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Extraction of Analytic Singular Values of a Polynomial Matrix

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***Abstract*—The proof of existence of an analytic singular value decomposition (SVD) has been formally established. This motivates the need to devise a singular value extraction algorithm which retrieves analytic singular values that are real-valued on the unit circle. We propose a frequency domain method which first computes a standard SVD of the given polynomial matrix in each discrete Fourier transform (DFT) bin. To re-establish their association across bins, the bin-wise singular values are permuted by assessing the orthogonality of singular vectors in adjacent DFT bins. In addition, the proposed algorithm determines whether bin-wise singular value may become negative, which is required for analyticity. The proposed algorithm is validated through an ensemble of polynomial matrices with known analytic SVD.**

I. Introduction

Signal processing problems are often formulated using matrix algebra. As such linear algebra factorisations like the singular value decomposition (SVD) have proven to be useful in devising solutions. This is particularly true in the field of narrowband array processing where any temporal correlation between signals reduces to a phase shift. In the case of broadband problems such correlations have to be modelled by proper time delays. If matrices are constructed containing delayed versions of a given signal, the resulting linear algebra decomposition effectively mixes the temporal and spacial dimensions. So, for example, signal enumeration using the number of non-zero singular values no longer works [1].

One approach to finding solutions to broadband problems is to switch to the z-transform domain. Here any matrix with a time delayed copy of a given signal gets transformed to one where the entries are polynomials (in z). It is therefore desirable to find analogues to the familiar linear algebra decompositions but for polynomial matrices. In last decade we have seen significant development in field of polynomial matrix algebra especially the polynomial eigenvalue decomposition (PEVD), polynomial singular value decomposition (PSVD) and the polynomial QR decomposition (PQRD). With a recent proof of analytic EVD existence, analytic eigenvalues and eigenvectors extraction algorithms [2], [3] has proposed. Subsequently, the analytic SVD existence proof has been formally established [4], [5], where the singular value are conventionally restricted to be real-valued on the unit circle similar to the standard SVD. Note however, in order to be analytic, the singular values must not be restricted to be non-negative on the unit circle.

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As state-of-the art, the SVD of a polynomial matrix can be computed either through time-domain approximate algorithms or via the application of two analytic EVDs. Both these methods produce singular values that are complex valued on the unit circle which is opposed to the convention of singular values being real-valued on the unit circle. The theoretical developments outlined above now motivate the development of new algorithms to extract analytic singular values, with an expected impact of higher precision and lower computational cost for applications such as MIMO communications [6], [7], [8], [9], paraunitary matrix completion and delay estimation [10], [11], or subspace detection [12]. Therefore, in this paper we propose an algorithm that extracts analytic singular values of a polynomial matrix which are real-valued on the unit circle.

II. Analytic SVD and State-of-the-Art Algorithms

A. Existence and Uniqueness of the Analytic SVD

An analytic matrix $\boldsymbol{A}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times N}$, which cannot be tied to any multiplexing operations, permits an analytic SVD [4], [5]

$$\boldsymbol{A}(z^{\kappa}) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\boldsymbol{V}^{\mathrm{P}}(z) \; , \qquad (1)$$

where $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$ are paraunitary matrices containing the analytic left- and right singular vectors, and $\boldsymbol{\Sigma}(z)$ is a diagonal matrix that holds the corresponding analytic singular values, and $\kappa = 1, 2$. In accordance with the standard SVD, we expect it to be real valued and non-negative on the unit circle. For $\boldsymbol{\Sigma}(z)$ to be analytic, the non-negativity on the unit circle must be relaxed, and we have to permit the singular values in $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})$ to take on negative values; this is unusual but similarly encountered for the analytic SVD in a real interval [13], [14]. Where singular values in $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})$ become negative, an odd number of zero-crossings will require $\kappa = 2$ in (1) for the analytic SVD to exist; we exclude this case for the purpose of this paper, but unlike e.g. [15], [16] do not restrict $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})$ to be real.

Below, we assume w.l.o.g. that $M \geq N$, as otherwise we can determine the analytic SVD as the parahermitian transpose of $\boldsymbol{A}$, i.e. of $\boldsymbol{A}^{\mathrm{P}} = \{\boldsymbol{A}(1/z^*)\}^{\mathrm{H}} : \mathbb{C} \rightarrow \mathbb{C}^{N\times M}$. The singular values in $\boldsymbol{\Sigma}(z)$ are unique upto a permutation. If distinct, their corresponding singular vectors can be modified by arbitrary allpass functions $\varphi_n(z)$, $n = 1, \ldots, N$, such that $\boldsymbol{U}(z) \cdot \mathrm{blockdiag}\{\boldsymbol{\Phi}(z), \boldsymbol{\Phi}'(z)\}$ and $\boldsymbol{V}(z)\boldsymbol{\Phi}(z)$ also hold valid analytic singular vectors of $\boldsymbol{A}(z)$, with $\boldsymbol{\Phi} = \mathrm{diag}\{\varphi_1(z), \ldots, \varphi_N(z)\}$ and $\boldsymbol{\Phi}'$ an arbitrary $(M-N)\times(M-N)$ paraunitary matrix, since there are $M-N$ left singular vectors that can form an arbitrary basis in the nullspace of $\boldsymbol{A}(z)$.

B. Polynomial/Analytic SVD Extraction Algorithms

In order to approximate (1), a number of algorithms exist. In the time domain, extensions of polynomial EVD algorithms — the second order sequential best rotation (SBR2) [17] and sequential matrix diagonalisation (SMD) [18] — have the resulted in the generalised SBR2 [19] and the generalised SMD [20] to iterative approximate a polynomial SVD (PSVD). These algorithms utilise elementary paraunitary operations [21] to transfer off-diagonal energy to the diagonal in each iteration. GSBR2 and GSMD often only result in an approximate diagonalisation at best, and the singular values they produce are complex valued on the unit circle, which somewhat violates the convention of real-valuedness of singular values. More profoundly, both approaches converge to a spectrally majorised solution with $|\sigma_n(\mathrm{e}^{\mathrm{j}\Omega})| \geq |\sigma_{n+1}(\mathrm{e}^{\mathrm{j}\Omega})$, $\forall\Omega$ and $n = 1, \ldots, (N-1)$, which will differ from $\boldsymbol{\Sigma}(z)$ in (1) in the case that singular values intersect, and therefore converge to SVD factors consisting of only piecewise analytic functions [5].

An alternative approach proposes the use of two PEVDs to compute one PSVD [9], [5], which utilising the SMD and SBR2 algorithms to compute the PEVDs of the parahermitian products $\boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z)$ and $\boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{A}(z)$. This approach produces approximate singular values similar to the GSBR2 and GSMD, that are generally complex-valued on the unit circle. In contrast, the generalised polynomial power method (GPPM) [15] extracts the dominant singular value in a way that it is real valued on the unit circle. However, it has several limitations. For instance, it is not applicable where a dominant singular value takes on negative values on the unit circle. Additionally, it can only extract the dominant singular value instead of all N singular values.

In the DFT domain, the analytic singular values of a polynomial matrix can be obtained by computing two analytic EVDs [22], [2], [3], [5] of again $\boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z)$ and $\boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{A}(z)$. Apart from being computationally expensive, the singular values obtained through this approach are also complex valued on the unit circle.

III. Proposed Method

A. Bin-Wise SVD of $\boldsymbol{A}(z)$

The proposed method evaluates $\boldsymbol{A}(z)$ on the unit circle at equispaced frequency points $\mathbf{A}_k = \boldsymbol{A}(z)|_{z=\mathrm{e}^{\mathrm{j}\Omega_k}}, \Omega_k = \frac{2\pi k}{K}$, $k = 0, \ldots, (K-1)$. These samples points are equivalent to a $K-$point DFT of $\mathbf{A}[\nu] \circ\!\!-\!\!\bullet\, \boldsymbol{A}(z)$, with $\nu \in \mathbb{Z}$ the discrete time index. Therefore, in the remainder of this paper, we refer to these frequency points on the unit circle as DFT bins. The singular values and singular vectors are computed in each DFT bin through the conventional SVD [23], yielding the sample points

$$\mathbf{A}_k = \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{V}_k^{\mathrm{H}} \ , \tag{2}$$

where $\mathbf{U}_k = [\mathbf{u}_{1,k}, \ldots, \mathbf{u}_{M,k}]$ and $\mathbf{V}_k = [\mathbf{v}_{1,k}, \ldots, \mathbf{v}_{N,k}]$ are unitary matrices of left and right singular vectors, respectively, and $\mathbf{\Sigma}_k = \mathrm{diag}\{\sigma_{1,k}, \ldots, \sigma_{N,k}\}$ contains the bin-wise real-valued singular values such that $\sigma_{i,k} \geq 0$.

The conventional SVD forces the singular values in each DFT bin to be positive even if the ground-truth singular values may be negative. Therefore, if a ground truth singular value of $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega_k})$ is negative, the corresponding bin-wise singular value will be positive due to convention of the ordinary SVD; the negative sign transfers itself onto one of the corresponding singular vectors. In the remainder of this paper, we assume that this negative sign is included in bin-wise right singular vectors.

With this, the analytic EVD factors in (1) at $z = \mathrm{e}^{\mathrm{j}\Omega_k}$ can be related to the SVD of $\mathbf{A}_k$ as

$$\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{P}_k \mathbf{S}_k \mathbf{\Sigma}_k \mathbf{P}_k^{\mathrm{T}} \ , \tag{3}$$

$$\boldsymbol{U}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{U}_k \begin{bmatrix} \mathbf{\Phi}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}'_k \end{bmatrix} \begin{bmatrix} \mathbf{P}_k \\ \mathbf{0} \end{bmatrix} , \tag{4}$$

$$\boldsymbol{V}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{V}_k \mathbf{\Phi}_k \mathbf{P}_k \mathbf{S}_k \ , \tag{5}$$

where $\mathbf{P}_k$ is an $N \times N$ permutation matrix, $\mathbf{S}_k = \mathrm{diag}\{s_{1,k}, \ldots, s_{N,k}\}$ is a diagonal matrix that corrects the negative sign, i.e. $s_{n,k} \in \{\pm 1\}$, $n = 1, \ldots, N$, $\mathbf{\Phi}_k = \mathrm{diag}\{\mathrm{e}^{\mathrm{j}\theta_{1,k}}, \ldots, \mathrm{e}^{\mathrm{j}\theta_{N,k}}\}$ is diagonal matrix for phase adjustments, and $\mathbf{\Phi}'_k$ and arbitrary unitary matrix.

B. Ordering of Singular Values: Permutation Matrices

Both the left and right singular vectors and corresponding singular values can be re-ordered in each DFT bin via the inner product based ordering mechanism proposed in [22]. Effectively this traces a smooth 1-D subspace through the frequency bins. This mechanism replaces the mth singular vectors in $\mathbf{V}_k$ with the nth singular vector in order to maximise the absolute value of the inner product with the mth singular vector of the previous bin,

$$n = \arg\max_{\nu} |\mathbf{v}_{m,k-1}^{\mathrm{H}} \mathbf{v}_{\nu,k}|, \quad m = 1, \ldots, N. \tag{6}$$

Note that the modulus operation in (6) removes the impact of any phase ambiguity. Therefore, for every value of $m = 1, \ldots, N$, all the N left singular vectors are re-ordered along the columns in $\mathbf{V}_k$; (6) thus determines the required permutations that define $\mathbf{P}_k$,

It must be noted that the inner product-based ordering mechanism in (6) becomes ambiguous if a DFT bin coincides with an algebraic multiplicity greater than one [3]. The reason is that the singular vectors corresponding to singular values which share a $\mathcal{C}-$fold algebraic multiplicity in any DFT bin can be selected to for an arbitrary orthonormal basis with a $\mathcal{C}$-dimensional subspace. If such an algebraic multiplicity occurs, it is possible to apply a modulation to the DFT bins until no

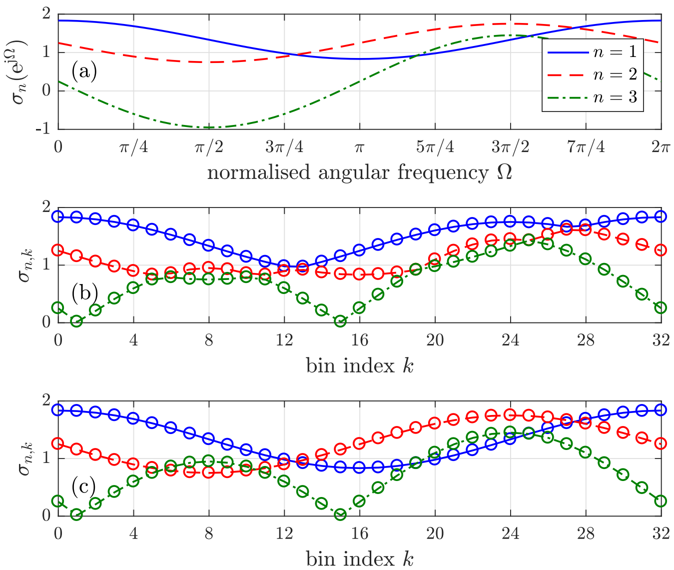

Fig. 1. Analytic singular values in (7) (a) evaluated on unit circle, (b) bin-wise singular values at a DFT size of 32, and (c) ordered singular values via the inner product mechanism in [22].

non-trivial algebraic multiplicities are encountered [24], and we therefore will not be commenting on this case further. Below, we present a simple example illustrating the inner product-based ordering mechanism.

Example 1: To demonstrate the efficacy of the sorting mechanism, we construct a simple polynomial matrix $\boldsymbol{A}(z) : \mathbb{C} \to \mathbb{C}^{3\times 3}$ from analytic eigenvectors

$$\sigma_1(z) = \frac{1}{4}z + \frac{4}{3} + \frac{1}{4}z^{-1} \ , \tag{7a}$$

$$\sigma_2(z) = \frac{\mathrm{j}}{4}z + \frac{5}{4} - \frac{\mathrm{j}}{4}z^{-1} \ , \tag{7b}$$

$$\sigma_3(z) = \frac{3\mathrm{j}}{5}z + \frac{1}{4} - \frac{3\mathrm{j}}{5}z^{-1} \ ; \tag{7c}$$

the left and right singular vectors, of polynomial order 4, are constructed through elementary paraunitary operations [21]. The evaluation of the singular values on the unit circle is illustrated in Fig. 1(a), and it is evident that $\sigma_3(\mathrm{e}^{\mathrm{j}\Omega})$ possesses two zero crossings and a negative segment. Applying a 32-point DFT to $\boldsymbol{A}(z)$ and computing conventional SVDs in each bin yields the bin-wise singular values plotted in Fig. 1(b); note that all singular values are now non-negative and spectrally majorised. The proposed sorting mechanism in (6) leads to the permuted arrangement in Fig. 1(c). It can be noticed that the bin-wise singular values are correctly sorted in each DFT however, the third singular value remains non-negative between bins $k = 2$ to $k = 14$, where it should be negative in order to admit an analytic $\sigma_3(z)$. △

C. Accounting for Negative Singular Values

Once the singular values are re-ordered in each DFT bin, the range of the DFT bins where the nth bin-wise singular values should be negative can be determined via the discontinuities in any component of the matrix

$$\mathbf{C}_k^{(n)} = \mathbf{u}_{n,k}\mathbf{v}_{n,k}^{\mathrm{H}} = \boldsymbol{u}_n(\mathrm{e}^{\mathrm{j}\Omega_k})s_{n,k}\boldsymbol{v}_n(\mathrm{e}^{\mathrm{j}\Omega_k})^{\mathrm{H}}. \tag{8}$$

If $s_{n,k} = 1 \ \forall \ k$, both the real and imaginary component of all elements of $\mathbf{C}_k^{(n)}$ will evolve smoothly across the DFT bins. However, if $s_{n,k}$ changes sign w.r.t. $s_{n,k-1}$, then a discontinuity between $\mathbf{C}_{k-1}^{(n)}$ and $\mathbf{C}_k^{(n)}$ will occur in every element of this matrix. Let $[\mathbf{C}_k^{(n)}]_{\{i,j\}}$ refer to the element in the ith row and jth column of this matrix.

Example 2: To illustrate the discontinuities and their detectability via (8), we plot the real and imaginary components a single element $[\mathbf{C}_k^{(n)}]_{\{1,1\}}$, for Example 1 for $n = 1, 2, 3$ in Fig. 2. Since both the first and second singular values are non-negative in all DFT bins, we see smooth variation of both real and imaginary components of $[\mathbf{C}_k^{(n)}]_{\{1,1\}}$ for $n = 1, 2$, whereas for $n = 3$ discontinuities arise between $k = 1$ and $k = 2$, and between bins $k = 14$ and $k = 15$, which reveal sign changes of $s_{3,k}$ between these bins . △

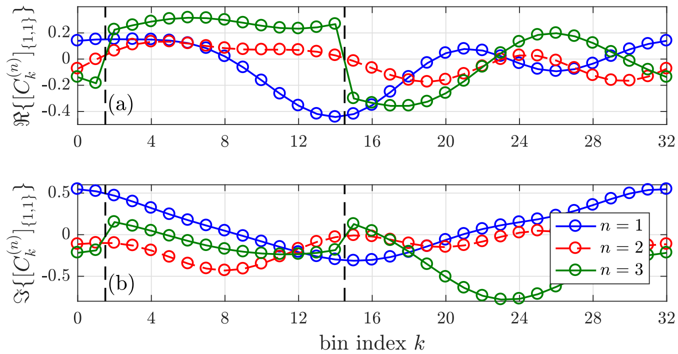

Fig. 2. The evolution of $[\mathbf{C}_k^{(n)}]_{\{1,1\}}$ for Example 1 for $n = 1, 2, 3$ with (a) illustrating the real component and, (b) the imaginary component.

Recall that the number of discontinuities must be even, otherwise the analytic SVD, where singular values are real on the unit circle, may not exist [5].

It may be possible to locate these discontinuities where phase jumps of π radians occur in $[\mathbf{C}_k^{(n)}]_{\{i,j\}}$. For the sake of robustness, it is important to note that such a phase jump may also occur around a spectral zero (with odd multiplicity) of any element $[\mathbf{C}_k^{(n)}]_{\{i,j\}}$. However, a phase discontinuity solely due to a sign change in $s_{n,k}$ will be common across all elements of $\mathbf{C}_k^{(n)}$. Under the assumption that spectral zeros of elements of both right and left singular vector do not coincide with spectral zeros of singular values on the unit circle, we can detect a sign change of $s_{n,k}$ through

$$\alpha_k^{(n)} = \min_{\substack{i\in\{1,\dots,N\}\\ j\in\{1,\dots,M\}}} \left|\angle[\mathbf{C}_k^{(n)}]_{\{i,j\}} - \angle[\mathbf{C}_{k-1}^{(n)}]_{\{i,j\}}\right| . \tag{9}$$

Therefore, we can check whether $\alpha_k^{(n)} > \frac{\pi}{2}$ to indicate a discontinuity in all elements of $\mathbf{C}_k^{(n)}$. Based on the condition $\alpha_k^{(n)} > \frac{\pi}{2}$, Algorithm 1 outlines the procedure of identifying the elements of the sign correcting matrices $\mathbf{S}_k$.

Algorithm 1: Sign Determination of $s_{n,k}$

for $n = 1 : N$ **do**
  $s_{n,0} = 1$;
  for $k = 1 : (K-1)$ **do**
    if $\alpha_k^{(n)} > \frac{\pi}{2}$ **then**
      $s_{n,k} = -s_{n,k-1}$;
    end
  end
end

D. Sufficiency of DFT Size

Once the bin-wise singular values are ordered and sign-corrected, i.e. we have associated and identified the sampl points of the analytic singular values, we need to evaluat whether the DFT size permits a sufficiently low time domai aliasing. For this assessment, we utilise the fact that the bes least squares approximation of an analytic function is b truncation [2], and compare the reconstruction from $K/2$ bin to those from K bins. A suitable metric for this assessment i defined in [3], [15] as

$$\zeta = \frac{\sum_\tau ||\mathbf{\Sigma}^{(K)}[\tau] - \mathbf{\Sigma}^{(K/2)}[\tau]||_\mathrm{F}^2}{\sum_\tau ||\mathbf{\Sigma}^{(K)}[\tau]||_\mathrm{F}^2}, \qquad (10$$

where $\mathbf{\Sigma}^{(K)}[\tau]$ and $\mathbf{\Sigma}^{(K/2)}[\tau]$ is the time-domain equiva lent obtained through a K and $K/2$ point IDFT applied t $\{\mathbf{\Sigma}_k | k = 0, \ldots, (K-1)\}$ and $\{\mathbf{\Sigma}_{2k} | k = 0, \ldots, K/2-1\}$, respectively. Generally, the metric ζ decreases as K increased and will eventually falls below a certain threshold $\epsilon > 0$ when the DFT size is sufficiently large to minimise both time-domain aliasing and truncation errors to an acceptable level.

E. Overall Algorithm

At a specific DFT size of K, the algorithm carries out bin-wise standard SVD computations on the sample points of $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega_k})$, $k = 0, \ldots, K$. If any DFT bin comprises repeated singular values, the algorithm employs modulation to prevent DFT bins from aligning with non-trivial algebraic multiplicities. Next, the inner product mechanism is implemented to arrange the bin-wise singular values in a sorted order. Thereafter, Algorithm 1 applies a sign adjustment, followed by the calculation of the ζ metric. If the ζ metric is below some threshold ϵ, the overall algorithm concludes; otherwise, K is increased (typically doubled, thereby saving half of the bin-wise SVD calculations), and the procedure is repeated. Due to the analyticity of the targeted singular values, the approach is expected to converge as K increases.

IV. Simulations and Results

A. Numerical Example

We compare the proposed algorithm with the GSBR2 [19] and the GSMD [20] algorithms applied to the polynomial matrix in Example 1. The moduli for the extracted time domain results are shown in Figs. 3 and 4. The GSMD

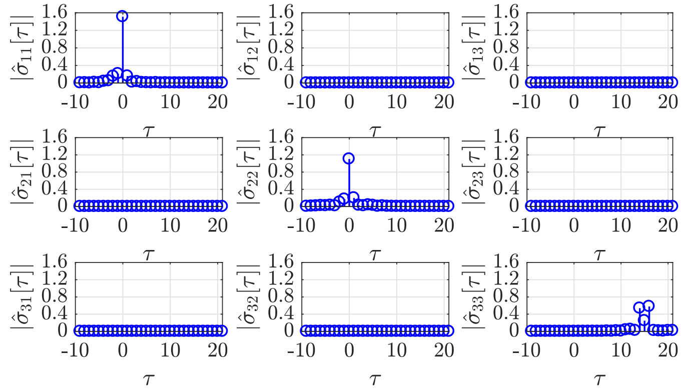

Fig. 3. Singular values of the polynomial matrix in Example 1 estimated through the GSMD [20] algorithm with 400 iterations.

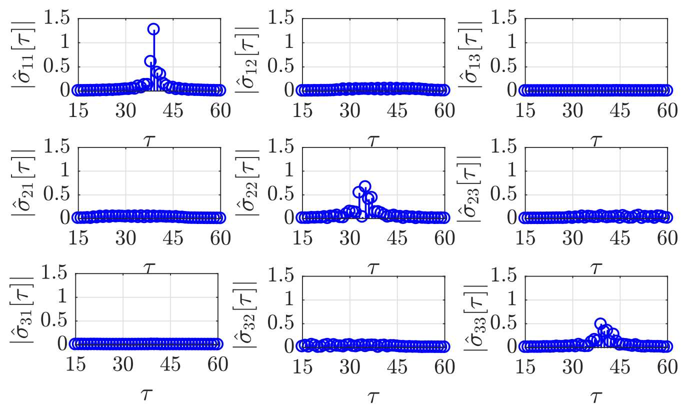

Fig. 4. Singular values of polynomial matrix in Example 1 approximated through computing two PEVDs via the SMD algorithm with 500 iterations.

approach reaches its result in Fig. 3 with a better diagonalisation compared to GSBR2, a direct Koglebtiantz polynomial SVD approach, in Fig. 4. Both approaches produce spectrally majorised singular values, which are complex valued on the unit circle, even though the ground truth singular values are unmajorised. In contrast, the proposed method produces compact order analytic singular values which are real valued on the unit circle as evident from para-Hermitian symmetry of the coefficients shown in Fig. 5. The PEVD approaches in [2], [3], [22] could be utilised via two PEVDs akin to GSMD, but are omitted here due to their comparably high computational cost; while expected to yield analytic solutions, like GSMD and GSBR2 these algorithms can be expected to return complex-valued singular values.

B. Ensemble Test

We next test the proposed algorithm on an ensemble containing 10^3 random instantiations of matrices $\boldsymbol{A}(z) : \mathbb{C} \rightarrow \mathbb{C}^{3\times 3}$ with known singular values of polynomial order 20 which are spectrally unmajorised on the unit circle. Each instantiation has known left- and right-singular vectors of polynomial order 10. In addition, each instance is created such that at least one of the singular value goes negative over some frequencies on the unit circle to exhaustively test the general

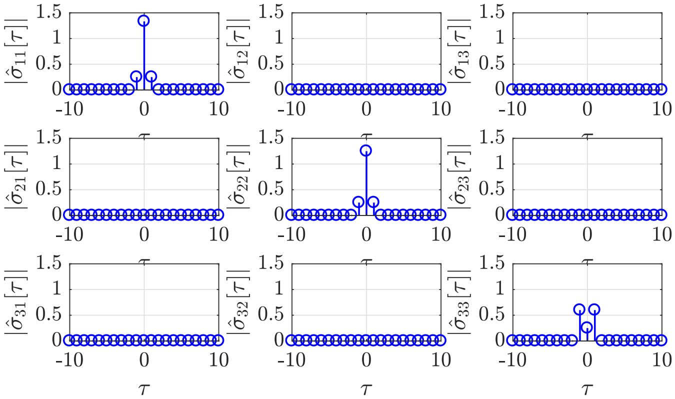

Fig. 5. Analytic singular values for the polynomial matrix in Example 1 approximated by the proposed method.

TABLE I
PERFORMANCE METRICS OF THE PROPOSED ALGORITHM

Comp. Time/$[s]$	**Accuracy**/ξ	$\mathcal{O}\{\boldsymbol{\Sigma}(z)\}$
0.9 ± 0.07	$2.02 \times 10^{-15} \pm 6.4 \times 10^{-14}$	20 ± 0

approach and in particular Algorithm 1. Since there current are no existing algorithms that are capable of extracting analytic singular values that are real on the unit circle, we only explore the proposed method without any benchmark algorithm.

In the ensemble test, we evaluate the proposed algorithm by the polynomial order of the estimated singular values $\mathcal{O}\{\hat{\boldsymbol{\Sigma}}(z)\}$, computation time and the accuracy of the estimated singular values measured as

$$\xi = \frac{\sum_\tau ||\hat{\boldsymbol{\Sigma}}[\tau] - \boldsymbol{\Sigma}[\tau]||_{\mathrm{F}}^2}{\sum_\tau ||\boldsymbol{\Sigma}[\tau]||_{\mathrm{F}}^2} . \quad (11)$$

These performance metrics are reported in Table I in form of mean and standard deviation of the ensemble results. From the table it is evident that the proposed algorithm estimates the analytic singular values accurately with a low ξ metric; the methods also recovers singular values with the exact polynomial order of the ground truth.

V. CONCLUSION

In this paper, we have proposed an analytic singular values extraction algorithm which extracts the analytic singular values of a polynomial matrix. These extracted singular values are real-valued on the unit circle, and are permitted to become negative in order to be analytic. The proposed method employs a DFT domain approach applying standard bin-wise SVDs. These are then smoothly associated across frequency bins in order to remove bin-wise permutations, and to identify over which bins singular values must carry a negative sign. The proposed method is validated through an ensemble of polynomial matrices. where the recovered singular values closely match those of the ground truth unmajorised functions.

REFERENCES

[1] V. Neo, S. Redif, J. McWhirter, J. Pestana, I. Proudler, S. Weiss, and P. Naylor, "Polynomial eigenvalue decomposition for multichannel broadband signal processing," *IEEE SP. Mag.*, **40**(7):18–37, Nov. 2023.

[2] S. Weiss, I.K. Proudler, and F.K. Coutts, "Eigenvalue decomposition of a Parahermitian matrix: Extraction of analytic eigenvalues," *IEEE Trans. SP*, **69**:722–737, Jan. 2021.

[3] S. Weiss, I.K. Proudler, F.K. Coutts, and F.A. Khattak, "Eigenvalue decomposition of a parahermitian matrix: Extraction of analytic eigenvectors," *IEEE Trans. SP*, **71**:1642–1656, Apr. 2023.

[4] G. Barbarino and V. Noferini, "On the Rellich eigendecomposition of para-Hermitian matrices and the sign characteristics of $*$-palindromic matrix polynomials," *Linear Algebra Appl.*, **672**:1–27, Sep. 2023.

[5] S. Weiss, I.K. Proudler, G. Barbarino, J. Pestana, and J.G. McWhirter, "On properties and structure of the analytic singular value decomposition," submitted to *IEEE Trans. SP*, 2023.

[6] C. H. Ta and S. Weiss, "A design of precoding and equalisation for broadband MIMO systems," in *44th Asilomar SSC Conf.*, Pacific Grove, CA, USA, Nov. 2007, pp. 1616–1620.

[7] N. Moret, A. Tonello, and S. Weiss, "MIMO precoding for filter bank modulation systems based on PSVD," in *IEEE 73rd Vehicular Technology Conference*, May 2011, pp. 1–5, (best paper award).

[8] A.I. Pérez-Neira, M. Caus, R. Zakaria, D.L. Ruyet, E. Kofidis, M. Haardt, X. Mestre, and Y. Cheng, "MIMO signal processing in offset-QAM based filter bank multicarrier systems," *IEEE Transactions on Signal Processing*, vol. 64, no. 21, pp. 5733–5762, November 2016.

[9] Z. Wang, A. Sandmann, J.G. McWhirter, and A. Ahrens, "Multiple shift SBR2 algorithm for calculating the SVD of broadband optical MIMO systems," in *2016 39th Int. Conf. Telecommunications and Signal Processing*, 2016, pp. 433–436.

[10] S. Weiss, S.J. Schlecht, O. Das, and E. De Sena, "Polynomial Procrustes problem: Paraunitary approximation of matrices of analytic functions," in *EUSIPCO*, Helsinki, Finland, 2023.

[11] O. Das, S.J. Schlecht, and E. De Sena, "Grouped feedback delay networks with frequency-dependent coupling," *IEEE/ACM Transactions on Audio, Speech, and Language Processing*, vol. 31, pp. 2004–2015, May 2023.

[12] S. Weiss, C. Delaosa, J. Matthews, I.K. Proudler, and B.A. Jackson, "Detection of weak transient signals using a broadband subspace approach," in *SSPD*, London, UK, Sep. 2021, pp. 1–5.

[13] B. De Moor and S. Boyd, "Analytic properties of singular values and vectors," KU Leuven, Tech. Rep., 1989.

[14] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols, "Numerical computation of an analytic singular value decomposition of a matrix valued function," *Numerische Mathematik*, vol. 60, no. 1, pp. 1–39, Dec. 1991.

[15] F. A. Khattak, I.K. Proudler, and S. Weiss, "Generalized polynomial power method," in *SSPD*, Edinburgh, Scotland, Sep. 2023, pp. 1–5.

[16] M. A. Bakhit, F.A. Khattak, I.K. Proudler, S. Weiss, and G.W. Rice, "Compact order polynomial singular value decomposition of a matrix of analytic functions," in *9th IEEE CAMSAP*, Los Sueños, Costa Rica, Dec. 2023.

[17] J.G. McWhirter, P.D. Baxter, T. Cooper, S. Redif, and J. Foster, "An EVD algorithm for para-Hermitian polynomial matrices," *IEEE Trans. SP*, **55**(5):2158–2169, May 2007.

[18] S. Redif, S. Weiss, and J. McWhirter, "Sequential matrix diagonalization algorithms for polynomial EVD of parahermitian matrices," *IEEE Trans. SP*, **63**(1):81–89, Jan. 2015.

[19] J.G. McWhirter, "An algorithm for polynomial matrix SVD based on generalised Kogbetliantz transformations," in *EUSIPCO*, Aalborg, Denmark, Aug. 2010, pp. 457–461.

[20] F.A. Khattak, I.K. Proudler, J.G. McWhirter, and S. Weiss, "Generalised sequential matrix diagonalisation for the SVD of polynomial matrices," in *SSPD*, Edinburgh, Scotland, 2023, pp. 1–5.

[21] P. Vaidyanathan, "Theory of optimal orthonormal subband coders," *IEEE Trans. SP*, **46**(6):1528–1543, 1998.

[22] M. Tohidian, H. Amindavar, and A. M. Reza, "A DFT-based approximate eigenvalue and singular value decomposition of polynomial matrices," *EURASIP JASP*, **2013**(1):1–16, 2013.

[23] G.H. Golub and C.F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore, Maryland: John Hopkins University Press, 1996.

[24] F.A. Khattak, "Advanced Polynomial Matrix Factorization Techniques," PhD thesis, University of Strathclyde, Glasgow, Scotland, Jan. 2024.