Formulating and Solving Broadband Multichannel Problems Using Matrices of Functions

Stephan Weiss

Centre for Signal & Image Processing
Department of Electronic & Electrical Engineering
University of Strathclyde, Glasgow, Scotland

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1 Background and Overview

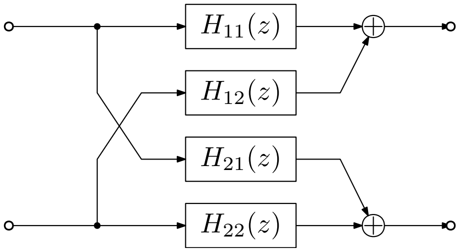
1. Background and Overview
2. Matrices of functions
3. Basic properties and operations
4. Formulation of broadband problems
5. Eigenvalue decomposition
6. Applications
7. Summary

2 Matrices of Functions

2.1 Transfer Function of a MIMO System

- A multiple-input multiple-output (MIMO) system comprises of individual transfer functions between source/sensor pairs:
- written as a matrix:

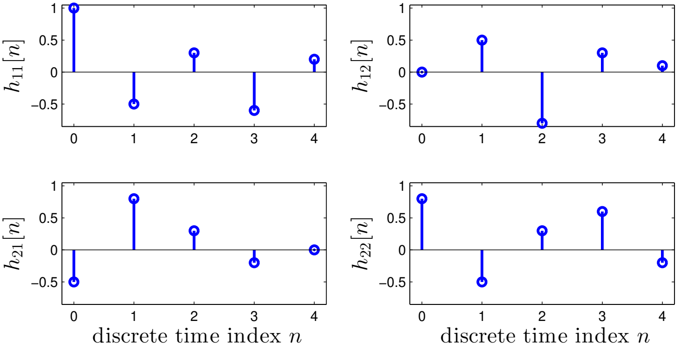
$$\boldsymbol{H}(z) = \begin{bmatrix} H_{11}(z) & H_{12}(z) \\ H_{21}(z) & H_{22}(z) \end{bmatrix} ;$$



- the terms $H_{mn}(z)$ can be general functions: (infinite) Laurent or power series, or (finite) Laurent polynomials or polynomials;
- Laurent series of polynomials are non-causal; power series or polynomials are strictly causal or anti-causal;
- analytic functions: series coefficients of $H_{mn}(z)$ decay at least exponentially.

Transfer Function Example

- Example for 2×2 MIMO system $\mathbf{H}[n]$ of finite impulse responses:



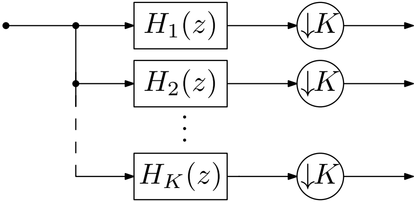
- the system

$$\boldsymbol{H}(z) = \sum_{n=-\infty}^{\infty} \mathbf{H}[n] z^{-1} \qquad \text{or} \qquad \boldsymbol{H}(z) \bullet\!\!-\!\!\circ\, \mathbf{H}[n] \tag{1}$$

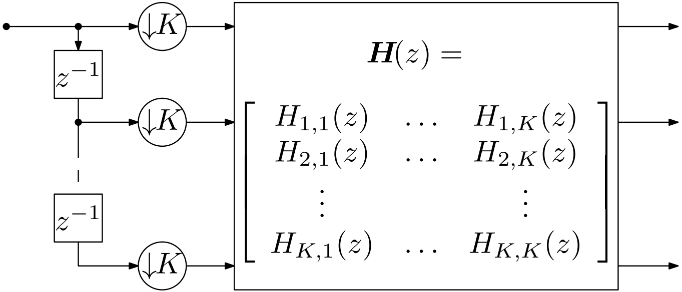
[45, 61, 66, 83].

2.2 Analysis Filter Bank

- Critically decimated K-channel analysis filter bank [88, 89, 41]:

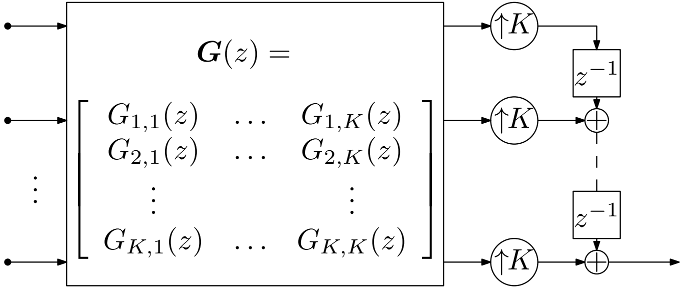


- equivalent polyphase representation:



Synthesis Filter Bank

- Polyphase representation of a critically decimated K-channel synthesis filter bank [36, 41, 89]:

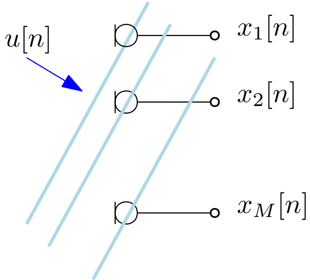


- operating analysis and synthesis back-to-back, perfect reconstruction is achieved if

$$\mathbf{G}(z)\mathbf{H}(z) = \mathbf{I}\,; \tag{2}$$

- for perfect reconstruction, we would like to find $\mathbf{G}(z) = \mathbf{H}^{-1}(z)$.

2.3 Space-Time Covariance Matrix



- Measurement vector obtained from M sensors:

$$\mathbf{x}^{\mathrm{T}}[n] = [x_1[n] \;\; x_2[n] \;\; \ldots \;\; x_M[n]] \;;$$

- with the expectation operator $\mathcal{E}\{\cdot\}$, the spatial correlation is captured by $\mathbf{R} = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n]\}$;
- for spatial and temporal correlation, we require a space-time covariance matrix [64, 70, 89, 101, 102, 107, 104]:

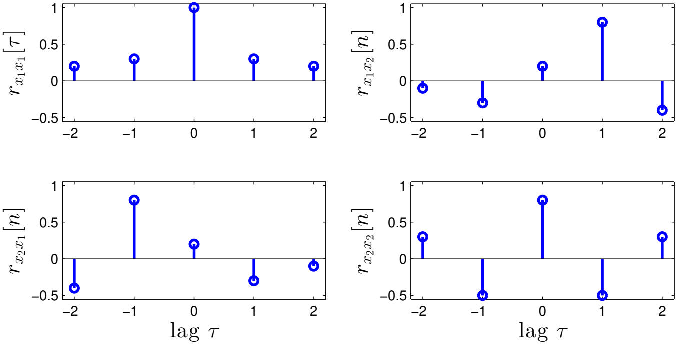
$$\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\} \;; \qquad (3)$$

- the space-time covariance contains auto- and cross-correlation terms; e.g. for $M = 2$

$$\mathbf{R}[\tau] = \begin{bmatrix} \mathcal{E}\{x_1[n]x_1^*[n-\tau]\} & \mathcal{E}\{x_1[n]x_2^*[n-\tau]\} \\ \mathcal{E}\{x_2[n]x_1^*[n-\tau]\} & \mathcal{E}\{x_2[n]x_2^*[n-\tau]\} \end{bmatrix} . \qquad (4)$$

Cross-Spectral Density Matrix

- example for a space-time covariance matrix $\mathbf{R}[\tau] \in \mathbb{R}^{2\times 2}$:



- the cross-spectral density (CSD) matrix contains Laurent series or polynomials:

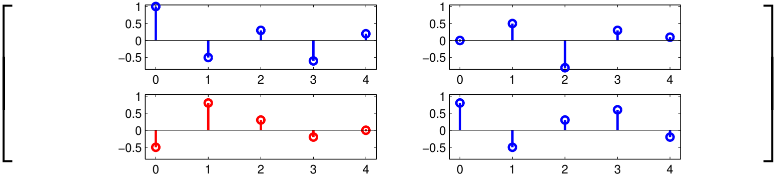
$$\boldsymbol{R}(z) = \sum_{\tau} \mathbf{R}[\tau] z^{-\tau} \qquad \text{or short} \qquad \boldsymbol{R}(z) \bullet\!\!-\!\!\circ\, \mathbf{R}[\tau] \ . \tag{5}$$

3 Some Basic Operations and Properties

3.1 Parahermitian Operator

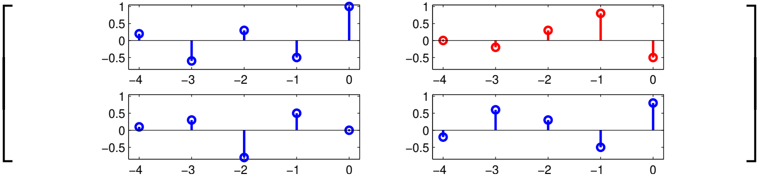
- A parahermitian operation is indicated by $\{\cdot\}^{\mathrm{P}}$, and compared to the Hermitian transposition of a matrix additionally performs a time-reversal;
- example:

$\boldsymbol{A}(z) =$



- parahermitian $\boldsymbol{A}^{\mathrm{P}}(z) = \{\boldsymbol{A}(1/z^{*})\}^{\mathrm{H}}$:

$\boldsymbol{A}^{\mathrm{P}}(z) =$



3.2 Parahermitian Property

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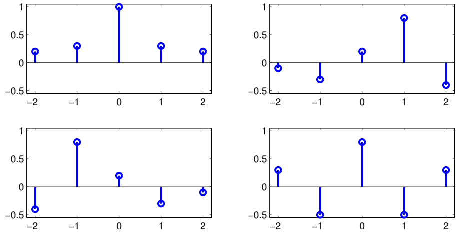
- A polynomial matrix $\boldsymbol{R}(z)$ is parahermitian if $\boldsymbol{R}^{\mathrm{P}}(z) = \boldsymbol{R}^{\mathrm{H}}(1/z^*) = \boldsymbol{R}(z)$;
- this is an extension of the symmetric (if $\mathbf{R} \in \mathbb{R}$) or or Hermitian (if $\mathbf{R} \in \mathbb{C}$) property to the polynomial case:
  transposition, complex conjugation and time reversal (in any order) do not alter a parahermitian $\boldsymbol{R}(z)$;
- any CSD matrix is parahermitian;
- example:

$$\boldsymbol{R}(z) = \left[\quad \right] = \boldsymbol{R}^{\mathrm{P}}(z) \quad .$$



Paraunitary Matrices

- Recall that $\mathbf{A} \in \mathbb{C}$ (or $\mathbf{A} \in \mathbb{R}$) is a unitary (or orthonormal) matrix if $\mathbf{A}\mathbf{A}^{\mathrm{H}} = \mathbf{A}^{\mathrm{H}}\mathbf{A} = \mathbf{I}$;
- in the polynomial case, $\mathbf{A}(z)$ is paraunitary if

$$\boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z) = \boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{A}(z) = \mathbf{I} \; ; \qquad (6)$$

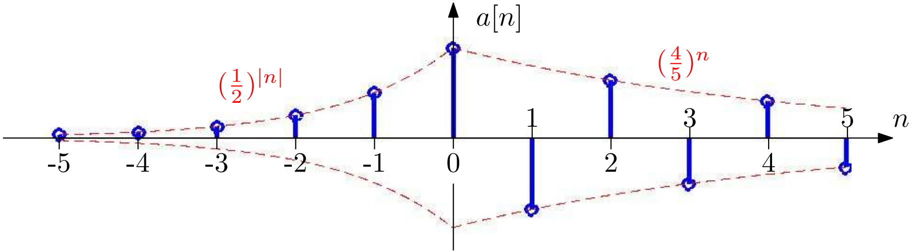
- therefore, if $\boldsymbol{A}(z)$ is paraunitary, then the polynomial matrix inverse is simple:

$$\boldsymbol{A}^{-1}(z) = \boldsymbol{A}^{\mathrm{P}}(z) \; ; \qquad (7)$$

- example: polyphase analysis or synthesis matrices of perfectly reconstructing (or lossless) filter banks are usually paraunitary.

3.3 Analytic Functions — Laurent Series

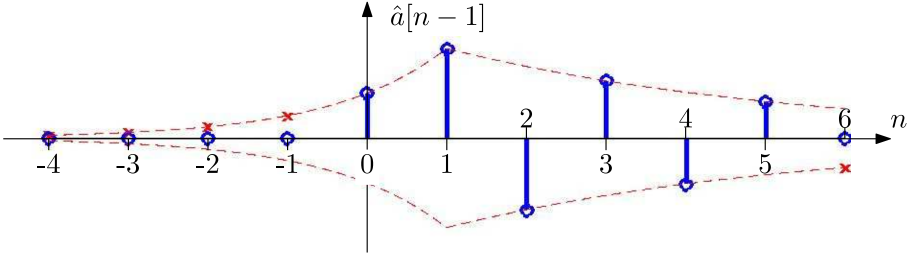
- A Laurent series $a[n]$ is potentially infinite, but can include non-negative terms for both $n \geq 0$ and $n < 0$;
- for $a(z) \bullet\!\!-\!\!\circ\; a[n]$ to exist, $a[n]$ needs to decay at least exponentially in both positive and negative time direction [1];



- if it possesses finite support, $a(z)$ is a Laurent polynomial.

Analyticity and Polynomial Approximation

- Absolute convergence of $a[n]$ implies analyticity of $a(z) \bullet\!\!-\!\!\circ\, a[n]$;
- the best approximation of an infinite order, analytic $a(z)$ in the least squares sense is by truncation (power series $\longrightarrow$ polynomial) [23, 24, 104];
- likewise, a Laurent series can be approximated by a polynomial through truncation ($\longrightarrow$ Laurent polynomial) and an appropriate delay ($\longrightarrow$ polymomial) [107];

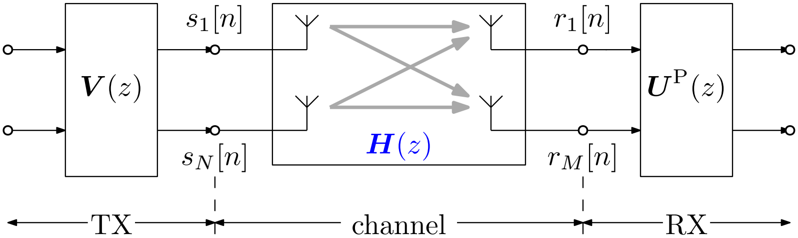


- hence polynomials can typically approximate any general analytic function well, and arbitrarily closely.

4. Polynomial Matrix Formulation of Broadband Problems

4.1 MIMO System Decoupling

- Aim: spatially decouple a channel by appropriate precoding and equalisation;



- narrowband case — SVD [46]:

$$\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}} \; ;$$

- spatial decoupling leads to optimality in various senses [91];

- broadband case [12, 106]:

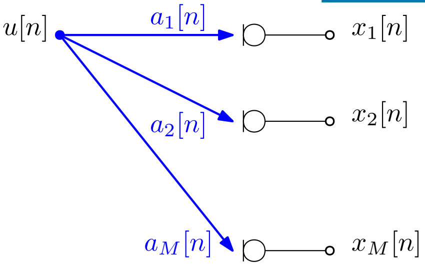
$$\boldsymbol{H}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\boldsymbol{V}^{\mathrm{P}}(z) \; ;$$

- diagonalisation for all values of z (or all values on the unit circle) [64, 67, 83, 82].

4.2 Broadband Steering Vector

- Assume an array of M sensors, and a single source $u[n]$:

$$\mathbf{x}[n] = \begin{bmatrix} a_1[n] \\ \vdots \\ a_M[n] \end{bmatrix} * u[n] \; ;$$

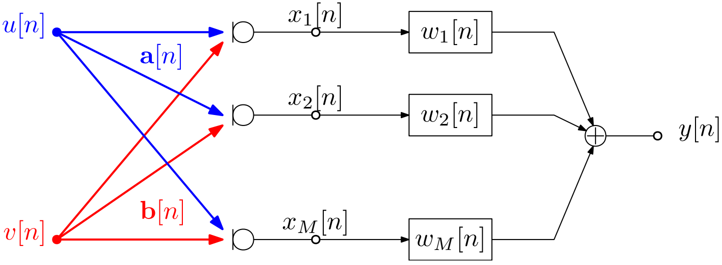


- $\mathbf{a}[n] \bullet\!\!-\!\!\circ \, \boldsymbol{a}(z)$ is a broadband steering vector;
- it can contain fractional delay filters [60] or general transfer functions;
- set of filters operating on the array signals:

$$\boldsymbol{w}^{\mathrm{P}}(z) = [w_1(z), \; w_2(z), \; \ldots \; w_M(z)] \; . \tag{8}$$

Simplistic Beamforming

- Filtering to coherently combine $u[n]$ and to suppress $v[n]$:



- we want $\boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{a}(z) = 1$ and $\boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{b}(z) = 0$;
- narrowband case:

$$\mathbf{w} = \begin{bmatrix} \mathbf{a}^{\mathrm{H}} \\ \mathbf{b}^{\mathrm{H}} \end{bmatrix}^{\dagger} \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

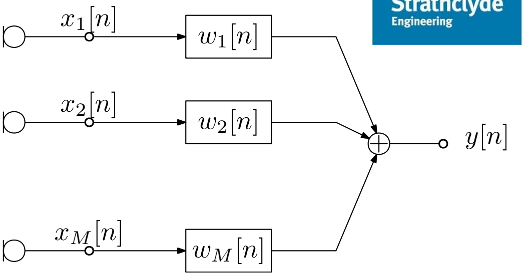
- broadband case:

$$\boldsymbol{w}(z) = \begin{bmatrix} \mathbf{a}^{\mathrm{P}}(z) \\ \mathbf{b}^{\mathrm{P}}(z) \end{bmatrix}^{\dagger} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Linearly Constrained Minimum Variance Beamforming

- To address unknown interferers, we want to minimize the output power subject to constraints (e.g. in look direction):

$$\left[\begin{array}{c} \mathbf{a}^{\mathrm{P}}(z) \\ \mathbf{b}^{\mathrm{P}}(z) \end{array}\right] \boldsymbol{w}(z) = \left[\begin{array}{c} 1 \\ 0 \end{array}\right] \; ;$$



- narrowband case [47]:

$$\min_{\mathbf{w}} \mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w} \qquad \text{s.t.} \;\; \mathbf{C}\mathbf{w} = \mathbf{f} \; ;$$

- broadband case [96]:

$$\min_{\boldsymbol{w}(z)} \oint_{|z|=1} \boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{R}(z)\boldsymbol{w}(z)\frac{\mathrm{d}z}{z}$$
$$\text{s.t.} \;\; \boldsymbol{C}(z)\boldsymbol{w}(z) = \boldsymbol{f}(z) \; .$$

LCMV Solution

- Narrowband formulation [47]:

$$\min_{\mathbf{w}} \mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w} \qquad \text{s.t. } \mathbf{C}\mathbf{w} = \mathbf{f} \; ;$$

- narrowband solution:
  Capon beamformer [81]:

$$\mathbf{w}_{\text{opt}} = \mathbf{R}^{-1}\mathbf{C}^{\mathrm{H}}\{\mathbf{C}\mathbf{R}^{-1}\mathbf{C}^{\mathrm{H}}\}^{-1}\mathbf{f} \; ;$$

- broadband formulation [96]:

$$\min_{\boldsymbol{w}(z)} \oint_{|z|=1} \boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{R}(z)\boldsymbol{w}(z)\frac{\mathrm{d}z}{z}$$
$$\text{s.t. } \boldsymbol{C}(z)\boldsymbol{w}(z) = \boldsymbol{f}(z) \; .$$

- broadband solution:
  Capon equivalent [96, 9]:

$$\boldsymbol{w}_{\text{opt}}(z) = \boldsymbol{R}^{-1}(z)\boldsymbol{C}^{\mathrm{P}}(z) \cdot \{\boldsymbol{C}(z)\boldsymbol{R}^{-1}(z)\mathbf{C}^{\mathrm{P}}(z)\}^{-1}\boldsymbol{f}(z) \; .$$

4.3 From Narrowband to Broadband Formulations

- "Polynomial matrices" is simplistic for what potentially are Laurent series; absolute convergence implies analyticity, and arbitrarily close approximations can be obtained by polynomials of sufficient order;
- operations and properties:

	real-valued	complex-valued	function-valued
transposition	$\mathbf{A}^{\mathrm{T}}$	$\mathbf{A}^{\mathrm{H}} = (\mathbf{A}^{\mathrm{T}})^*$	$\boldsymbol{A}^{\mathrm{P}}(z) = \{\boldsymbol{A}(1/z^*)\}^{\mathrm{H}}$
energy preservation	orthonormal $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$	unitary $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{H}}$	para-unitary $\boldsymbol{A}^{-1}(z) = \boldsymbol{A}^{\mathrm{P}}(z)$
structure	symmetric $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$	Hermitian $\mathbf{A}^{\mathrm{H}} = \mathbf{A}$	para-Hermitian $\boldsymbol{A}^{\mathrm{P}}(z) = \boldsymbol{A}(z)$

- using polynomial notation, broadband formulations generally just extend from the narrowband case;
- to access solutions to polynomial matrix formulations, the eigenvalue decomposition of a parahermitian $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ \mathbf{R}[\tau]$ will be key;
- such an EVD must provide a diagonalisation for every value of z or for every lag τ [64, 78, 101, 108, 103].

5. Eigenvalue Decomposition

5.1 Existence of an Analytic EVD on a Real Interval

- A standard EVD can diagonalise $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ \mathbf{R}[\tau]$ only for one specific value of z or of τ, respectively;
- Franz Rellich (1939, [79]) for a self-adjoint, analytic $\boldsymbol{R}(t) = \boldsymbol{R}^{\mathrm{H}}(t)$, $t \in \mathbb{R}$:

$$\boldsymbol{R}(t) = \boldsymbol{Q}(t)\boldsymbol{\Lambda}(t)\boldsymbol{Q}^{\mathrm{H}}(t) \; ;$$

- $\boldsymbol{Q}(t)$ and $\boldsymbol{\Lambda}(t)$ *can* be chosen analytic;
- similarly for an arbitrary (i.e. not necessarily Hermitian or square) analytic matrix, de Moor & Boyd (1989, [37]) and Bunse-Gerstner (1991, [14]) established an analytic SVD.

Analytic EVD on the Unit Circle

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- Analyticity of $\boldsymbol{R}(z)$ permits a restricted evaluation on the unit circle $z = \mathrm{e}^{\mathrm{j}\Omega}$;
- due to Rellich [79]:

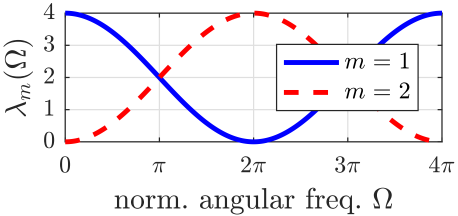
$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{Q}(\Omega)\ \boldsymbol{\Lambda}(\Omega)\ \boldsymbol{Q}^{\mathrm{H}}(\Omega)\ , \qquad (9)$$

- unfortunately, while analytic in $\Omega \in \mathbb{R}$, $\boldsymbol{\Lambda}(\Omega)$ and $\boldsymbol{Q}(\Omega)$ can be $2\pi L$-periodic, with some $L \in \mathbb{Z}$ [102, 12];
- example [28, 86, 102]:

$$\boldsymbol{R}(z) = \frac{1}{2}\begin{bmatrix} 2 & 1+z^{-1} \\ 1+z & 2 \end{bmatrix}\ ,$$

$$\rightarrow \lambda_{1,2}(z) = 2 \pm (z^{\frac{1}{2}} + z^{-\frac{1}{2}})\ ,$$

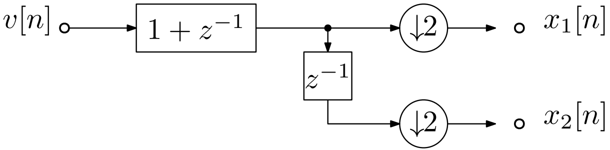
$$\lambda_{1,2}(\mathrm{e}^{\mathrm{j}\Omega}) = 2 \pm \cos(\Omega/2)\ ;$$



- while $\cos(\Omega/2)$ is analytic in Ω, a fractional delay $z^{-\frac{1}{2}}$ is not analytic: its time domain equivalent decays too slowly [60].

Multiplexing and Pseudo-Circulant Property

- The previous example of $\boldsymbol{R}(z) = [2 \;\; 1+z^{-1}; 1+z \;\; 2]$ arises from the following arrangement with uncorrelated $v[n] \in \mathcal{N}(0,1)$:



- therefore we require oversampling by $L = 2$:

$$\boldsymbol{R}(z^2) = \begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix} \begin{bmatrix} z+2+z^{-1} & \\ & -z+2-z^{-1} \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 1 & -z^{-1} \end{bmatrix};$$

- if linked to block filtering, $\boldsymbol{R}(z)$ is pseudo-circulant [89], but this property may be obscured by paraunitary operations [102].

Analytic EVD of a Parahermitian Matrix

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- For an analytic parahermitian matrix $\boldsymbol{R}(z)$, $z \in \mathbb{C}$, that is connected to L-fold multiplexing, we can find [101, 102]

$$\boldsymbol{R}(z^L) = \boldsymbol{Q}(z)\ \boldsymbol{\Lambda}(z)\ \boldsymbol{Q}^{\mathrm{P}}(z)\ , \tag{10}$$

  with analytic factors;
- $\boldsymbol{Q}(z) = [\boldsymbol{q}_1(z), \ldots, \boldsymbol{q}_M(z)]$ must be paraunitary [89, 90], such that

$$\boldsymbol{Q}(z)\boldsymbol{Q}^{\mathrm{P}}(z) = \boldsymbol{Q}^{\mathrm{P}}(z)\boldsymbol{Q}(z) = \mathbf{I}\ ; \tag{11}$$

- $\boldsymbol{\Lambda}(z) = \mathrm{diag}\{\lambda_1(z), \ldots, \lambda_M\}$ must be diagonal and parahermitian;
- the parahermitian property implies that on the unit circle, $\lambda(\mathrm{e}^{\mathrm{j}\Omega}) = \lambda(z)|_{z=\mathrm{e}^{\mathrm{j}\Omega}} \in \mathbb{R}$.

Uniqueness and Ambiguities

- For the analytic EVD [101, 102, 12]

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z) \cdot \boldsymbol{\Lambda}(z) \cdot \boldsymbol{Q}^{\mathrm{P}}(z) \; ; \tag{12}$$

- the eigenvalues in $\boldsymbol{\Lambda}(z)$ are unique up to a permutation;
- if eigenvalues are distinct, then eigenvectors are unique up to an allpass filter $\psi_\ell(z)$;
- with $\boldsymbol{\Psi}(z) = \mathrm{diag}\{\psi_1(z), \ldots, \psi_M(z)\}$,

$$\begin{aligned} \boldsymbol{R}(z) &= \boldsymbol{Q}(z)\boldsymbol{\Psi}(z)\boldsymbol{\Lambda}(z)\boldsymbol{\Psi}^{\mathrm{P}}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \\ &= \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{\Psi}(z)\boldsymbol{\Psi}^{\mathrm{P}}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \\ &= \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \; ; \end{aligned}$$

- an analytic allpass $\psi_m(z)$ does not affect analyticity, but will affects the support of $\mathbf{Q}[n] \circ\!\!-\!\!\bullet \boldsymbol{Q}(z)$.

5.2 Polynomial EVD and Spectral Majorisation

- Polynomial EVD or McWhirter decomposition [64] of the CSD matrix

$$\boldsymbol{R}(z) \approx \boldsymbol{U}(z)\, \boldsymbol{\Gamma}(z)\, \boldsymbol{U}^{\mathrm{P}}(z) \qquad (13)$$

- with paraunitary, polynomial $\boldsymbol{U}(z)$, s.t. $\boldsymbol{U}(z)\boldsymbol{U}^{\mathrm{P}}(z) = \mathbf{I}$;
- diagonal Laurent polynomial matrix

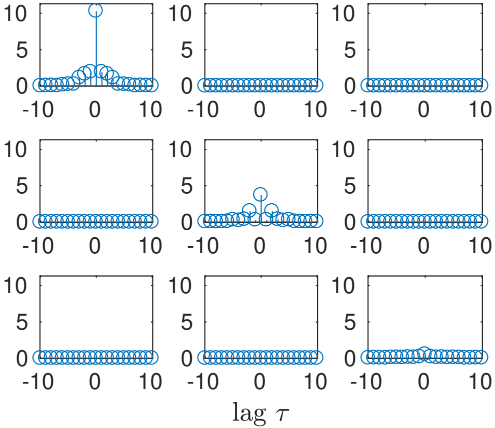
$$\boldsymbol{\Gamma}(z) = \mathrm{diag}\{\gamma_1(z),\ \ldots,\ \gamma_M(z)\}\ , \qquad (14)$$

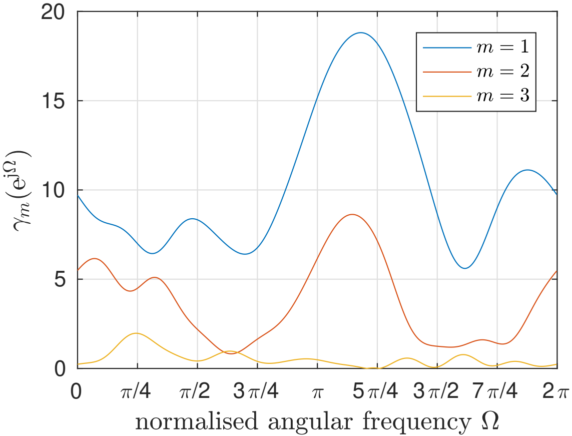
- approximation sign due to restriction to polynomials [49];
- the eigenvalues are spectrally majorised [87], i.e. on the unit circle must satisfy

$$\gamma_m(\mathrm{e}^{\mathrm{j}\Omega}) \geq \gamma_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})\ , \qquad \forall\Omega, \quad m = 1, \ldots (M-1)\ . \qquad (15)$$

Polynomial Eigenvalues and Spectral Majorisation

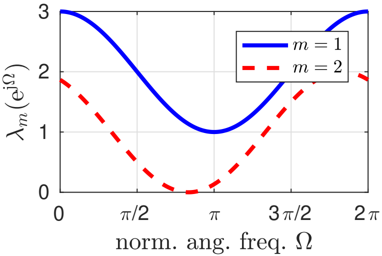
- Example for polynomial eigenvalues $\gamma_m[\tau] \circ\!\!-\!\!\bullet \gamma_m(e^{j\Omega})$ of a 3×3 matrix:

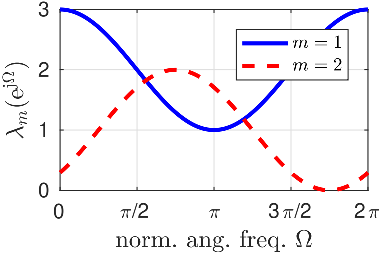




Relation to Analytic EVD

- If the analytic eigenvalues do not intersect on the unit circle, then analytic EVD and polynomial EVD (with sufficiently high order) are 'identical';
- the polynomial EVD has a strict ordering of eigenvalues;
- specific polynomial/analytic eigenvector solutions may differ — recall the allpass ambiguity;





- if analytic eigenvalues intersect, then the solutions of analytic EVD and polynomial EVD differ;
- we explore by way of an example ...

Numerical Example

- We pick our own eigenvalues (order 2) and eigenvectors (order 1):

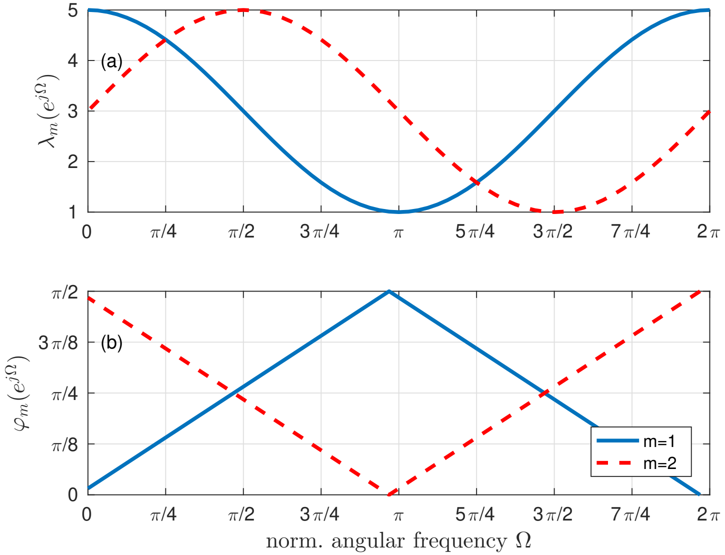
$$\boldsymbol{\Lambda}(z) = \begin{bmatrix} z+3+z^{-1} & \\ & -jz+3+jz^{-1} \end{bmatrix}$$

$$\boldsymbol{Q}(z) = [\boldsymbol{q}_1(z),\ \boldsymbol{q}_2(z)] \qquad \text{with} \qquad \boldsymbol{q}_{1,2}(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm z^{-1} \end{bmatrix} ;$$

- parahermitian matrix $\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z)$:

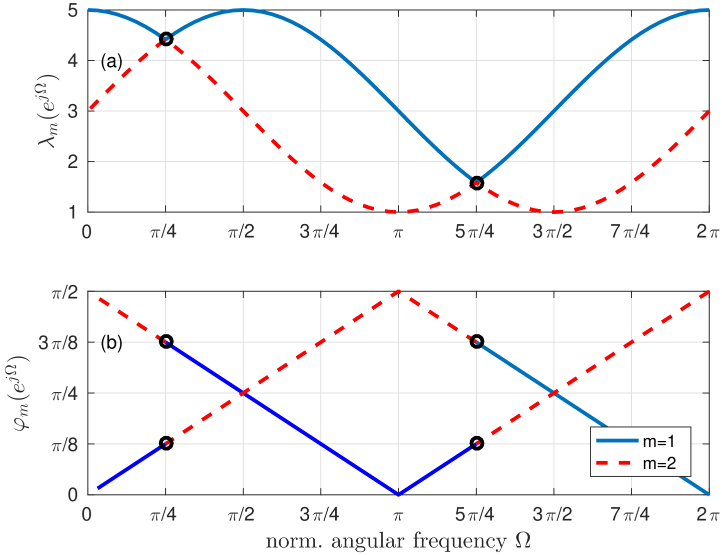
$$\boldsymbol{R}(z) = \begin{bmatrix} \frac{1-j}{2}z+3+\frac{1+j}{2}z^{-1} & \frac{1+j}{2}z^{2}+\frac{1-j}{2} \\ \frac{1+j}{2}+\frac{1-j}{2}z^{-2} & \frac{1-j}{2}z+3+\frac{1+j}{2}z^{-1} \end{bmatrix} .$$

Numerical Example — Analytic Solution



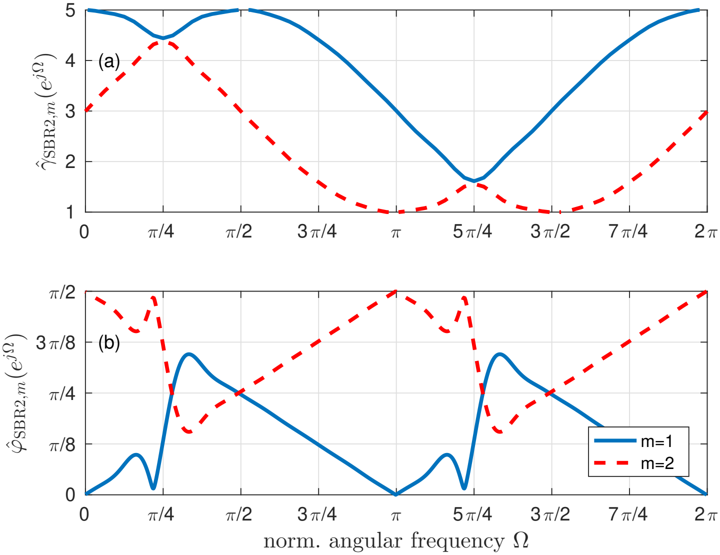
- Analytic (and therefore infinitely differentiable) eigenvalues $\lambda_m(\mathrm{e}^{\mathrm{j}\Omega})$;
- smooth Hermitian angles $\cos\varphi_m = |\boldsymbol{q}_1^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}0}) \cdot \boldsymbol{q}_m(\mathrm{e}^{\mathrm{j}\Omega})|$.

Numerical Example — Ideal Spectral Majorisation



- Analytic eigenvalues are permuted where they intersect;
- resulting spectrally majorised eigenvalues are piecewise analytic but not differentiable;
- corresponding eigenvectors are piecewise analytic but not continuous.

Numerical Example — PEVD Algorithmic Solution



- Using the SBR2 algorithm in [64] to approximate the McWhirter factorisation;
- trimming is applied to PEVD factors [23, 44, 24, 85];
- spectrally majorised eigenvalues $\mathbf{\Gamma}(z)$ of order 24;
- corresponding eigenvectors in $\boldsymbol{U}(z)$ of order 84.

5.3 Iterative PEVD Approaches

- Second order sequential best rotation (SBR2, McWhirter 2007, [64, 76, 78, 93]);
- iterative approach based on an elementary paraunitary operation:

$$\begin{aligned} \boldsymbol{S}^{(0)}(z) &= \boldsymbol{R}(z) \\ &\vdots \\ \boldsymbol{S}^{(i)}(z) &= \left\{\boldsymbol{H}^{(i)}(z)\right\}^{\mathrm{P}} \boldsymbol{S}^{(i-1)}(z)\boldsymbol{H}^{(i)}(z) \end{aligned}$$

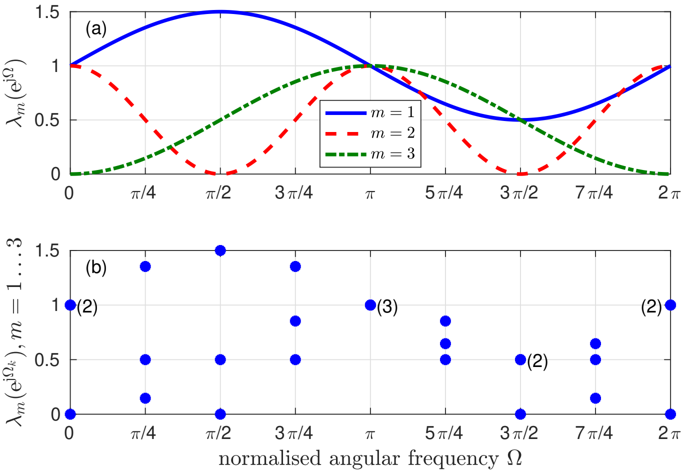
- $\boldsymbol{H}^{(i)}(z)$ is an elementary paraunitary operation, which at the ith step eliminates the largest off-diagonal element in $\boldsymbol{S}^{(i-1)}(z) \bullet\!\!-\!\!\circ\ \mathbf{S}^{(i=1)}[\tau]$;
- stop after I iterations:

$$\hat{\boldsymbol{\Gamma}}(z) = \boldsymbol{S}^{(I)}(z) \qquad , \qquad \hat{\boldsymbol{U}}(z) = \prod_{i=1}^{I} \boldsymbol{H}^{(i)}(z)$$

- sequential matrix diagonalisation (SMD) [19, 20, 17, 23, 22, 18, 30, 26, 25, 27, 77, 73] follows a similar scheme, but performs a complete diagonalisation of $\mathbf{S}^{(i-1)}[0]$.

DFT Domain Algorithms

- Idea for DFT-based algorithms: calculate an EVD in every DFT bin;



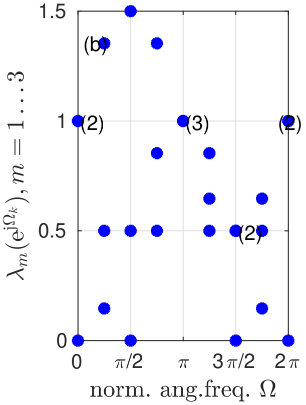
- spectral coherence must be re-established across bins;
- we exploit that the solution must be analytic, i.e. infinitely differentiable;
- we first extract eigenvalues, which are less volatile under perturbation [51];

Analytic Eigenvalue Extraction Algorithm I

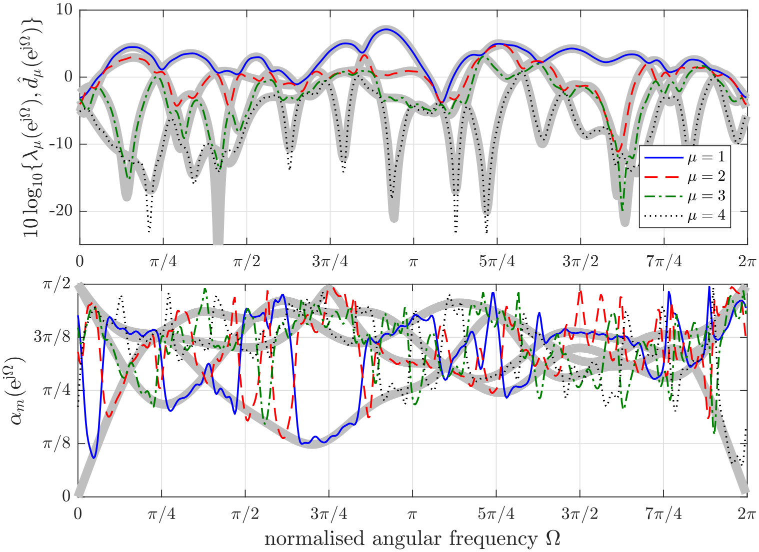
- Bin-wise EVD yields:

$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{Q}_k\boldsymbol{\Lambda}_k\mathbf{Q}_k^{\mathrm{H}} = \underbrace{\mathbf{Q}_k\boldsymbol{\Psi}_k\mathbf{P}_k}_{\boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{P}_k}_{\boldsymbol{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}}\boldsymbol{\Psi}_k^{\mathrm{H}}\mathbf{Q}_k^{\mathrm{H}}}_{\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega_k})}$$

- $\mathbf{P}_k$ is a permutation matrix, since in the analytic EVD, eigenvalues can intersect and are not necessarily majorised;
- for distinct eigenvalues: $\boldsymbol{\Psi}_k$ is a diagonal matrix that accounts for the phase ambiguity of eigenvectors;
- in case of a C-fold algebraic multiplicity: $\boldsymbol{\Psi}_K$ is block diagonal, with a $C \times C$ unitary matrix accounting for eigenvectors forming an arbitrary basis within a C-dimensional subspace;
- a predecessor algorithm [86] can fail on this;

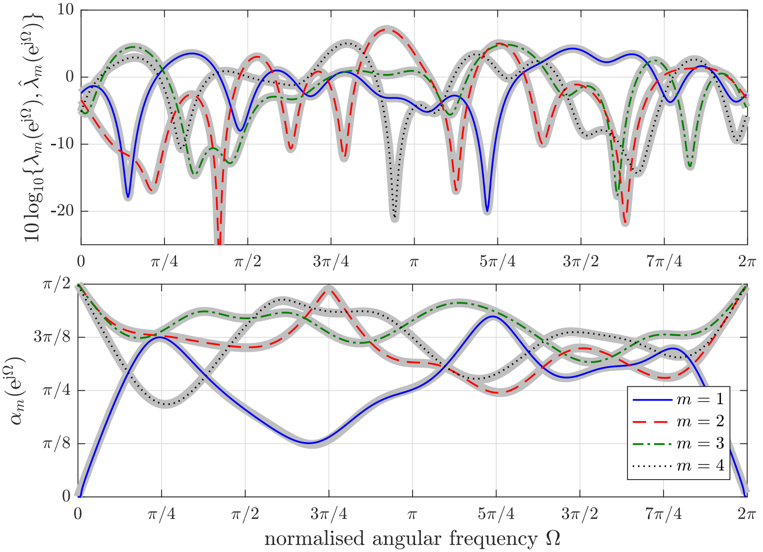


Comparison — SMD Algorithm Example



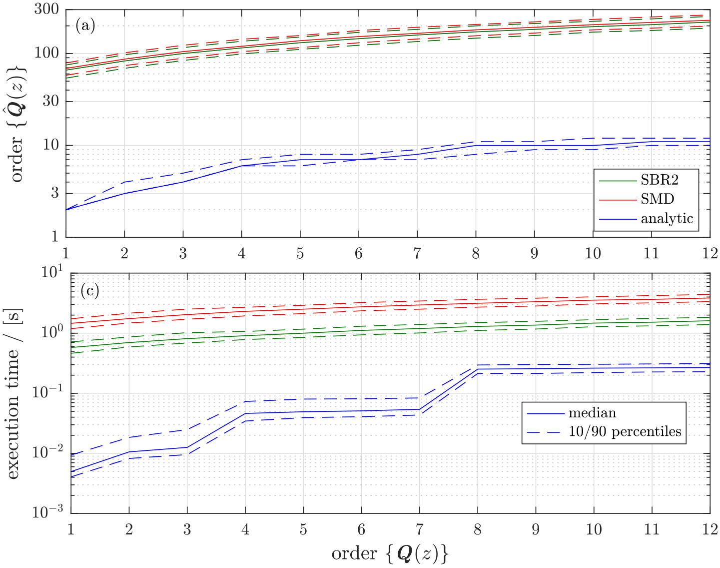
- $\boldsymbol{R}(z) : \mathbb{C} \rightarrow \mathbb{C}^{4\times 4}$ of order 47;
- SMD algorithm [77] yields approximate spectral majorisation [65];
- Hermitian angles of eigenvectors to a reference vector indicate approximation of piecewise analytic functions.

Analytic EVD Extraction Example



- same matrix, but utilising analytic eigen-value [107] and -vector extraction [104];
- extracted analytic EVD factors are close to ground truth;
- lower order compared to SMD result.

Comparison — Ensemble Results



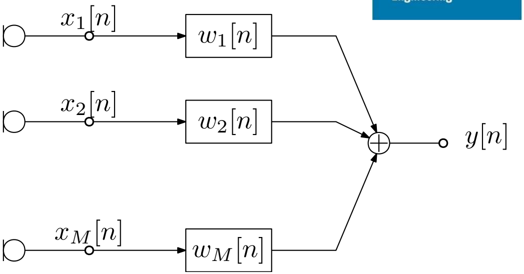
- ensemble results over matrices with different ground truth, and for various orders;
- above: application cost — the order of the extracted paraunitary matrices, required e.g. for a subspace projection;
- below: execution time of the algorithms.

6. Applications

6.1 Linearly Constrained Minimum Variance Beamforming

- Recall narrowband case [47]:

$$\min_{\mathbf{w}} \mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w} \qquad \text{s.t.} \;\; \mathbf{C}\mathbf{w} = \mathbf{f} \;;$$



- broadband case [96]:

$$\min_{\boldsymbol{w}(z)} \oint_{|z|=1} \boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{R}(z)\boldsymbol{w}(z)\frac{\mathrm{d}z}{z}$$
$$\text{s.t.} \;\; \boldsymbol{C}(z)\boldsymbol{w}(z) = \boldsymbol{f}(z) \;.$$

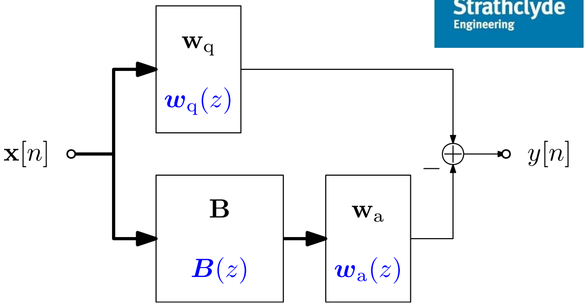
- standard processing with tap delay line of length L:

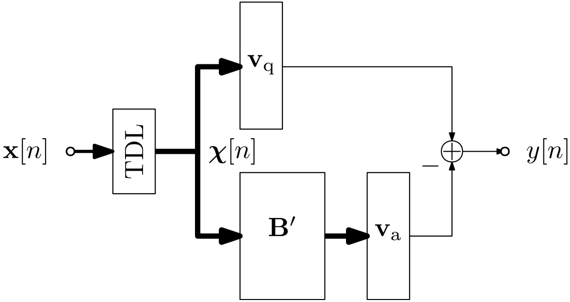
$$\boldsymbol{\chi}_n^{\mathrm{H}} = [\mathbf{x}_1^{\mathrm{H}}, \ldots, \mathbf{x}_M^{\mathrm{H}}] \in \mathbb{C}ML \;, \qquad \mathbf{v}^{\mathrm{H}} = [\mathbf{w}_1^{\mathrm{H}}, \ldots, \mathbf{w}_M^{\mathrm{H}}] \in \mathbb{C}^{ML} \;. \qquad (16)$$

- the constraint equation $\mathbf{C}'\mathbf{v} = \mathbf{f}'$ can be bulky.

LCMV — Generalised Sidelobe Canceller

- unconstrained optimisation problem via a quiescent beamformer $\mathbf{w}_\mathrm{q} = \mathbf{C}^\dagger \mathbf{f} \in \mathbb{C}^M$, a blocking matrix $\mathbf{B} \in \mathbb{C}^{(M-1)\times M}$, and adaptive filter $\mathbf{w}_\mathrm{a} \in \mathbb{C}^{M-1}$ [47]
- equivalent polynomial GSC [96];





- for the standard TDL approach: quiescent beamformer $\mathbf{v}_\mathrm{q} \in \mathbb{C}^{ML}$, blocking matrix $\mathbf{B}' \in \mathbb{C}^{(M-1)L\times ML}$, and adaptive filter vector $\mathbf{v}_\mathrm{a}^{(M-1)L}$ [13].

Broadband GSC — Computational Cost

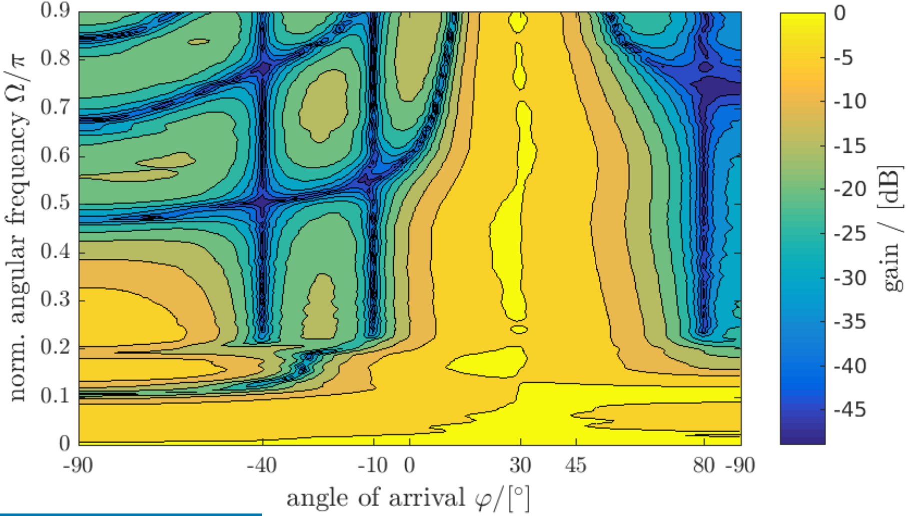
- With M sensors and a TDL length of L, the standard TDL GSC is determined by $\{M, L\}$;
- polynomial GSC: the adaptive filter $\boldsymbol{w}_{\mathrm{a}}(z)$ has order $(L-1)$, but the orders $(T-1)$ and $(N-1)$ of $\boldsymbol{w}_{\mathrm{q}}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M}$ and blocking matrix $\boldsymbol{B}(z) : \mathbb{C} \rightarrow \mathbb{C}^{(M-1)\times M}$ can be selected to satisfy accuracy (although generally $T \approx N \approx L$);
- cost comparison in multiply-accumulate (MAC) operations:

component	GSC cost	
	polynomial	standard
quiescent beamformer	MT	ML
blocking matrix	$M(M-1)N$	$M(M-1)L^2$
adaptive filter (NLMS)	$2(M-1)L$	$2(M-1)L$

- blocking matrix (particularly of standard GSC) dominates the cost.

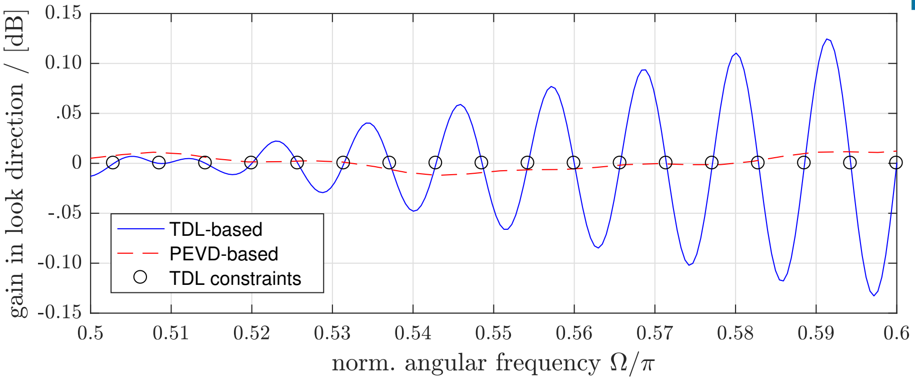
Directivity Pattern

- ULA with $M = 8$ sensors, $L = 176$, look direction $30°$, adapted GSC:



Response in Look Direction

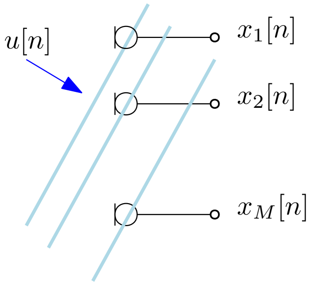
- Inspecting the directivity pattern for $\varphi = 30^\circ$ (excerpt):



- the standard TDL GSC requires $J > L$ point constraints along the frequency axis;
- polynomial GSC uses a single constraint equation using fractional delay filters [60].

- Multiple signal classification (MUSIC, [80]) can be used for narrowband angle of arrival estimation;
- for broadband signals: subspace decomposition of the cross-spectral density matrix

$$\mathbf{R}(z) = \underbrace{[\boldsymbol{Q}_{\mathrm{s}}(z) \ \ \boldsymbol{Q}_{\mathrm{n}}(z)]}_{\boldsymbol{Q}(z)} \underbrace{\begin{bmatrix} \boldsymbol{\Lambda}_{\mathrm{s}}(z) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_{\mathrm{n}}(z) \end{bmatrix}}_{\boldsymbol{\Lambda}(z)} \begin{bmatrix} \boldsymbol{Q}_{\mathrm{s}}^{\mathrm{P}}(z) \\ \boldsymbol{Q}_{\mathrm{n}}^{\mathrm{P}}(z) \end{bmatrix} \tag{17}$$

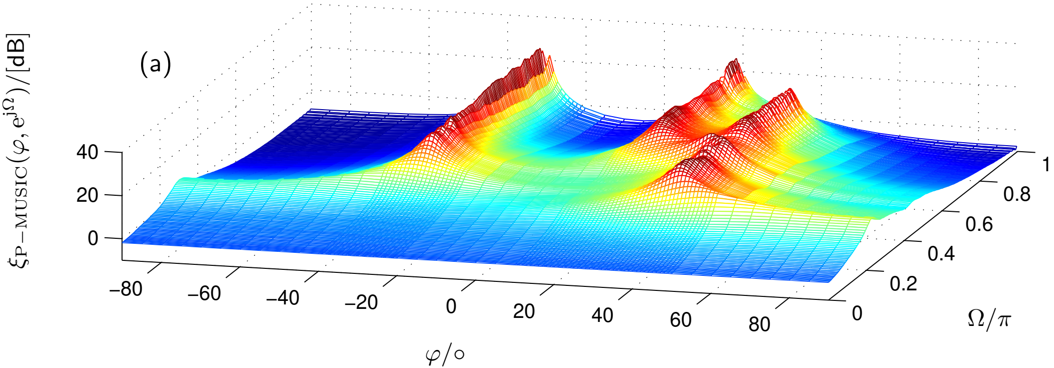


- potential steering vectors for a known array configuration are defined by (fractional) delay filters $\boldsymbol{a}_{\varphi}(z)$;
- scan for φ which steering vectors least fit the noise-only subspace:

$$\xi^{-1}_{\mathrm{P-MUSIC}}(\varphi, z) = \boldsymbol{a}^{\mathrm{P}}_{\varphi}(z)\boldsymbol{Q}^{\mathrm{P}}_{\mathrm{n}}(z)\boldsymbol{Q}_{\mathrm{n}}(z)\boldsymbol{a}_{\varphi}(z)\ . \tag{18}$$

Polynomial MUSIC Example

- ULA with $M = 8$ sensors, 3 sources, $\xi_{\mathrm{P-MUSIC}}(\varphi, z)$ evaluated on the unit circle, $z = \mathrm{e}^{\mathrm{j}\Omega}$:



- a number of variations and applications exist [6, 95, 35, 48, 6, 5, 8, 7, 4].

6.3 Signal Compaction

- Recall subspace decomposition of $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ \mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\rm H}[n-\tau]\}$:

$$\mathbf{R}(z) = \underbrace{[\boldsymbol{Q}_{\rm s}(z) \;\; \boldsymbol{Q}_{\rm n}(z)]}_{\boldsymbol{Q}(z)} \underbrace{\begin{bmatrix} \boldsymbol{\Lambda}_{\rm s}(z) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_{\rm n}(z) \end{bmatrix}}_{\boldsymbol{\Lambda}(z)} \begin{bmatrix} \boldsymbol{Q}_{\rm s}^{\rm P}(z) \\ \boldsymbol{Q}_{\rm n}^{\rm P}(z) \end{bmatrix} ; \qquad (19)$$

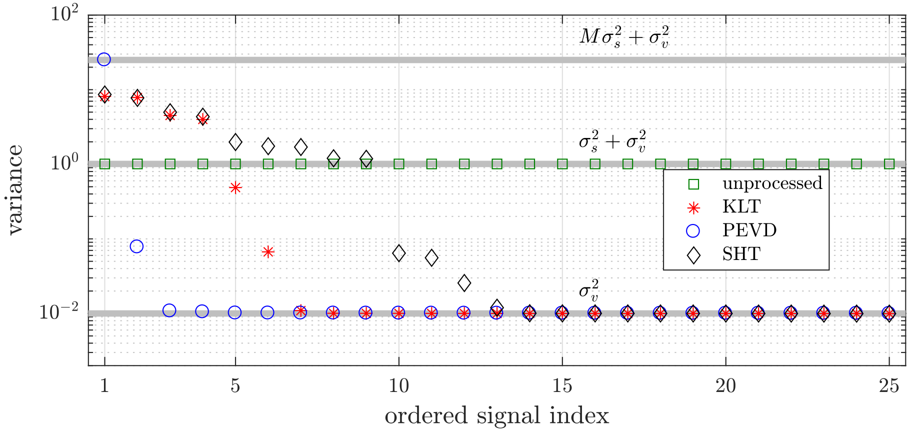
- if we have $N < M$ dominant eigenvalues, we can compact the signal components by means of $\mathbf{Q}_{\rm s}[n] \circ\!\!-\!\!\bullet \boldsymbol{Q}_{\rm s}(z)$ into

$$\mathbf{y}[n] = \sum_{\nu} \mathbf{Q}_{\rm s}^{\rm H}[-\nu]\mathbf{x}[n-\nu] ; \qquad (20)$$

- spectral majorisation and strong decorrelation ensure that the coding gain is maximised [87];
- in the context of subband coding such compaction filters had not been possible beyond $M = 2$ without the PEVD [76, 78].

Signal Compaction Example

- Example of an $M = 25$ element spherical array, single directional signal with power σ_s^2 in uncorrelated noise with variance σ_v^2; for $\sigma_s^2/\sigma_v^2 = 100$ [69]:

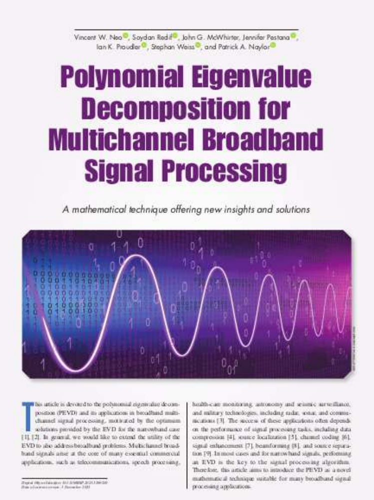


7. Summary

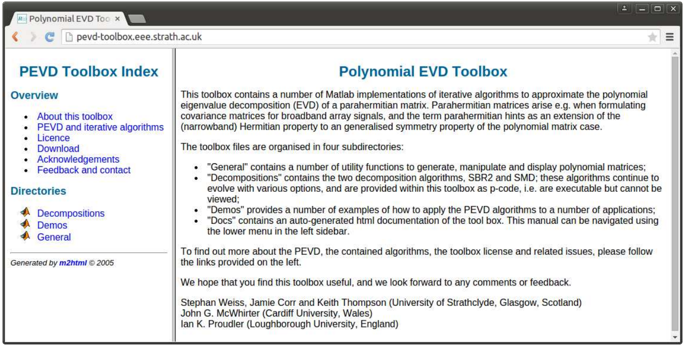
- Matrices of analytic functions are useful to formulate broadband multichannel signal processing challenges;
- solutions are often enabled by matrix factorisations — such as the analytic / polynomial EVD;
- key applications show strengths of the approach:
- beamforming/angle of arrival estimation: simple generalisation of well-known narrowband solutions, potential savings, and performance improvements;
- coding & compaction: approaches theoretical optimal data encoding system, and works for more than two channels;
- speech enhancement: preservation of spectral coherence avoids audible artefacts experienced with DFT domain algorithms.
- "Can I found out more? Can I try this myself?" — yes you can!

Additional Information and Resources

- V. Neo *et al.*, IEEE Signal Processing Magazine, **40**(7):18–37, Nov. 2023.



- Vincent Neo's audio demos and code: https://vwn09.github.io/portfolio/
- Matlab PEVD toolbox: pevd-toolbox.eee.strath.ac.uk



Some Further Material I

- Polynomial EVD algorithm variations; second order sequential best rotation (SBR2) algorithm [63, 64], with optimised coding gain [76, 78]; multiple shift SBR2 [93, 92]; efficient implementation [50, 58]; sequential matrix diagonalisation (SMD) algorithm [77], and various SMD family versions to undertake multiple shifts [19, 17, 18], apply search space reduction [22, 19, 26, 27], numerical efficiencies [16, 23, 24, 21, 30, 34, 25, 29, 31, 84]; a Householder approach to SMD [73];
- DFT domain algorithms to extract analytic solution — separate extraction of eigenvalues [108, 107] and eigenvectors [103, 106] based on smoothness criteria [99, 109, 112]; a similar attempt had been undertaken. in [86] with analysis in [28, 32, 33]; a principal eigenpair can be extracted via the power method [56]

Some Further Material II

- Support estimation [53] and trimming of polynomial matrices [44, 85, 23, 24];
- estimation of the space-time covariance matrix [38, 39, 40, 55, 59];
- applications in coding [94, 110, 76], angle of arrival estimation [6, 4, 48, 95], beamforming [105, 9, 10, 71, 96], subspace detection [74, 75, 97], speech enhancement [72, 69], communications [66, 113, 68, 84, 83, 100] and others [114, 98, 111];
- implementations [30, 26, 25, 29, 28, 31]
- extension to other decompositions, such as e.g. SVD [2, 3, 42, 52, 62, 64, 106, 11, 57], QRD [30, 42, 43, 54], etc. [15].

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