessarily rigid).

Theorem 6.4. *δ-Event algebras and their maps form a category isomorphic to the category of Eilenberg-Moore coalgebras of* $\mathrm{Rel}_\delta(\_)$.

Coalgebras have a simpler characterisation when δ is copycat:

Corollary 6.5. *Let G be a signature game with homomorphic winning condition. Let G_V be its projection to V-moves. A æ_G-event algebra corresponds to a pair $(\mathcal{A}, f_0)$ consisting of a Σ-structure $\mathcal{A}$ and a rigid map of event structures*

$$f_0 : (|\mathcal{A}|, \leq_{\mathcal{A}}, \#_{\mathcal{A}}) \to G_V,$$

whose domain is an event structure with events elements of $\mathcal{A}$, such that whenever $R_{\mathcal{A}}(a_1, \cdots, a_k)$, for a Σ-relation R,

$$\begin{aligned} &\exists x \in C(expn(G, \mathcal{A})).\ x \models W \ \& \\ &\qquad f(a_1), \cdots, f(a_k) \in \mathrm{last}_{expn(G,\mathcal{A})}(x). \end{aligned} \qquad (ii)'$$

Above, $f : |\mathcal{A}| \to |expn(G, \mathcal{A})|$ is defined by $f(a) = (f_0(a), \gamma)$, where $\gamma(e_1)$ is that unique $a_1 \leq_{\mathcal{A}} a$ for which $f_0(a_1) = e_1$.

A benefit of game comonads is their coalgebraic characterisation of combinatorial parameters such as treewidth [1] and pathwidth [19]. We can use Corollary 6.5 to extract the same results. As an example, consider the width of a tree-decomposition from æ_G, where G is the k-pebble game. Condition $(ii)'$ implies that $\mathcal{A}$ has a k-pebble forest cover. From the fact that tree-decomposition of width $< k$ is equivalent to k-pebble forest cover [3], we obtain a coalgebraic characterisation of treewidth similarly to [1]. Theorem 6.4 and its corollary open up the means to analogous investigations based on properties of event structures—their width, concurrency width, degree of confusion and causal depth [27]. For instance, there are variations of Example 4.4, *e.g.* where in G we have k concurrent assignments and compare two undirected graphs. In this case, the coalgebras are colourings of graphs with chromatic number $\leq k$.

To our knowledge, all the game comonads mentioned in the literature, based on instantiations of moves as single elements of relational structures, fit within the scheme above. As do new game comonads, such as that associated with the game for trace-inclusion, Example 5.1; as well as those exploiting the explicit concurrency of event structures. Note, not all categories of coalgebras we obtain are *arboreal categories*: that of Example 4.4 will not be; its coalgebras can take a non-treelike shape.

6.3 Characterisation of one-sided strategies. Even in the one-sided case, not all strategies in $\mathbf{SD}_\delta$ are coKleisli maps of game comonads. Here we characterise all $\mathbf{SD}_\delta$ in the one-sided case.

Consider now a strategy $(\sigma, \rho) : \mathcal{A} \twoheadrightarrow_\delta \mathcal{B}$ in $\mathbf{SD}_\delta$ where $\mathcal{A}$ and $\mathcal{B}$ are Σ-structures. In this one-sided case the situation of Section 5.2 simplifies. Again, σ factors through the expansion of $G^\perp \| G$ with respect to $\mathcal{A}$ and $\mathcal{B}$ as

$$\sigma : S \xrightarrow{\sigma_0} expn(G^\perp, \mathcal{A}) \| expn(G, \mathcal{B}) \to G^\perp \| G.$$

The expansion w.r.t. $\mathcal{A}$ and $\mathcal{B}$ factors into the expansion w.r.t. $\mathcal{A}$ followed by the expansion w.r.t. $\mathcal{B}$ as

$$expn(G^\perp, \mathcal{A}) \| expn(G, \mathcal{B}) \to expn(G^\perp, \mathcal{A}) \| G \to G^\perp \| G.$$

Now we obtain the commuting diagram

$$\begin{tikzcd} S \arrow[rrr] \arrow[d, "\sigma_0"'] \arrow[drr, dotted, "\theta"] & & & D \arrow[d, "\delta"] \\ expn(G^\perp, \mathcal{A}) \| expn(G, \mathcal{B}) \arrow[drr] & & D(\mathcal{A}) \arrow[ur, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner"] & G^\perp \| G \\ & & expn(G^\perp, \mathcal{A}) \| G \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

where the dotted arrow represents the unique mediating map θ to the pullback. As a direct corollary of Lemma 6.6 we get that:

Corollary 6.6. *The map θ is an isomorphism.*

Directly from θ being an isomorphism, we can reformulate the one-sided strategies of $\mathbf{SD}_\delta$.

Theorem 6.7. *Strategies $(\sigma_0, \rho_0) \in \mathbf{SD}_\delta(\mathcal{A}, \mathcal{B})$ bijectively correspond to strategies $(\sigma(\mathcal{A}), \rho(\mathcal{A}) \cup h)$, where $h : |D(\mathcal{A})|_{V_2} \to |\mathcal{B}|$ is a sort-respecting function, $\rho(\mathcal{A}) \cup h$ acts as $\rho(\mathcal{A})$ on V_1-moves of $D(\mathcal{A})$ and as h on V_2-moves, and for all +-maximal $x \in C(D(\mathcal{A}))$, $x \models_{\sigma(\mathcal{A}), \rho(\mathcal{A}) \cup h} W_1 \to W_2$.*

(As earlier, W_1 and W_2 are copies of the winning condition W of G with renamings by constants and variables of respectively the left and right of $(\Sigma + \Sigma, C + C, V + V)$.)

The theorem makes precise how one-sided strategies of $\mathbf{SD}_\delta$ are determined by the extra part of the instantiation to V_2-moves, given by h above.

Strategies $\mathbf{SD}_\delta(\mathcal{A}, \mathcal{B})$ bijectively correspond to sort-respecting functions $h : |D(\mathcal{A})|_{V_2} \to |\mathcal{B}|$ such that for each +-maximal configuration $x \in C(D(\mathcal{A}))$ we have:

$$x_{G^\perp} \models W \text{ implies } (x[h])_G \models W. \tag{1}$$

Above, $x[h]$ is the configuration of $expn(D(\mathcal{A}), \mathcal{B})$ in which the V_2-moves of x are instantiated to the elements of $\mathcal{B}$ prescribed by h. We also use the fact that x projects to a configuration $x_{G^\perp}$ of $expn(G^\perp, \mathcal{A})$ and $x[h]$ projects to a configuration $(x[h])_G$ of $expn(G, \mathcal{B})$.

We call a function h satisfying (1) a *δ-homomorphism* from $\mathcal{A}$ to $\mathcal{B}$. We turn this characterisation into a direct presentation of $\mathbf{SD}_\delta$ as relational structures with δ-homomorphisms. Recall from Section 5.1 that the candidates for identity strategies, *viz.* $(\iota_\mathcal{A}, \rho_\mathcal{A}) : \mathcal{A} \mathrel{\longrightarrow\!\!\!\!\!+\!\!\!>}_\delta \mathcal{A}$, only become so when they are winning. We say that $\mathbf{SD}_\delta$ *has identities* when the strategies $(\iota_\mathcal{A}, \rho_\mathcal{A})$ are winning for each relational structures $\mathcal{A}$.

Theorem 6.8 (one-sided characterisation)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and C be Σ-structures.*

(1) Assuming $\mathbf{SD}_\delta$ has identities, the function $j_\mathcal{A} : |D(\mathcal{A})|_{V_2} \to \mathcal{A}$ such that $j_\mathcal{A}(e) = \rho(\mathcal{A})(\bar{e})$ is a δ-homomorphism.
(2) Suppose h is a δ-homomorphism from $\mathcal{A}$ to $\mathcal{B}$, and k is a δ-homomorphism from $\mathcal{B}$ to C. Then, the composition $k \circ h^\dagger : |D(\mathcal{A})|_{V_2} \to |C|$ is a δ-homomorphism from $\mathcal{A}$ to C.

Further, Σ-structures and δ-homomorphisms form a category with identities given by (1) and composition given by (2). Provided $\mathbf{SD}_\delta$ has identities, the category $\mathbf{SD}_\delta$ is isomorphic to the category of Σ-structures with δ-homomorphisms.

The coKleisli categories of the game comonads of Section 6.1, coincide with special Spoiler-Duplicator categories $\mathbf{SD}_\delta$:

Proposition 6.9. *Assume G has homomorphic winning condition. Assume $\mathbf{SD}_\delta$ has identities. Then, $\mathbf{SD}_\delta$ is the coKleisli category of the comonad $\mathrm{Rel}_\delta(\_)$ iff every δ-homomorphism $h : |D(\mathcal{A})|_{V_2} \to |\mathcal{B}|$ is a Σ-homomorphism $h : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathcal{B}$.*

The role of coKleisli maps in [1] is replaced by the more general notion of δ-homomorphisms. Categories $\mathbf{SD}_\delta$ are strictly more general than coKleisli categories of game comonads: G need not have homomorphic winning condition, *e.g.* if its winning condition involves negation; even when G has homomorphic winning condition, a δ-homomorphism need not be a Σ-homomorphism. In replacing the role of a map in a coKleisli category of a game comonad, it is now the presence of a δ-homomorphism which is equivalent to truth preservation in a logic characterised by its Spoiler-Duplicator game; for instance, k-variable logic in the case of the k-pebble game [1] and its restricted conjunction fragment in the case of the all-in-one game [19] —*cf.* Section 8.

7 Two-sided games

Theorem 5.3 expresses how a two-sided strategy in a Spoiler-Duplicator game reduces to a pair of functions. It entails:

Corollary 7.1. *Strategies $\mathbf{SD}_\delta(\mathcal{A}, \mathcal{B})$ bijectively correspond to pairs of sort-respecting functions*

$$k : |D(\mathcal{A}, \mathcal{B})|_{V_1^+} \to |\mathcal{A}| \text{ and } h : |D(\mathcal{A}, \mathcal{B})|_{V_2^+} \to |\mathcal{B}|$$

such that for all +-maximal $x \in C(D(\mathcal{A}, \mathcal{B}))$,

$$(x[k])_{G^\perp} \models W \implies (x[h])_G \models W.$$

Does the representation in terms of functions enable us to express the composition of two-sided Spoiler-Duplicator strategies more simply, as happened in the one-sided case? In general, there is a description of composition in the intricate manner of Geometry of Interaction [35]. There is however a simpler description when the comonad δ is copycat. When $\delta = \mathbb{C}_G$, Spoiler-Duplicator strategies correspond to *G-spans* of event structures, and their composition is simplified to the standard composition of spans via pullbacks.

Throughout this section, we adopt the notation G^+ for the event structure obtained as the projection of an esp G to its Player moves. The conversion of strategies to spans hinges on the map

$$\mathrm{F} : \mathbb{C}_G^+ \to G,$$

given by taking $\mathrm{F}(1, g) = \bar{g}$ and $\mathrm{F}(2, g) = g$. It is an isomorphism of event structures which folds the Player moves on the left of $\mathbb{C}_G$ to their corresponding Opponent moves on the right. Precomposing F with the projection $\mathbb{C}_G(\mathcal{A}, \mathcal{B})^+ \to \mathbb{C}_G^+$ associated with the partial expansion, we define the map of event structures

$$\mathrm{f}_{\mathcal{A},\mathcal{B}} : \mathbb{C}_G(\mathcal{A}, \mathcal{B})^+ \to G.$$

We construct the G-span of event structures corresponding to a strategy in $\mathbf{SD}_{\mathbb{C}_G}(\mathcal{A}, \mathcal{B})$ given by functions h and k. Its vertex, $\mathbb{C}_G(\mathcal{A}, \mathcal{B})^+$ is the projection of the partial expansion $\mathbb{C}_G(\mathcal{A}, \mathcal{B})$ to its Player moves. Its two maps are got by extending k and h of Corollary 7.1 to maps of event structures. First define functions

$$K : |\mathbb{C}_G(\mathcal{A}, \mathcal{B})^+|_{V+V} \to |\mathcal{A}| \text{ and } H : |\mathbb{C}_G(\mathcal{A}, \mathcal{B})^+|_{V+V} \to |\mathcal{B}|,$$

both from the variable-labelled Player moves of $\mathbb{C}_G(\mathcal{A}, \mathcal{B})$, by

$$K(s) = \begin{cases} k(s) & \text{if } s \in |\mathbb{C}_G(\mathcal{A}, \mathcal{B})|_{V_1^+}, \\ \rho(\mathcal{A})(\bar{s}) & \text{if } s \in |\mathbb{C}_G(\mathcal{A}, \mathcal{B})|_{V_2^+} \end{cases}$$

and

$$H(s) = \begin{cases} h(s) & \text{if } s \in |\mathbb{C}_G(\mathcal{A}, \mathcal{B})|_{V_2^+}, \\ \rho(\mathcal{B})(\bar{s}) & \text{if } s \in |\mathbb{C}_G(\mathcal{A}, \mathcal{B})|_{V_1^+}. \end{cases}$$

Every $s \in |\mathbb{C}_G(\mathcal{A}, \mathcal{B})^+|_{V+V}$ is now associated with a pair of values $(K(s), H(s))$ in the product of Σ-structures $\mathcal{A} \times \mathcal{B}$.

From the map $\mathrm{f}_{\mathcal{A},\mathcal{B}}$ and the functions K and H, we obtain an instantiation $(\mathrm{f}_{\mathcal{A},\mathcal{B}}, K)$ of G in $\mathcal{A}$, and an instantiation $(\mathrm{f}_{\mathcal{A},\mathcal{B}}, H)$ of G in $\mathcal{B}$. By the universality of the expansions of G w.r.t. $\mathcal{A}$ and $\mathcal{B}$ —Lemma 4.2, we get maps of event structures

$$\begin{tikzcd} & \mathbb{C}_G(\mathcal{A}, \mathcal{B})^+ \arrow[dl, "l_\mathcal{A}"'] \arrow[dr, "r_\mathcal{B}"] & \\ expn(G, \mathcal{A}) & & expn(G, \mathcal{B}) \end{tikzcd}$$

—the G-span corresponding to the original strategy in $\mathbf{SD}_{\mathbb{C}_G}(\mathcal{A}, \mathcal{B})$. From the original strategy being winning we derive: if $l_{\mathcal{A}} x \models W$ then $r_{\mathcal{B}} x \models W$, for any $x \in C(\mathbb{C}_G(\mathcal{A}, \mathcal{B})^+)$.

Two G-spans compose via pullback to form a G-span:

$$\begin{tikzcd}[column sep=small]
& & \mathbb{C}_G(\mathcal{A}, C)^+ \arrow[dl, dotted] \arrow[dr, dotted] \arrow[dd, phantom, "\vee" near start] & & \\
& \mathbb{C}_G(\mathcal{A}, \mathcal{B})^+ \arrow[dl, "l_{\mathcal{A}}"'] \arrow[dr, "r_{\mathcal{B}}"] & & \mathbb{C}_G(\mathcal{B}, C)^+ \arrow[dl, "l_{\mathcal{B}}"'] \arrow[dr, "r_C"] & \\
expn(G, \mathcal{A}) & & expn(G, \mathcal{B}) & & expn(G, C)
\end{tikzcd}$$

Theorem 7.2. $\mathbf{SD}_{\mathbb{C}_G}$ *is isomorphic to the category of G-spans.*

We were looking for a characterisation, so correspondence, with (deterministic) delta-strategies, and could only get that via spans when delta is copycat. Bisimulations describe a form of nondeterministic strategy—they do not fix Player to a single answering move—and this is why there is only an equivalence, and not a correspondence, between bisimulations and deterministic strategies in [3]; and why bisimulations do not figure more centrally in our account.

8 Final remarks

We have introduced a general method for generating comonads from one-sided Spoiler-Duplicator games, provided a characterisation of strategies for one-sided games in general, delineated when they reduce to coKleisli maps, and captured an important subclass of two-sided strategies as spans.

As a next step, we will focus on generalising the framework to allow moves assigning subsets, rather than individual elements of the relational structure—extending the free logic to allow second order variables and quantifiers in winning conditions. This will bring *bijective games* within the framework [1, 3]. Their interest is illustrated through the bijective k-pebble game which corresponds to indistinguishability by the k-dimensional Weisfeiler-Lehman algorithm [6]. Extending moves to assign subsets is a crucial avenue for future work, making other games accessible to structural, categorical methods. Notable challenges are games characterising (fragments of) Monadic Second Order Logic and μ-calculus. We are optimistic because of the adaptable nature of the games here.

A comonad δ in signature games specifies Spoiler-Duplicator strategies $\mathbf{SD}_\delta$. Imagine a logic extending the free logic. Within it a δ-*assertion* is an assertion preserved by all strategies in $\mathbf{SD}_\delta$; it is preserved by a strategy in the same manner as the winning condition. A task is that of finding a structural characterisation of the δ-assertions corresponding to δ, and *vice versa*; how tuning the rich array of combinatorial parameters encapsulated in δ correlates with changes in the structural properties of δ-assertions. We can potentially study how such correlations vary with δ, using the fact that such comonads themselves form a category.

Finally, since strategies are operational refinements of presheaves [33] they could potentially inform Lovász-type theorems, as resource-aware refinements of the Yoneda Lemma. Another advantage of concurrent games is that they already accommodate extensions, for instance to games with imperfect information [31], and enrichments with probability or quantum effects [8, 9, 32], appropriate to the study of probabilistic and quantum advantage.

9 Addendum: Free logic

A winning condition for a (Σ, C, V)-game is an assertion in the free logic over (Σ, C, V). We explain the syntax and semantics of the free logic. A good reference for free logic is Dana Scott's article [24].

The free logic uses an explicit existence predicate $\mathbf{E}(t)$ to specify when a term t denotes something existent. The existence of elements is highly relevant when exploring relational structures with bounded resources as in pebbling games where the number of pebbles bounds the existent part of the relational structure, so what assertions it satisfies, at any stage.

Throughout this section, assume a (Σ, C, V)-*game over a* Σ-*structure* $(G, \mathcal{A})$, comprising a (Σ, C, V)-game G and a Σ-structure $\mathcal{A}$. In providing the semantics to the free logic it will be convenient to name the elements of the Σ-structure $\mathcal{A}$ as terms. *Terms* of the free logic are either variables, constants or elements of the relational structure:

$$t ::= \alpha \in V \mid c \in C \mid a \in |\mathcal{A}|$$

Assertions of the free logic are given by:

$$\varphi ::= R(t_1, \ldots, t_k) \mid t_1 = t_2 \mid \mathbf{E}(t) \mid t_1 \lessdot t_2 \mid t_1 \eqcirc t_2$$
$$\mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi \mid \forall\alpha.\varphi \mid \exists\alpha.\varphi \mid \bigwedge_{i \in I} \varphi_i \mid \bigvee_{i \in I} \varphi_i$$

with t_i and t ranging over terms of correct sorts w.r.t. the arity of relation symbols and the indexing sets I are countable. The predicate $\mathbf{E}(t)$ asserts the existence of the value denoted by term t. An assertion $t_1 \lessdot t_2$ is an *order constraint* between terms. It expresses that t_1 is a latest constant or assignment move on which t_2 depends w.r.t. G. The *equality constraint* $t_1 \eqcirc t_2$ ensures in addition that the two terms denote equal values in $\mathcal{A}$. Equality constraints are used in the "all-in-one game", Example 5.1. We adopt the usual abbreviations. *e.g.* an implication $\varphi \to \varphi'$ abbreviates $\neg\varphi \vee \varphi'$.

The semantics of assertions is given w.r.t. an instantiation (σ, ρ) of G in $\mathcal{A}$, with $\sigma : S \to G$ and $\rho : |S|_V \to |\mathcal{A}|$. Given an instantiation (σ, ρ), we define the semantics of terms by:

$$\begin{aligned}
[\![\alpha]\!]_{(\sigma,\rho)} &= \{(x, s, a) \in C(S) \times S \times |\mathcal{A}| \mid s \in \mathrm{last}_S(x) \ \& \\
&\qquad \mathrm{vc}(\sigma(s)) = \alpha \ \& \ \rho(s) = a\} \\
[\![c]\!]_{(\sigma,\rho)} &= \{(x, s, c) \in C(S) \times S \times C \mid s \in x \ \& \ \mathrm{vc}(\sigma(s)) = c \ \& \\
&\qquad \forall s' \in x.\ \sigma(s) \leq_G \sigma(s') \ \& \ \mathrm{vc}(\sigma(s')) \in C \ \& \\
&\qquad pol_S(s) = pol_S(s') \Rightarrow s' = s\} \\
[\![a]\!]_{(\sigma,\rho)} &= \{(x, s, a) \in C(S) \times S \times |\mathcal{A}| \mid s \in \mathrm{last}_S(x) \ \& \ \rho(s) = a\}
\end{aligned}$$

A term is denoted by a set of triples (x, s, v) consisting of a configuration x, an event $s \in \mathrm{last}_S(x)$, at which its value v becomes existent. Notice, in particular, that a value in the algebra, $a \in |A|$, is only regarded as existing at a configuration x if there is a last move in x, *viz.* some $s \in \mathrm{last}_S(x)$, which is instantiated to a: accordingly, the denotation of $\mathbf{E}(a)$, $a \in |\mathcal{A}|$, below will be

$$[\![\mathbf{E}(a)]\!]_{(\sigma,\rho)} = \{x \in C(S) \mid a \in \rho\, \mathrm{last}_S(x)\},$$

i.e. the set of configurations at which a exists. We have chosen to say that a constant c exists in a configuration if an event bearing it has occurred there and there are no other *constant events of the same polarity* which causally depend on it in that configuration; for a constant to exist in a configuration, it has to be a latest (causally maximal) constant event there w.r.t. all other constant events of the same polarity. (This interpretation is critical in Examples 4.6, 5.1.)

Define the size of assertions as an ordinal:

- $size(R(t_1,\ldots,t_k)) = size(t_1 = t_2) = size(\mathbf{E}(t))$
  $= size(t_1 < t_2) = size(t_1 \prec\!\!\!= t_2) = 1$;
- $size(\neg\varphi) = size(\exists\alpha.\varphi) = size(\varphi) + 1$;
- $size(\varphi_1 \wedge \varphi_2) = sup\{size(\varphi_1), size(\varphi_2)\} + 1$;
- $size(\bigwedge_{i\in I}\varphi_i) = sup\{size(\varphi_i) \mid i \in I\} + 1$.

Definition 9.1 (semantics). Given an instantiation (σ,ρ), with $\sigma : S \to G$, the semantics of an assertion specifies the set of configurations of S which satisfy it. The key clauses of the semantics are defined below by induction on the size of assertions:

- $[\![R(t_1,\ldots,t_k)]\!]_{(\sigma,\rho)} =$

$$\{x \in \mathcal{C}(S) \mid \exists (a_1,\cdots,a_k) \in R_{\mathcal{A}}, s_1,\cdots,s_k \in S.\ (x,s_1,a_1) \in [\![t_1]\!]_{(\sigma,\rho)} \ \&\ldots\ \&\ (x,s_k,a_k) \in [\![t_k]\!]_{(\sigma,\rho)}\}$$

- $[\![t_1 = t_2]\!]_{(\sigma,\rho)} =$

$$\{x \in \mathcal{C}(S) \mid \exists a \in |A|, s_1,s_2 \in S.\ (x,s_1,a) \in [\![t_1]\!]_{(\sigma,\rho)} \ \&\ (x,s_2,a) \in [\![t_2]\!]_{(\sigma,\rho)}\}$$

- $[\![\mathbf{E}(t)]\!]_{(\sigma,\rho)} =$

$$\{x \in \mathcal{C}(S) \mid \exists v \in |\mathcal{A}| \cup C, s \in S.\ (x,s,v) \in [\![t]\!]_{(\sigma,\rho)}\}$$

- $[\![t_1 < t_2]\!]_{(\sigma,\rho)} =$

$$\{x \in \mathcal{C}(S) \mid \exists s_2,s_1 \in S, v_1,v_2 \in |\mathcal{A}| \cup C.\ \sigma(s_1) <_G \sigma(s_2)\ \&\ (x,s_2,v_2) \in [\![t_2]\!]_{(\sigma,\rho)} \ \&\ ([s_2]_S,s_1,v_1) \in [\![t_1]\!]_{(\sigma,\rho)}\}$$

- $[\![t_1 \prec\!\!\!= t_2]\!]_{(\sigma,\rho)} =$

$$\{x \in \mathcal{C}(S) \mid \exists s_2,s_1 \in S, a \in |\mathcal{A}|.\ \sigma(s_1) <_G \sigma(s_2)\ \&\ (x,s_2,a) \in [\![t_2]\!]_{(\sigma,\rho)} \ \&\ ([s_2]_S,s_1,a) \in [\![t_1]\!]_{(\sigma,\rho)}\}$$

- $[\![\varphi_1 \wedge \varphi_2]\!]_{(\sigma,\rho)} = [\![\varphi_1]\!]_{(\sigma,\rho)} \cap [\![\varphi_2]\!]_{(\sigma,\rho)}$
- $[\![\neg\varphi]\!]_{(\sigma,\rho)} = \mathcal{C}(S) \setminus [\![\varphi]\!]_{(\sigma,\rho)}$
- $[\![\exists\alpha.\varphi]\!]_{(\sigma,\rho)} =$

$$\{x \in \mathcal{C}(S) \mid \exists a \in |\mathcal{A}|. sort(a) = sort(\alpha)\ \&\ x \in [\![\mathbf{E}(a)]\!]_{(\sigma,\rho)} \ \&\ x \in [\![\varphi[a/\alpha]]\!]_{(\sigma,\rho)}\}$$

- $[\![\bigwedge_{i\in I}\varphi_i]\!]_{(\sigma,\rho)} = \bigcap_{i\in I}[\![\varphi_i]\!]_{(\sigma,\rho)}$

We write $x \models_{\sigma,\rho} \varphi$ iff $x \in [\![\varphi]\!]_{(\sigma,\rho)}$. Note an assertion $R(t_1,\ldots,t_k)$ is only true at a configuration at which all the terms $t_1,\ldots,t_k$ denote existent elements.

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A Proofs for Section 2 (Preliminaries)

A.1 Composition in concurrent games

1.1 Hiding—the defined part of a map. Let E be an event structure. Let $V \subseteq |E|$ be a subset events we call *visible*. Define the *projection* on V, by $E{\downarrow}V := (V, \leq_V, \#_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $v \#_V v'$ iff $v \#_E v'$ and $v, v' \in V$. The operation $\downarrow$ *hides* all events in $E \backslash V$. It is associated with a *partial-total factorisation system.* Consider a map of event structures $f : E \to E'$ and let

$$V := \{e \in E \mid f(e) \text{ is defined}\}.$$

Then f clearly factors into the composition

$$E \xrightarrow{f_0} E{\downarrow}V \xrightarrow{f_1} E'$$

of f_0, which is a partial map of event structures taking $e \in |E|$ to itself if $e \in V$ and undefined otherwise, and f_1, which is a total map of event structures acting like f on V. Each $x \in \mathcal{C}(E{\downarrow}V)$ is the image under f_0 of a *minimum configuration, viz.* $[x]_E \in \mathcal{C}(E)$. We call f_0 a *projection* and f_1 the *defined part* of the map f.

1.1 Pullbacks. The *pullback* of total maps of event structures is essential in composing strategies. We can build it out of computation paths. A computation path is described by a partial order $(p, \leq_p)$ for which the set $\{e' \in p \mid e' \leq_p e\}$ is finite for all $e \in p$. We can identify such a path with an event structure with no conflict. Between two paths $p = (p, \leq_p)$ and $q = (q, \leq_q)$, we write $p \hookrightarrow q$ when $p \subseteq q$ and the inclusion is a rigid map of event structures.

Proposition A.1 ([Win16]). *A* rigid family $\mathcal{R}$ *comprises a non-empty subset of finite partial orders which is down-closed w.r.t. rigid inclusion, i.e.* $p \hookrightarrow q \in \mathcal{R}$ *implies* $p \in \mathcal{R}$. *It is* coherent *when a pairwise-compatible finite subfamily is compatible. A coherent rigid family determines an event structure* $\mathrm{Pr}(\mathcal{R})$ *whose order of finite configurations is isomorphic to* $(\mathcal{R}, \hookrightarrow)$*. The event structure* $\mathrm{Pr}(\mathcal{R})$ *has events those elements of* $\mathcal{R}$ *with a top event; its causal dependency is given by rigid inclusion; and its conflict by incompatibility w.r.t. rigid inclusion. The order isomorphism* $\mathcal{R} \cong \mathcal{C}(\mathrm{Pr}(\mathcal{R}))^o$ *takes* $q \in \mathcal{R}$ *to* $\{p \in \mathrm{Pr}(\mathcal{R}) \mid p \hookrightarrow q\}$.

We can define pullback via a rigid family of *secured bijections.* Let $\sigma : S \to B$ and $\tau : T \to B$ be total maps of event structures. There is a composite bijection

$$\theta : x \cong \sigma x = \tau y \cong y,$$

between $x \in \mathcal{C}(S)^o$ and $y \in \mathcal{C}(T)^o$ such that $\sigma x = \tau y$; because σ and τ are total they induce bijections between configurations and their image. The bijection is *secured* when the transitive relation generated on θ by $(s, t) \leq (s', t')$ if $s \leq_S s'$ or $t \leq_T t'$ is a partial order.

Theorem A.2 ([Win16]). *Let* $\sigma : S \to B$ *and* $\tau : T \to B$ *be total maps of event structures. The family* $\mathcal{R}$ *of secured bijections between* $x \in \mathcal{C}(S)^o$ *and* $y \in \mathcal{C}(T)^o$ *such that* $\sigma x = \tau y$ *is a rigid family. The functions* $\pi_1 : \mathrm{Pr}(\mathcal{R}) \to S$ *and* $\pi_2 : \mathrm{Pr}(\mathcal{R}) \to T$*, taking a secured bijection with top to, respectively, the left and right components of its top, are maps of event structures.* $\mathrm{Pr}(\mathcal{R})$ *with* π_1 *and* π_2 *is the pullback of* σ *and* τ *in the category of event structures.*

W.r.t. $\sigma : S \to B$ and $\tau : T \to B$, define $x \wedge y$ to be the configuration of their pullback which corresponds via this isomorphism to a secured bijection between $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$, necessarily with $\sigma x = \tau y$; any configuration of the pullback takes the form $x \wedge y$ for unique x and y.

1.1 Composition of strategies. Two strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ compose via pullback and hiding, summarised below.

$$\begin{tikzcd}
 & T \circledast S \arrow[ld, "\pi_1"'] \arrow[rd, "\pi_2"] \arrow[d, dotted, "\tau \circledast \sigma"] \arrow[rr, dashed] & & T \odot S \arrow[dd, dashed, "\tau \odot \sigma"] \\
S \| C \arrow[r, "\sigma \| C"'] & A \| B \| C & A \| T \arrow[l, "A \| \tau"] & \\
 & & & A \| C
\end{tikzcd}$$

$$A \| B \| C \longrightarrow A \| C .$$

Ignoring polarities, by forming the pullback of $\sigma \| C$ and $A \| \tau$ we obtain the synchronisation of complementary moves of S and T over the common game B; subject to the causal constraints of S and T, the effect is to instantiate the Opponent moves of T in $B^\perp$ by the corresponding Player moves of S in B, and *vice versa.* Reinstating polarities we obtain the *interaction* of σ and τ

$$\tau \circledast \sigma : T \circledast S \to A^\perp \| B^0 \| C,$$

where we assign neutral polarities to all moves in or over B. Neutral moves over the common game B remain unhidden. The map $A^\perp \| B^0 \| C \rightharpoonup A^\perp \| C$ is undefined on B^0 and otherwise mimics the identity. Pre-composing this map with $\tau \circledast \sigma$ we obtain a partial map $T \circledast S \rightharpoonup A^\perp \| C$; it is undefined on precisely the neutral events of $T \circledast S$. The defined parts of its partial-total factorisation yields

$$\tau \odot \sigma : T \odot S \to A^\perp \| C.$$

On reinstating polarities; this is the *composition* of σ and τ. We obtain a bicategory **Strat** where the objects are games, arrows are strategies, and 2-cells are maps between strategies [RW11, CGW12]. Identities are given by copycat strategies.

It is useful to introduce notation for configurations of the interaction and composition of strategies σ and τ. For $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$, let $\sigma x = x_A \| x_B$ and $\tau y = y_B \| y_C$ where $x_A \in \mathcal{C}(A)$, $x_B, y_B \in \mathcal{C}(B)$, $y_C \in \mathcal{C}(C)$. Define $y \circledast x = (x \| y_C) \wedge (x_A \| y)$. This is a partial operation only defined if the $\wedge$-expression is. It is defined and glues configurations x and y together at their common overlap over B provided $x_B = y_B$ and a finitary partial order of causal dependency results. Any configuration of $T \circledast S$ has the form $y \circledast x$, for unique $x \in \mathcal{C}(S), y \in \mathcal{C}(T)$. Accordingly, any finite configuration of $T \odot S$ is given as

$$y \odot x = (y \circledast x) \cap \{e \in T \circledast S \mid pol_{T \circledast S}(e) \neq 0\},$$

for some $x \in \mathcal{C}(S)^o$ and $y \in \mathcal{C}(T)^o$. The configurations x and y need not be unique. However, w.r.t. any configuration z of $T \odot S$ there are *minimum* x, y such that $y \odot x = z$, *viz.* those for which $y \circledast x = [z]_{T \circledast S}$.

B Proofs for Section 3 (Games over relational structures)

B.1 Multi-sorted signature

A *relational many-sorted signature* Σ is built from a set, whose elements are called *sorts*, which specifies for each string of sorts $\vec{s} = s_1 \dots s_k$ a set $\Sigma_{\vec{s}}$ of *relation symbols* of *arity* $\vec{s}$. A many-sorted Σ-structure $\mathcal{A}$ provides sets and relations as interpretations of the sorts and relation symbols. Each element $a \in |A|$ of a many-sorted Σ-structure $\mathcal{A}$ has a unique sort, $sort(a)$; we write $|\mathcal{A}|_s$ for those elements of $\mathcal{A}$ of sort s; we insist on *nonemptiness*, *i.e.* $|\mathcal{A}|_s \neq \emptyset$ for all sorts s. A relation symbol $R \in \Sigma_{\vec{s}}$, of arity $\vec{s} = s_1 \dots s_k$, is interpreted in $\mathcal{A}$ by $R_{\mathcal{A}} \subseteq |\mathcal{A}|_{s_1} \times \cdots \times |\mathcal{A}|_{s_k}$. The *sum* of many-sorted relational structures $\mathcal{A}_1$, with signature Σ_1, and $\mathcal{A}_2$, with signature Σ_2, denoted $\mathcal{A}_1 + \mathcal{A}_2$ with signature $\Sigma_1 + \Sigma_2$, has sorts given by the disjoint union of the sorts of $\mathcal{A}_1$ and $\mathcal{A}_2$, and relations lifted from those of the components.

Proposition 3.6. *Let f be a map of instantiations of G in $\mathcal{A}$, from (σ, ρ) to (σ', ρ'). Let φ be an assertion in free logic. Then, $x \models_{\sigma,\rho} \varphi$ iff $fx \models_{\sigma',\rho'} \varphi$, for every configuration x.*

Proof. By structural induction on φ. Assume $\sigma : S \to G$ and $\sigma' : S' \to G$ with f a map of instantiations from (σ, ρ) to (σ', ρ'). The proof rests on $f\,\mathrm{last}_S(x) = \mathrm{last}_{S'}(fx)$, for any configuration of S, by Proposition 3.5. ◻

C Proofs for Section 4 (Strategies over structures)

Lemma 4.2 (universal instantiation). *The pair $(\epsilon_{G,\mathcal{A}}, \rho_{G,\mathcal{A}})$ forms an* instantiation *of G in $\mathcal{A}$. For each instantiation $(\sigma : S \to G, \rho)$ of G in $\mathcal{A}$, there is a unique map σ_0 of instantiations from (σ, ρ) to $(\epsilon_{G,\mathcal{A}}, \rho_{G,\mathcal{A}})$ such that the following diagrams commute:*

$$\begin{tikzcd} S \arrow[r, dashed, "\sigma_0"] \arrow[dr, "\sigma"'] & expn(G,\mathcal{A}) \arrow[d, "\epsilon_{G,\mathcal{A}}"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} |S|_V \arrow[r, dashed, "\sigma_0"] \arrow[dr, "\rho"'] & |expn(G,\mathcal{A})|_V \arrow[d, "\rho_{G,\mathcal{A}}"] \\ & |\mathcal{A}| \end{tikzcd}$$

Moreover, $\epsilon_{G,\mathcal{A}}$ is open; σ is rigid iff σ_0 is rigid; and σ is open iff σ_0 is open.

Proof. For $s \in |S|$, define $\sigma_0(s) = (g, \gamma)$ where $g = \sigma(s)$ and for $g' \leq_G g$ with $\mathrm{vc}(g') \in V$, $\gamma(g') = \rho(s')$ for the unique $s' \leq_S s$ such that $\sigma(s') = g'$. Then, σ_0 can be checked to be a map of event structures $\sigma_0 : S \to expn(G, \mathcal{A})$ and that it is the unique map of instantiations from (σ, ρ) to $(\epsilon_{G,\mathcal{A}}, \rho_{G,\mathcal{A}})$. The map $\epsilon_{G,\mathcal{A}}$ is open because of the nonemptiness of $\mathcal{A}$. Because $\epsilon_{G,\mathcal{A}}$ is open, it is easy to check that σ_0 is rigid/open according as σ is rigid/open, and *vice versa.* ◻

C.1 Explicit characterisation of strategies on games over structures

Below, we use $x \overset{s}{\longrightarrow\!\subset}$ to mean that an event s is enabled at a configuration x, *i.e.* that $s \notin x$ and $x \cup \{s\}$ is a configuration.

Theorem C.1 (explicit characterisation)**.** *An instantiation $(\sigma : S \to G, \rho)$ is a strategy in $(G, \mathcal{A})$ iff*

- *for $x \in \mathcal{C}(S)$, $g \in |G|$, with $\sigma x \overset{g}{\longrightarrow\!\subset}$ and $pol(g) = -$,*
  - *if $\mathrm{vc}(g) \in C$ then there is a unique $s \in |S|$ such that $x \overset{s}{\longrightarrow\!\subset}$ and $\sigma(s) = g$;*
  - *if $\mathrm{vc}(g) = \alpha \in V$ then for all $a \in |A|$ with $sort(a) = sort(\alpha)$ there is a unique $s \in |S|$ such that $x \overset{s}{\longrightarrow\!\subset}$ and $\sigma(s) = g$ and $\rho(s) = a$;*
- *if $s \rightarrowtriangle_S s'$ and [$pol(s) = +$ or $pol(s') = -$], then $\sigma(s) \rightarrowtriangle_G \sigma(s')$.*

Proof. This follows directly from a strategy in a (G, A) being the composition of a strategy in the expansion $expn(G, \mathcal{A})$ and $\epsilon_{G,\mathcal{A}}$, and properties of the latter [HTW24]. ◻

C.2 Red, the bicategory of games over structures

We explain how composition in **Red** is inherited from composition in **Strat**. It is easy to see that the expansion preserves the dual and parallel composition of games over relational structures. Below, $(\emptyset, \emptyset)$ describes the empty game over the empty structure, with winning condition *false*, the unit w.r.t. the parallel composition of games over structures.

Lemma C.2. *Let $(G, \mathcal{A})$ and $(H, \mathcal{B})$ be games over relational structures. Then,*

$$expn(\emptyset, \emptyset) = \emptyset\,,\ expn((G,\mathcal{A})^{\perp}) = expn(G,\mathcal{A})^{\perp} \text{ and } expn((G,\mathcal{A})\|(H,\mathcal{B})) \cong expn(G,\mathcal{A})\|expn(H,\mathcal{B})\,.$$

From Lemma C.2 and Lemma 4.2, which provides an isomorphism between strategies in games over relational structures and traditional concurrent strategies, we have

$$\mathbf{Red}((G,\mathcal{A}),(H,\mathcal{B})) \cong \mathbf{Strat}(expn(G,\mathcal{A}), expn(H,\mathcal{B}))\,.$$

Write $(\sigma,\rho) : (G,\mathcal{A}) \mathrel{-\!\!\!+\!\!\!\!\!\twoheadrightarrow} (H,\mathcal{B})$ when $(\sigma,\rho) \in \mathbf{Red}((G,\mathcal{A}),(H,\mathcal{B}))$.

We derive composition in **Red** from the composition of **Strat**. Let $(\sigma, \rho_S) : (G,\mathcal{A}) \mathrel{-\!\!\!+\!\!\!\!\!\twoheadrightarrow} (H,\mathcal{B})$ and $(\tau,\rho_T) : (H,\mathcal{B}) \mathrel{-\!\!\!+\!\!\!\!\!\twoheadrightarrow} (K,\mathcal{C})$. By Lemmas 4.2 and C.2, there are associated traditional strategies $\sigma_0 : expn(G,\mathcal{A}) \mathrel{-\!\!\!+\!\!\!\!\!\twoheadrightarrow} expn(H,\mathcal{B})$ and $\tau_0 : expn(H,\mathcal{B}) \mathrel{-\!\!\!+\!\!\!\!\!\twoheadrightarrow} expn(K,\mathcal{C})$. Their composition $\tau_0 \odot \sigma_0 : expn(G,\mathcal{A}) \mathrel{-\!\!\!+\!\!\!\!\!\twoheadrightarrow} expn(K,\mathcal{C})$ is a strategy in the game

$$expn(G,\mathcal{A})^{\perp}\|expn(K,\mathcal{C}) \cong expn((G,\mathcal{A})^{\perp}\|(K,\mathcal{C}))\,,$$

by Lemma C.2; hence, by Lemma 4.2, $\tau_0 \odot \sigma_0$ determines a strategy

$$(\tau,\rho_T)\odot(\sigma,\rho_S) : (G,\mathcal{A}) \mathrel{-\!\!\!+\!\!\!\!\!\twoheadrightarrow} (K,\mathcal{C})\,.$$

The composition $(\tau,\rho_T)\odot(\sigma,\rho_S)$ can be expressed directly, essentially mimicking the composition of traditional concurrent strategies, though with one key difference. Instead of just forming the pullback of $\sigma\|K$ and $G\|\tau$ we must restrict its configurations to those $y \circledast x$ for which ρ_S and ρ_T agree over the common part $x_H = y_{H^{\perp}}$. Having done this, the construction of composition proceeds as for the composition of concurrent strategies, finally defining the resulting instantiation to be agree with ρ_S over the game $G^{\perp}$ and ρ_T over the game K.

D Proofs for Section 5 (Spoiler-Duplicator games)

D.1 Composition in $\mathbf{SD}_\delta$

Composition in $\mathbf{SD}_\delta$ is the composition of strategies over games over relational structures; that the composition factors openly through δ relies on horizontal composition preserving open 2-cells and the idempotence of δ. Identity strategies $(\iota_B, \rho_B) : \mathcal{B} \mathrel{-\!\!\!+\!\!\!\!\!\twoheadrightarrow}_\delta \mathcal{B}$ must also factor openly through δ. They are constructed as pullbacks of δ along $\text{æ}_{G,B}$, where $(\text{æ}_{G,B}, \gamma_{G,B}) : (G,\mathcal{B}) \mathrel{-\!\!\!+\!\!\!\!\!\twoheadrightarrow} (G,\mathcal{B})$ is the copycat strategy of $(G,\mathcal{B})$:

$$\begin{tikzcd}
I_B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\delta"] \\
\mathbb{C}_{G,B} \arrow[r, "\text{æ}_{G,B}"] & G^{\perp}\|G\,.
\end{tikzcd}$$

Define $\iota_B \coloneqq \delta \circ \pi_2 = \text{æ}_{G,B} \circ \pi_1$ and $\rho_B \coloneqq \gamma_{G,B} \circ \pi_1$. As the pullback of the strategy δ, the map π_1 defines a strategy in $\mathbb{C}_{G,B}$; thus ι_B, its composition with the strategy $\text{æ}_{G,B}$, is a strategy. That I_B is deterministic—so the strategy ι_B is deterministic—follows from both D and $\mathbb{C}_{G,B}$ being deterministic. To see that (ι_B, ρ_B) forms a strategy in $\mathbf{SD}_\delta$, fill in the diagram above to the commuting diagram

$$\begin{tikzcd}
I_B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "c"] \arrow[dr, bend left, "\delta"] & \\
\mathbb{C}_{G,B} \arrow[r, "\epsilon_0"'] \arrow[rr, bend right, "\text{æ}_{G,B}"'] & \mathbb{C}_G \arrow[r, "\text{æ}_G"'] & G^{\perp}\|G\,.
\end{tikzcd}$$

The map ι_B factors openly through δ as π_2 is open, being the pullback of the open map ϵ_0.

Provided (ι_B, ρ_B) is winning, the pair will together constitute an identity in the category $\mathbf{SD}_\delta$. In general $\mathbf{SD}_\delta$ need not be a *subcategory* of **Red** as their identities may differ.

Definition D.1 (Scott order). Let A be a game and let $x, y \in \mathcal{C}(A)$. We define the *Scott order* $\sqsubseteq_A$ as $y \sqsubseteq_A x \iff [\exists z \in \mathcal{C}(A).\ y \supseteq^{-} z \text{ and } z \subseteq^{+} x]$.

It is not hard to show that z is unique and equal to $x \cap y$. The Scott order is also characterised by $y \sqsubseteq_A x \iff [y^{-} \supseteq x^{-} \text{ and } y^{+} \subseteq x^{+}]$, which makes clear why it is a partial order. The Scott order is so named because it reduces to Scott's order on functions in special cases and plays a central role in relating games to Scott domains and "generalised domain theory" [Hyl10].

Lemma D.2. *Let $\delta : D \to G^\perp \| G$ be the deterministic idempotent comonad $\delta : G \longrightarrow\!\!\!\!\!+\!\!\!> G$ in* **Sig** *with comultiplication $d : D \cong D\odot D$. Suppose $x, y, z \in C(D)$. If $dz = y\odot x$, then*

$$z \uparrow_D x \;\&\; z \sqsubseteq_D x \;\&\; z \uparrow_D y \;\&\; z \sqsubseteq_D y.$$

Proof. The comultiplication d makes the diagram

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, "\delta"'] & D\odot D \arrow[dl, "\delta\odot\delta"] \\ G^\perp \| G & \end{tikzcd}$$

commute and is an isomorphism. Suppose $x, y, z \in C(D)$ with $dz = y\odot x$.

We first show $z \uparrow_D x$. As δ is deterministic it suffices to show no pair of $-$ve events in $x \cup z$ are in conflict. It also suffices to consider a pair $e_1 \in x$ and $e_2 \in z$ with $pol_D(e_1) = pol_D(e_2) = -$. As δ is a strategy so receptive, if $e_1 \#_D e_2$ then $\delta(e_1) \#_{G^\perp\|G} \delta(e_2)$. We show that $\delta(e_1) \#_{G^\perp\|G} \delta(e_2)$ is impossible. Assume otherwise, that $\delta(e_1) \#_{G^\perp\|G} \delta(e_2)$. Then $\delta(e_1)$ and $\delta(e_2)$ have to be in the same parallel component $G^\perp$ or G. Consider the two cases.

1. *Case where $\delta(e_1) \in \{1\} \times G^\perp$ and $\delta(e_2) \in \{1\} \times G^\perp$.* From the construction of $y\odot x$ we see that $\delta(e_1), \delta(e_2) \in \delta x$, a configuration of $G^\perp\|G$, which contradicts their being in conflict.
2. *Case where $\delta(e_1) \in \{2\} \times G$ and $\delta(e_2) \in \{2\} \times G$.* Then from the construction of $y \circledast x$ we see that $e_1 \in x$ synchronises with a +ve event $\bar{e}_1 \in y$. As the counit factorises δ through α_G there is $e_1' \in y$ with $\delta(e_1) = \delta(e_1') \in \delta z$. But then $\delta(e_1), \delta(e_2) \in \delta z$, a configuration, contradicting their being in conflict.

In all cases, $\delta(e_1)$ and $\delta(e_2)$ are not in conflict—a contradiction, from which we deduce that neither are e_1, e_2. It follows that $x \cup z$ is a configuration and that $z \uparrow_D x$.

Now, to show $z \sqsubseteq_D x$ it suffices to show $\delta z \sqsubseteq_{G^\perp\|G} \delta x$. Because then

$$\delta z \supseteq^- ((\delta z) \cap (\delta x)) \subseteq^+ \delta x,$$

making $\delta(z \cap x) = (\delta z) \cap (\delta x)$, as $z \uparrow_D x$, so

$$\delta z \supseteq^- \delta(z \cap x) \subseteq^+ \delta x\,;$$

whence, as δ is deterministic, it reflects inclusions between configurations, to yield

$$z \supseteq^- (z \cap x) \subseteq^+ x, \quad \textit{i.e.}\; z \sqsubseteq_D x.$$

To show $\delta z \sqsubseteq_{G^\perp\|G} \delta x$ we require

$$\delta z^+ \subseteq \delta x^+ \;\;\&\;\; \delta z^- \supseteq \delta x^-.$$

Both these inclusions follow by considering over which component of $G^\perp\|G$ events lie. For example, to show $\delta z^+ \subseteq \delta x^+$ the trickier case is where $e \in z^+$ lies over G, *i.e.* $\delta(e) \in \{2\} \times G$. Then, from the construction of $y \circledast x$, we have $\delta(e) \in \delta y$. But δ factors through α_G, from which there must be a matching $-$ve event $\bar{e} \in y$ which synchronises with a +ve event $e' \in x$ with $\delta(e) = \delta(e')$, as required. Similarly, $z \uparrow_D y$ and $z \sqsubseteq_D y$. ◻

Theorem D.3. *Suppose that the strategy $(\iota_A, \rho_A) : (G, \mathcal{A}) \longrightarrow\!\!\!\!\!+\!\!\!> (G, \mathcal{A})$ is winning for all Σ-structures $\mathcal{A}$. Then $\mathbf{SD}_\delta$ is a category with identities $(\iota_A, \rho_A) : \mathcal{A} \longrightarrow\!\!\!\!\!+\!\!\!>_\delta \mathcal{A}$.*

Moreover, the category $\mathbf{SD}_\delta$ is a subcategory of **Red** *in the case where δ is copycat $\alpha_G : \mathbb{C}_G \to G^\perp\|G$; then identities are copycat strategies $(\alpha_{G,A}, \gamma_{G,A}) : \mathcal{A} \longrightarrow\!\!\!\!\!+\!\!\!>_\delta \mathcal{A}$.*

Proof. Strategies $(\sigma, \rho_S) : \mathcal{A} \longrightarrow\!\!\!\!\!+\!\!\!>_\delta \mathcal{B}$ and $(\tau, \rho_T) : \mathcal{B} \longrightarrow\!\!\!\!\!+\!\!\!>_\delta C$ are associated with open strategies $\sigma_1 : S \to D$ and $\tau_1 : T \to D$ and their composition with $\tau_1\odot\sigma_1 : T\odot S \to D\odot D \cong D$, which is also open as horizontal composition of strategies $\odot$ preserves open 2-cells.

That $(\iota_\mathcal{A}, \rho_\mathcal{A}) : \mathcal{A} \longrightarrow\!\!\!\!\!+\!\!\!>_\delta \mathcal{A}$ and $(\iota_\mathcal{B}, \rho_\mathcal{B}) : \mathcal{B} \longrightarrow\!\!\!\!\!+\!\!\!>_\delta \mathcal{B}$ are identities in composition with $(\sigma, \rho_S) : \mathcal{A} \longrightarrow\!\!\!\!\!+\!\!\!>_\delta \mathcal{B}$ relies on isomorphisms

$$I_\mathcal{B}\odot S \cong S \;\text{ and }\; S\odot I_\mathcal{A} \cong S,$$

where $\sigma : S \to G^\perp\|G$ and $\iota_\mathcal{B} : I_\mathcal{B} \to G^\perp\|G$. We build the isomorphism

$$\theta : I_\mathcal{B}\odot S \cong S$$

as the composite map

$$I_B\odot S \xrightarrow{\pi_1\odot S} \mathbb{C}_{G,B}\odot S \cong S,$$

which relies on $\alpha_{G,B}$ being identity in **Red**. (The map names are those used in the definition of ι_B above.) To show that θ is an isomorphism it suffices to show that it is rigid and both surjective and injective on configurations (Lemma 3.3 of [Win07]). Injectivity on configurations is a consequence of $\pi_1 \odot S$ being a deterministic strategy in $\mathbb{C}_{G,B} \odot S$—it is the composition of deterministic strategies. To establish rigidity and surjectivity on configurations, consider the commuting diagram

$$\begin{tikzcd} I_B \odot S \arrow[r, "\pi_1 \odot S"] \arrow[d, "\pi_2 \odot \sigma_1"'] & \mathbb{C}_{G,B} \odot S \arrow[r, phantom, "\cong"] \arrow[d, "\alpha_{G,B} \odot \sigma_1"] & S \arrow[d, "\sigma_1"] \\ D \odot D \arrow[r, "c \odot D"] & \mathbb{C}_G \odot D \arrow[r, phantom, "\cong"] & D \arrow[ll, bend left, "d"] \end{tikzcd},$$

where θ is the upper horizontal composite map. The lower horizontal composite map is inverse to d —this follows from the comonad law for the counit c —and the map $c \odot D$ an isomorphism.

The vertical maps are open—being the composition of open 2-cells, so rigid. Consequently $\sigma_1 \circ \theta$ is rigid which, through σ_1 reflecting causal dependency locally, implies the rigidity of θ.

To see that θ is surjective on configurations is more involved. Suppose that $z_1 \in \mathcal{C}(S)$. Let $z \coloneqq \sigma_1 z_1 \in \mathcal{C}(D)$. As d is an isomorphism, there are $x, y \in \mathcal{C}(D)$ such that $dz = y \odot x$. By Lemma D.2,

$$z \sqsubseteq_D x \text{ and } z \sqsubseteq_D y.$$

Because σ_1 is deterministic and open there is a unique $x_1 \in \mathcal{C}(S)$ with $z_1 \sqsubseteq_S x_1$ and $\sigma_1 x_1 = x$. We now construct $y_1 \in \mathcal{C}(I_B)$ so that $\theta(y_1 \odot x_1) = z_1$. It is defined via an instantiation.

Let $y_0 = cy$, a configuration of $\mathbb{C}_G$; as $\alpha_G : \mathbb{C}_G \to G^\perp \| G$ is given as the identity function on events, and $\delta = \alpha_G c$ we also have $y_0 \in G^\perp \| G$. Define $\rho : y_0 \to |\mathcal{B}|$ by cases, according as $e \in y_0$ is in the left or right component of $G^\perp \| G$. If $e \in \{1\} \times G^\perp$, then $\rho(e) \coloneqq \rho_S(e_1)$ when $e_1 \in x_1$ & $\sigma(e_1) = \bar{e}$. If $e \in \{2\} \times G$, then $\rho(e) = \rho_S(e_1)$ when $e_1 \in z_1$ & $\sigma(e_1) = e$. From this definition it can be checked that $\rho(e) = \rho(\bar{e})$ when $e, \bar{e} \in y_0$. For this reason, y_0 and ρ define a configuration y' of $\mathbb{C}_{G,\mathcal{B}}$. The required y_1 is defined by $y_1 = y' \wedge y$. By construction, $\theta(y_1 \odot x_1) = z_1$, as required for θ to be surjective on configurations.

We deduce that θ is an isomorphism $I_{\mathcal{B}} \odot S \cong S$. Similarly, $S \odot I_{\mathcal{A}} \cong S$. This makes (ι_B, ρ_B) the identity w.r.t. composition in $\mathbf{SD}_\delta$. □

E Partial expansion

Definition E.1. Let G be a (Σ, V, C)-game. Let A be a Σ-algebra. Let $V_0 \subseteq V$. The V_0*-expansion* of $(G, \mathcal{A})$ is the event structure $expn^{V_0}(G, \mathcal{A})$ with

- *events* (g, γ) where $\gamma : [g]_{V_0} \to ||A$ assigns an element of $\mathcal{A}$ of the correct sort to each V_0-move on which g causally depends;
- *causal dependency* $(g', \gamma') \leq (g, \gamma)$ iff $g' \leq_G g$ & $\gamma' = \gamma \upharpoonright [g']_{V_0}$;
- *conflict* $(g, \gamma) \# (g', \gamma')$ iff $g \#_G g'$ or $\exists g'' \leq_G g, g' . \, \gamma(g'') \neq \gamma'(g'')$; and
- *polarity* $pol(g, \gamma) = pol_G(g)$, inherited from G.

The function $\epsilon^{V_0}_{G,\mathcal{A}} : |expn_{V_0}(G, \mathcal{A})| \to |G|$ acts so $\epsilon^{V_0}_{G,\mathcal{A}} : (g, \gamma) \mapsto g$; it is an open map of event structures (because each sort of $\mathcal{A}$ is nonempty). The function $\rho^{V_0}_{G,\mathcal{A}} : |expn^{V_0}(G, \mathcal{A})|_{V_0} \to |\mathcal{A}|$ acts so $\rho^{V_0}_{G,\mathcal{A}}(g, \gamma) = \gamma(g)$ for $g \in |G|$ with $\mathrm{vc}(g) \in V_0$.

The pair $(\epsilon^{V_0}_{G,\mathcal{A}}, \rho^{V_0}_{G,\mathcal{A}})$ forms a universal V_0-*instantiation* of G in $\mathcal{A}$:

Proposition E.2. *Let $\sigma : S \to G$ be a total map of event structures with polarity. Let $\rho : |S|_{V_0} \to |\mathcal{A}|$ be a sort-respecting function from $|S|_{V_0} \coloneqq \{s \in |S| \mid \mathrm{vc}(\sigma(\mathcal{A})(s)) \in V_0\}$. Then there is a unique map of instantiations $\sigma_0 : S \to expn^{V_0}(G, \mathcal{A})_{V_0}$ from (σ, ρ) to $(\epsilon^{V_0}_{G,\mathcal{A}}, \rho^{V_0}_{G,\mathcal{A}})$.*

The above follows from the earlier Lemma 4.2 characterising expansion as giving the universal instantiation.

Definition E.3. The *partial expansion* $expn^-(G, \mathcal{A})$ of $(G, \mathcal{A})$ is $expn^{V_0}(G, \mathcal{A})$ where $V_0 \subseteq V$ is the subset of Opponent variables.

Lemma E.4. *The partial expansion $D(\mathcal{A})$ is deterministic. Moreover, if $y \longrightarrow\!\!\!\subset^+ y'$ and $y \longrightarrow\!\!\!\subset^+ y''$ in $\mathcal{C}(D(\mathcal{A}))^o$ and $\pi_1 y' = \pi_1 y''$, then, $y' = y''$.*

Proof. From the construction of the pullback via secured families, and the fact that: if $z \longrightarrow\!\!\!\subset^+ z'$ and $z \longrightarrow\!\!\!\subset^+ z''$ in $\mathcal{C}(expn(G^\perp, \mathcal{A}))^o$ and $\epsilon_{G,\mathcal{A}} \, z' = \epsilon_{G,\mathcal{A}} \, z''$, then $z' = z''$. □

Lemma E.5. *The following diagram is a pullback:*

$$\begin{tikzcd} D(\mathcal{A}, \mathcal{B}) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\delta"] \\ expn^-(G^\perp \| G, \mathcal{A} + \mathcal{B}) \arrow[r, "\epsilon^-_{G^\perp \| G, \mathcal{A}+\mathcal{B}}"'] & G^\perp \| G \end{tikzcd}$$

where $\pi_1 \coloneqq \epsilon^-_{D,\mathcal{A}+\mathcal{B}}$ and π_2 is the unique map of instantiations from $(\delta\pi_1, \rho^-_{D,\mathcal{A}+\mathcal{B}})$ to $(\epsilon^-_{G^\perp \| G,\mathcal{A}+\mathcal{B}}, \rho^-_{G^\perp \| G,\mathcal{A}+\mathcal{B}})$.

Proof. To show it is a pullback consider the following diagram (initially without h)

$$\begin{tikzcd}
S \arrow[dr, dotted, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& D(\mathcal{A},\mathcal{B}) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & D \arrow[d, "\delta"] \\
& expn^-(G^\perp\|G, \mathcal{A}+\mathcal{B}) \arrow[r, "\epsilon^-_{G^\perp\|G,\mathcal{A}+\mathcal{B}}"'] & G^\perp\|G
\end{tikzcd}$$

where f and g are maps such that $\delta f = g\,\epsilon^-_{G^\perp\|G,\mathcal{A}+\mathcal{B}}$. Equip S with $\rho \coloneqq g\,\rho_{G^\perp\|G,\mathcal{A}+\mathcal{B}}$ to form an instantiation (f,ρ) in D. From universality of the Opponent-expansion of D, there is a unique map of instantiations $h : S \to D(\mathcal{A},\mathcal{B})$ from (f,ρ) to $(\pi_1, \rho^-_{D,\mathcal{A}+\mathcal{B}})$. This ensures $f = \pi_1 h$. Hence $\delta f = \epsilon^-_{G^\perp\|G,\mathcal{A}+\mathcal{B}}\,\pi_2 h$. Both g and $\pi_2 h$ are maps of instantiations from $(\delta f, \rho)$ to $(\epsilon^-_{G^\perp\|G,\mathcal{A}+\mathcal{B}}, \rho_{G^\perp\|G,\mathcal{A}+\mathcal{B}})$. From the latter's universality, the two maps must be equal, $g = \pi_2 h$. This provides a mediating map $h : S \to D(\mathcal{A},\mathcal{B})$, as shown in the diagram above. We also require its uniqueness. Suppose $h' : S \to D(\mathcal{A},\mathcal{B})$ is another mediating map, *i.e.* so $f = \pi_1 h'$ and $g = \pi_2 h'$. Then those two conditions together ensure that h' is a map of instantiations from (f,ρ) to $(\pi_1, \rho^-_{D,\mathcal{A}+\mathcal{B}})$. From the latter's universality $h' = h$. □

As $\pi_1 = \epsilon^-_{D,\mathcal{A}+\mathcal{B}}$, it is open. As δ is a strategy, its pullback π_2 along $\epsilon^-_{G^\perp,\mathcal{A}}\|\epsilon^-_{G,\mathcal{B}}$ a strategy in $expn^-(G^\perp,\mathcal{A})\|expn^-(G,\mathcal{B})$. Write $\sigma(\mathcal{A},\mathcal{B})$ for the composite map

$$\sigma(\mathcal{A},\mathcal{B}) = \delta \circ \pi_1 = (\epsilon^-_{G^\perp,\mathcal{A}}\|\epsilon^-_{G,\mathcal{B}}) \circ \pi_2 : D(\mathcal{A},\mathcal{B}) \to G^\perp\|G\,.$$

Write $\rho(\mathcal{A})$ for the sort-respecting function $\rho(\mathcal{A}) : |D(\mathcal{A},\mathcal{B})|_{V_1^-} \to |\mathcal{A}|$ sending $s \in |D(\mathcal{A},\mathcal{B})|_{V_1^-}$ to $\rho_{G^\perp,\mathcal{A}}(e)$ when $\pi_2(s) = (1,e)$; similarly, $\rho(\mathcal{B})$ is the sort-respecting function $\rho(\mathcal{B}) : |D(\mathcal{A},\mathcal{B})|_{V_2^-} \to |\mathcal{B}|$ sending $s \in |D(\mathcal{A},\mathcal{B})|_{V_2^-}$ to $\rho_{G,\mathcal{B}}(e)$ when $\pi_2(s) = (2,e)$. Then $\rho_{D,\mathcal{A}+\mathcal{B}} = \rho(\mathcal{A}) \cup \rho(\mathcal{B})$ as it acts as $\rho(\mathcal{A})$ on V_1^--moves and as $\rho(\mathcal{B})$ on V_2^--moves.

Lemma 5.2. *The map θ in the diagram above is an isomorphism.*

Proof. It suffices to show that θ is rigid and both injective and surjective on configurations (Lemma 3.3 of [Win07]). As (σ,ρ) is a strategy in $\mathbf{SD}_\delta$, the map $\pi_1\theta : S \to D$ is open so rigid. Consequently θ is rigid. The composite map

$$S \xrightarrow{\sigma_0} expn(G^\perp,\mathcal{A})\|expn(G,\mathcal{B}) \to expn^-(G^\perp,\mathcal{A})\|expn^-(G,\mathcal{B})$$

is a (nondeterministic) strategy—this relies on $expn^-(G^\perp,\mathcal{A})\|expn^-(G,\mathcal{B})$ being an Opponent-expansion to ensure that further expansion, necessarily by values for Player moves, doesn't violate receptivity. As remarked earlier, π_2 is also a strategy. The map θ, as a mediating 2-cell between these two strategies, is itself a strategy, so receptive as well as rigid. As σ is deterministic so is S, making θ a deterministic strategy and ensuring its injectivity on configurations. To show that θ is surjective on configurations consider a covering chain

$$\emptyset {-\!\!\subset} y_1 {-\!\!\subset} \cdots {-\!\!\subset} y_i {-\!\!\subset} y_{i+1} {-\!\!\subset} \cdots {-\!\!\subset} y_n = y$$

to an arbitrary y in $\mathcal{C}(D(\mathcal{A},\mathcal{B}))^o$. We show by induction along the chain that there are

$$\emptyset {-\!\!\subset} x_1 {-\!\!\subset} \cdots {-\!\!\subset} x_i {-\!\!\subset} x_{i+1} {-\!\!\subset} \cdots {-\!\!\subset} x_n = x$$

in $\mathcal{C}(S)^o$ such that each $y_i = \theta x_i$. Establishing this inductively for steps $y_i {-\!\!\subset}^- y_{i+1}$ is a direct consequence of the receptivity of θ. Consider a step $y_i {-\!\!\subset}^+ y_{i+1}$ where inductively we assume $y_i = \theta x_i$. Then $\pi_1\theta\,x_i = \pi_1\,y_i {-\!\!\subset}^+ \pi_1\,y_{i+1}$. From the openness of $\pi_1\theta$ there is x_{i+1} with $x_i {-\!\!\subset}^+ x_{i+1}$ such that $\pi_1\theta\,x_{i+1} = \pi_1\,y_{i+1}$. Now, $y_i {-\!\!\subset}^+ \theta\,x_{i+1}$ and $y_i {-\!\!\subset} y_{i+1}$ in $\mathcal{C}(D(\mathcal{A},\mathcal{B}))^o$ and $\pi_1\theta\,x_{i+1} = \pi_1 y_{i+1}$. By Lemma E.4, $y_{i+1} = \theta\,x_{i+1}$, □

F Proofs for Section 6 (One-sided games)

Lemma F.1. *(i)(a) If $e_1 \in D_1$, $e_2 \in D_2$ and $e_1 \le_D e_2$, then there is a unique companion $\bar{e}_1 \in D_2$ of e_1 ;*
(b) if $e_1 \in D(\mathcal{A})_1$, $e_2 \in D(\mathcal{A})_2$ and $e_1 \le_{D(\mathcal{A})} e_2$, then there is a unique companion $\bar{e}_1 \in D(\mathcal{A})_2$ of e_1 ;
(ii) if $e_0, e_1 \in D_1$, $e_2 \in D_2$ and $e_0 \le_D \bar{e}_1$ and $e_1 \le_D e_2$, then $e_0 \le_D e_2$.

Proof. Follows from the idempotency of δ. (W.l.o.g. here we assume the isomorphism $d : \delta \Rightarrow \delta \odot \delta$ is given by the identity function on events.)

(i) (a) Otherwise the causal dependency of $D \odot D$ would lack $e_1 \le e_2$ which needs the synchronisation of companions e_1 and $\bar{e}_1$, and not coincide with that of D. (b) The analogous fact for $D(\mathcal{A})$ follows from the openness of $\pi_1 : D(\mathcal{A}) \to D$. The uniqueness of +ve companions of −ve moves, if they exist, follows from D and $D(\mathcal{A})$ being deterministic.

(ii) If $e_0, e_1 \in D_1$, $e_2 \in D_2$ and $e_0 \le_D \bar{e}_1$ and $e_1 \le_D e_2$, then $e_0 \le_{D \odot D} \bar{e}_2$ via the synchronisation of e_1 and $\bar{e}_1$ in the interaction, and hence $e_0 \le_D \bar{e}_2$ by idempotency.

□

Theorem 6.2 (comonadic characterisation). *Assume G has homomorphic winning condition. The operation $\mathrm{Rel}_\delta(\_)$ extends to a unique comonad on $\mathfrak{R}(\Sigma)$, which*

- *maps every Σ-structure $\mathcal{A}$ to $\mathrm{Rel}_\delta(\mathcal{A})$;*
- *has counit $\rho_\mathcal{A} : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathcal{A}$ acting as $e \mapsto \rho(\mathcal{A})(\bar{e})$;*
- *has coextension mapping a homomorphism $h : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathcal{B}$ to a homomorphism $h^\dagger : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathrm{Rel}_\delta(\mathcal{B})$.*

Proof. We check that a homomorphism $h : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathcal{B}$ is taken to a homomorphism $h^\dagger : \mathrm{Rel}_\delta(\mathcal{A}) \to \mathrm{Rel}_\delta(\mathcal{B})$ by the construction above. Suppose $R(e_1, \ldots, e_k)$ in $\mathrm{Rel}_\delta(\mathcal{A})$, *i.e.*

$$e_1, \ldots, e_k \in \mathrm{last}(x) \ \&\ R_\mathcal{A}(\rho(\mathcal{A})(\bar{e}_1), \ldots, \rho(\mathcal{A})(\bar{e}_k)) \ \&\ \ \&\ x_{G^\perp} \models W\,,$$

for some +-maximal configuration x of $D(\mathcal{A})$. We require that $R(h^\dagger(e_1), \ldots, h^\dagger(e_k))$ in $\mathrm{Rel}_\delta(\mathcal{B})$, *i.e.* for some +-maximal configuration y of $D(\mathcal{B})$,

$$h^\dagger(e_1), \ldots, h^\dagger(e_k) \in \mathrm{last}(y) \ \&\ R_\mathcal{B}(\rho(\mathcal{B})(\overline{h^\dagger(e_1)}), \ldots, \rho(\mathcal{B})(\overline{h^\dagger(e_k)})) \ \&\ \ \&\ y_{G^\perp} \models W.$$

By virtue of G having homomorphic W, we achieve this by taking $y \coloneqq h^\dagger x$.

To establish the comonad we need to verify the laws

$$\rho^\dagger_{G,\mathcal{A}} = \mathrm{id}_{\mathrm{Rel}_\delta(\mathcal{A})}, \quad \rho_{G,\mathcal{A}} \circ h^\dagger = h, \quad (k \circ h^\dagger)^\dagger = k^\dagger \circ h^\dagger.$$

They follow routinely. ◻

Corollary 6.3. *Assume G has homomorphic winning condition. When $\delta = \mathbb{c}_G$, under $e \mapsto \bar{e}$ the Σ-structure $\mathrm{Rel}_\delta(\mathcal{A})$ is isomorphic to $|expn(G, \mathcal{A})|_V$ with Σ-relations*

$$R(e_1, \cdots, e_k) \textit{ in } |\exp(G, \mathcal{A})|_V \textit{ iff } \exists x \in C(\exp(G, \mathcal{A})).$$
$$e_1, \cdots, e_k \in \mathrm{last}(x) \ \&\ R_\mathcal{A}(\rho_{G,\mathcal{A}}(e_1), \cdots, \rho_{G,\mathcal{A}}(e_k)) \ \&\ x \models W.$$

Under this isomorphism the counit acts as $\rho_{G,\mathcal{A}}$ and coextension takes $h : |expn(G, \mathcal{A})|_V \to |\mathcal{B}|$ to $h^\dagger : |expn(G, \mathcal{A})|_V \to |expn(G, \mathcal{B})|_V$, where $h^\dagger(g, \gamma) = (g, \gamma')$ with $\gamma'(g') = h(g', \gamma \restriction [g']_G)$, for all $g' \leq_G g$.

Proof. For $x \in C(\exp(G, \mathcal{A}))$, write

$$R_x(e_1, \cdots, e_k) \ \text{ iff } \ e_1, \cdots, e_k \in \mathrm{last}(x) \ \&\ R_\mathcal{A}(\rho_{G,\mathcal{A}}(e_1), \cdots, \rho_{G,\mathcal{A}}(e_k)) \ \&\ x \models W.$$

By definition, we have $R(e_1, \cdots, e_k)$ in $\mathrm{Rel}_{\mathbb{c}_G}(\mathcal{A})$ iff there exists +-maximal $y \in C(\mathbb{C}_G(\mathcal{A}))$ with $e_1, \cdots, e_k \in \mathrm{last}(y)$ and $R_\mathcal{A}(\rho(\mathcal{A})(\bar{e}_1), \cdots, \rho(\mathcal{A})(\bar{e}_k))$ and $y_{G^\perp} \models W$. Taking $x = y_{G^\perp} \in C(expn(G, \mathcal{A}))$ we obtain $R_x(\bar{e}_1, \cdots, \bar{e}_k)$. Then, $R(\bar{e}_1, \cdots, \bar{e}_k)$ in $|\exp(G, \mathcal{A})|_V$.

Conversely, if $R(\bar{e}_1, \cdots, \bar{e}_k)$ in $|\exp(G, \mathcal{A})|_V$ then $R_{\bar{x}}(\bar{e}_1, \cdots, \bar{e}_k)$ for some configuration $\bar{x}$ of $expn(G, \mathcal{A})$. Letting $x \coloneqq \epsilon_{G,\mathcal{A}}\bar{x}$ we construct a +-maximal configuration $y \in C(\mathbb{C}_G(\mathcal{A}))$ as follows. The map $\epsilon_{G,\mathcal{A}} \| \mathrm{id}_x : \bar{x} \| x \to G^\perp \| G$ together with $\rho : (\bar{x} \| x)_{V_1} \to |\mathcal{A}|$, taking $(1, \bar{e})$ to $\rho_{G,\mathcal{A}}(\bar{e})$, form an instantiation of (the left part of) $G^\perp \| G$ in $\mathcal{A}$. Via the universality of partial expansion E.2 we obtain a map $\bar{x} \| x \to \mathbb{C}_G$. Its image provides $y \coloneqq (\bar{x} \| x)[\rho] \in C(\mathbb{C}_G(\mathcal{A}))$. Then, $y_{G^\perp} = \bar{x}$ so $y_{G^\perp} \models W$ and—writing e_i for the companion of $\bar{e}_i$ in y —we have $e_1, \cdots, e_k \in \mathrm{last}(y)$ with $R_\mathcal{A}(\rho(\mathcal{A})(\bar{e}_1), \cdots, \rho(\mathcal{A})(\bar{e}_k))$. Hence $R(e_1, \cdots, e_k)$ in $\mathrm{Rel}_{\mathbb{c}_G}(\mathcal{A})$.

The re-expression of the counit and extension follow directly from the bijection $|\mathrm{Rel}_\delta(\mathcal{A})| \cong |expn(G, \mathcal{A})|_V$ given by $e \mapsto \bar{e}$. ◻

6.0 Eilenberg-Moore coalgebras. In proving the characterisation of coalgebras of Rel_δ it will be convenient to use the following proposition [Win82, Win86]:

Proposition F.2. *Let A and B be event structures. A partial function $f : |A| \rightharpoonup |B|$ is a map of event structures iff*

- *$f(a) = b \ \&\ b' \leq b \implies \exists a' \leq a.\ f(a') = b'$, for all $a \in |A|, b'b' \in |B|$; and*
- *$f(a) = f(a')$ or $f(a) \# f(a') \implies a = a'$ or $a \# a'$, for all $a'a' \in |A|$.*

Theorem 6.4. *δ-Event algebras and their maps form a category isomorphic to the category of Eilenberg-Moore coalgebras of $\mathrm{Rel}_\delta(\_)$.*

Proof. Abbreviate Rel_δ to $\mathcal{R}$. Let $f : \mathcal{A} \to \mathcal{R}(\mathcal{A})$ be an Eilenberg-Moore coalgebra of the comonad $\mathcal{R}(\_)$. By definition, we have the commuting diagrams

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d, "f"'] & \mathcal{R}(\mathcal{A}) \arrow[d, "\nu_\mathcal{A}"] \\
\mathcal{R}(\mathcal{A}) \arrow[r, "\mathcal{R}(f)"'] & \mathcal{R}(\mathcal{R}(\mathcal{A}))
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[dr, "\mathrm{id}_\mathcal{A}"'] & \mathcal{R}(\mathcal{A}) \arrow[d, "\rho_\mathcal{A}"] \\
 & \mathcal{A}\,.
\end{tikzcd}$$

Comultiplication $\nu_\mathcal{A}$ is obtained as $(\mathrm{id}_{\mathcal{R}(\mathcal{A})})^\dagger$ from which we derive

$$\nu_\mathcal{A}(e, \gamma) = (e, \gamma')$$

where, for all $e_1 \in [e]_{V_1}$,

$$\gamma'(e_1) = (\bar{e}_1, \gamma \upharpoonright [\bar{e}_1]) .$$

Let $a \in |A|$ and suppose $f(a) = (e, \gamma)$. From the second commuting diagram we get

$$\gamma(\bar{e}) = a . \tag{1}$$

It follows directly that f is injective. From the first commuting diagram we obtain

$$\mathcal{R}(f) \circ f(a) = \nu_{\mathcal{A}} \circ f(a) .$$

But

$$\mathcal{R}(f) \circ f(a) = \mathcal{R}(f)(e, \gamma) = (e, f \circ \gamma)$$

while

$$\nu_{\mathcal{A}} \circ f(a) = \nu_{\mathcal{A}}(e, \gamma) = (e, \gamma')$$

where, for all $e_1 \in [e]_{V_1}$,

$$\gamma'(e_1) = (\bar{e}_1, \gamma \upharpoonright [\bar{e}_1]) .$$

Thus, from the commuting first diagram we obtain

$$\forall e_1 \in [e]_{V_1}\ f(\gamma(e_1)) = (\bar{e}_1, \gamma \upharpoonright [\bar{e}_1]) . \tag{2}$$

Construct an event structure $(|\mathcal{A}|, \leq_{\mathcal{A}}, \#_{\mathcal{A}})$ on the elements of $\mathcal{A}$. For $a, a' \in |\mathcal{A}|$, take

$$a' \leq_{\mathcal{A}} a \text{ iff } f(a') \leq f(a) \text{ and } a' \#_{\mathcal{A}} a \text{ iff } f(a') \# f(a) , \text{ in } D(\mathcal{A}) .$$

Now, from a special case of (2), if $f(a) = (e, \gamma)$ and $(e', \gamma') \leq (e, \gamma)$ in $D(\mathcal{A})_{V_2}$ then $\bar{e}' \in [e]_{V_1}$, so taking $a' = \gamma(\bar{e}')$ we obtain $a' \leq_{\mathcal{A}} a$ and $f(a') = (e', \gamma')$. As f is also injective and reflects conflict it is a map of event structures, by Proposition F.2. The function f is an injective and a rigid map of event structures, *i.e.* a rigid embedding

$$f : (|\mathcal{A}|, \leq_{\mathcal{A}}, \#_{\mathcal{A}}) \to \mathcal{R}(\mathcal{A}) ,$$

which additionally satisfies (1) and (2).

Moreover, suppose $R_{\mathcal{A}}(a_1, \cdots, a_k)$. Then

$$\exists x \in C(D(\mathcal{A})).\ x \text{ is } +\text{-maximal } \&\ f(a_1), \cdots, f(a_k) \in \mathrm{last}_{D(\mathcal{A})}(x)\ \&\ x_{G^\perp} \models W \tag{3}$$

—this follows from f being a homomorphism and the interpretation of R in $\mathcal{R}(\mathcal{A})$.

Conversely, a rigid map which satisfies (1) and (2) checks out to be a coalgebra.

The category of Eilenberg-Moore coalgebras of $\mathcal{R}(\_)$ has maps homomorphisms h making

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "h"'] & \mathcal{R}(\mathcal{A}) \arrow[d, "\mathcal{R}(h)"] \\ \mathcal{B} \arrow[r, "g"'] & \mathcal{R}(\mathcal{B}) , \end{tikzcd}$$

commute, a fact which is left undisturbed when lifted to the required maps of event structures. □

F.1 One-sided characterisation

Theorem 6.8 (one-sided characterisation). *Let $\mathcal{A}$, $\mathcal{B}$ and C be Σ-structures.*

(1) Assuming $\mathbf{SD}_\delta$ has identities, the function $j_{\mathcal{A}} : |D(\mathcal{A})|_{V_2} \to \mathcal{A}$ such that $j_{\mathcal{A}}(e) = \rho(\mathcal{A})(\bar{e})$ is a δ-homomorphism.

(2) Suppose h is a δ-homomorphism from $\mathcal{A}$ to $\mathcal{B}$, and k is a δ-homomorphism from $\mathcal{B}$ to C. Then, the composition $k \circ h^\dagger : |D(\mathcal{A})|_{V_2} \to |C|$ is a δ-homomorphism from $\mathcal{A}$ to C.

Further, Σ-structures and δ-homomorphisms form a category with identities given by (1) and composition given by (2). Provided $\mathbf{SD}_\delta$ has identities, the category $\mathbf{SD}_\delta$ is isomorphic to the category of Σ-structures with δ-homomorphisms.

Proof. (i) Assume $\mathbf{SD}_\delta$ has identities, *i.e.* the strategy $(i_{\mathcal{A}}, \rho_{\mathcal{A}})$ of Section 5.1 is winning so an identity strategy in $\mathbf{SD}_\delta$. From Lemma 6.6, the identity strategy $(\iota_{\mathcal{A}}, \rho_{\mathcal{A}})$ is isomorphic to the strategy $(\iota_{\mathcal{A}}(\mathcal{A}), \rho(\mathcal{A}) + j_{\mathcal{A}})$. Because $(\iota_{\mathcal{A}}, \rho_{\mathcal{A}})$ is winning, so is $(\iota_{\mathcal{A}}(\mathcal{A}), \rho(\mathcal{A}) + j_{\mathcal{A}})$. Thus if x is a +-maximal configuration of $D(\mathcal{A})$ such that $x_{G^\perp} \models W_G$, then $(x[j_{\mathcal{A}}])_G \models W_G$, as required for $j_{\mathcal{A}}$ to be a δ-homomorphism.
(ii) From (i), when $\mathbf{SD}_\delta$ has identities $(\iota_{\mathcal{A}}, \rho_{\mathcal{A}})$ they correspond with the δ-homomorphisms $j_{\mathcal{A}}$. Let $\mathcal{A}$, $\mathcal{B}$ and C be Σ-structures. Let h be a δ-homomorphism from $\mathcal{A}$ to $\mathcal{B}$, and k a δ-homomorphism from $\mathcal{B}$ to C; they correspond to strategies $\mathcal{A} \mathrel{+\!\!\!\rightarrow}_\delta \mathcal{B}$ and $\mathcal{B} \mathrel{+\!\!\!\rightarrow}_\delta C$. We need to check the composition $k \circ h^\dagger : |D(\mathcal{A})|_{V_2} \to |C|$ is identical to the δ-homomorphism, say l, corresponding to the composition $\mathcal{A} \mathrel{+\!\!\!\rightarrow}_\delta C$ of the two strategies.

The δ-homomorphism h from $\mathcal{A}$ to $\mathcal{B}$ corresponds to a concurrent strategy

$$\sigma_0 : D(\mathcal{A}) \to expn(G^\perp, \mathcal{A}) \| expn(G, \mathcal{B})$$

while k from $\mathcal{B}$ to C corresponds to a concurrent strategy

$$\sigma_0 : D(\mathcal{B}) \to expn(G^{\perp}, \mathcal{B}) \| expn(G, C).$$

It is convenient to identify the configurations of $D(\mathcal{A})$ with their images in $expn(D(\mathcal{A}), \mathcal{B})$ given by the instantiation h : a configuration then has the form $(y[\rho_A])[h]$ for some $y \in C(D)$ and sort-respecting $\rho_A : |D|_{V_1} \to |\mathcal{A}|$. Similarly, a configuration of the concurrent strategy for k has the form $(z[\rho_B])[k]$ for some $z \in C(D)$ and sort-respecting $\rho_B : |D|_{V_1} \to |\mathcal{B}|$.

Consider an arbitrary $(e_0, \gamma_0) \in |D(\mathcal{A})|_{V_2}$. There exist $x \in C(D)$ and $\rho_A : |D|_{V_1} \to |\mathcal{A}|$ such that $(e_0, \gamma_0) \in x[\rho_A]$; then $e_0 \in x$ and $\gamma_0 = \rho_A(e')$ for all V_1-moves $e' \leq_D e_0$.

From the idempotency of δ there are $y, z \in C(D)$ for which $dx = z \odot y$. Consider a specific configuration of the interaction of the two strategies, *viz.*

$$((z[\rho_B])[k]) \circledast (y[\rho_A])[h]),$$

where $(y[\rho_A])[h] \in expn(D(\mathcal{A}), \mathcal{B})$ and $(z[\rho_B])[k] \in expn(D(\mathcal{B}), C)$; in order for the interaction to be defined, for all V_1-moves $e_1 \in z$, we must have

$$\rho_B(e_1) = h((\bar{e}_1, \gamma_1))$$

with $\gamma_1(e') = \rho_A(e')$ for all V_1-moves $e' \leq_D \bar{e}_1$.

Consider a V_2-move $(e, \gamma) \in z[\rho_B]$. Then,

$$\gamma(e_1) = \rho_B(e_1) \text{ for all } V_1\text{-moves } e_1 \leq_D e, \ \textit{i.e.}$$

$$\gamma(e_1) = h(\bar{e}_1, \gamma_1) \text{ for all } V_1\text{-moves } e_1 \leq_D e,$$

where γ_1 is defined above. In other words, recalling the definition of $h^{\dagger}$,

$$(e, \gamma) = h^{\dagger}((e, \gamma')),$$

where $\gamma'(e') = \rho_A(e')$ for all $e' \leq_D e$ with e' a V_1-move. Consequently, the V_2-move $(e, \gamma) \in z[\rho_B]$ is instantiated to $k(h^{\dagger}((e, \gamma')))$ in $(z[\rho_B])[k]$. Hence $k \circ h^{\dagger}((e, \gamma')) = l((e, \gamma'))$.

In particular, there is a V_2-move $(e, \gamma) \in z[\rho_B]$ for which $(e, \gamma) = h^{\dagger}((e, \gamma_0)))$, ensuring $k \circ h^{\dagger}((e, \gamma_0)) = l((e, \gamma_0))$ for an arbitrary $(e_0, \gamma_0) \in |D(\mathcal{A})|_{V_2}$. Thus l, the δ-homomorphism corresponding to the composition of strategies, and $k \circ h^{\dagger}$ are the same functions from $|D(\mathcal{A})|_{V_2}$ to $|C|$. □

G Proofs for Section 7 (Two-sided games)

We explain in more detail how G-spans correspond to two-sided strategies in $\mathbf{SD}_{\mathbb{C}_G}$. By Corollary 7.1 such a strategy corresponds to a pair of functions h and k from which we construct a span

$$\begin{tikzcd} & \mathbb{C}_G(\mathcal{A}, \mathcal{B})^{+} \arrow[dl, "l_{\mathcal{A}}"'] \arrow[dr, "r_{\mathcal{B}}"] & \\ expn(G, \mathcal{A}) & & expn(G, \mathcal{B})\,. \end{tikzcd}$$

The span is "winning" in the sense that $l_{\mathcal{A}} x \models W$ implies $r_{\mathcal{B}} x \models W$, for any $x \in C(\mathbb{C}_G(\mathcal{A}, \mathcal{B})^{+})$. By the universality of expansions involved in their definitions, the maps $l_{\mathcal{A}}$ and $r_{\mathcal{B}}$ are unique such that

$$\epsilon_{G,\mathcal{A}} \circ l_{\mathcal{A}} = \mathrm{f}_{\mathcal{A},\mathcal{B}} \quad \text{and} \quad \rho_{G,\mathcal{A}} \circ l_{\mathcal{A}}(s) = \rho(\mathcal{A})(\bar{s})\,, \text{ for } s \in |\mathbb{C}_G(\mathcal{A}, \mathcal{B})|_{V_2^+} \tag{a}$$

and

$$\epsilon_{G,\mathcal{B}} \circ r_{\mathcal{B}} = \mathrm{f}_{\mathcal{A},\mathcal{B}} \quad \text{and} \quad \rho_{G,\mathcal{B}} \circ r_{\mathcal{B}}(s) = \rho(\mathcal{B})(\bar{s})\,, \text{ for } s \in |\mathbb{C}_G(\mathcal{A}, \mathcal{B})|_{V_1^+}\,. \tag{b}$$

The original functions h and k are recovered as

$$k(s) = \rho_{G,\mathcal{A}} \circ l_{\mathcal{A}}(s)\,, \text{ for } s \in |\mathbb{C}_G(\mathcal{A}, \mathcal{B})|_{V_1^+}\,,$$

and

$$h(s) = \rho_{G,\mathcal{B}} \circ r_{\mathcal{B}}(s)\,, \text{ for } s \in |\mathbb{C}_G(\mathcal{A}, \mathcal{B})|_{V_2^+}\,.$$

Thus winning G-spans $l_{\mathcal{A}}$, $r_{\mathcal{B}}$, satisfying (a) and (b), bijectively correspond to two-sided strategies in $\mathbf{SD}_{\mathbb{C}_G}$.

Theorem 7.2. $\mathrm{SD}_{\mathbb{C}_G}$ *is isomorphic to the category of G-spans.*

Proof. (Outline) A strategy in $\mathbf{SD}_{\mathbb{C}_G}(\mathcal{A}, \mathcal{B})$ takes the form of an instantiation $(\sigma(\mathcal{A}, \mathcal{B}), \rho_{\mathcal{A},\mathcal{B}})$ where recall $\sigma(\mathcal{A}, \mathcal{B}) : \mathbb{C}_G(\mathcal{A}, \mathcal{B}) \to G^{\perp} \| G$ and $\rho_{\mathcal{A},\mathcal{B}} : |\mathbb{C}_G(\mathcal{A}, \mathcal{B})^{+}|_{V+V} \to |\mathcal{A} + \mathcal{B}|$. Similarly, a strategy in $\mathbf{SD}_{\mathbb{C}_G}(\mathcal{B}, C)$ is an instantiation $(\sigma(\mathcal{B}, C), \rho_{\mathcal{B},C})$. From Lemma 5.2, in their composition,

$$\mathbb{C}_G(\mathcal{A}, C) \cong \mathbb{C}_G(\mathcal{B}, C) \odot \mathbb{C}_G(\mathcal{A}, \mathcal{B})\,.$$

Furthermore,

$$(\mathbb{C}_G(\mathcal{B}, C) \odot \mathbb{C}_G(\mathcal{A}, \mathcal{B}))^{+} \cong \mathbb{C}_G(\mathcal{A}, \mathcal{B})^{+} \wedge \mathbb{C}_G(\mathcal{B}, C)^{+}\,,$$

the associated pullback in the composition of spans. To see this, consider a configuration w of the lhs. Its downclosure $[w]$ in the interaction $\mathbb{C}_G(\mathcal{B},\mathcal{C}) \circledast \mathbb{C}_G(\mathcal{A},\mathcal{B})$, of the form $y \circledast x$, is such that for both x and y their images in $G^\perp \| G$,

$$\sigma(\mathcal{A},\mathcal{B})\, x = z\|z \ \text{ and } \ \sigma(\mathcal{B},\mathcal{C})\, y = z\|z\,,$$

for some $z \in \mathcal{C}(G)$ —this is because $y \circledast x$ is the downclosure of +-moves; with the instantiations $\rho_{\mathcal{A},\mathcal{B}}$ and $\rho_{\mathcal{B},\mathcal{C}}$ agreeing over $\mathcal{B}$. From this, the arbitrary configuration w of the lhs bijectively corresponds to a configuration $x^+ \wedge y^+$ of the pullback on the rhs.

This shows the agreement between the composition of spans and that of the strategies with which they correspond. $\square$

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