Properties and Structure of the Analytic Singular Value Decomposition

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***Abstract*—We investigate the singular value decomposition (SVD) of a rectangular matrix $A(z)$ of functions that are analytic on an annulus that includes at least the unit circle. Such matrices occur, e.g., as matrices of transfer functions representing broadband multiple-input multiple-output systems. Our analysis is based on findings for the analytic SVD applicable to continuous time systems, and on the analytic eigenvalue decomposition. Using these, we establish two potentially overlapping cases where analyticity of the SVD factors is denied. Firstly, from a structural point of view, multiplexed systems require oversampling by the multiplexing factor in order to admit an analytic solution. Secondly, from an algebraic perspective, we state under which condition spectral zeroes of any singular value require additional oversampling by a factor of two if an analytic solution is to be found. In all other cases, an analytic matrix admits an analytic SVD, whereby the singular values are unique up to a permutation, and the left- and right-singular vectors are coupled through a joint ambiguity w.r.t. an arbitrary allpass function. We demonstrate how qsome state-of-the-art polynomial matrix decomposition algorithms approximate this solution, motivating the need for dedicated algorithms.**

I. Introduction

Following its inception and proof of existence by various mathematicians in the 18th and 19th century [1] and the development of powerful algorithms [2], the singular value decomposition (SVD) has been playing a central role in providing various optimum solutions to signal processing problems [3]–[5]. This includes, e.g., applications such as rank determination for source enumeration, subspace identification for angle of arrival and frequency-estimation tasks, or source separation. In the context of communications, the construction of precoder and equaliser matrices that are optimal in various senses for a multiple-input multiple-output (MIMO) channel matrix of complex gain factors [6]. For broadband problems, where, instead of complex gain factors, impulse responses form the channel matrix entries, such a matrix decomposition can only decouple a matrix for one particular time instance, or one frequency if operating in the Fourier domain. Generally for any of the above applications, while it is possible to decompose a broadband problem into a number of discrete Fourier transform (DFT)-bins, independent bin-wise processing neglects spectral coherence and typically leads to suboptimal solutions [7]–[9].

To address problems such as H_∞ control [10] for matrices $\boldsymbol{A}(t)$ that are functions in a real, continuous variable t, de Moor & Boyd [11] and Bunse-Gerstner *et al.* [12] have proven the existence of an SVD, where for an analytic $\boldsymbol{A}(t)$, the factorisation $\boldsymbol{A}(t) = \boldsymbol{U}(t)\boldsymbol{\Sigma}(t)\boldsymbol{V}^{\mathrm{H}}(t)$ on some real interval $t_1 < t < t_2$ is satisfied by analytic left- and right-singular vectors in $\boldsymbol{U}(t)$ and $\boldsymbol{V}(t)$. For $\boldsymbol{\Sigma}(t)$ to be analytic, the singular values on its diagonal must not be restricted to be nonnegative. A number of algorithms have been developed to address such an analytic SVD [13]–[16].

More recently, several algorithms for a polynomial SVD (PSVD) have been developed for a rectangular matrix $\mathbf{A}[n]$, where $n \in \mathbb{Z}$ is the discrete time index. This was initially performed by first computing two polynomial matrix eigenvalue decompositions (PEVD) of parahermitian (or palindromic) matrices. The PEVD performs an approximate factorisation into Laurent-polynomial matrices. Two algorithm families for approximating the PEVD have been developed — second order sequential best rotation (SBR2) [17], [18] and sequential matrix diagonalisation (SMD) algorithms [19]–[21]. Subsequently, a number of related decomposition algorithms have emerged, whereby a PSVD is evaluated either via a number of polynomial QR operations [22] or directly [23]. These algorithms possess proven convergence in the sense that they monotonically minimise a given cost function, even though it is unclear what solution they attain. These algorithms behave similarly in that they (i) approximate a diagonalisation, and (ii) encourage (or even guarantee [24]) spectral majorisation, such that the identified singular values are ordered at every frequency [25]. While spectral majorisation can be useful in the context of e.g. optimal coding [19] or communications [26], it can lead to functions that are only piece-wise analytic and therefore require a much higher approximation order than their analytic counterparts [27], [28]. The use of PSVD algorithms extends from generic problems [17], [29] to a variety of practical applications including, e.g., MIMO communications [26], [30]–[34], the equalisation of filter bank-based multicarrier systems [35], [36], broadband beamforming [37] or the construction of paraunitary matrices and lossless filter banks [38], [39].

Despite the fact that the algorithms in [17]–[21] are proven to converge to a diagonalised matrix, it is unclear what this matrix is, if an exact solution exists, and whether potential solutions are unique. Therefore, the purpose of this paper is to investigate the existence and uniqueness of the SVD

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of a matrix $\mathbf{A}[n] \in \mathbb{C}^{M\times N}$, whose z-transform $\boldsymbol{A}(z) = \sum_n \mathbf{A}[n]z^{-n}$, or for short $\boldsymbol{A}(z) \bullet\!\!-\!\!\circ\, \mathbf{A}[n]$ [40], is analytic in z within some region of convergence that includes $|z| = 1$. We are interested in whether the SVD

$$\boldsymbol{A}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\,\boldsymbol{V}^{\mathrm{P}}(z) \tag{1}$$

exists with a diagonal, analytic $\boldsymbol{\Sigma}(z)$ that is real-valued on the unit circle, i.e., $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$, and paraunitary, analytic matrices $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$. Paraunitarity implies that $\boldsymbol{U}(z)\boldsymbol{U}^{\mathrm{P}}(z) = \boldsymbol{U}^{\mathrm{P}}(z)\boldsymbol{U}(z) = \mathbf{I}$, with the parahermitian operator $\{\cdot\}^{\mathrm{P}}$ performing a time reversal and complex conjugation, $\boldsymbol{U}^{\mathrm{P}}(z) = \boldsymbol{U}^{\mathrm{H}}(1/z^*)$ [41]. The existence of analytic factors in (1) is important as this guarantees the absolute convergence of their time domain equivalents. This in turn means that $\mathbf{U}[n] \circ\!\!-\!\!\bullet\, \boldsymbol{U}(z)$ and $\mathbf{V}[n] \circ\!\!-\!\!\bullet\, \boldsymbol{V}(z)$ can be well approximated by the polynomial matrices (or filter banks with finite impulse responses) that the PSVD algorithms in [17], [22], [23], [29] are seeking.

To investigate the existence of an analytic solution, we restrict ourselves to an analysis on the unit circle for $z = \mathrm{e}^{\mathrm{j}\Omega}$, where a matrix in a real-valued parameter Ω results. Exploiting findings in [11], [12], we must permit singular values to become negative in order to maintain analyticity. Further, we investigate the particular implications that a periodicity in the continuous parameter Ω brings. We will show that the demand for $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})$ to be restricted to real-valued singular values potentially has algebraic consequences for the analyticity; in contrast, we will see that if $\boldsymbol{A}(z)$ emerges from multiplexing — such as in multiplexed transmission or block filtering [41], [45]–[47] — a structural loss of analyticity emerges. To understand the existence and uniqueness of the left- and right-singular vectors in $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$, we rely on combining the existence of analytic singular values with the findings on the analytic parahermitian matrix EVD in [42]–[44].

The contribution of this paper — against the backdrop of milestones in developing the analytic SVD on a real interval [11], [12] accompanied by algorithms, and the design of practical algorithms for a polynomial SVD in the complex domain [17] — therefore lies in providing the theoretical foundations for an analytic SVD. While the design of algorithms is beyond the scope of this paper, we demonstrate how these results relate to the PSVD, for which the spectral majorisation property can obstruct finding the analytic solution to (1). While recent analytic parahermitian matrix EVD algorithms in [27] provide a solution that is closer to (1), the findings motivate the need for analytic SVD extraction algorithms akin to [23], [27], [28]. These could then address many of the broadband extensions of the SVD applied to or sought for the narrowband problems above.

In the following, Sec. II reviews the standard SVD, and uses the analytic SVD of a matrix in a continuous, real parameter [11], [12] to define a general approach to the analytic SVD of a matrix $\boldsymbol{A}(z)$ with $z \in \mathbb{C}$. Two exceptions to the analyticity of the SVD factors are elaborated in Secs. III and IV for algebraic and structural reasons, respectively. The existence of an analytic SVD of $\boldsymbol{A}(z)$, and the uniqueness and ambiguity of its analytic factors are defined in Sec. V. Sec. VI demonstrates some results based on existing algorithms before Sec. VII draws conclusions.

II. Analytic Singular Value Decomposition

A. Singular Value Decomposition

Any matrix $\mathbf{A} \in \mathbb{C}^{M\times N}$ admits a singular value decomposition [48]

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}} \ , \tag{2}$$

where $\mathbf{\Sigma} \in \mathbb{R}^{M\times N}$ is a diagonal matrix containing the non-negative, unique singular values σ_i, $i = 1, \dots, K$ with $K = \min(M, N)$, and $\mathbf{U} \in \mathbb{C}^{M\times M}$ and $\mathbf{V} \in \mathbb{C}^{N\times N}$ are unitary matrices containing the left- and right-singular vectors of $\mathbf{A}$.

The majorised ordering of the singular values,

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_K \geq 0 \ , \tag{3}$$

ensures that $\mathbf{\Sigma}$ is unique. If there are C identical, non-zero singular values $\sigma_i = \ldots = \sigma_{i+C-1} > 0$, with associated left- and right-singular vectors, $\mathbf{u}_i, \ \ldots, \ \mathbf{u}_{i+C-1}$ and $\mathbf{v}_i, \ \ldots, \ \mathbf{v}_{i+C-1}$, then $\mathbf{u}'_i, \ldots \mathbf{u}'_{i+C-1}$ and $\mathbf{v}'_i, \ldots \mathbf{v}'_{i+C-1}$ are also valid left- and right-singular vectors where

$$[\mathbf{u}'_i, \ldots \mathbf{u}'_{i+C-1}] = [\mathbf{u}_i, \ldots \mathbf{u}_{i+C-1}] \cdot \mathbf{\Phi} \tag{4}$$

$$[\mathbf{v}'_i, \ldots \mathbf{v}'_{i+C-1}] = [\mathbf{v}_i, \ldots \mathbf{v}_{i+C-1}] \cdot \mathbf{\Phi} \ , \tag{5}$$

and $\mathbf{\Phi} \in \mathbb{C}^{C\times C}$ is an arbitrary unitary matrix. This means that either the C left- or right-singular vectors can be selected arbitrarily within a C-dimensional subspace, which then ties down the corresponding C right- or left-singular vectors, respectively. In case $C = 1$, $\mathbf{\Phi} = \mathrm{e}^{\mathrm{j}\varphi}$ with φ arbitrary; if $\mathbf{A}$ is restricted to be real-valued, this leads to the well-known sign-ambiguity of the SVD, see e.g. [49].

Let S be the number of singular values equalling zero. If $M \leq N$ and $J = N - M$, then the rightmost $S + J$ right-singular vectors span the nullspace of $\mathbf{A}$, and can form an arbitrary orthonormal basis within this $(S + J)$-dimensional space. Further, the S rightmost left-singular vectors can also be arbitrarily selected within an S-dimensional subspace. For $M > N$, the same considerations apply by inspecting $\mathbf{A}^{\mathrm{T}}$ instead of $\mathbf{A}$.

One way to calculate the SVD is via two EVDs of $\mathbf{R}_1 = \mathbf{A}\mathbf{A}^{\mathrm{H}}$ and $\mathbf{R}_2 = \mathbf{A}^{\mathrm{H}}\mathbf{A}$, such that

$$\mathbf{R}_1 = \mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^{\mathrm{H}} = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^{\mathrm{H}} \ , \tag{6}$$

$$\mathbf{R}_2 = \mathbf{Q}_2\mathbf{\Lambda}_2\mathbf{Q}_2^{\mathrm{H}} = \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^{\mathrm{H}} \ . \tag{7}$$

The singular values can therefore be obtained as the square root of the eigenvalues in $\mathbf{\Lambda}_1$ or $\mathbf{\Lambda}_2$. In case of distinct eigenvalues, from (6) we can deduce that $\mathbf{U} = \mathbf{Q}_1\mathbf{\Phi}_1$ where $\mathbf{\Phi}_1$ is a diagonal unitary matrix. Similarly we can write $\mathbf{V}^{\mathrm{H}} = \mathbf{\Phi}_2^{\mathrm{H}}\mathbf{Q}_2^{\mathrm{H}}$. Hence $\mathbf{A} = \mathbf{Q}_1\mathbf{\Phi}_1\mathbf{\Lambda}_1^{1/2}\mathbf{\Phi}_2^{\mathrm{H}}\mathbf{Q}_2^{\mathrm{H}} = \mathbf{Q}_1\mathbf{\Lambda}_1^{1/2}\left(\mathbf{\Phi}_1\mathbf{\Phi}_2^{\mathrm{H}}\right)\mathbf{Q}_2^{\mathrm{H}}$. Since $\mathbf{\Lambda}_1^{1/2}$ is real we require $\left(\mathbf{\Phi}_1\mathbf{\Phi}_2^{\mathrm{H}}\right)$ to also be real. In the case of a C-fold algebraic multiplicity of eigenvalues, within a C-dimensional invariant subspace there may be different but equivalent bases — and therefore unitary matrices as in (4) and (5) — implied in (6) and (7) that need to be reconciled.

B. Analytic SVD on a Real Interval

As established in [11], [12], we have:

Theorem 1 (Analytic SVD on a real interval): For an $M \times N$ matrix $\boldsymbol{A}(t)$ that is analytic in $t \in \mathbb{R}$ on some interval, a decomposition $\boldsymbol{A}(t) = \boldsymbol{U}(t)\boldsymbol{\Sigma}(t)\boldsymbol{V}^{\rm H}(t)$ exists with unitary $\boldsymbol{U}(t)$ and $\boldsymbol{V}(t)$, and diagonal, real-valued $\boldsymbol{\Sigma}(t)$. The singular values in $\boldsymbol{\Sigma}(t)$ can be analytic if (i) they are permitted to become negative and (ii) their ordering is relaxed from the majorisation in (3) for every t. If the singular values are chosen to be analytic, then the left- and right-singular vectors in $\boldsymbol{U}(t)$ and $\boldsymbol{V}(t)$ can also be selected to be analytic.

Proof. For the case of a real-valued matrix, please see the proof of Theorem 1 in [12]. The complex-valued case is covered in [11]. ■

The key difference between the analytic SVD and (2) lies in permitting singular values to become negative. Enforcing a positive semi-definite constraint may lead to singular values $\sigma_i(t)$, $i = 1, \ldots, K$, that are continuous but not differentiable. Therefore, implementations of the analytic SVD are based on, e.g., checking the first derivative [11], the arc length [12], or on Chebychev polynomials to enforce smoothness of functions [16].

C. Periodicity of a Singular Value

For the analytic SVD of a matrix $\boldsymbol{A}(z)$ that is analytic on an annulus containing at least the unit circle, we can restrict our investigation to $z = \mathrm{e}^{\mathrm{j}\Omega}$ [40]. We thus work with $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})$, which now is analytic in $\Omega \in \mathbb{R}$. Therefore, Theorem 1 guarantees a decomposition $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{U}'(\Omega)\boldsymbol{\Sigma}'(\Omega)\boldsymbol{V}'^{\rm H}(\Omega)$ with analytic but not necessarily periodic factors $\boldsymbol{U}'(\Omega)$, $\boldsymbol{\Sigma}'(\Omega)$, and $\boldsymbol{V}'(\Omega)$. We will first contemplate the singular values $\sigma'_i(\Omega)$, $i = 1, \ldots, K$, on the diagonal of $\boldsymbol{\Sigma}'(\Omega)$.

The 2π-periodicity of $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})$ means that, similar to the case of the eigenvalue decomposition of a self-adjoint $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega})$ in [43], [44], $\boldsymbol{\Sigma}'(\Omega)$ has to equal, up to some permutation and negation for individual singular values, $\boldsymbol{\Sigma}'(\Omega + 2\pi n)$ and its derivatives w.r.t. Ω with $n \in \mathbb{Z}$. Thus, the singular values are composed of repeats of 2π segments. Let a 'segment' be that part of a singular value between 0 and 2π, defined as $\varsigma_i(\Omega) = \sigma'_i(\Omega)$, $\Omega \in [0; 2\pi)$, $i = 1, \ldots, K$.

To record the repetition of segments that make up an analytic singular value $\sigma_i(\Omega)$, we utilise an index sequence such as, e.g., $\{1, 2, -3, \ldots\}$ to indicate a concatenation of segments $\sigma'_i = [\varsigma_1, \varsigma_2, -\varsigma_3, \ldots]$. A negative index refers to a negated segment. We assume that there are K distinct singular values. Then, due to the uniqueness theorem for analytic functions, there is only one possibility for analytic continuation from one segment to another. Hence, for a sequence $\{\ldots, i, j, \ldots\}$ with a fixed index i, there is exactly one unique index j, which can be either positive or negative. Therefore, some simple rules follow:

- (R1) If an index is repeated such as i in $\{\ldots, i, j, \ldots, i, \ell, \ldots\}$, analytic continuation demands that ς_i can only be followed by one unique segment, such that $\ell = j$.
- (R2) Analytic continuation also demands that e.g. for $\{\ldots, j, i, \ldots, \ell, i, \ldots\}$ the segment preceding ς_i must be unique, i.e. $j = \ell$.
- (R3) For an index sequence $\{\ldots, i, j, \ldots, -i, \ell, \ldots\}$ with distinct indices i and j, ς_i is analytically continued by ς_j; therefore, $-\varsigma_i$ must be analytically continued by $-\varsigma_j$, i.e. $\ell = -j$.

Note that there are only a maximum number of K distinct segments that can be concatenated, hence two consequences result:

- (C1) The repeat of an index according to (R1) and (R2) after k segments, $1 \leq k \leq K$, implies a $2k\pi$ periodicity of a singular value.
- (C2) The repeat of a negated index after k segments means that a sequence $\{\ldots, i, j, \ldots, \ell, -i, \ldots\}$ must continue according to rules (R3) and (R1) as $\{i, j, \ldots, \ell, -i, -j, \ldots, -\ell, i, \ldots\}$, and thus leads to a $4k\pi$-periodic singular value.

Therefore analytic singular values must be periodic, with a maximum possible period of $4\pi K$.

For a more detailed analysis of the periodicity, below we will extract one analytic singular value, without loss of generality $\sigma'_1(\Omega)$, from these segments. To start, for $\Omega \in [0; 2\pi)$, we set $\sigma'_1(\Omega) = \varsigma_1(\Omega)$, with an index sequence $\{1\}$. On the interval $\Omega \in [2\pi; 4\pi)$, we find three possibilities to analytically continue $\sigma'_1(\Omega)$, with the notation (Pξ.n) indicating the nth possibility for segment $(\xi + 1)$:

- (P1.1) For $\sigma'_1(\Omega) = \varsigma_1(\Omega - 2\pi)$, we have an index sequence $\{1, 1\}$, and due to (C1) have established 2π-periodicity.
- (P1.2) For $\sigma'_1(\Omega) = -\varsigma_1(\Omega - 2\pi)$, we have $-\sigma'_1(\Omega) = \sigma'_i(\Omega - 2\pi)$ and an index sequence $\{1, -1\}$; due to (C2), the periodicity is 4π;
- (P1.3) otherwise, $\varsigma_1(\Omega)$ must be followed by a different segment; w.l.o.g. we assume this to be $\sigma'_1(\Omega) = \pm\varsigma_2(\Omega - 2\pi)$, and record an index sequence $\{1, \pm 2\}$.

We have established periodicity with cases (P1.1) and (P1.2). For case (P1.3), we continue to investigate the analytic continuation.

In general, for the interval $\Omega \in [2\xi\pi; 2(\xi + 1)\pi)$ with $\xi = 1, 2, \ldots, (K - 1)$, assuming we have not yet encountered a periodicity, we will assume w.l.o.g. that we have already used the segments $[1, \pm 2, \ldots, \pm\xi]$. Then we find the following possibilities for an analytic continuation:

- (Pξ.1) With $\sigma'_1(\Omega) = \varsigma_1(\Omega - 2\xi\pi)$, the index sequence $\{1, \ldots, \pm\xi, 1\}$ implies a periodicity of $2\pi\xi$ according to (C1).
- (Pξ.2) For $\sigma'_1(\Omega) = -\varsigma_1(\Omega - 2\xi\pi)$, the index sequence is $\{1 \ldots, \pm\xi, -1\}$, and $\sigma'_1(\Omega)$ is $4\pi\xi$ periodic according to (C2).

Continuations $\sigma'_1(\Omega) = \pm\varsigma_i(\Omega - 2\xi\pi)$, $i \in \{2, \ldots, \xi\}$ would imply the repeat of a previous segment or negated segment without repeating index 1. E.g. the index sequence $\{1, 2, 3, 2\}$ violates rule (R2), and cannot represent an analytic continuation. Therefore:

- (Pξ.3) The singular value $\sigma'_1(\Omega)$ must be followed by a different, yet unused segment. Without loss of generality we assume this to be $\sigma'_1(\Omega) = \pm\varsigma_{\xi+1}(\Omega - 2\xi\pi)$, with a resulting index sequence of $\{1, \pm 2, \dots, \pm\xi, \pm(\xi+1)\}$.

Unless a periodic repetion has been established via the cases (Pξ.1) or (Pξ.2), $\xi = 1, 2, \dots, (K-1)$, via (Pξ.3) we finally reach the interval $\Omega \in [2K\pi; 2(K+1)\pi)$. We find:

(PK.1) Analytic continuation with $\sigma'_1(\Omega) = \varsigma_1(\Omega)$ leads to an index sequence $\{1, \pm 2, \dots, \pm K, 1\}$ and $2K\pi$ periodicity due to (C1).

(PK.2) Analytic continuation with $\sigma'_1(\Omega) = -\varsigma_1(\Omega - 2K\pi)$ implies a sequence $\{1, \pm 2, \dots, \pm K, -1\}$ with $4K\pi$ periodicity due to (C2).

Any other continuation violates rules (R1) and (R2), and cannot yield an analytic function. Thus, the above scheme for analytic continuation either ends with (Pk.1) and a periodicity of $2k\pi$, or with (Pk.2) and a periodicity of $4k\pi$, $k = 1, \dots, K$, for this singular value.

D. Properties of the Singular Value Matrix

Sec. II-C has established the periodicity of a singular value. We now explore the properties of $\boldsymbol{\Sigma}'(\Omega)$, and therefore of all the singular values.

Let us assume that a first analytic singular value $\sigma'_1(\Omega)$ has been investigated as outlined in Sec. II-C. Assume it has a repeat pattern via case (Pk_1.1), with a sequence $\{1, \pm 2, \dots, \pm k_1, 1\}$ and a periodicity of $2k_1\pi$. Since all segments have to be used in a given $[2n\pi, 2(n+1)\pi)$ range, then another analytic singular value starts with the segment $\pm k_1$. Because of the arguments in Sec. II-C, its sequence must be $\{\pm k_1, 1, \pm 2, \pm 3, \dots, \pm k_1, 1, \dots\}$ i.e. a shifted version of $\sigma'_1(\Omega)$. This consideration continues similarly for the singular values that start with the segment $\pm\ell$ $(2 \le \ell < k_1)$. Therefore, $\sigma'_1(\Omega)$ is part of a set of k_1 frequency-shifted singular values, all with a $2k_1\pi$-periodicity. If we establish a pattern via (Pk_1.2), then the investigated singular value is also part of a set of k_1 singular values related by a frequency shift but with a periodicity of $4k_1\pi$. Note in this case that a shift by more than k_1 in the index sequence only leads to a negated value, e.g. $\{1, \dots, \pm k_1, -1, \dots, \mp k_1, 1\}$ would produce $\{-1, \dots, \mp k_1, 1, \dots, \pm k_1, -1\}$. Overall, we therefore find that

$$\sigma'_\nu(\Omega) = \sigma'_1(\Omega - 2(\nu-1)\pi), \qquad \nu = 1, \dots, k_1 \; . \tag{8}$$

The periodicity of these singular values is $2k_1\kappa_1\pi$, whereby $\kappa_1 = 1$ in case of (Pk_1.1) and $\kappa_1 = 2$ for (Pk_1.2).

From the remaining $(K - k_1)$ singular values of $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})$, we pick another singular value and iterate the investigation of Sec. II-C. This will lead to a pattern via (Pk_2.1) or (Pk_2.2), with $k_2 \in \{1, \dots, K - k_1\}$, establishing a set of k_2 singular values of periodicity $2k_2\kappa_2\pi$. We repeat this until all K analytic singular values are addressed, whereby we have P sets of frequency-shifted singular values. The p-th set contains k_p frequency-shifted singular values with a periodicity $2k_p\kappa_p\pi$ where $\kappa_p \in \{1, 2\}$ depending on the inclusion of a negated segment in the singular value. Note that $\sum_{\nu=1}^{P} k_p = K$. Since all singular values are periodic, we can write $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega/L}) = \sigma'_m(\Omega)$ or overall

$$\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L}) = \boldsymbol{\Sigma}'(\Omega) \; , \tag{9}$$

whereby $L = \mathrm{lcm}\{k_1\kappa_1, k_2\kappa_2, \dots\}$, i.e. the period of $\boldsymbol{\Sigma}'(\Omega)$ is the least common multiple (lcm) of the periods of all the singular values of $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})$.

E. Analytic SVD on the Unit Circle

Theorem 2 (Analytic SVD on the unit circle): For an analytic matrix $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})$, $\Omega \in \mathbb{R}$, the analytic SVD on the unit circle can be formulated as

$$\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{U}(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{V}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega/L}) \; , \tag{10}$$

where the diagonal matrix $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})$ and unitary matrices $\boldsymbol{U}(\mathrm{e}^{\mathrm{j}\Omega/L})$ and $\boldsymbol{V}(\mathrm{e}^{\mathrm{j}\Omega/L})$ can be analytic in Ω for some $L \in \mathbb{N}$.

Proof: We can state the SVD of $\boldsymbol{A}(z)$ on the unit circle generally as $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{U}'(\Omega)\boldsymbol{\Sigma}'(\Omega)\boldsymbol{V}'^{\mathrm{H}}(\Omega)$. Starting with the singular values, based on Theorem 1 and the reasoning in Secs. II-C and II-D, we know that $\boldsymbol{\Sigma}'(\Omega)$ must be $2L\pi$-periodic for some $L \in \mathbb{N}$, see (9).

For the left- and right-singular vectors in $\boldsymbol{U}'(\Omega)$ and $\boldsymbol{V}'(\Omega)$, Theorem 1 guarantees analyticity of their elements, but their periodicity still needs to be shown. For this, we formulate the parahermitian matrices $\boldsymbol{R}_1(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z)$ and $\boldsymbol{R}_2(z) = \boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{A}(z)$. For the analytic EVD on the unit circle we potentially require oversampling such that analytic 2π-periodic eigenvalues [43] exist for $\boldsymbol{R}_i(z^{L_i})$, $i = 1, 2$, $L_i \in \mathbb{N}$. Further, analytic eigenvectors that match the periodicity of the eigenvalues also exist [44]. We therefore have

$$\boldsymbol{R}_1(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{Q}_1(\mathrm{e}^{\mathrm{j}\Omega/L_1})\boldsymbol{\Lambda}_1(\mathrm{e}^{\mathrm{j}\Omega/L_1})\boldsymbol{Q}_1^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega/L_1}) \tag{11}$$

$$\boldsymbol{R}_2(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{Q}_2(\mathrm{e}^{\mathrm{j}\Omega/L_2})\boldsymbol{\Lambda}_2(\mathrm{e}^{\mathrm{j}\Omega/L_2})\boldsymbol{Q}_2^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}/L_2}) \; . \tag{12}$$

Inserting the SVD $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{U}'(\Omega)\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{V}'^{\mathrm{H}}(\Omega)$ into the definition of $\boldsymbol{R}_1(\mathrm{e}^{\mathrm{j}\Omega})$ and $\boldsymbol{R}_2(\mathrm{e}^{\mathrm{j}\Omega})$, we obtain

$$\boldsymbol{R}_1(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{U}'(\Omega)\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{\Sigma}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{U}'^{\mathrm{H}}(\Omega) \tag{13}$$

$$\boldsymbol{R}_2(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{V}'(\Omega)\boldsymbol{\Sigma}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{V}'^{\mathrm{H}}(\Omega) \; . \tag{14}$$

The analytic eigenvalues of a parahermitian matrix are unique up to a permutation [42]. Assuming appropriate ordering, we find that

$$\boldsymbol{\Lambda}_1(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega L_1/L})\boldsymbol{\Sigma}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega L_1/L}) \tag{15}$$

$$\boldsymbol{\Lambda}_2(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{\Sigma}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega L_2/L})\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega L_2/L}) \; , \tag{16}$$

and hence $L_1 = L_2 = L$ [1]. Comparing (11) with (13) and (12) with (14), and allowing for the ambiguities discussed in Sec. II-A, we may set $\boldsymbol{U}'(\Omega) = \boldsymbol{Q}_1(\mathrm{e}^{\mathrm{j}\Omega/L})$ and $\boldsymbol{V}'(\Omega) = \boldsymbol{Q}_2(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{\Psi}'(\Omega)$. The unitary matrix $\boldsymbol{\Psi}'(\Omega)$ links the ambiguities of the left- and right-singular vectors as per Sec. II-A. For distinct singular values it is a diagonal matrix of allpass functions, i.e. functions with unit magnitude but variable phase response [40]. Inserting these expressions for $\boldsymbol{U}'(\Omega)$ and $\boldsymbol{V}'(\Omega)$ into the SVD of $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})$, we find

$$\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{Q}_1(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})\boldsymbol{\Psi}'^{\mathrm{H}}(\Omega)\boldsymbol{Q}_2^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega/L}) \; . \tag{17}$$

This leaves $\boldsymbol{\Psi}'(\Omega)$ as the only factor on the r.h.s. of (17) whose periodicity is unknown. Since $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}(\Omega+2L\pi)}) =$

[1] If the singular values contain sign changes, it may be possible to even halve the periodicity, such that $L_1 = L_2 = L/2$.

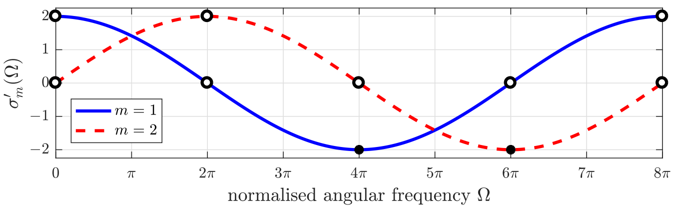

Fig. 1. Example of 8π-periodic singular values of $\boldsymbol{A}_1(\mathrm{e}^{\mathrm{j}\Omega})$ that are analytic in $\Omega \in \mathbb{R}$; $\circ$ indicates the repeat of the singular values at $\Omega = 0$, $\bullet$ signifies a repeat of such a singular value with a sign change. To be analytic in $z \in \mathbb{C}$, these functions have to be oversampled by $L = 4$.

$\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})$, expanding both sides according to (17) and exploiting the paraunitarity and hence invertibility of $\boldsymbol{Q}_1(\mathrm{e}^{\mathrm{j}\Omega/L}) = \boldsymbol{Q}_1(\mathrm{e}^{\mathrm{j}(\Omega/L+2\pi)})$ and $\boldsymbol{Q}_2(\mathrm{e}^{\mathrm{j}\Omega/L}) = \boldsymbol{Q}_2(\mathrm{e}^{\mathrm{j}(\Omega/L+2\pi)})$, we arrive at $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}(\Omega/L+2\pi)})\,\boldsymbol{\Psi}'^{\mathrm{H}}(\Omega + 2L\pi) = \boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})\,\boldsymbol{\Psi}'^{\mathrm{H}}(\Omega)$. Since $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}(\Omega/L+2\pi)}) = \boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})$, we obtain

$$\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L}) \left\{ \boldsymbol{\Psi}'^{\mathrm{H}}(\Omega + 2L\pi) - \boldsymbol{\Psi}'^{\mathrm{H}}(\Omega) \right\} = 0 \tag{18}$$

Therefore, assuming that $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})$ has only the trivial null space, $\boldsymbol{\Psi}'(\Omega) = \boldsymbol{\Psi}'^{H}(\Omega + 2L\pi)$ and is thus $2L\pi$-periodic i.e. we can write $\boldsymbol{\Psi}'(\Omega) = \boldsymbol{\Psi}(\mathrm{e}^{\mathrm{j}\Omega/L})$. If $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})$ has a nontrivial null space, the properties of $\boldsymbol{\Psi}'(\Omega)$ in that null space are moot as it always appears multiplied by a zero portion of $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})$. Then we also have that $\boldsymbol{V}'(\Omega)$ is at least $2L\pi$-periodic as well. Hence $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})$ has the SVD as in (10) where $\boldsymbol{U}(\mathrm{e}^{\mathrm{j}\Omega/L}) = \boldsymbol{Q}_1(\mathrm{e}^{\mathrm{j}\Omega/L})$ and $\boldsymbol{V}(\mathrm{e}^{\mathrm{j}\Omega/L}) = \boldsymbol{Q}_2(\mathrm{e}^{\mathrm{j}\Omega/L})\,\boldsymbol{\Psi}(\mathrm{e}^{\mathrm{j}\Omega/L})$. ■

Example 1: As an example for the case $L > 1$, consider the matrix $\boldsymbol{A}_1(z) = [1,\ 1; z^{-1},\ 1]$. This matrix can be shown to possess on the unit circle singular values $\sigma'_1(\Omega) = 2\cos(\Omega/4)$ and $\sigma'_2(\Omega) = 2\sin(\Omega/4) = \sigma'_1(\Omega - 2\pi)$ that are analytic in $\Omega \in \mathbb{R}$ and 8π periodic as shown in Fig. 1, such that w.r.t. (10), $L = 4$. Note that for e.g. $\Omega = 0$, $\Omega = 2\pi$, $\Omega = 4\pi$ and $\Omega = 6\pi$ the moduli of the singular values are identical, as indicated in Fig. 1.

The considerations according to Sec. II-C end with case (2.2), and we find that the sequences of segment indices are $\{1, 2, -1, -2, 1 \dots\}$ and $\{-2, 1, 2, -1, -2, \dots\}$, and so $k_1 = \kappa_1 = 2$ and $L = 4$. The singular values in Fig. 1 are 2π-shifted versions of each other and are $2L\pi = 8\pi$ periodic. △

The discussion in Sec. II-D was based on the extraction of functions that are analytic in $\Omega \in \mathbb{R}$ with a $2\pi L$ periodicity. Thus, while e.g. $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega/L})$ in (9) is analytic in the real parameter Ω, the same function may not be analytic in the complex parameter z [44]. W.r.t. Example 1, while $\sigma'_1(\Omega) = 2\cos(\Omega/4) = \mathrm{e}^{\mathrm{j}\Omega/4} + \mathrm{e}^{-\mathrm{j}\Omega/4}$ is analytic in Ω on the unit circle, the resubstitution with $z = \mathrm{e}^{\mathrm{j}\Omega}$ leading to $\sigma_1(z) = z^{1/4} + z^{-1/4}$ is not analytic in the complex plane. E.g. the term $z^{-1/4}$ represents a fractional delay; its time domain equivalent is a sampled sinc function [50], which is not absolutely convergent, and hence $z^{-1/4}$ is not analytic. We can however obtain analytic singular values in the complex plane if we analyse $\boldsymbol{A}_1(z^4)$ instead of $\boldsymbol{A}_1(z)$, i.e. if we oversample the matrix $\boldsymbol{A}_1(z)$ by a factor of $L = 4$, and thus avoid fractional delays.

In the following two sections, we will explore in more detail when and why cases with $L > 1$ occur; first, in Sec. III we consider the algebraic aspect that causes a sign change in a singular value. Thereafter, we focus on the case of frequency-shifted —or in the time domain modulated— singular values in Sec. IV.

III. Unmodulated Singular Values

This section addresses the case where singular values $\sigma'_k(\Omega)$, $k = 1, \dots, K$ are not frequency-shifted — or equivalently, $\sigma_k[n] \circ\!\!-\!\!\bullet\, \sigma'_k q(\Omega)$ are not modulated — versions of one another, but can be 4π-periodic instead of the 2π-periodic functions required for analyticity of the singular values. For this, we initially focus on the degenerate case of a 1×1 matrix $\boldsymbol{A}(z) = \gamma(z) : \mathbb{C} \to \mathbb{C}$ in Sec. III-A. We then explore how for multiple distinct singular values, their zero-crossings determine their periodicity in Secs. III-B and III-C.

A. Factorising 2π-Periodic Complex-Valued Functions

For a function $\gamma(z) : \mathbb{C} \to \mathbb{C}$ that is analytic in z, we initially evaluate the SVD on the unit circle for $z = \mathrm{e}^{\mathrm{j}\Omega}$,

$$\gamma(\mathrm{e}^{\mathrm{j}\Omega}) = u'(\Omega) \cdot \sigma'(\Omega) \cdot v'^{*}(\Omega) \;, \tag{19}$$

which factorises $\gamma(\mathrm{e}^{\mathrm{j}\Omega})$ into a real-valued and two complex-valued components. The function $\sigma'(\Omega)$ represents the real-valued analytic singular value and the complex-valued analytic left- and right-singular vectors $u'(\Omega)$ and $v'(\Omega)$ reduce to allpass filters. W.l.o.g., we set $v'(\Omega) = 1$, i.e., we mandate that the frequency-dependent phase change has to be performed by $u'(\Omega)$.

As long as $\sigma'(\Omega) > 0\ \forall\Omega$, the SVD performs a split into a magnitude and a phase term. Since we only have one singular value, no shifted versions can appear; further, since there is no sign change, and given that $\sigma'(\Omega) = \sigma'(\Omega - 2\pi)$, then $\sigma'(\Omega)$ must be 2π-periodic. Therefore, according to Theorem 2, the terms in (19) can be analytic, such that $\gamma(z) = u(z)\sigma(z)v^{\mathrm{P}}(z)$. The same argument applies if $\sigma'(\Omega) < 0\ \forall\Omega$, as the sign can be incorporated into $u'(\Omega)$, and an analytic 2π-periodic solution is also possible.

Example 2: The function $\gamma(z) = 1 + \mathrm{j}z^{-1} + \frac{1}{2}z$ is not parahermitian, i.e. $\gamma^{\mathrm{P}}(z) \neq \gamma(z)$, and therefore $\gamma(\mathrm{e}^{\mathrm{j}\Omega}) \notin \mathbb{R}$, but satisfies $|\gamma(\mathrm{e}^{\mathrm{j}\Omega})| > 0$. Its singular value decomposition leads to the analytic singular value $\sigma(z)$ and an allpass $u(z)$ characterised in Fig. 2. Note from Fig. 2(a) that $\sigma[n] = \sigma^{*}[-n]$ is Hermitian. Both $\sigma[n]$ and $u[n]$ are (potentially infinite) Laurent series, but due to their analyticity are absolutely convergent and hence decay at least exponentially, as seen in Fig. 2(b). While thus the support of $\sigma(z)$ has increased w.r.t. $\gamma(z)$, the exponential decay is sufficient for analyticity, which in turn guarantees that $\sigma(z)$ can be approximated arbitrarily closely by a finite length Laurent polynomial [27], [28]. △

The remainder of Sec. III considers the case where $\sigma'(\Omega)$ is not strictly positive or negative, i.e. $\sigma'(\Omega) = 0$ for some Ω, and we aim to understand under which circumstances a 2π or a 4π periodicity arises. We further expand the scope from a single function $\sigma'(\Omega)$ to multiple singular values $\sigma'_k(\Omega)$, $k = 1, \dots, K$, that are not shifted versions of each other.

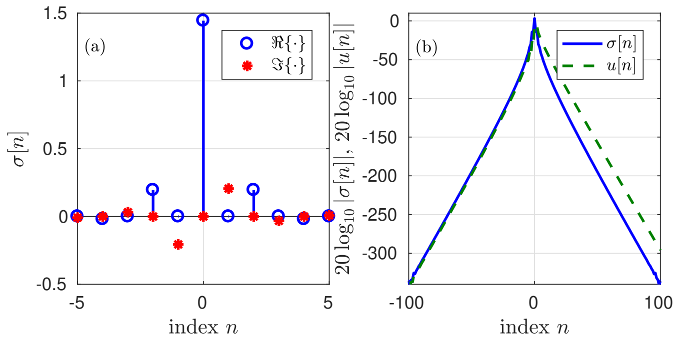

Fig. 2. For $\gamma(z) = 1 + \mathrm{j}z^{-1} - \frac{1}{2}z^{-2} = u(z)\sigma(z)$ in Example 2, (a) singular value $\sigma[n] \circ\!\!-\!\!\bullet\ \sigma(z)$, and (b) decay of both $\sigma[n]$ and $u[n] \circ\!\!-\!\!\bullet\ u(z)$.

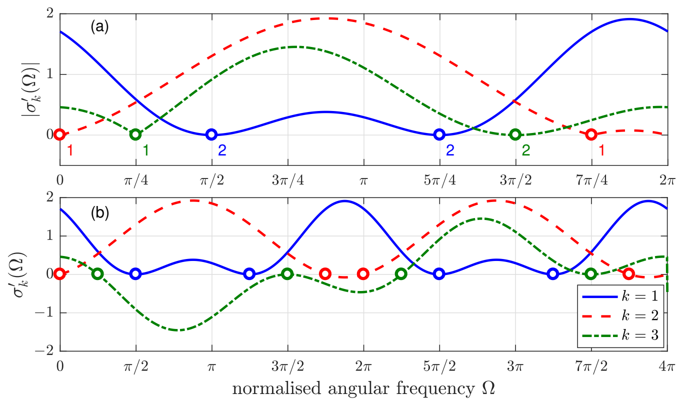

Fig. 3. (a) Moduli of singular values $\sigma'_k(\Omega)$, with zeros indicated by circles together with their multiplicities C_i; (b) singular values with 2π-periodicity for $k = 1, 2$ and 4π periodicity for $k = 3$ that are analytic in $\Omega \in \mathbb{R}$.

B. Sign Changes of Singular Values

Since we have excluded the case of modulated singular values, following the discussions of Sec. II-C we only look at repetitions of a single segment. The index sequence $\{k, k, \dots\}$ for the kth singular value means that $\sigma'_k(\Omega) = \sigma'_k(\Omega - 2\pi)$, i.e. a 2π periodicity. In contrast for the sequence $\{k, -k, k, \dots\}$ we have $\sigma'_k(\Omega) = -\sigma'_k(\Omega - 2\pi)$ and 4π periodicity. The latter must include at least a sign change within a 2π interval of $\sigma'_k(\Omega)$, which must be connected with zeros , i.e. values where $\sigma'_k(\Omega) = 0$. Such zeros on the unit circle are also referred to as spectral zeros.

C. Numbers and Multiplicities of Spectral Zeros

We next explore under which conditions a singular value does or does not experience a sign change over a 2π-interval, i.e. whether $\sigma'_k(\Omega)$ is 2π-periodic. Assume that a function $\sigma'_k(\Omega)$ possesses zeros at I distinct frequencies Ω_i, $i = 1, \dots, I$, within the interval $0 \le \Omega < 2\pi$, where each has multiplicity C_i. We will show below that the absence or presence of a sign change over a 2π interval can be tied to whether the condition

$$\mathrm{mod}_2\Big\{\sum_{i=1}^{I} C_i\Big\} = 0 \tag{20}$$

holds true. This motivates the following theorem.

Theorem 3 (Non-existence of an analytic singular value due to spectral zeros): If a singular value $\sigma'_k(\Omega)$ is not a frequency shifted version of another singular value, and possesses zeros at I distinct frequencies in $[0, 2\pi)$ with multiplicities C_i, $i = 1, \dots, I$, and (20) is satisfied, then $\sigma'_k(\Omega)$ is 2π-periodic, and an analytic $\sigma'_k(z)$ exists. If (20) does not hold, then $\sigma'_k(\Omega)$ must be 4π-periodic.

Proof: For an even C_i, $\sigma'_k(\Omega)$ will only tangentially touch zero at $\Omega = \Omega_i$ without crossing. Hence there will not be a sign change if all the C_i are even. If there is one odd C_i then the singular value will change sign once within $\Omega_0 \le \Omega < \Omega_0 + 2\pi$, with $\Omega_0 \in [0, 2\pi)$ and $\sigma'_k(\Omega_0) \ne 0$. This means that $\sigma'_k(\Omega_0)$ and $\sigma'_k(\Omega_0 + 2\pi)$ will have different signs and so $\sigma_k(\Omega)$ is 4π periodic. If, on the other hand, there are two odd C_i then the singular value will change sign twice within $\Omega_0 \le \Omega < \Omega_0 + 2\pi$. So although $\sigma'_k(\Omega)$ becomes negative at some point, we have that $\sigma'_k(\Omega_0)$ and $\sigma'_k(\Omega_0 + 2\pi)$ will have identical signs and $\sigma'_k(\Omega)$ is 2π periodic. More generally, 2π-periodicity results if the number of zeros with an odd order of multiplicities is even, as established by (20). ■

Example 3: Fig. 3(a) shows the moduli of three singular values $\sigma'_k(\Omega)$, $k = 1, 2, 3$, that all take on a value of zero for some Ω. The multiplicity C_i of these zeros is also indicated in the graph. With two zeros, each of multiplicity two, $\sigma'_1(\Omega)$ satisfies (20) and hence is 2π-periodic and analytic. The function $\sigma'_2(\Omega)$ possesses two single zeros, and the modulus is not differentiable at $\Omega = 0$ and $\Omega = \frac{7}{4}\pi$. An analytic, 2π-periodic function $\sigma'_2(\Omega)$ can be created by allowing the singular value to be negative on the interval $\frac{7}{4}\pi \le \Omega < 2\pi$, as shown in Fig. 3(b). The singular value $\sigma'_3(\Omega)$ in Fig. 3(a) has two zeros, one with an even multiplicity and one odd, thus violating (20). Simply letting $\sigma'_3(\Omega)$ be negative on the interval $\frac{1}{4}\pi \le \Omega < \frac{3}{2}\pi$ will result in an analytic continuation at $\Omega = \frac{1}{4}\pi$, but causes the non-existence of derivatives above first order at $\Omega = \frac{3}{2}\pi$ (even though this may not be directly evident from Fig. 3(b)), and therefore does not create an analytic function. An analytic function can only be created with a 4π periodicity, as shown in Fig. 3(b). Overall, Fig. 3(b) illustrates the resulting analytic functions, which for zeros of even multiplicity C_i only tangentially touch zero, but which for odd multiplicities C_i possess a crossing point at zero. △

In summary, spectral zeros can lead to a 4π-periodicity instead of a 2π periodicity of the singular values on the unit circle. If condition (20) is not met by all singular values, and if these singular values are not modulated, then analytic singular values can only be obtained for $\boldsymbol{A}(z^2)$.

IV. Modulated Singular Values

This section explores under which circumstances an analytic matrix $\boldsymbol{A}(z)$ will not admit an analytic SVD due to modulated singular values. Following preliminaries in Sec. IV-A, we first show how particular matrix structures are necessarily linked to modulated singular values: a single set of modulated singular values is linked to pseudo-circulant matrices in Sec. IV-B, while block-pseudo circulant matrices are connected to multiple sets of modulated singular values of the same cardinality in Sec. IV-C. For general sets of modulated singular values, Sec. IV-D shows that the corresponding matrices necessarily and sufficiently must be linked to multiplexing operations, but that structural evidence such as the pseudo-circulant property of a matrix, can be obscured by arbitrary paraunitary operations.

A. Preliminaries

This section explores under which circumstances an analytic matrix $\mathbf{A}(z)$ will not admit an analytic SVD due to modulated singular values. As a basic building block, we want to consider a matrix $\boldsymbol{A}(z) : \mathbb{C} \to \mathbb{C}^{M\times N}$, whose singular values at a κF-times oversampled rate are F frequency-shifted or modulated versions of a single function $\sigma(z)$,

$$\boldsymbol{\Sigma}(z)=\mathrm{diag}\Big\{\sigma(z),\, \sigma(z\mathrm{e}^{\mathrm{j}\frac{2\pi}{\kappa F}}),\, \dots,\, \sigma(z\mathrm{e}^{\mathrm{j}(F-1)\frac{2\pi}{\kappa F}})\Big\} . \quad (21)$$

The parameter F is introduced for later consistency but here we have $F = K$, and κ accounts for Theorem 3: in case (20) is satisfied, we have $\kappa = 1$, otherwise we require $\kappa = 2$ for $\boldsymbol{\Sigma}(z)$ to be analytic. Note that any matrix produced under left- and right-multiplication of $\boldsymbol{\Sigma}(z)$ with arbitrary paraunitary operators $\boldsymbol{U}(z) : \mathbb{C} \to \mathbb{C}^{M\times M}$ and $\boldsymbol{V}^{\mathrm{P}}(z) : \mathbb{C} \to \mathbb{C}^{N\times N}$, respectively, will have the same singular values.

B. Pseudo-Circulant Matrices

We initially restrict ourselves to a square matrix $\boldsymbol{A}(z) : \mathbb{C} \to \mathbb{C}^{M\times M}$ and thus $F = K = M$, and investigate under which conditions it will possess modulated singular values as defined in (21). From multirate signal processing, it is known that for a pseudo-circulant matrix [25]

$$\boldsymbol{A}(z) = \begin{bmatrix} H_0(z) & H_1(z) & \dots & H_{F-1}(z) \\ z^{-1}H_{F-1}(z) & H_0(z) & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z^{-1}H_1(z) & \dots & z^{-1}H_{F-1}(z) & H_0(z) \end{bmatrix} , \quad (22)$$

the expanded matrix $\boldsymbol{A}(z^F)$ is diagonalised by a paraunitary operation $\boldsymbol{W}(z)$,

$$\boldsymbol{W}(z) = \boldsymbol{D}(z)\mathbf{T} , \quad (23)$$

where $\boldsymbol{D}(z) = \mathrm{diag}\{1 \,, z^{-1} \,, \dots\, z^{-F+1}\}$ and $\mathbf{T}$ represents an F-point DFT matrix scaled to be unitary [18], [52]. This yields a factorisation $\boldsymbol{A}(z^F) = \boldsymbol{W}(z)\boldsymbol{\Gamma}(z)\boldsymbol{W}^{\mathrm{P}}(z)$, where

$$\boldsymbol{\Gamma}(z) = \mathrm{diag}\Big\{H(z),\, H(z\mathrm{e}^{\mathrm{j}\frac{2\pi}{F}}), \dots,\, H(z\mathrm{e}^{\mathrm{j}\frac{2\pi(F-1)}{F}})\Big\} . \quad (24)$$

The elements $H_m(z)$, $m = 0, \dots, (F-1)$, in (22) are commonly known as the F polyphase components of some system $H(z) = \sum_{m=0}^{F-1} z^{-m}H_m(z^F)$. This system and its $(F-1)$ modulated versions form the diagonal entries of the matrix $\boldsymbol{\Gamma}(z)$ in (24). Importantly, the connection between pseudo-circulance in (22) and the diagonalisation involving modulated functions in (24) is both necessary and sufficient [45].

The elements $H(z\mathrm{e}^{\mathrm{j}\frac{2\pi i}{F}})$ relate to the entries of $\boldsymbol{\Sigma}(z)$ as follows. As per (19) and allowing for a spectral zero (see Sec. III-C) we can write the SVD of $H(z^\kappa)$ as $H(z^\kappa) = \varphi(z)\sigma(z)$ where $\varphi(z)$ is an allpass filter and $\sigma(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$. Then overall we have

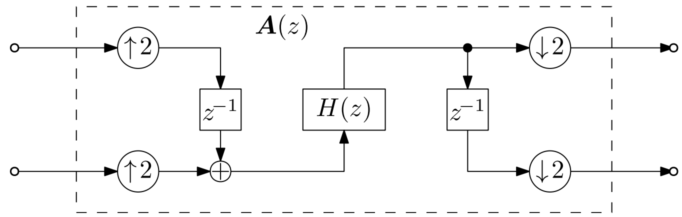

Fig. 4. System matrix $\boldsymbol{A}(z) : \mathbb{C} \to \mathbb{C}^{2\times 2}$ created by multiplexing across a single-input single-output system $H(z)$.

$$\boldsymbol{A}(z^{\kappa F}) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\boldsymbol{V}^{\mathrm{P}}(z) , \quad (25)$$

where

$$\boldsymbol{U}(z) = \boldsymbol{W}(z^\kappa)\mathrm{diag}\Big\{\varphi(z),\, \dots,\, \varphi(z\mathrm{e}^{\mathrm{j}\frac{2\pi(F-1)}{\kappa F}})\Big\} \quad (26)$$

$$\boldsymbol{\Sigma}(z) = \mathrm{diag}\Big\{\sigma(z),\, \dots,\, \sigma(z\mathrm{e}^{\mathrm{j}\frac{2\pi(F-1)}{\kappa F}})\Big\} \quad (27)$$

$$\boldsymbol{V}(z) = \boldsymbol{W}(z^\kappa) . \quad (28)$$

This necessarily and sufficiently connects a set of F modulated singular values to an $F \times F$ pseudo-circulant matrix $\boldsymbol{A}(z)$, as also observed in [44]. Hence, an analytic SVD does not exist for a pseudo-circulant matrix $\boldsymbol{A}(z)$. In contrast, the oversampled version $\boldsymbol{A}(z^{\kappa F})$ does have an analytic SVD.

Example 4: The matrix $\boldsymbol{A}_1(z)$ from Example 1 is a pseudo-circulant system arising from multiplexing data across a transfer function $H(z) = 1 + z^{-1}$ with $F = 2$ as shown in Fig. 4. In this multiplexed operation, the input of $H(z)$ arises from interleaving of two lower rate signals by means of expansion and delay, while at the output a deinterleaver or serial-to-parallel converter extracts two signals sampled at a lower rate via delay and decimation operations [41], [45]–[47]. The paraunitary $\boldsymbol{W}(z)$ in (23) indeed diagonalises $\boldsymbol{A}_1(z^2)$ with $F = 2$, such that

$$\boldsymbol{A}_1(z^2) = \boldsymbol{W}(z)\mathrm{diag}\big\{H(z),\, H(z\mathrm{e}^{\mathrm{j}\pi})\big\}\, \boldsymbol{W}^{\mathrm{P}}(z) , \quad (29)$$

with $H(z\mathrm{e}^{\mathrm{j}\pi}) = 1 - z^{-1}$. Since $H(z)$ possesses a single spectral zero at $z = -1$, (20) is violated and we have to oversample by a further factor $\kappa = 2$ in order to extract analytic singular values. By therefore expanding both $H(z)$ and $H(z\mathrm{e}^{\mathrm{j}\pi})$ by a factor of $\kappa = 2$, we have

$$H(z^2) = 1 + z^{-2} = \underbrace{z^{-1}}_{\varphi(z)} \cdot \underbrace{(z + z^{-1})}_{\sigma_1(z)} \quad (30)$$

$$H(z^2\mathrm{e}^{\mathrm{j}\pi}) = 1 - z^{-2} = -\mathrm{j}z^{-1}(\mathrm{j}z - \mathrm{j}z^{-1}) \quad (31)$$

$$= \underbrace{(z\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})^{-1}}_{\varphi(z\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})} \cdot \underbrace{\big(z\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} + (z\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})^{-1}\big)}_{\sigma_2(z)=\sigma_1(z\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})} . \quad (32)$$

Therefore, overall we have $\boldsymbol{A}_1(z^4) = \boldsymbol{U}_1(z)\boldsymbol{\Sigma}_1(z)\boldsymbol{V}_1^{\mathrm{P}}(z)$ with

$$\boldsymbol{U}_1(z) = \boldsymbol{W}(z^2)\begin{bmatrix} z^{-1} & \\ & -\mathrm{j}z^{-1} \end{bmatrix} , \quad (33)$$

$\boldsymbol{V}_1(z) = \boldsymbol{W}(z^2)$ and $\boldsymbol{\Sigma}_1(z) = \mathrm{diag}\{\sigma_1(z),\, \sigma_2(z)\} = \mathrm{diag}\big\{\sigma_1(z),\, \sigma_1(z\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})\big\}$. Thus, the terms $\varphi(z\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}) = -\mathrm{j}z^{-1}$

and $\sigma_1(z\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})$ are indeed modulated versions of $\varphi(z) = z^{-1}$ and $\sigma_1(z)$ as per (26) and (27) with $F = \kappa = 2$. △

C. Block-Pseudo-Circulant Systems

As a first generalisation of Sec. IV-B, we consider non-square matrices that possess modulated singular values. Assume a multiple-input multiple-output system $\boldsymbol{H}(z) : \mathbb{C} \to \mathbb{C}^{M_H \times N_H}$, whereby its inputs and outputs are multiplexed and demultiplexed by a factor of F. Thus, in analogy to Fig. 4, a system matrix $\mathbf{A}'(z) : \mathbb{C} \to \mathbb{C}^{M \times N}$ with $M = FM_H$ and $N = FN_H$ results, which for block sampling yields a matrix-valued version of (22),

$$\boldsymbol{A}'(z) = \begin{bmatrix} \boldsymbol{H}_0(z) & \boldsymbol{H}_1(z) & \ldots & \boldsymbol{H}_{F-1}(z) \\ z^{-1}\boldsymbol{H}_{F-1}(z) & \boldsymbol{H}_0(z) & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z^{-1}\boldsymbol{H}_1(z) & \ldots & z^{-1}\boldsymbol{H}_{F-1}(z) & \boldsymbol{H}_0(z) \end{bmatrix},$$

that is sometimes referred to as a generalised pseudo-circulant matrix [53]. With a permutation matrix $\boldsymbol{P}_Q = [\mathbf{I}_F \odot \mathbf{e}_1, \mathbf{I}_F \odot \mathbf{e}_2, \ldots, \mathbf{I}_F \odot \mathbf{e}_Q]$, where $\mathbf{I}_F$ is an $F \times F$ identity matrix, $\odot$ the Kronecker product, and $\mathbf{e}_q \in \mathbb{Z}^Q$ a vector of zeros except of a value of one as the qth element, we can define $\boldsymbol{A}(z) = \boldsymbol{P}_{M_H}\boldsymbol{A}'(z)\boldsymbol{P}^{\mathrm{T}}_{N_H}$,

$$\boldsymbol{A}(z) = \begin{bmatrix} \boldsymbol{A}_{1,1}(z) & \ldots & \boldsymbol{A}_{1,N_H}(z) \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_{M_H,1}(z) & \ldots & \boldsymbol{A}_{M_H,N_H}(z) \end{bmatrix} . \tag{34}$$

The matrix $\boldsymbol{A}(z)$ in (34) possesses a block-pseudo-circulant structure: the submatrices $\boldsymbol{A}_{i,j}(z) : \mathbb{C} \to \mathbb{C}^{F \times F}$, with $i = 1, \ldots, M_H$ and $j = 1, \ldots, N_H$, are pseudo-circulant, and emerge from F-fold multiplexing the element $H_{i,j}(z)$ in the ith row and jth column of $\boldsymbol{H}(z)$ analogously to (22).

While the diagonalisation of pseudo-circulant matrices [45] and of block-circulant (i.e. non-polynomial) matrices [54], [55] has been addressed in the literature, we are not aware of any discussions of block-pseudo-circulant matrices. We therefore state:

Theorem 4 (Singular values of block-pseudo-circulant matrices): If a system $\boldsymbol{A}(z)$ emerges from F-fold multiplexing of a system $\mathbf{H}(z)$, then the singular values of $\mathbf{A}(z)$ will be F-fold modulated versions of those of $\boldsymbol{H}(z)$.

Proof: To investigate the singular values of $\boldsymbol{A}(z)$, we define

$$\boldsymbol{W}_1(z) = \mathrm{blockdiag}\{\underbrace{\boldsymbol{W}(z), \ldots, \boldsymbol{W}(z)}_{M_H}\} \tag{35}$$

$$\boldsymbol{W}_\mathrm{r}(z) = \mathrm{blockdiag}\{\underbrace{\boldsymbol{W}(z), \ldots, \boldsymbol{W}(z)}_{N_H}\} , \tag{36}$$

with $\boldsymbol{W}(z)$ as in (23). With paraunitary matrices $\boldsymbol{Q}_1(z) : \mathbb{C} \to \mathbb{C}^{M_H F \times M_H F}$ and $\boldsymbol{Q}_\mathrm{r}(z) : \mathbb{C} \to \mathbb{C}^{N_H F \times N_H F}$, we initially postulate the decomposition

$$\boldsymbol{A}(z^F) = \boldsymbol{W}_1(z)\boldsymbol{Q}_1(z)\boldsymbol{\Gamma}(z)\boldsymbol{Q}^{\mathrm{P}}_\mathrm{r}(z)\boldsymbol{W}^{\mathrm{P}}_\mathrm{r}(z) , \tag{37}$$

and let $\boldsymbol{\Gamma}(z^\kappa) = \boldsymbol{\Psi}(z)\boldsymbol{\Sigma}(z)$, where $\boldsymbol{\Psi}(z) : \mathbb{C} \to \mathbb{C}^{M_H F \times M_H F}$ and $\boldsymbol{\Gamma}(z^\kappa), \boldsymbol{\Sigma}(z) : \mathbb{C} \to \mathbb{C}^{M_H F \times N_H F}$ are diagonal matrices. First consider the structure of

$$\boldsymbol{S}(z) = \boldsymbol{W}^{\mathrm{P}}_1(z)\boldsymbol{A}(z^F)\boldsymbol{W}_\mathrm{r}(z) . \tag{38}$$

This matrix $\boldsymbol{S}(z)$ can be subdivided into $F \times F$ subblocks $\boldsymbol{S}_{i,j}(z)$, $i = 1, \ldots, M_H$, $j = 1, \ldots, N_H$, such that

$$\boldsymbol{S}(z) = \begin{bmatrix} \boldsymbol{S}_{1,1}(z) & \ldots & \boldsymbol{S}_{1,N_H}(z) \\ \vdots & \ddots & \vdots \\ \boldsymbol{S}_{M_H,1}(z) & \ldots & \boldsymbol{S}_{M_H,N_H}(z) \end{bmatrix} , \tag{39}$$

whereby

$$\boldsymbol{S}_{i,j}(z) = \boldsymbol{W}^{\mathrm{P}}(z)\boldsymbol{A}_{i,j}(z^F)\boldsymbol{W}(z) . \tag{40}$$

Recall from (34) that $\boldsymbol{A}_{i,j}(z^F)$ is a pseudo-circulant matrix derived from F-fold multiplexing a system $H_{i,j}(z)$. Thus $\boldsymbol{S}_{i,j}(z)$ is diagonal with modulated entries, such that

$$\boldsymbol{S}_{i,j}(z) = \mathrm{diag}\Big\{H_{i,j}(z), \ldots, H_{i,j}(z\mathrm{e}^{\mathrm{j}2\pi(F-1)/F})\Big\} . \tag{41}$$

We now focus on the structure of

$$\boldsymbol{S}(z) = \boldsymbol{Q}_1(z)\boldsymbol{\Gamma}(z)\boldsymbol{Q}^{\mathrm{P}}_\mathrm{r}(z) , \tag{42}$$

which is sparse with only every Fth sub-diagonal occupied by potentially non-zero values. Using the earlier defined permutation matrices, the operation

$$\boldsymbol{S}'(z) = \mathbf{P}_{M_\mathrm{H}}\boldsymbol{S}(z)\mathbf{P}^{\mathrm{T}}_{N_\mathrm{H}} \tag{43}$$
$$= \mathrm{blockdiag}\Big\{\boldsymbol{H}(z), \boldsymbol{H}(z\mathrm{e}^{\mathrm{j}2\pi\frac{1}{F}}), \ldots, \boldsymbol{H}(z\mathrm{e}^{\mathrm{j}2\pi\frac{F-1}{F}})\Big\} \tag{44}$$

changes $\boldsymbol{S}(z)$ into a block-diagonal matrix. From Theorem 2, $\boldsymbol{H}(z^J)$ admits an analytic SVD for some $J \in \mathbb{Z}$, and the same can be said for all its shifted versions as

$$\boldsymbol{H}(z^J\mathrm{e}^{\mathrm{j}2\pi\frac{s}{F}}) = \boldsymbol{H}([z\mathrm{e}^{\mathrm{j}2\pi\frac{s}{FJ}}]^J)$$
$$= \boldsymbol{U}'(z\mathrm{e}^{\mathrm{j}\frac{2\pi s}{JF}})\boldsymbol{\Sigma}'(z\mathrm{e}^{\mathrm{j}\frac{2\pi s}{JF}})\boldsymbol{V}'^{\mathrm{P}}(z\mathrm{e}^{\mathrm{j}\frac{2\pi s}{JF}}) \tag{45}$$

$s = 0, \ldots, (F-1)$. Therefore, the paraunitary matrices

$$\boldsymbol{Q}'_1(z) = \mathrm{blockdiag}\Big\{\boldsymbol{U}'(z), \boldsymbol{U}'(z\mathrm{e}^{\mathrm{j}\frac{2\pi}{FJ}}), \ldots, \boldsymbol{U}'(z\mathrm{e}^{\mathrm{j}\frac{2\pi(F-1)}{FJ}})\Big\} \tag{46}$$

$$\boldsymbol{Q}'_\mathrm{r}(z) = \mathrm{blockdiag}\Big\{\boldsymbol{V}'(z), \boldsymbol{V}'(z\mathrm{e}^{\mathrm{j}2\frac{2\pi}{FJ}}), \ldots, \boldsymbol{V}'(z\mathrm{e}^{\mathrm{j}\frac{2\pi(F-1)}{FJ}})\Big\} \tag{47}$$

are block-diagonal and $\boldsymbol{\Sigma}(z^{\frac{1}{J}}) = \mathrm{blockdiag}\{\boldsymbol{\Sigma}'([z\mathrm{e}^{\mathrm{j}2\pi\frac{s}{F}}]^{\frac{1}{J}})\}_{s=0,\ldots,F-1}$ contains the singular values of $\boldsymbol{H}(z)$ and its F-fold modulated versions. From (37), (38), (44), and (45), we have that

$$\boldsymbol{A}(z^F) = \boldsymbol{W}_1(z)\mathbf{P}^{\mathrm{T}}_{M_\mathrm{H}}\boldsymbol{Q}'_1(z^{\frac{1}{J}})\boldsymbol{\Sigma}(z^{\frac{1}{J}})\boldsymbol{Q}'^{\mathrm{P}}_\mathrm{r}(z^{\frac{1}{J}})\mathbf{P}_{N_\mathrm{H}}\boldsymbol{W}^{\mathrm{P}}_\mathrm{r}(z) , \tag{48}$$

thus proving that the singular values of $\boldsymbol{A}(z)$ are the F-fold modulated versions of those of $\boldsymbol{H}(z)$, even if the latter contains modulated singular values. If $\boldsymbol{H}(z)$ does not contain any further modulated singular values, then indeed $J = 1$. The result in (48) relates back to the postulated (37), with $\boldsymbol{\Psi}(z)$ absorbed into the either left- or right-singular vectors. ■

Example 5: Consider the system $\boldsymbol{H}(z) : \mathbb{C} \to \mathbb{C}^{2\times 3}$,

$$\boldsymbol{H}(z) = \begin{bmatrix} 1 + 2z^{-1} & \mathrm{j} & \mathrm{j}z^{-1} \\ z^{-1} & -2 & \mathrm{j} - \mathrm{j}z^{-1} \end{bmatrix} . \tag{49}$$

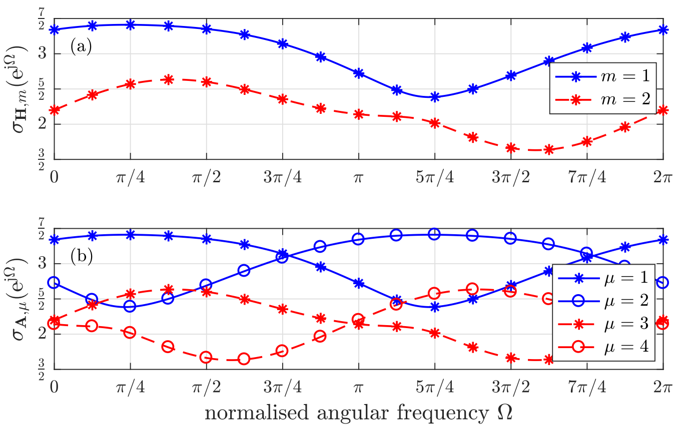

Fig. 5. Singular values of (a) $\boldsymbol{H}(z)$ and (b) $\boldsymbol{A}(z^2)$ in Example 5.

By evaluating individual SVDs along the unit circle, we find the singular values $\sigma_{\mathbf{H},m}(\mathrm{e}^{\mathrm{j}\Omega})$, $m = 1, 2$ of $\boldsymbol{H}(\mathrm{e}^{\mathrm{j}\Omega})$ as shown in Fig. 5(a). Note that both singular values satisfy (20), such that $\kappa = 1$. For a system $\boldsymbol{A}(z) : \mathbb{C} \rightarrow \mathbb{C}^{4\times 6}$ obtained by 2-fold multiplexing $\boldsymbol{H}(z)$, we need to evaluate the twice oversampled system $\boldsymbol{A}(z^2)$. Its singular values $\sigma_{\mathbf{A},\mu}(\mathrm{e}^{\mathrm{j}\Omega})$ are shown in Fig. 5(b). Note that in addition to matching singular values $\sigma_{\mathbf{A},1}(\mathrm{e}^{\mathrm{j}\Omega}) = \sigma_{\mathbf{H},1}(\mathrm{e}^{\mathrm{j}\Omega})$ and $\sigma_{\mathbf{A},3}(\mathrm{e}^{\mathrm{j}\Omega}) = \sigma_{\mathbf{H},2}(\mathrm{e}^{\mathrm{j}\Omega})$, the modulated versions $\sigma_{\mathbf{A},2}(\mathrm{e}^{\mathrm{j}\Omega}) = \sigma_{\mathbf{H},1}(\mathrm{e}^{\mathrm{j}(\Omega-\pi)})$ and $\sigma_{\mathbf{A},4}(\mathrm{e}^{\mathrm{j}\Omega}) = \sigma_{\mathbf{H},2}(\mathrm{e}^{\mathrm{j}(\Omega-\pi)})$ appear due to the multiplexing operation. △

D. General Multiplexed Systems

Sec. II-D has shown that for an analytic matrix $\boldsymbol{A}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times N}$, $M \leq N$, we can generally have P sets of modulated singular values, each of cardinality k_p, $p = 1, \ldots, P$, where $\sum_p k_p = M$. We now want to explore under which conditions such singular values may occur in systems of transfer functions. For simplicity, we exclude singular values with zero crossings, so that $\kappa_p = 1$, resulting in (see after (9)) an overall $2\pi L$ periodicity of singular values with $L = \mathrm{lcm}\{k_1, \ldots, k_P\}$.

Different from the preceding subsections, where we assumed particular matrix structures, we start with the singular values of some matrix $\boldsymbol{A}(z)$. Assume that the k_p modulated singular values of periodicity $2\pi k_p$ form the diagonal of the $k_p \times k_p$ matrix $\bar{\boldsymbol{\Sigma}}_p(z^{1/k_p})$; the notation z^{1/k_p} reminds us that this function will only be analytic once oversampled by integer multiples of k_p. The reasoning in Sec. II-D yields an overall $M \times M$ matrix $\bar{\boldsymbol{\Sigma}}(z)$ of singular values, such that

$$\bar{\boldsymbol{\Sigma}}(z^{1/L}) = \mathrm{blockdiag}\Big\{\bar{\boldsymbol{\Sigma}}_1(z^{1/k_1}), \ldots, \bar{\boldsymbol{\Sigma}}_P(z^{1/k_P})\Big\} \ . \quad (50)$$

Based on the analysis in Sec. IV-B we know that $\bar{\boldsymbol{\Sigma}}_1(z^{1/k_1})$ can be related to an analytic pseudo-circulant matrix via a paraunitary matrix $\boldsymbol{W}_{k_p}(z)$ (cf. (23)). Specifically we have that $\boldsymbol{W}_{k_p}(z) = \boldsymbol{D}_{k_p}(z)\mathbf{T}_{k_p}$ with $\boldsymbol{D}_{k_p}(z) = \mathrm{diag}\{1, z^{-1}, \ldots, z^{-k_p-1}\}$ and $\mathbf{T}_{k_p}$ a k_p-point DFT matrix. By forming a block-diagonal matrix

$$\bar{\boldsymbol{W}}(z^{1/L}) = \mathrm{blockdiag}\Big\{\boldsymbol{W}_{k_1}(z^{1/k_1}), \ldots \boldsymbol{W}_{k_P}(z^{1/k_P})\Big\} \ , \quad (51)$$

we find that

$$\bar{\boldsymbol{A}}(z) = \bar{\boldsymbol{W}}(z^{1/L})\bar{\boldsymbol{\Sigma}}(z^{1/L})\bar{\boldsymbol{W}}^{\mathrm{P}}(z^{1/L}) \quad (52)$$

is analytic in z, and is an $M \times M$ block-diagonal matrix consisting of P pseudo-circulant subblocks.

We now want to utilise these results to factorise the general matrix $\boldsymbol{A}(z)$, such that sets of modulated singular values as in (50) arise. We are therefore looking for a decomposition of

$$\boldsymbol{A}(z) = \boldsymbol{U}(z^{1/L})[\boldsymbol{\Sigma}(z^{1/L}) \ \ \mathbf{0}_{L\times(N-L)}]\boldsymbol{V}^{\mathrm{P}}(z^{1/L}) \ , \quad (53)$$

where $\boldsymbol{U}(z^{1/L}) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ and $\boldsymbol{V}(z^{1/L}) : \mathbb{C} \rightarrow \mathbb{C}^{N\times N}$ are matrices of left- and right-singular vectors that potentially must be oversampled by integer multiples of L in order to be analytic. Inserting nugatory factors $\bar{\boldsymbol{W}}^{\mathrm{P}}(z^{1/L})\bar{\boldsymbol{W}}(z^{1/L})$ and substituting (52), we obtain

$$\boldsymbol{A}(z) = \underbrace{\boldsymbol{U}(z^{\frac{1}{L}})\bar{\boldsymbol{W}}^{\mathrm{P}}(z^{\frac{1}{L}})}_{\bar{\boldsymbol{U}}(z^{1/L})}\bar{\boldsymbol{A}}(z)\underbrace{[\bar{\boldsymbol{W}}(z^{\frac{1}{L}}) \ \ \mathbf{0}]\boldsymbol{V}^{\mathrm{P}}(z^{\frac{1}{L}})}_{\bar{\boldsymbol{V}}^{\mathrm{P}}(z^{1/L})} \ . \quad (54)$$

This means that $\boldsymbol{A}(z)$ consists of an inner system $\bar{\boldsymbol{A}}(z)$, representing P multiplexed systems of the type analysed in Sec. IV-B, each with potentially different multiplexing factors k_p, $p = 1, \ldots, P$. Outer system components, the paraunitary $\bar{\boldsymbol{U}}(z^{1/L}) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ and the matrix $\bar{\boldsymbol{V}}(z^{1/L}) : \mathbb{C} \rightarrow \mathbb{C}^{N\times M}$ containing M columns of a paraunitary matrix, are as defined in (54). These outer components may involve further sampling rate changes, and can generally obscure the pseudo-circulant property when inspecting $\boldsymbol{A}(z)$.

With the analysis of general multiplexed systems concluded with (54), we briefly show how this result can be reconciled with the block-pseudo-circulant systems of Sec. IV-C. In this case, the cardinality of the different sets of modulated singular values are the same, so that $k_1 = \ldots = k_P = L = F$, such that $M = LP$, i.e. we have P sets of L-fold modulated singular values. Without any further modulations across the therefore distinct sets of singular values, we can write the analytic SVD with (48) as

$$\boldsymbol{A}(z^L) = \boldsymbol{W}_1(z)\mathbf{P}^{\mathrm{T}}_{M_{\mathrm{H}}}\boldsymbol{Q}'_1(z)\boldsymbol{\Sigma}(z)\boldsymbol{Q}'^{\mathrm{P}}_{\mathrm{r}}(z)\mathbf{P}_{N_{\mathrm{H}}}\boldsymbol{W}^{\mathrm{P}}_{\mathrm{r}}(z) \ . \quad (55)$$

In contrast, with (52) and (54),

$$\boldsymbol{A}(z^L) = \bar{\boldsymbol{U}}(z)\bar{\boldsymbol{W}}(z)\bar{\boldsymbol{\Sigma}}(z)\bar{\boldsymbol{W}}^{\mathrm{P}}(z)\bar{\boldsymbol{V}}^{\mathrm{P}}(z) \ . \quad (56)$$

The square matrix $\bar{\boldsymbol{\Sigma}}(z)$ contains potentially permuted singular values, such that $\boldsymbol{\Sigma}(z) = \mathbf{P}_{\Sigma,1}[\bar{\boldsymbol{\Sigma}}(z) \ \ \mathbf{0}_{M\times(N-M)}]\mathbf{P}^{\mathrm{T}}_{\Sigma,2}$, with $\mathbf{P}_{\Sigma,1}$ and $\mathbf{P}_{\Sigma,2}$ suitable permutation matrices. Comparing (55) and (56), and noting that $\bar{\boldsymbol{W}}(z) = \boldsymbol{W}_1(z)$, we find for the outer systems as defined in (54)

$$\bar{\boldsymbol{U}}(z) = \boldsymbol{W}_1(z)\mathbf{P}^{\mathrm{T}}_{M_{\mathrm{H}}}\boldsymbol{Q}'_1(z)\mathbf{P}_{\Sigma,1}\boldsymbol{W}^{\mathrm{P}}_1(z) \ , \quad (57)$$

$$\bar{\boldsymbol{V}}(z) = \boldsymbol{W}_{\mathrm{r}}(z)\mathbf{P}^{\mathrm{T}}_{N_{\mathrm{H}}}\boldsymbol{Q}'_{\mathrm{r}}(z)\mathbf{P}_{\Sigma,2}\begin{bmatrix}\mathbf{I}_M \\ \mathbf{0}_{(N-M)\times M}\end{bmatrix}\boldsymbol{W}^{\mathrm{P}}_1(z) \ . \quad (58)$$

Thus, a block-pseudo-circulant matrix can be viewed as a special case of a more general matrix $\boldsymbol{A}(z)$ containing — potentially hidden by outer operations, i.e. without obvious pseudo-circulant or block-pseudo-circulant properties — multiplexing operations.

The following theorem summarises the findings of Sec. IV:

Theorem 5 (Modulated singular values): A matrix $\boldsymbol{A}(z)$ analytic in $z \in \mathbb{C}$ that can be tied to a multiplexing operation via paraunitary operations necessarily and sufficiently possesses modulated singular values. Such singular values have a periodicity of $2\pi L$, $L \in \mathbb{N}$ that only become analytic for $\boldsymbol{A}(z^L)$, i.e. if $\boldsymbol{A}(z)$ is oversampled by a factor L.

Proof: The necessary and sufficient link between multiplexing and modulated singular values has been established by (54). ■

If $\boldsymbol{A}(z)$ is not pseudo-circulant or block-pseudo-circulant then there is no known way of determining whether it can be tied to a multiplexing operation by paraunitary operations $\bar{\boldsymbol{U}}(z)$ and $\bar{\boldsymbol{V}}(z)$ as in (54), except for the causality dilemma: we would need to determine the analytic SVD, but it does not exist unless $\boldsymbol{A}(z)$ is expanded. Interestingly, with some effort that is beyond the scope of this paper, it can be shown that the outer, paraunitary factors in (54) can be selected to be analytic, such that any $\boldsymbol{A}(z)$ can be brought into a block-diagonal pseudo-circulant representation without oversampling.

Example 6: Consider $\boldsymbol{B}(z) = [1+z^{-1},\ 2;\ 1-z^{-1},\ 0]$. This matrix is not pseudo-circulant, but can be obtained as $\boldsymbol{B}(z) = \sqrt{2}\mathbf{T}_2\boldsymbol{A}(z)$, with $\mathbf{A}(z)$ the pseudo-circulant system of Example 1, with which it therefore shares its modulated, 8π-periodic singular values. In terms of an implementation, $\boldsymbol{B}(z)$ can be constructed by attaching a 2-point DFT matrix to the output of the system in Fig. 4. △

V. Existence of the Analytic SVD

A. Existence

Recall that we want to establish under which conditions a matrix $\boldsymbol{A}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times N}$ that is analytic in $z \in \mathbb{C}$ within some region including the unit circle admits, without oversampling, an analytic SVD $\boldsymbol{A}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\boldsymbol{V}^{\mathrm{P}}(z)$ as in (1) with analytic factors. The matrices $\boldsymbol{U}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ and $\boldsymbol{V}(z) : \mathbb{C} \rightarrow \mathbb{C}^{N\times N}$ are paraunitary matrices containing the left- and right-singular vectors. For the diagonal matrix $\boldsymbol{\Sigma}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times N}$, we only demand that $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}^{M\times N}$, i.e., for the singular values to be real-valued on the unit circle but not necessarily positive.

Theorem 6 (Existence of the analytic SVD): The decomposition of an analytic matrix $\boldsymbol{A}(z)$ with analytic factors in (1) exists if and only if $\boldsymbol{A}(z)$ cannot be tied to a multiplexing operation, and if on the unit circle the spectral zeros of its singular values all satisfy (20).

Proof: We first evaluate on the unit circle. According to Theorem 5, a matrix $\boldsymbol{A}(z)|_{z=\mathrm{e}^{\mathrm{j}\Omega}}$ cannot be tied to a multiplexing operation if and only if its analytic singular values are not modulated. Additionally, Theorem 3 guarantees 2π-periodic singular values if and only if (20) is satisfied for all singular values. Thus, we have $L = 1$, which with Theorem 2 also establishes 2π-periodic left- and right-singular vectors. Resubstituting $z = \mathrm{e}^{\mathrm{j}\Omega}$ means that the SVD factors in (1) are analytic within a region of convergence that includes at least the unit circle.

To show that the region of convergence extends beyond the unit circle, we utilise (1) to write the EVDs

$$\boldsymbol{R}_1(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\boldsymbol{\Sigma}^{\mathrm{P}}(z)\boldsymbol{U}^{\mathrm{P}}(z) \tag{59}$$

$$\boldsymbol{R}_2(z) = \boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{A}(z) = \boldsymbol{V}(z)\boldsymbol{\Sigma}^{\mathrm{P}}(z)\boldsymbol{\Sigma}(z)\boldsymbol{V}^{\mathrm{P}}(z)\ . \tag{60}$$

Since $\boldsymbol{A}(z)$ is unmultiplexed, $\boldsymbol{R}_1(z)$ and $\boldsymbol{R}_2(z)$ are neither tied to multiplexing operations, and their EVDs are guaranteed to exist with analytic eigenvalues and eigenvectors [43] whose region of convergence extends beyond the unit circle. Hence the factors $\boldsymbol{\Sigma}(z)$, $\boldsymbol{U}(z)$, and $\boldsymbol{V}(z)$, due to the uniqueness theorem of analytic functions [51], must also have a region of convergence extending beyond the unit circle, within which they are analytic in z. ■

If $\boldsymbol{A}(z)$ emerges from multiplexing and/or (20) is not satisfied, an analytic SVD can only be found for $\mathbf{A}(z^L)$, for some suitable integer $L > 1$.

Example 7: The two causes for a loss of analyticity can occur simultaneously, as seen in Example 1, which involved multiplexing by $F = 2$ and where the singular value violated (20), such that $\kappa = 2$. An analytic SVD is only possible for $\boldsymbol{A}(z^L)$ with $L = \kappa F = 4$. △

Corollary 1 (Existence of the Analytic SVD without Oversampling): For an analytic matrix $\boldsymbol{A}(z)$ that neither can be tied to multiplexing operations nor possesses any singularities for $z = \mathrm{e}^{\mathrm{j}\Omega}$, the analytic SVD as defined in (1) exists without the need for oversampling of $\boldsymbol{A}(z)$.

Proof: If $\boldsymbol{A}(z)$ is not tied to any multiplexing operations, then $\boldsymbol{A}(z)$ only denies the existence of an analytic SVD if and only if its singular values violate (20). However, $\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})$ not possessing any singularies implies that its singular values are free of zero crossings, which is sufficient albeit not necessary for (20) to be satisfied. ■

B. Ambiguities

For the singular values, Sec. II-D has established that the analytic solution is unique up to some ordering. While the standard SVD is defined with majorised singular values according to (3), such ordering is not meaningful for functions that can intersect. For the following, we do however assume that any identical singular values are ordered in groups, s.t. $\sigma_i(\mathrm{e}^{\mathrm{j}\Omega}) = \ldots = \sigma_{i+C_-1}(\mathrm{e}^{\mathrm{j}\Omega})\ \forall\Omega$ in the case that there are C identical singular values.

For the singular vectors, we assume w.l.o.g. that $M \leq N$, as otherwise we can operate on the transpose matrix. Let $\boldsymbol{\Phi}_1(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ and $\boldsymbol{\Phi}_2(z) : \mathbb{C} \rightarrow \mathbb{C}^{(N-M)\times(N-M)}$ be two paraunitary matrices. The matrix $\boldsymbol{\Phi}_1(z)$ is block-diagonal with the size of the blocks reflecting the groups of identical singular values on the diagonal of the square matrix $\bar{\boldsymbol{\Sigma}}(z)$. For M distinct singular values, $\boldsymbol{\Phi}_1(z)$ is a diagonal matrix of arbitrary allpass filters. Note that $\bar{\boldsymbol{\Sigma}}(z)$ and $\boldsymbol{\Phi}_1$ commute. The matrix $\boldsymbol{\Phi}_2(z)$ is an arbitrary paraunitary matrix. If $\boldsymbol{A}(z)$ is rectangular with $M < N$ and admits an analytic SVD then

$\boldsymbol{A}(z) = \boldsymbol{U}(z)\left[\bar{\boldsymbol{\Sigma}}(z), \mathbf{0}\right] \boldsymbol{V}^{\mathrm{P}}(z)$. Inserting a nugatory factor we have

$$\begin{aligned}\boldsymbol{A}(z) &= \boldsymbol{U}(z)\left[\bar{\boldsymbol{\Sigma}}(z), \mathbf{0}\right]\begin{bmatrix} \boldsymbol{\Phi}_1(z) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}_2(z) \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \boldsymbol{\Phi}_1^{\mathrm{P}}(z) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}_2^{\mathrm{P}}(z) \end{bmatrix} \boldsymbol{V}^{\mathrm{P}}(z) \\ &= \boldsymbol{U}(z)\boldsymbol{\Phi}_1(z)\boldsymbol{\Sigma}(z)\begin{bmatrix} \boldsymbol{\Phi}_1^{\mathrm{P}}(z) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}_2^{\mathrm{P}}(z) \end{bmatrix} \boldsymbol{V}^{\mathrm{P}}(z) \ .\end{aligned}$$

The matrix $\boldsymbol{\Phi}_1(z)$ represents a coupled ambiguity between the left- and right-singular vectors is equivalent to (4) and (5) in the case of a standard SVD. The matrix $\boldsymbol{\Phi}_2(z)$ permits the right-singular vectors to form an arbitrary orthonormal basis within the $(N-M)$-dimensional nullspace of $\boldsymbol{A}(z)$ without affecting the decomposition.

VI. Computation Using Some Existing Algorithms

This section explores the computation of the analytic SVD in (1) by means of existing algorithms. There are currently three types of algorithm for calculating a polynomial SVD. The first involves repeatedly applying a polynomial QRD algorithm [22]. The second method is an SBR2-style, direct diagonalisation of a matrix $\boldsymbol{A}(z)$ [23]. The third one uses two EVDs as in section V. There are two classes of EVD algorithms: time domain and frequency domain. The time-domain approaches include the SBR2 [17], [18] or SMD [19]–[21] families of algorithms. The SBR2 and SMD algorithms have either been explicitly proven to converge to, or may encourage, a spectrally majorised solution. In contrast, the frequency domain approaches [27]–[29] seek to compute the analytic solution. The QRD and direct diagonalization approaches are also based in the time domain and yield results similar to SBR2- and SMD-based methods. In the following, in order to compare the majorised vs analytic approaches we utilise the "two EVD" approach using either the SMD [19] or DFT domain algorithms in [27], [28] to approximate (1).

A. Approach and Challenges

To attempt to calculate an analytic SVD via two parahermitian matrix EVDs, we first compute

$$\boldsymbol{R}_1(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z) = \boldsymbol{Q}_1(z)\boldsymbol{\Lambda}_1(z)\boldsymbol{Q}^{\mathrm{P}}{}_1(z) \ , \qquad (61)$$

$$\boldsymbol{R}_2(z) = \boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{A}(z) = \boldsymbol{Q}_2(z)\boldsymbol{\Lambda}_2(z)\boldsymbol{Q}^{\mathrm{P}}{}_2(z) \ . \qquad (62)$$

Assuming that the eigenvalues in $\boldsymbol{\Lambda}_1(z)$ and $\boldsymbol{\Lambda}_2(z)$ are distinct and similarly ordered, then comparing (61) to (59), we set $\boldsymbol{U}(z) = \boldsymbol{Q}_1(z)$. Since the ambiguity of the eigenvectors in $\boldsymbol{Q}_2(z)$ is not coupled with the ambiguity of those in $\boldsymbol{Q}_1(z)$, with (61) and (60) we find that $\boldsymbol{V}(z) = \boldsymbol{Q}_2(z)\boldsymbol{\Psi}(z)$, where $\boldsymbol{\Psi}(z)$ is some diagonal matrix of allpass filters. Thus, equivalently to (17), we have

$$\hat{\boldsymbol{\Sigma}}(z) = \boldsymbol{Q}^{\mathrm{P}}{}_1(z)\boldsymbol{A}(z)\boldsymbol{Q}_2(z) = \boldsymbol{\Sigma}(z)\boldsymbol{\Psi}(z) \ . \qquad (63)$$

As a result, (63) yields a $\hat{\boldsymbol{\Sigma}}(z)$ with diagonal components $\hat{\sigma}_m(z)$,

$$\hat{\sigma}_m(z) = \sigma_m(z)\psi_m(z), \qquad m = 1, \ldots, K \ , \qquad (64)$$

where $\psi_m(z)$ is some allpass filter. Thus, on the unit circle we find that $\hat{\sigma}_m(\mathrm{e}^{\mathrm{j}\Omega}) \notin \mathbb{R}$ in violation of the desired real-valuedness of the singular values in (1). When using SBR2- or SMD-based time domain methods, due to spectral majorisation the resulting PEVD differs from the analytic solution in (61) and (62) if singular values intersect. Then SBR2- and SMD-based algorithms aim to approximate piecewise analytic functions, resulting in approximation errors and incomplete diagonalisation of $\hat{\boldsymbol{\Sigma}}(z)$. Nonetheless, the latter approach has been applied widely, see e.g. [17], [26], [30]–[33], [37].

To demonstrate some of the challenges using the above approach, we now consider two examples: one where the singular values are majorised, and one where they are not.

B. Spectrally Majorised Singular Values

We want to diagonalise a matrix $\boldsymbol{A}(z) : \mathbb{C} \rightarrow \mathbb{C}^{3\times 4}$ with a known ground truth analytic SVD. The singular values in $\boldsymbol{\Sigma}(z) : \mathbb{C} \rightarrow \mathbb{C}^{3\times 4}$ are given by

$$\sigma_1(z) = \tfrac{1}{4}z + 2 + \tfrac{1}{4}z^{-1} \ , \qquad (65a)$$

$$\sigma_2(z) = \tfrac{\mathrm{j}}{4}z + \tfrac{5}{4} - \tfrac{\mathrm{j}}{4}z^{-1} \ , \qquad (65b)$$

$$\sigma_3(z) = \tfrac{\mathrm{j}}{2}z + \tfrac{1}{4} - \tfrac{\mathrm{j}}{2}z^{-1} \ , \qquad (65c)$$

and are spectrally majorised, such that $\sigma_1(\mathrm{e}^{\mathrm{j}\Omega}) > \sigma_2(\mathrm{e}^{\mathrm{j}\Omega}) \geq \sigma_3(\mathrm{e}^{\mathrm{j}\Omega})$ (as shown later by the grey underlaid lines in Fig. 8(a) and (b)). The left-singular vectors are determined via paraunitary elementary operations [41],

$$\boldsymbol{U}(z) = \prod_{i=1}^{2}\left\{\mathbf{I} - (1 - z^{-1})\mathbf{u}_i\mathbf{u}_i^{\mathrm{H}}\right\} \ , \qquad (66)$$

where $\mathbf{u}_i \in \mathbb{C}^3$, $i = 1, 2$ are random unit-norm vectors. The matrix of right-singular vectors $\boldsymbol{V}(z) : \mathbb{C} \rightarrow \mathbb{C}^{4\times 4}$ of order 2 is defined analogously to (66) with a different set of random unit-norm vectors. Thus, we assemble $\boldsymbol{A}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\boldsymbol{V}^{\mathrm{P}}(z)$.

Performing two polynomial EVDs using the SMD algorithm [19] for a maximum of 400 iterations yields the matrix $\hat{\boldsymbol{\Sigma}}(z)$ via (63) shown in Fig. 6. The notation $\hat{\sigma}_{i,k}[n] \circ\!\!-\!\!\bullet\, \hat{\sigma}_{i,k}(z)$ refers to the element in the ith row and kth column of $\hat{\boldsymbol{\Sigma}}(z)$. Firstly, note that the diagonal components $\hat{\sigma}_{i,i}[n]$, $i = 1, 2, 3$, are not symmetric, which implies that $\hat{\boldsymbol{\Sigma}}(\mathrm{e}^{\mathrm{j}\Omega})$ is not real valued. Secondly, there are small non-zero components remaining in off-diagonal elements. The diagonalisation metric

$$\rho = \frac{\sum_{i,k,n}|\hat{\sigma}_{i,k}[n]|^2 \ - \ \sum_{i,n}|\hat{\sigma}_{i,i}[n]|^2}{\sum_{i,k,n}|\hat{\sigma}_{i,k}[n]|^2} \qquad (67)$$

measures the ratio between the energy in the off-diagonal terms and the overall energy, which in the case of complete diagonalisation is zero. For the SMD approach, we obtain a value of $\rho = 1.1 \cdot 10^{-2}$. For the left- and right-singular vectors, the matrices extracted by SMD are of orders 8 and 10, respectively, after trimming tails in these polynomials containing less than 0.01% of the energy in these paraunitary matrices [20].

Fig. 7 shows the result for $\hat{\boldsymbol{\Sigma}}[n] \circ\!\!-\!\!\bullet\, \hat{\boldsymbol{\Sigma}}(z)$ when using the parahermitian matrix EVD algorithm in [27], which aims to

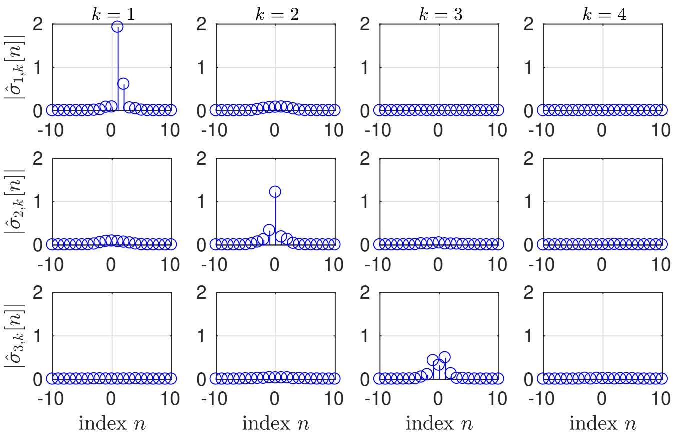

Fig. 6. Elements of $\hat{\boldsymbol{\Sigma}}(z)$ in (63) obtained via two polynomial EVDs implemented using the SMD algorithm [19] on a matrix $\boldsymbol{A}(z)$ with ground truth spectrally majorised singular values.

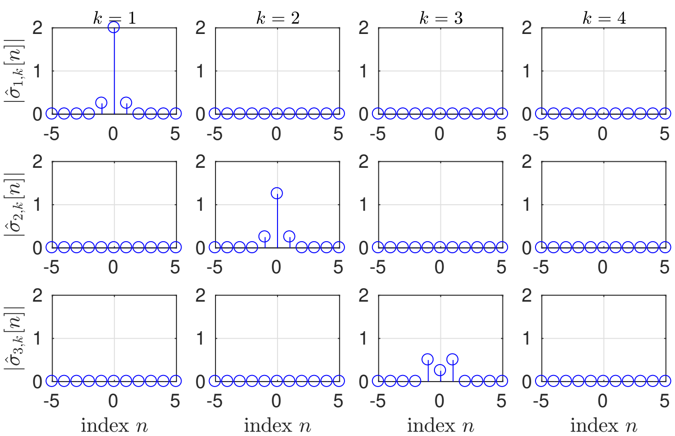

Fig. 7. Elements of $\hat{\boldsymbol{\Sigma}}(z)$ in (63) obtained via two parahermitian matrix EVDs implemented using the analytic EVD algorithm in [27], [28] on a matrix $\boldsymbol{A}(z)$ with ground truth spectrally majorised singular values.

extract analytic factors. In this case, $\rho = 9.0 \cdot 10^{-10}$, and at least the moduli of the extracted singular values show symmetry. On inspection, because the analytic eigenvector extraction pursues a minimum support for its solution, the matrices $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$, extracted with the correct ground truth polynomial order of two, appear closely coupled: the allpass filters $\psi_m(z)$ that couple the left- and right-singular vectors in (60) are approximately constant with $\psi_1(z) \approx 1$, $\psi_2(z) \approx \mathrm{e}^{\mathrm{j}0.4755\pi}$, and $\psi_3(z) \approx \mathrm{e}^{-\mathrm{j}0.1386\pi}$.

The evaluation of the singular values on the unit circle is provided in Figs. 8(a) and (b) for the SMD [19] and the analytic parahermitian matrix EVD [27], respectively. Both methods approximate the ground truth singular values, underlaid in grey, well, with the SMD approach demonstrating a slight deviation for $\hat{\sigma}(\mathrm{e}^{\mathrm{j}\Omega})$ at around $\Omega = \frac{5}{4}\pi$.

C. Spectrally Unmajorised Singular Values

We repeat the experiment in Sec. VI-B for a matrix $\boldsymbol{A}(z)$ that possesses the same ground truth left- and right-singular vectors in $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$, but that now has the following

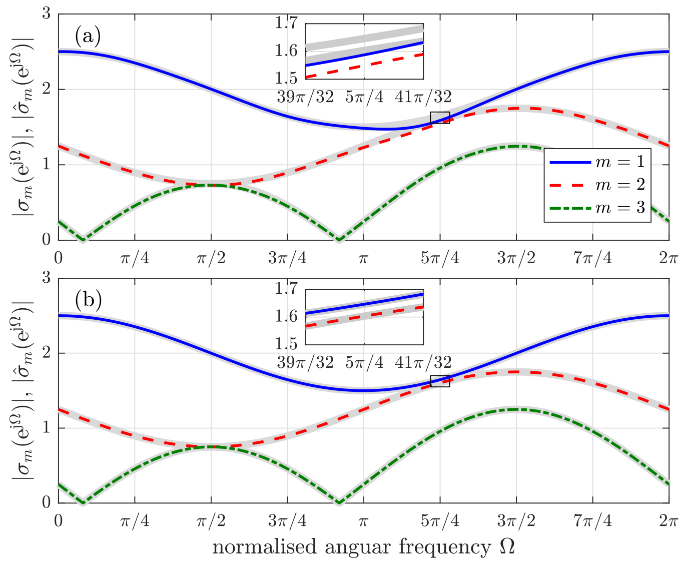

Fig. 8. Moduli of ground truth singular values $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega})$ and of the estimated quantities $\hat{\sigma}_m(\mathrm{e}^{\mathrm{j}\Omega})$, $m = 1, 2, 3$, using (a) SMD [19] and (b) the Fourier domain approach in [27], [28], when applied to a matrix $\boldsymbol{A}(z)$ with ground truth spectrally majorised singular values.

spectrally unmajorised singular values:

$$\sigma_1(z) = \frac{1}{2}z + \frac{5}{4} + \frac{1}{2}z^{-1} , \quad (68a)$$

$$\sigma_2(z) = -\frac{1}{2}z + \frac{5}{4} - \frac{1}{2}z^{-1} , \quad (68b)$$

$$\sigma_3(z) = \mathrm{j}z + \frac{1}{2} - \mathrm{j}z^{-1} . \quad (68c)$$

Their evaluation on the unit circle, $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$, is depicted in Fig. 11 as grey underlaid curves.

The extracted matrices $\hat{\boldsymbol{\Sigma}}(z)$ are characterised in Figs. 9 and 10 for SMD [19] and the Fourier domain approach in [27], [28], respectively. For SMD, the polynomial order of $\hat{\boldsymbol{\Sigma}}(z)$ has significantly increased w.r.t. the spectrally majorised case in Fig. 6, and the diagonalisation in Fig. 9, with a metric of $\rho = 0.7 \cdot 10^{-2}$, looks incomplete. Similarly, the polynomial orders of the extracted left- and right-singular vectors have significantly increase w.r.t. the spectrally majorised case and now are 37 and 40, respectively. This is caused by the SMD algorithm encouraging a spectrally majorised solution, which causes permutations in the extracted singular values in Fig. 11(a). In these points, the SMD algorithm tries to approximate non-differentiable functions, which requires high polynomial orders and incurs poor convergence of an approximation [42].

The Fourier domain approach in [27], [28], extracting the analytic solution, provides the $\hat{\boldsymbol{\Sigma}}[n]$ in Fig. 10 with diagonalisation metric $\rho = 9.0 \cdot 10^{-16}$. This matrix is at least symmetric w.r.t. the moduli of its coefficients. The extracted left- and right-singular vectors match the polynomial order of the ground truth. Further we find that $\psi_1(z) \approx 1$, but have the 2nd and 3rd singular values permuted w.r.t. (68b) and (68c), with $\hat{\sigma}_2(z) \approx \mathrm{e}^{-\mathrm{j}0.0334\pi}\sigma_3(z)$ and $\hat{\sigma}_3(z) \approx \mathrm{e}^{\mathrm{j}0.0103\pi}\sigma_2(z)$.

Figs. 11(a) and (b) characterise the extracted singular values on the unit circle. The SMD approach, which favours spectral majorisation, yields singular values that are indeed approximately spectrally majorised and hence deviate from the ground truth. This has the benefit of concentrating as much

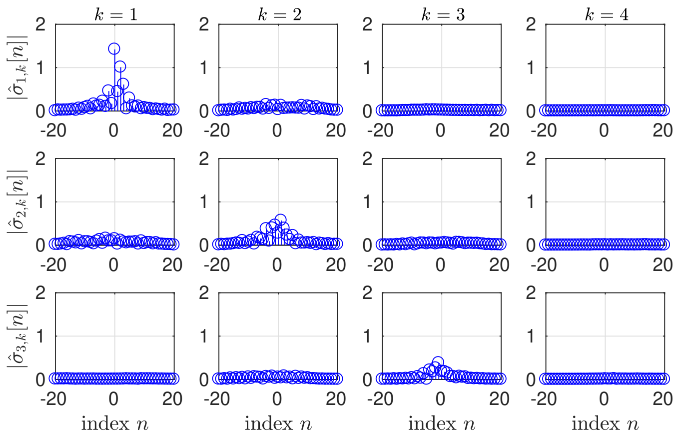

Fig. 9. Elements of $\hat{\boldsymbol{\Sigma}}(z)$ in (63) obtained via two polynomial EVDs implemented using the SMD algorithm [19] on a matrix $\boldsymbol{A}(z)$ with ground truth spectrally unmajorised singular values.

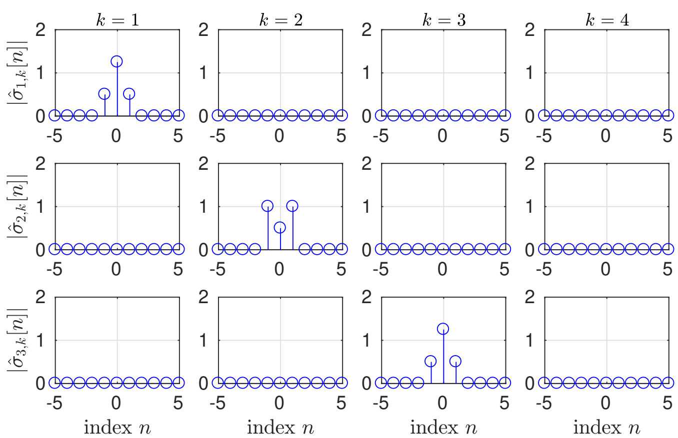

Fig. 10. Elements of $\hat{\boldsymbol{\Sigma}}(z)$ in (63) obtained via two parahermitian matrix EVDs implemented using the analytic EVD algorithm in [27] on a matrix $\boldsymbol{A}(z)$ with ground truth spectrally unmajorised singular values.

energy as possible in as few subchannels as possible. Note that at the permutation points, i.e. where the ground truth singular vectors intersect, the spectrally majorised solution attempts to approximate a piecewise analytic solution, which can be rather poor. In contrast, the Fourier domain approach in Fig. 11(b), targetting the analytic solution, provides a very accurate extraction of the singular values save for the allpass filters $\psi_m(z)$, $m = 1, 2, 3$ in (64).

VII. Discussion and Conclusions

In this paper, we have established under which circumstances an analytic matrix $\boldsymbol{A}(z)$, for example consisting of transfer functions of a multiple-input multiple-output system, admits an analytic SVD, such that the extracted singular values, as well as the left- and right-singular vectors, can be selected as analytic functions. An analytic solution is guaranteed to exist for the oversampled $\boldsymbol{A}(z^{\kappa F})$, with $\kappa, F \in \mathbb{N}$. There are two situations that lead to $\kappa F \neq 1$: (i) structurally, if $\boldsymbol{A}(z)$ can be tied to a multiplexing operation by a factor F, such as in the case of block filtering or multiplexed transmission; (ii) algebraically, if any of the singular values of $\boldsymbol{A}(z^F)$ possesses spectral zeros whose multiplicities sum to an odd integer. In the latter case, we have $\kappa = 2$; otherwise, we have

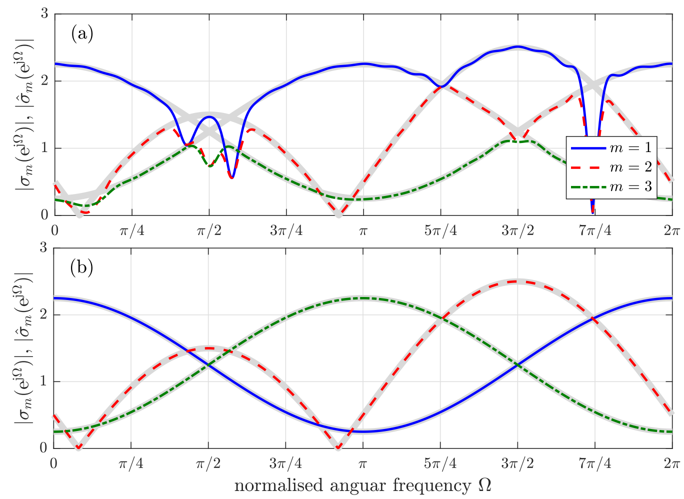

Fig. 11. Moduli of the ground truth singular values $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega})$ and of the estimated quantities $\hat{\sigma}_m(\mathrm{e}^{\mathrm{j}\Omega})$, $m = 1, 2, 3$, using (a) SMD [19] and (b) the Fourier domain approach in [27], when applied to a matrix $\boldsymbol{A}(z)$ with ground truth spectrally unmajorised singular values.

$\kappa = 1$. In the absence of multiplexing, and as long as none of the singular values have zeros with multiplicities that sum to be odd, we have proven that an analytic singular value decomposition of $\boldsymbol{A}(z)$ exists. While the analytic singular values are unique up to a permutation, there is an ambiguity for the analytic singular vectors: corresponding left- and right-singular vectors can be modified by the same allpass function.

The implications of the existence of an analytic solution for the singular values and the left- and right-singular vectors are profound. Firstly, previous polynomial SVD algorithms have been proven to converge in terms of yielding a diagonalisation and spectral majorisation, but it was unclear to what values these algorithms would converge. The analysis in this paper provides this answer. Secondly, since the time domain equivalents of analytic SVD factors are absolutely convergent, they can be well approximated by Laurent polynomials. This favours DFT-domain algorithms such as [27]–[29] over their time domain counterparts [17]–[21]. The former algorithms pursue the analytic solutions for the singular values, even if they are not spectrally majorised on the unit circle.

The algorithmic pursuit of real-valued singular values may be tempting in ‘correctly’ generalising the ordinary SVD, and can be built into dedicated DFT-domain algorithms that not only avoid a spectrally majorised solution in favour of the analytic one, but may also yield singular values that are real-valued on the unit circle. However, a real-valued rather than a complex-valued solution for the singular values may come at the cost of (i) an increased order of the analytic SVD factors (see Example 2 and Fig. 2), and (ii) the need for oversampling by $\kappa = 2$ in case of spectral zeros whose multiplicities sum to an odd value. It may therefore be advantageous to perform a modified analytic SVD which permits complex-valued diagonal entries as e.g. contemplated in [57]. Of particular interest may be the combination of a dedicated SVD algorithm in [23] with an analyticity-enforcing DFT-domain approach [27], [28]. Thus, in addition to describing the existence, properties, and structure of the analytic SVD, this paper motivates a number of new algorithmic developments.

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