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SEMI-ANALYTICAL LONG-TERM ATTITUDE PROPAGATION

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A semi-analytical model is developed to study the long-term attitude motion of space objects, with different geometries and in different orbital regimes, under the effect of gravity, magnetic, light pressure, and aerodynamic torque. The problem is formulated in Sadov variables and a semi-analytical model is derived by averaging the equations of attitude motion over the fast Sadov angles and the orbital mean anomaly. A technique based on Lie transformations is used to compute an error estimate and improve the accuracy of the solution. We also propose a coordinate transformation that removes one of the singularities in the formulation in Sadov variables.

INTRODUCTION

The scope of this work is to develop a general semi-analytical framework to address the long-term attitude propagation of Earth satellites with different geometries in different orbital regimes, from low orbits (LEO) to geosynchronous orbits (GEO), considering the main perturbations affecting the dynamics, i.e. the geopotential, the magnetic residual torque, the solar radiation pressure torque, and the aerodynamics torque.

While analytical and semi-analytical theories are widely used to study and predict the orbital dynamics of space objects, the application of similar techniques to rotational dynamics is rarer. It's worth recalling the works by Benson and Scheeres,[1,2] who compute a semi-analytical model for GEO objects perturbed by the solar radiation pressure, averaging the attitude equations of motion over fast attitude variables. Instead, in Ref. 3 4, 5, 6, 7, 8, and, 9, the rotational problem perturbed by conservative torques is treated. When only conservative perturbations are considered the problem becomes Hamiltonian, so that the classical Lie series approach[10] can be applied. The Lie series technique consists in the iterative application of Lie transformations leading to the desired averaged semi-analytical model. By this method, the transformation from the "original" osculating attitude variables to mean variables, necessary to increase the accuracy of the propagation, can be analytically determined. Despite the evident advantages of the Lie series approach, the computation of the generating functions of the Lie transformations can be very complex. Typically, to simplify the procedure, action-angle variables are employed. While for axisymmetric bodies, these are the well-known Andoyer-Serret variables,[11] in the more general case of triaxial bodies, the action-angle variables are the so-called *Sadov variables*, introduced almost contemporary by Sadov[12] and Kinoshita.[13] The use of action-angle variables is advantageous not only for the application of the Lie series method but also because it becomes straightforward to identify the fast variables of the problem, over which the equations of motion can be averaged.

We express the attitude problem in Sadov variables and develop the semi-analytical model by averaging the equations of motion over the fast Sadov angles and the orbital mean anomaly, introduced in the problem by the perturbations. The dependence of the system on the fast Sadov angles

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occurs through Jacobi elliptic functions. This makes it extremely difficult to determine the variables transformation leading to the semi-analytical model. However, this transformation can be analytically estimated after performing Fourier expansions of the Jacobi elliptic functions, as suggested by Elipe and Vallejo.[3,14] To handle all the perturbations, a method similar to the one described by Barrio and Palacián in Ref. 15 is applied. It consists in a generalized Lie series approach suitable for vector fields with both Hamiltonian and non-Hamiltonian terms.

REPRESENTATION OF THE ATTITUDE DYNAMICS

In this work, we consider an equatorial inertial reference frame $EX_iY_iZ_i$ centered at the Earth's center of mass, with the Z_i axis pointing towards the Earth's north pole, and a rotating reference frame $Oxyz$ attached to the satellite, centered in the body center of mass. The rotating frame is characterized by principal axes of inertia. The z axis is the axis of maxima inertia and the x axis is the axis of minima inertia. Thus, the moments of inertia are $A \leq B \leq C$ with

$$A = \int (y^2 + z^2)\mathrm{d}m, \quad B = \int (x^2 + z^2)\mathrm{d}m, \quad C = \int (x^2 + y^2)\mathrm{d}m.$$

The classical sets of attitude variables, such as the Andoyer-Serret variables, describe the orientation of the rotating reference frame with respect to the inertial reference frame. On the contrary, the physical meaning of the Sadov variables is less clear. They are determined from a canonical transformation from the Andoyer-Serret variables with the purpose to obtain a Hamiltonian for the free-torque problem depending only on momenta. After recalling the physical meaning of the Andoyer-Serret variables, the main steps of the canonical transformation leading to Sadov variables are revised, following Ref. 5.

Andoyer variables

The Andoyer-Serret variables (L, G, H, l, g, h) are canonical variables for the rotational problem when conservative torques are considered. In particular, (L, G, H) are the momenta conjugated to the coordinates (l, g, h). Let $\boldsymbol{G}$ be the satellite's angular momentum. L is the projection of $\boldsymbol{G}$ on the z axis, $G = |\boldsymbol{G}|$ and H is the projection of $\boldsymbol{G}$ on the Z_i axis. Calling σ the inclination between the rotating reference plane and the plane perpendicular to $\boldsymbol{G}$ and δ the inclination of this latter plane with respect to the inertial frame, relations

$$\cos\sigma = \frac{L}{G}, \qquad \cos\delta = \frac{H}{G}$$

hold. Let $\hat{\mathbf{e}}_{Z_i}$ and $\hat{\mathbf{e}}_z$ be unit vectors with the same direction of the Z_i and z axes, respectively, and consider

$$\hat{\mathbf{e}}_{iG} = \frac{\hat{\mathbf{e}}_{Z_i} \times \boldsymbol{G}}{||\hat{\mathbf{e}}_{Z_i} \times \boldsymbol{G}||}, \qquad \hat{\mathbf{e}}_{bG} = \frac{\boldsymbol{G} \times \hat{\mathbf{e}}_z}{||\boldsymbol{G} \times \hat{\mathbf{e}}_z||}.$$

The unit vector $\hat{\mathbf{e}}_{iG}$ lies along the intersection between the OX_iY_i plane and the plane perpendicular to $\boldsymbol{G}$. Similarly, $\hat{\mathbf{e}}_{bG}$ lies along the intersection between the Oxy plane and the plane perpendicular to $\boldsymbol{G}$. The coordinate l is the angle defined between $\hat{\mathbf{e}}_{bG}$ and the x axis; g is the angle between $\hat{\mathbf{e}}_{iG}$ and $\hat{\mathbf{e}}_{bG}$; h is the angle between the X_i axis and $\hat{\mathbf{e}}_{iG}$. Thus, the Andoyer-Serrets variables describe a set of five rotations leading from the inertial to the rotating reference frame: a rotation around the Z_i axis by the angle h; a rotation around $\hat{\mathbf{e}}_{iG}$ by the angle δ; a rotation around $\boldsymbol{G}$ by the angle g; a rotation around $\hat{\mathbf{e}}_{bi}$ by the angle σ; a rotation around the z axis by the angle l.

In the case of axisymmetric bodies, the Andoyer-Serret variables are action-angle variables for the free-torque problem. The Hamiltonian is

$$\mathcal{H}_{\mathrm{AS}} = \frac{1}{2}\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right)\left(G^2 - L^2\right) + \frac{L^2}{2C}. \tag{1}$$

Thus, it is evident that $\mathcal{H}_{\mathrm{AS}}$ becomes a function of the only momenta if $A = B$. The same would hold true also if $C = B$ by considering the alternative rotating frame $Ox'y'z'$, with the x' axis of maxima inertia and the z' axis of minima inertia, such that

$$A' = C = \int (y'^2 + z'^2)\mathrm{d}m, \quad B' = B = \int (x'^2 + z'^2)\mathrm{d}m, \quad C' = A = \int (x'^2 + y'^2)\mathrm{d}m. \tag{2}$$

Sadov variables

The generating function of the canonical transformation from the Andoyer-Serret variables to the Sadov variables $(J_l, J_g, J_h, \psi_l, \psi_g, \psi_h)$ is

$$\mathcal{S}(J_l, J_g, J_h, l, g, h, t) = -t\Phi(J_l, J_g) + hJ_h + gJ_g + \mathcal{W}(l, J_l, J_g),$$

where t is the time variable and $\mathcal{W}(l, J_l, J_g)$ is a function fulfilling the Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right)\left(J_g^2 - \left(\frac{\partial \mathcal{W}}{\partial l}\right)^2\right) + \frac{1}{2C}\left(\frac{\partial \mathcal{W}}{\partial l}\right)^2 - \Phi(J_l, J_g) = 0,$$

equal to

$$\mathcal{W} = J_g \int \sqrt{\mathcal{Q}}\mathrm{d}l, \quad \mathcal{Q} = \frac{\sin^2 l/A + \cos^2 l/B - 1/J_d}{\sin^2 l/A + \cos^2 l/B - 1/C}, \quad J_d = \frac{J_g^2}{2\Phi}.$$

In the above formula, J_d is the dynamic moment of inertia.[1] Consider the change of coordinates

$$\sin l = \frac{\cos\lambda}{\sqrt{1 + \kappa \sin^2\lambda}}, \quad \cos l = -\frac{\sqrt{1+\kappa}\sin\lambda}{\sqrt{1 + \kappa \sin^2\lambda}},$$

and the function

$$m(J_l, J_g) = \frac{A}{C}\kappa\frac{C - J_d}{J_d - A}, \tag{3}$$

with

$$\kappa = \frac{C(B - A)}{A(C - B)}. \tag{4}$$

It results

$$\mathcal{Q} = \frac{\kappa(1 - m\sin^2\lambda)}{\kappa + m}, \quad \frac{\partial \mathcal{Q}}{\partial(2\Phi/J_g^2)} = \frac{AC(1 + k\sin^2\lambda)}{A - C}, \quad \Phi(J_g, J_l) = \frac{J_g^2}{2AC}\frac{\kappa A + mC}{\kappa + m},$$

so that

$$
\begin{aligned}
L &= \frac{\partial \mathcal{S}}{\partial l} = J_g \sqrt{\frac{\kappa}{\kappa+m}} \sqrt{1 - m \sin^2 \lambda}, \\
G &= \frac{\partial \mathcal{S}}{\partial g} = J_g, \\
H &= \frac{\partial \mathcal{S}}{\partial h} = J_h, \\
\psi_l &= \frac{\partial \mathcal{S}}{\partial J_l} = \frac{J_g}{2} \frac{\sqrt{k(1+\kappa}}{(\kappa+m)^{3/2}} F(\lambda, m) \frac{\partial m}{\partial J_l}, \\
\psi_g &= \frac{\partial \mathcal{S}}{\partial J_g} = g + \sqrt{\kappa+m} \sqrt{\frac{(1+\kappa)}{\kappa}} \left(\Pi(-\kappa, \lambda, m) - \frac{F(\lambda, m)}{\kappa+m} \left(m + \frac{J_g}{2} \frac{\kappa}{\kappa+m} \frac{\partial m}{\partial J_g} \right) \right), \\
\psi_h &= \frac{\partial \mathcal{S}}{\partial J_h} = h,
\end{aligned}
\tag{5}
$$

where

$$
F(\lambda, m) = \int_0^\lambda \frac{\mathrm{d}\vartheta}{\sqrt{1 - m \sin^2 \vartheta}}, \quad \Pi(-\kappa, \lambda, m) = \int_0^\lambda \frac{\mathrm{d}\vartheta}{(1 + k \sin^2 \vartheta) \sqrt{1 - m \sin^2 \vartheta}}
$$

are the incomplete elliptic integrals of first and third kind. The variables ψ_l and ψ_g are angles if the conditions $\oint \psi_l = 2\pi$ and $\oint \psi_g = 2\pi$ are fulfilled. Since $F(0, m) = \Pi(-\kappa, 0, m) = 0$, $F(2\pi, m) = 4K(m)$ and $\Pi(-\kappa, 2\pi, m) = 4\Pi(-\kappa, m)$, with $K(m)$ and $\Pi(-\kappa, m)$ the complete integrals of first and third kind,

$$
K(m) = F(\pi/2, m), \quad \Pi(-\kappa, m) = \Pi(-\kappa, \pi/2, m),
$$

it follows

$$
\begin{aligned}
2\pi &= -2 J_g \frac{\sqrt{\kappa(1+\kappa}}{(\kappa+m)^{3/2}} K(m) \frac{\partial m}{\partial J_l}, \\
0 &= 4\sqrt{\kappa+m} \sqrt{\frac{(1+\kappa)}{\kappa}} \left(\Pi(-\kappa, m) - \frac{1}{\kappa+m} \left(m + \frac{J_g}{2} \frac{k}{\kappa+m} \frac{\partial m}{\partial J_g} \right) K(m) \right),
\end{aligned}
$$

so that

$$
\frac{\partial m}{\partial J_l} = -\frac{\pi}{J_g} \frac{1}{\zeta^{\frac{3}{2}}} \frac{\kappa}{\sqrt{1+\kappa}} \frac{1}{K(m)}, \quad \frac{\partial m}{\partial J_g} = \frac{2\kappa \left(\Pi(-\kappa, m) - (1-\zeta) K(m) \right)}{J_g K(m) \zeta^2},
$$

where

$$
\zeta = \frac{\kappa}{\kappa+m}. \tag{6}
$$

One can assume $m = m(J_l/J_g)$, which implies

$$
J_l \frac{\partial m}{\partial J_l} + J_g \frac{\partial m}{\partial J_g} = 0,
$$

and relation

$$
\frac{J_l}{J_g} = \frac{2}{\pi} \sqrt{\frac{1+\kappa}{\zeta}} \left(\Pi(-\kappa, m) - (1-\zeta) K(m) \right) \tag{7}
$$

holds. Thus, the transformation from the Andoyer-Serret variables to the Sadov variables is

$$\begin{aligned}
J_l &= \frac{2G}{\pi}\sqrt{\frac{1+\kappa}{\zeta}}\left(\Pi(-\kappa,m)-(1-\zeta)K(m)\right),\\
J_g &= G,\\
J_h &= H,\\
\psi_l &= \frac{\pi}{2}\frac{F(\lambda,m)}{K(m)},\\
\psi_g &= g+\sqrt{\frac{1+\kappa}{\zeta}}\left(\Pi(-\kappa,\lambda,m)-\frac{\Pi(-\kappa,m)}{K(m)}F(\lambda,m)\right),\\
\psi_h &= h,
\end{aligned} \tag{8}$$

where the value of m is given by Eq. (3). The Hamiltonian of the torque-free problem expressed in Sadov variables becomes

$$\mathcal{H}_{\mathrm{S}} = \frac{J_g^2}{2AC}\frac{\kappa A+mC}{\kappa+m}. \tag{9}$$

The rotation matrix from the inertial frame to the rotating frame is

$$R_{\mathrm{i2b}} = \begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}\begin{bmatrix} \cos\psi_h & \sin\psi_h & 0\\ -\cos\tilde{\delta}\sin\psi_h & \cos\tilde{\delta}\cos\psi_h & \sin\tilde{\delta}\\ \sin\tilde{\delta}\sin\psi_h & -\sin\tilde{\delta}\cos\psi_h & \cos\tilde{\delta}\end{bmatrix} \tag{10}$$

with

$$\tilde{\delta} = \delta(J_h,J_g) = \arccos\left(J_h/J_g\right) \tag{11}$$

and

$$b_{11} = -\left(\sqrt{\zeta}\sin\left(\psi_g+\delta g\right)\mathrm{cn}(u,m)\mathrm{dn}(u,m)+\sqrt{1+\kappa}\cos\left(\psi_g+\delta g\right)\mathrm{sn}(u,m)\right)\mathrm{dnk}^{-\frac{1}{2}}, \tag{12}$$

$$b_{12} = -\left(\sqrt{1+\kappa}\sin\left(\psi_g+\delta g\right)\mathrm{sn}(u,m)-\sqrt{\zeta}\,\cos\left(\psi_g+\delta g\right)\mathrm{dn}(u,m)\mathrm{cn}(u,m)\right)\mathrm{dnk}^{-\frac{1}{2}}, \tag{13}$$

$$b_{13} = \sqrt{1-\zeta}\,\mathrm{cn}(u,m), \tag{14}$$

$$b_{21} = -\left(\cos\left(\psi_g+\delta g\right)\mathrm{cn}(u,m)-\sqrt{1+\kappa}\,\sqrt{\zeta}\sin\left(\psi_g+\delta g\right)\mathrm{sn}(u,m)\,\mathrm{dn}(u,m)\right)\mathrm{dnk}^{-\frac{1}{2}}, \tag{15}$$

$$b_{22} = -\left(\sqrt{1+\kappa}\,\sqrt{\zeta}\cos\left(\psi_g+\delta g\right)\mathrm{sn}(u,m)\,\mathrm{dn}(u,m)+\sin\left(\psi_g+\delta g\right)\mathrm{cn}(u,m)\right)\mathrm{dnk}^{-\frac{1}{2}}, \tag{16}$$

$$b_{23} = -\sqrt{1-\zeta}\,\sqrt{1+\kappa}\,\mathrm{sn}(u,m), \tag{17}$$

$$b_{31} = \sqrt{1-\zeta}\sin\left(\psi_g+\delta g\right)\mathrm{dnk}^{\frac{1}{2}}, \tag{18}$$

$$b_{32} = -\sqrt{1-\zeta}\,\cos\left(\psi_g+\delta g\right)\mathrm{dnk}^{\frac{1}{2}}, \tag{19}$$

$$b_{33} = \sqrt{\zeta}\,\mathrm{dn}(u,m), \tag{20}$$

where

$$\delta g = -\sqrt{\frac{1+\kappa}{\zeta}}\left(\Pi(-\kappa,\lambda,m)-u\frac{\Pi(-\kappa,m)}{K(m)}\right),\quad \mathrm{dnk} = 1+\kappa\mathrm{sn}^2(u,m),\quad u = \frac{2K(m)\psi_l}{\pi},$$

and $\mathrm{cn}(u,m)$, $\mathrm{sn}(u,m)$, $\mathrm{dn}(u,m)$ are the Jacobi elliptic cosine, the Jacobi elliptic sine, and the Jacobi elliptic delta amplitude, respectively.

Let us remark that the transformation from the Andoyer-Serrets variables to the Sadov variables is suitable only if both the following conditions are fulfilled, as it results from Eq. (5):

1. since $|L| < G$, $\mathcal{Q}$ has to belong to $(0,1)$ for all λ, i.e.
$$B \leq J_d \leq C; \tag{21}$$
2. since $J_g > 0$, it is also clear that $L > 0$.

In case Eq. (21) is fulfilled, the satellite is said to be in *short axis mode (SAM)*. On the contrary if $A < J_d < B$, the satellite is in *long axis mode (LAM)*. In this case, it is sufficient to consider the alternative reference frame $Ox'y'z'$ with the principal moments of inertia in Eq. (2). All the above relations hold by employing the Andoyer-Serret variables defined with respect to $Ox'y'z'$ and substituting A, B, C with the moments of inertia A', B', C'. Similarly, if the second condition is not fulfilled, one can simply consider an alternative rotating reference frame $Ox_oy_oz_o$ with the z_o axis in the opposite direction of the z axis.

EQUATIONS OF MOTION FOR THE ATTITUDE DYNAMICS

Given $\mathcal{H}_\mathrm{S}$ in EQ. 9, the attitude equations of motion for the free-torque problem are

$$\begin{gathered}\frac{\mathrm{d}J_l}{\mathrm{d}t} = -\frac{\mathrm{d}\mathcal{H}_\mathrm{S}}{\mathrm{d}\psi_l} = 0, \quad \frac{\mathrm{d}J_g}{\mathrm{d}t} = -\frac{\mathrm{d}\mathcal{H}_\mathrm{S}}{\mathrm{d}\psi_g} = 0, \quad \frac{\mathrm{d}J_h}{\mathrm{d}t} = -\frac{\mathrm{d}\mathcal{H}_\mathrm{S}}{\mathrm{d}\psi_h} = 0, \\ \frac{\mathrm{d}\psi_l}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{H}_\mathrm{S}}{\mathrm{d}J_l} = n_{\psi_l}, \quad \frac{\mathrm{d}\psi_g}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{H}_\mathrm{S}}{\mathrm{d}J_g} = n_{\psi_g}, \quad \frac{\mathrm{d}\psi_h}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{H}_\mathrm{S}}{\mathrm{d}J_h} = 0,\end{gathered} \tag{22}$$

with

$$n_{\psi_l} = -\pi\sqrt{\frac{\zeta}{1+\kappa}}\frac{J_g}{2AC}\frac{C-A}{K(m)}, \quad n_{\psi_g} = \left(\Pi(-\kappa,m) + \frac{A}{C-A}K(m)\right)\frac{C-A}{K(m)}\frac{J_g}{AC}. \tag{23}$$

Since the equations of motion depend explicitly on m and ζ and not on J_l, it is more convenient to use one of these variables in place of J_l. We select ζ, whose time derivative is

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = -\frac{\mathrm{d}\zeta}{\mathrm{d}Jl}\frac{\mathrm{d}\mathcal{H}_\mathrm{S}}{\mathrm{d}\psi_l} - \frac{\mathrm{d}\zeta}{\mathrm{d}Jg}\frac{\mathrm{d}\mathcal{H}_\mathrm{S}}{\mathrm{d}\psi_g} = 0,$$

where

$$\frac{\mathrm{d}\zeta}{\mathrm{d}Jl} = -\frac{\zeta^2}{\kappa}\frac{\mathrm{d}m}{\mathrm{d}Jl} = \frac{\pi}{J_g}\frac{\zeta^{\frac{1}{2}}}{\sqrt{1+\kappa}}\frac{1}{K(m)}, \quad \frac{\mathrm{d}\zeta}{\mathrm{d}Jg} = -\frac{\zeta^2}{\kappa}\frac{\mathrm{d}m}{\mathrm{d}Jg} = -\frac{2\left(\Pi(-\kappa,m) - (1-\zeta)K(m)\right)}{J_gK(m)}.$$

This choice does not give any real advantage in the framework of the free-torque problem, as both m and ζ are constant. However, it becomes convenient when external torques perturb the satellite, and m and ζ may vary. In the case of external torques, setting $\boldsymbol{s} = (\zeta, J_g, J_h, \psi_l, \psi_g, \psi_h)$, the equations of motion become

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \mathscr{A}\nabla_{\boldsymbol{s}}\mathcal{H}_\mathrm{S} + \mathscr{A}\nabla_{\boldsymbol{s}}\mathcal{V} + \mathscr{B}\boldsymbol{M} \tag{24}$$

with $\nabla_{\boldsymbol{s}}$ the gradient operator,

$$\mathscr{A} = \begin{bmatrix} 0 & \tilde{\mathscr{I}} \\ -\tilde{\mathscr{I}}^{\mathrm{T}} & 0 \end{bmatrix}, \quad \tilde{\mathscr{I}} = \begin{bmatrix} -\frac{\pi}{J_g}\frac{\zeta^{\frac{1}{2}}}{\sqrt{1+\kappa}}\frac{1}{K(m)} & \frac{2(\Pi(-\kappa,m)-(1-\zeta)K(m))}{J_g K(m)} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{25}$$

and

$$\mathscr{B} = \begin{bmatrix} -\frac{2\zeta}{J_g}b_{13} & -\frac{2\zeta(1-m)}{J_g(1+\kappa)}b_{23} & \frac{2(1-\zeta)}{J_g}b_{33} \\ b_{13} & b_{23} & b_{33} \\ \cos\tilde{\delta}b_{13} + \sin\tilde{\delta}b_{12} & \cos\tilde{\delta}b_{23} + \sin\tilde{\delta}b_{22} & \cos\tilde{\delta}b_{33} + \sin\tilde{\delta}b_{32} \\ -\frac{\pi S_x}{2J_g K(m)(1-m)} & -\frac{\pi S_y}{2J_g K(m)} & -\frac{\pi S_z}{2J_g K(m)(1-m)} \\ \mathcal{T}S_x - \frac{b_{11}\cos\tilde{\delta}}{J_g \sin\tilde{\delta}} & \mathcal{T}(1-m)S_y - \frac{b_{21}\cos\tilde{\delta}}{J_g \sin\tilde{\delta}} & \mathcal{T}S_z - \frac{b_{31}\cos\tilde{\delta}}{J_g \sin\tilde{\delta}} \\ \frac{b_{11}}{J_g \sin\tilde{\delta}} & \frac{b_{21}}{J_g \sin\tilde{\delta}} & \frac{b_{31}}{J_g \sin\tilde{\delta}} \end{bmatrix}, \tag{26}$$

where $\tilde{\delta}$ is given in Eq. (11), b_{ij} are given in Eq. (12)-(20),

$$\mathcal{T} = \frac{(\Pi(-\kappa,m) - (1-\zeta)K(m))\sqrt{1+\kappa}}{J_g K(m)\sqrt{\zeta}\,(1-m)}, \quad S_x = \frac{\mathrm{dn}(u,m)\mathrm{sn}(u,m) - \mathrm{cn}(u,m)\mathrm{zn}(u,m)}{\sqrt{1-\zeta}},$$

$$S_y = \frac{\mathrm{dn}(u,m)\mathrm{cn}(u,m) + \mathrm{sn}(u,m)\mathrm{zn}(u,m)}{\sqrt{1+\kappa}\sqrt{1-\zeta}}, \quad S_z = \frac{\mathrm{dn}(u,m)\mathrm{zn}(u,m) - m\mathrm{cn}(u,m)\mathrm{sn}(u,m)}{\sqrt{\zeta}},$$

and $\mathrm{zn}(u,m)$ is the Jacobi zeta function. $\boldsymbol{M}$ is an external non-conservative torque and $\mathcal{V}$ is the potential energy of an external conservative torque. In the equations of motion, m has to be considered a function of the only variable ζ, while u is a function of both ζ and ψ_l.

EXTERNAL TORQUES

The most significant external torques acting on an Earth satellite are the gravity, the magnetic, and the solar radiation pressure torques.[16] In particular, in comparison with the gravity and the magnetic torques, the solar radiation pressure torque becomes more significant at higher altitudes. Instead, if the satellite is on a LEO, it is necessary to consider also the effects of the aerodynamic torque, induced by the atmospheric drag.

Gravity torque

As suggested in Ref. 17, the gravity torque can be modeled by approximating the Earth as a perfectly spherical body. In particular, the potential energy corresponding to this torque can be approximated as

$$\mathcal{V}_{\mathrm{g}} = \frac{3\mu_{\oplus}}{2r^3}\left(A\alpha_1^2 + B\alpha_2^2 + C\alpha_3^2\right). \tag{27}$$

where $\mu_{\oplus}$ is the Earth's gravitational parameters, $r = |\boldsymbol{r}|$, with $\boldsymbol{r}$ the geocentric position vector of the satellite's center of mass, and $(\alpha_1, \alpha_2, \alpha_3)$ are the direction cosine of $\boldsymbol{r}$, i.e.

$$(\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}} = R_{\mathrm{i2b}}\frac{\boldsymbol{r}}{r}.$$

Magnetic Torque

As suggested in Ref. 17, the magnetic flux density of the Earth can be described through a simplified model, assuming the geomagnetic dipole aligned with the Earth's polar axis, i.e.

$$\boldsymbol{B}_{\oplus} = \frac{\mu_{\mathrm{m}}}{r^3}\left(\hat{\mathbf{e}}_{Z_i} - 3\left(\hat{\mathbf{e}}_{Z_i}\cdot\frac{\boldsymbol{r}}{r}\right)\frac{\boldsymbol{r}}{r}\right),$$

where μ_{m} is the Earth's magnetic dipole strength. Given the intrinsic magnetic moment of the satellite $\boldsymbol{I}_{\mathrm{m}}$, the potential energy corresponding to the magnetic torque is equal to

$$\mathcal{V}_{\mathrm{m}} = -\boldsymbol{I}_{\mathrm{m}}\cdot R_{\mathrm{i2b}}\boldsymbol{B}_{\oplus}. \tag{28}$$

Solar radiation pressure torque

The solar radiation pressure torque is modeled as suggested in Ref. 1. The satellite surface is subdivided into n_f facets and the solar radiation pressure force acting on the satellite is computed as the sum of the forces acting on each single facet, i.e.

$$\boldsymbol{f}_{\mathrm{srp}} = \sum_{i=1}^{n_f}\boldsymbol{f}_{\mathrm{srp},i}, \quad \boldsymbol{f}_{\mathrm{srp},i} = -P_{\mathrm{srp}}S_i\left(c_{\mathrm{ai}}\hat{\boldsymbol{u}} + c_{\mathrm{di}}\hat{\boldsymbol{n}}_i + c_{\mathrm{di}}(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_i)\,\hat{\boldsymbol{n}}_i\right)\max\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_i, 0\right),$$

where P_{srp} is the solar radiation pressure, S_i is the surface area of each facets, $\hat{\boldsymbol{u}}$ is the satellite to Sun unit vector and $\hat{\boldsymbol{n}}_i$ is the facet outer-pointing normal unit vector. Moreover, $c_{\mathrm{ai}} = 1-\rho_i s_i$, $c_{\mathrm{di}} = 2/3(1-s_i)\rho_i$ and $c_{\mathrm{si}} = 2\rho_i s_i$, with ρ_i the total reflectivity of the facet and s_i the fraction of ρ_i that is specular. Some simplifications are adopted. The solar radiation pressure is assumed constant, equal to its value at 1 astronomical unit, and

$$\hat{\boldsymbol{u}} = R_{\mathrm{i2b}}\frac{\boldsymbol{r}_{\odot}-\boldsymbol{r}}{|\boldsymbol{r}_{\odot}-\boldsymbol{r}|} \sim \tilde{\boldsymbol{u}} = R_{\mathrm{i2b}}\left(\frac{\boldsymbol{r}_{\odot}}{|\boldsymbol{r}_{\odot}|} + \frac{(\boldsymbol{r}\cdot\boldsymbol{r}_{\odot})\,\boldsymbol{r}_{\odot}}{|\boldsymbol{r}_{\odot}|^3} - \frac{\boldsymbol{r}}{|\boldsymbol{r}_{\odot}|}\right),$$

with $\boldsymbol{r}_{\odot}$ the geocentric position vector of the Sun. Furthermore, as proposed in Ref. 2,

$$\max\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_i, 0\right) \sim g_i = \frac{1}{3\pi} + \frac{1}{2}(\tilde{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_i) + \frac{4}{3\pi}(\tilde{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_i)^2.$$

Thus, the solar radiation pressure torque is given by

$$\boldsymbol{M}_{\mathrm{srp}} \sim -P_{\mathrm{srp}}\sum_{i=1}^{n_f} S_i g_i \boldsymbol{\rho_i}\times\left(c_{\mathrm{ai}}\tilde{\boldsymbol{u}} + c_{\mathrm{di}}\hat{\boldsymbol{n}}_i + c_{\mathrm{si}}(\tilde{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_i)\,\hat{\boldsymbol{n}}_i\right). \tag{29}$$

with $\boldsymbol{\rho_i}$ the center of mass to facet centroid position vector.

Over long periods of time, the Earth's shadow may have an effect on the evolution of the attitude variables. The effect can be taken into account by multiplying the torque by the shadow function

$$v_s(\Upsilon) = \begin{cases} 0 & \text{if} \quad \Upsilon_1 \le \Upsilon \le \Upsilon_2, \\ 1 & \text{otherwise,} \end{cases}$$

with Υ the satellite's eccentric longitude, and Υ_1, Υ_2 the values of Υ at which the satellite enters and exits from the shadow region. Actually, since it is more convenient to deal with smooth functions, $\boldsymbol{M}_{\mathrm{srp}}$ is multiplied by the Fourier expansion of $v_s(\Upsilon)$ truncated at the 20-th harmonic, i.e.

$$v_s(\Upsilon) \sim 1 - \frac{\Upsilon 2 - \Upsilon 1}{2\pi} - \sum_{k=1}^{20}\frac{2}{\pi k}\sin\left(k\frac{\Upsilon 2-\Upsilon 1}{2}\right)\cos\left(k\Upsilon - k\frac{\Upsilon 2+\Upsilon 1}{2\pi}\right). \tag{30}$$

The values Υ_1, Υ_2 can be computed by the algorithm described in Ref. 18.

Aerodynamic torque

The aerodynamic force acting on a surface element $\mathrm{d}S$ of the satellite is

$$\mathrm{d}\boldsymbol{f}_{\mathrm{d}} = -\frac{1}{2} c_D \rho_a \tilde{V} \max\left(\hat{\boldsymbol{n}}_s \cdot \frac{\tilde{\boldsymbol{V}}}{\tilde{V}}, 0\right) \tilde{\boldsymbol{V}} \mathrm{d}S,$$

where c_D is a coefficient, ρ_a is the stream density, $\tilde{\boldsymbol{V}}$ is the velocity of the surface element relative to the incident stream, $\tilde{V} = |\tilde{\boldsymbol{V}}|$ and $\hat{\boldsymbol{n}}_s$ is the surface element outer-pointing unit normal vector, see Ref. 16. In particular,

$$\tilde{\boldsymbol{V}} = \boldsymbol{\omega} \times \boldsymbol{\rho}_s + \boldsymbol{V}_0,$$

where $\boldsymbol{\omega}$ is the satellite's angular velocity and $\boldsymbol{V}_0$ is the velocity of the mass center relative to the incident stream. Let $\dot{\boldsymbol{r}}$ be the velocity vector of the satellite's center of mass. Then,

$$\boldsymbol{V}_0 = R_{\mathrm{i2b}} \left(\dot{\boldsymbol{r}} - n_{\oplus} \hat{\mathbf{e}}_{Z_i} \times \boldsymbol{r} \right),$$

with $n_{\oplus}$ the mean angular speed of the Earth's rotation around its polar axis.[19] Since, typically $|\boldsymbol{\omega} \times \boldsymbol{\rho}_s| < V_0$, with $V_0 = |\boldsymbol{V}_0|$, we introduce the approximation $\tilde{\boldsymbol{V}} \sim \boldsymbol{V}_0$. If the satellite surface is subdivided into n_f facets and the approximation

$$\max\left(\hat{\boldsymbol{n}}_s \cdot \frac{\boldsymbol{V}_0}{V_0}, 0\right) = d_i \sim \frac{1}{3\pi} + \frac{1}{2}\left(\hat{\boldsymbol{n}}_i \cdot \frac{\boldsymbol{V}_0}{V_0}\right) + \frac{4}{3\pi}\left(\hat{\boldsymbol{n}}_i \cdot \frac{\boldsymbol{V}_0}{V_0}\right)^2$$

is employed, it results

$$\boldsymbol{M}_{\mathrm{d}} = -\frac{1}{2} c_D \rho_a V_0^2 \sum_i^{n_f} S_i d_i \, \boldsymbol{\rho}_i \times \frac{\boldsymbol{V}_0}{V_0}. \tag{31}$$

SEMI-ANLYTICAL MODEL

We aim to determine a semi-analytical model suitable to study the dynamics over long timescales. Typically, the effects of the external torques can be considered as perturbations of the satellite's free-torque motion. Let us shorty write the equations of motion in Eq. 24 as

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \mathscr{A} \nabla_{\boldsymbol{s}} \mathcal{H}_{\mathrm{S}} + \epsilon \boldsymbol{F}_{\mathrm{et}}(\boldsymbol{s}), \tag{32}$$

where $\epsilon \boldsymbol{F}_{\mathrm{et}}(\boldsymbol{s})$ is the total external torque, with ϵ its magnitude. The semi-analytical model is computed by retaining only the long-period terms in the equations of motion. This can be achieved by averaging them with respect to the fast variables of the problem. The operation is equivalent to performing a transformation of variables close to the identity, i.e.

$$\boldsymbol{s}(t) = \overline{\boldsymbol{s}}(t) + \boldsymbol{\chi}\left(\overline{\boldsymbol{s}}(t), t\right), \tag{33}$$

which transforms (32) into

$$\frac{\mathrm{d}\overline{\boldsymbol{s}}(t)}{\mathrm{d}t} = \mathscr{A} \nabla_{\boldsymbol{s}} \mathcal{H}_{\mathrm{S}} + \overline{\epsilon \boldsymbol{F}_{\mathrm{et}}}(\overline{\boldsymbol{s}}) + \mathcal{O}(\epsilon^2), \tag{34}$$

where the remainder of the second order in ϵ is neglected:

$$\frac{\mathrm{d}\overline{\boldsymbol{s}}(t)}{\mathrm{d}t} \sim \overline{\frac{\mathrm{d}\boldsymbol{s}(t)}{\mathrm{d}t}} = \mathscr{A} \nabla_{\boldsymbol{s}} \mathcal{H}_{\mathrm{S}} + \overline{\epsilon \boldsymbol{F}_{\mathrm{et}}}. \tag{35}$$

The resulting semi-analytical model gives the evolution of $\overline{\boldsymbol{s}}$, which correspond to mean attitude variables over the largest of the periods of the fast angles. The number of terms in (35) is reduced in comparison to (32), which implies a shorter computational time for its integration. From Eq. (22), it is clear that the fast variables of the problem are the angles ψ_l and ψ_g. Furthermore, because of the perturbations, the equations of motion depend also on the orbital mean anomaly M, which is a fast angle as well. Supposing that the periods of the fast angles are not resonant, the semi-analytical model (35) is computed as

$$\overline{\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t}} = \mathscr{A}\nabla_{\boldsymbol{s}}\mathcal{H}_{\mathrm{S}} + \frac{1}{8\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \epsilon\boldsymbol{F}_{\mathrm{et}}\,\mathrm{d}\psi_g\mathrm{d}\psi_l\mathrm{d}M.$$

The term $\epsilon\boldsymbol{F}_{\mathrm{et}}$ is the sum of different components, i.e.,

$$\epsilon\boldsymbol{F}_{\mathrm{et}}(\boldsymbol{s}) = \mathscr{A}\nabla_{\boldsymbol{s}}\left(\mathcal{V}_{\mathrm{g}} + \mathcal{V}_{\mathrm{m}}\right) + \mathscr{B}\left(\boldsymbol{M}_{\mathrm{srp}}v_s(\Upsilon) + \boldsymbol{M}_{\mathrm{d}}\right),$$

see Eq. (25)-(31). In particular, the average of the terms due to conservative torques can be computed from the average of the corresponding potential energies as $\mathscr{A}\nabla_{\boldsymbol{s}}\left(\overline{\mathcal{V}}_{\mathrm{g}} + \overline{\mathcal{V}}_{\mathrm{m}}\right)$, with

$$\overline{\mathcal{V}}_{\mathrm{g}} = \frac{1}{8\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\mathcal{V}_{\mathrm{g}}\mathrm{d}\psi_g\mathrm{d}\psi_l\mathrm{d}M, \quad \overline{\mathcal{V}}_{\mathrm{m}} = \frac{1}{8\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\mathcal{V}_{\mathrm{m}}\mathrm{d}\psi_g\mathrm{d}\psi_l\mathrm{d}M.$$

The averaging process is conducted assuming that the satellite is along a Keplerian orbit.

Averaging process

The terms contained in $\mathscr{B}\boldsymbol{M}_{\mathrm{srp}}v_{\mathrm{s}}(\Upsilon)$, $\mathscr{B}\boldsymbol{M}_{\mathrm{d}}$, $\mathcal{V}_{\mathrm{g}}$ or $\mathcal{V}_{\mathrm{m}}$ have the form

$$f = \mathcal{F}(J_l, J_g, J_h, \psi_h)f_{\mathrm{a}}(J_l, J_g, \psi_l, \psi_g)f_{\mathrm{o}}(a, P_1, P_2, Q_1, Q_2, M),$$

with (a, P_1, P_2, Q_1, Q_2) the orbital equinoctial elements. The perturbing torques depend on (ψ_l, ψ_g) through the components of the rotation matrix R_{i2b} in Eq. 10. From the expression of $\mathscr{B}$ in Eq. (26), f_{a} can depend on the Sadov fast angles also through the terms $\tilde{S}_x, \tilde{S}_y, \tilde{S}_z$. It follows that f_{a} has the form

$$\begin{aligned} f_{\mathrm{a}} = \mathcal{F}_{\mathrm{a}}(\zeta, J_g)\sin^i(\psi_g + \delta g)\cos^j(\psi_g + \delta g) \\ \cdot\,\mathrm{dn}^o(u,m)\mathrm{sn}^s(u,m)\mathrm{cn}^k(u,m)\mathrm{dnk}^{\pm\frac{i}{2}\pm\frac{j}{2}}\mathrm{zn}^w(u,m), \end{aligned} \tag{36}$$

with $w \in [0, 1]$, and i, j, o, s, k natural numbers. It is convenient to compute the average of f performing, in the given order, of the following operations:

1. Computation of the average over ψ_g, i.e.

$$\langle f\rangle_{\psi_g} = \mathcal{F}\,f_{\mathrm{o}}\,\langle f_{\mathrm{a}}\rangle_{\psi_g} = \mathcal{F}f_{\mathrm{o}}\frac{1}{2\pi}\int_0^{2\pi} f_{\mathrm{a}}\mathrm{d}\psi_g.$$

If either the natural exponents i or j in Eq. (36) are odd numbers, $\langle f_{\mathrm{a}}\rangle_{\psi_g} = 0$. Otherwise,

$$\langle f_{\mathrm{a}}\rangle_{\psi_g} = \tilde{\mathcal{F}}_{\mathrm{a}}(J_g, \zeta)\mathrm{dn}^o(u,m)\mathrm{sn}^s(u,m)\mathrm{cn}^k(u,m)\mathrm{dnk}^p\mathrm{zn}^w(u,m), \tag{37}$$

with p an integer number and

$$\tilde{\mathcal{F}}_{\mathrm{a}}(J_g,\zeta)=\mathcal{F}_{\mathrm{a}}(J_g,\zeta)\sum_{d=0}^{\frac{i}{2}}\left(\begin{array}{c}\frac{i}{2}\\ d\end{array}\right)\frac{(-1)^d}{2d+j}\prod_{v=1}^{\frac{2d+j}{2}-1}\frac{2d+j-2v+1}{2d+j-2v}.$$

Thus, the term δg disappears and the dependence on ψ_l is only through Jacobi elliptic functions. This makes $\langle f_{\mathrm{a}}\rangle_{\psi_g}$ easier to be averaged over ψ_l than f_{a}.

2. Computation of the average over ψ_l, i.e.

$$\langle f\rangle_{\psi_g,\psi_l}=\mathcal{F}f_{\mathrm{o}}\overline{f_{\mathrm{a}}}=\mathcal{F}f_{\mathrm{o}}\frac{1}{2\pi}\int_0^{2\pi}\langle f_{\mathrm{a}}\rangle_{\psi_g}\mathrm{d}\psi_l=\frac{\mathcal{F}f_{\mathrm{o}}}{4K(m)}\tilde{\mathcal{F}}_{\mathrm{a}}(J_l,J_g)\mathcal{J},$$

with

$$\mathcal{J}=\int_0^{4K(m)}\mathrm{dn}^o(u,m)\mathrm{sn}^s(u,m)\mathrm{cn}^k(u,m)\mathrm{dnk}^p\mathrm{dn}(u,m)\mathrm{zn}^w(u,m)\mathrm{d}u.$$

If $w=1$,

$$\mathcal{J}=\int_0^{4K(m)}\left(\int\mathrm{dn}^o(u,m)\mathrm{sn}^s(u,m)\mathrm{cn}^k(u,m)\mathrm{dnk}^p\mathrm{d}u\right)\left(\frac{E(m)}{K(m)}-\mathrm{dn}^2(u,m)\right)\mathrm{d}u,$$

with $E(m)$ the complete elliptic integral of second kind. Thus, computing $\mathcal{J}$ implies to compute integrals of the type

$$\int\mathrm{dn}^v(u,m)\mathrm{sn}^s(u,m)\mathrm{cn}^k(u,m)\mathrm{dnk}^p\mathrm{d}u,$$

which can be easily done with the support of a symbolic manipulator, such as MAPLE.

3. Computation of the average over M. The dependence of the external torques on M is implicit through geometrical angles, such as the true longitude Λ or the eccentric longitude Υ. Thus, the average is computed using the formulas

$$\overline{f}=\mathcal{F}\overline{f_{\mathrm{a}}}\frac{1}{2\pi}\int_0^{2\pi}f_{\mathrm{o}}\mathrm{d}M=\mathcal{F}\overline{f_{\mathrm{a}}}\frac{1}{2\pi}\int_0^{2\pi}f_{\mathrm{o}}\frac{r^2}{a^2\eta}\mathrm{d}\Lambda=\mathcal{F}\overline{f_{\mathrm{a}}}\frac{1}{2\pi}\int_0^{2\pi}f_{\mathrm{o}}\frac{r}{a}\mathrm{d}\Upsilon,\qquad(38)$$

with $\eta=\sqrt{1-P_1^2-P_2^2}$.

The average over M is performed analytically for all torques except for the aerodynamic one, for which it is computed numerically. This allows to use any desired model for the atmospheric density.

Variables transformation

As previously explained, the semi-analytical model gives the evolution of mean attitude variables $\overline{\boldsymbol{s}}$ related to the 'original' osculating attitude variables by the transformation in Eq. (33). Determining this transformation is necessary to compute the initial values of mean attitude variables. This

can be done by employing a method similar to the one described in Ref. 15, based on the combination of Lie transformations. A Lie transformation is a transformation between two sets of variables $\boldsymbol{s}^{(0)}$ and $\boldsymbol{s}^{(1)}$ given by

$$\boldsymbol{s}^{(0)} = \sum_{i=0}^{\infty} \frac{1}{i!} \mathscr{L}_{\boldsymbol{W}}^{i} \boldsymbol{s}^{(1)}, \tag{39}$$

with $\mathscr{L}_{\boldsymbol{W}} \boldsymbol{s}^{(1)}$the Lie derivative of $\boldsymbol{s}^{(1)}$ with respect to a vector field $\boldsymbol{W}$, called generator, and

$$\mathscr{L}_{\boldsymbol{W}}^{i}(\boldsymbol{s}^{(1)}) = \underbrace{\mathscr{L}_{\boldsymbol{W}} \left(\mathscr{L}_{\boldsymbol{W}} \left(\ldots \left(\mathscr{L}_{\boldsymbol{W}} \boldsymbol{s}^{(1)} \right) \right) \right)}_{j \text{ times}}.$$

As shown by Henrard,[20] following this change of variables, a vector field $\boldsymbol{F}(\boldsymbol{s}^{(0)})$ is transformed into

$$\boldsymbol{F}'(\boldsymbol{s}^{(1)}) = \sum_{i=0}^{\infty} \frac{1}{i!} \tilde{\mathscr{L}}_{\boldsymbol{W}}^{i} \boldsymbol{F}(\boldsymbol{s}^{(0)}) \,|_{\boldsymbol{s}^{(0)}=\boldsymbol{s}^{(1)}} \,, \quad \tilde{\mathscr{L}}_{\boldsymbol{W}}^{i} \boldsymbol{F}(\boldsymbol{s}^{(0)}) = \underbrace{\tilde{\mathscr{L}}_{\boldsymbol{W}} \left(\tilde{\mathscr{L}}_{\boldsymbol{W}} \left(\ldots \left(\tilde{\mathscr{L}}_{\boldsymbol{W}} \boldsymbol{F} \left(\boldsymbol{s}^{(0)} \right) \right) \right) \right)}_{i \text{ times}},$$

where

$$\tilde{\mathscr{L}}_{\boldsymbol{W}} \cdot = \frac{\partial \cdot}{\partial \boldsymbol{s}^{(0)}} \boldsymbol{W} - \frac{\partial \boldsymbol{W}}{\partial \boldsymbol{s}^{(0)}} \cdot$$

is the so-called *Lie operator*. In the following, we will omit the subscripts here used to distinguish variables before and after the Lie transformation to simplify the notation.

Because of the dependence of the equations of motion in Eq. (24) on the Sadov fast angles through the Jacobi elliptic function, determining the generators of the Lie transformations leading to the desired vector field is extremely complex. However, an analytical estimation can be determined after performing Fourier expansions of the Jacobi elliptic functions as explained in Ref. 14. Indeed, trigonometric functions are easier to handle. Before applying any transformation, it is convenient to extend the set of variables $\boldsymbol{s}$ to

$$\tilde{\boldsymbol{s}} = (\zeta, J_g, J_h, \psi_l, \psi_g, \psi_h, J_M, M),$$

with J_M a dummy action conjugated to the mean anomaly M so that the system of equations of motion becomes autonomous:

$$\frac{\mathrm{d}\tilde{\boldsymbol{s}}}{\mathrm{d}t} = \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \left(\mathcal{H}_{\mathrm{S}} + n J_M + \mathcal{V}_{\mathrm{g}} + \mathcal{V}_{\mathrm{m}} \right) + \tilde{\mathscr{B}} \boldsymbol{M}_{\mathrm{srp}}, \quad \tilde{\mathscr{A}} = \begin{bmatrix} \mathscr{A} & 0 \\ 0 & \mathscr{J} \end{bmatrix}, \tilde{\mathscr{B}} = \begin{bmatrix} \mathscr{B} & 0 \\ 0 & 0 \end{bmatrix},$$

with n the orbital mean motion and

$$\mathscr{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In particular, assuming a fast-rotating satellite, the above system can be re-written as

$$\frac{\mathrm{d}\tilde{\boldsymbol{s}}}{\mathrm{d}t} = \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \mathcal{H}_{\mathrm{S}} + \varepsilon \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \left(n J_M \right) + \varepsilon^2 \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \left(\mathcal{V}_{\mathrm{g}} + \mathcal{V}_{\mathrm{m}} \right) + \varepsilon^2 \tilde{\mathscr{B}} \boldsymbol{M}_{\mathrm{srp}}, \tag{40}$$

where ε is a formal parameter called *book-keeping parameter*, whose numerical value is equal to one and whose powers are used to keep track of the different relative sizes of the terms characterizing the vector field. At this point, three successive Lie transformations can be performed:

1. The first transformation has generator $\boldsymbol{W}_1 = \boldsymbol{W}_{1_{\mathrm{NH}}} + \boldsymbol{W}_{1_{\mathrm{H}}}$, where $\boldsymbol{W}_{1_{\mathrm{NH}}}$ fulfils

$$\tilde{\mathscr{L}}_{\boldsymbol{W}_{1_{\mathrm{NH}}}} \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \mathcal{H}_{\mathrm{S}} + \varepsilon^2 \tilde{\mathscr{B}} \boldsymbol{M}_{\mathrm{srp}} = \varepsilon^2 \boldsymbol{Z}_2, \quad \boldsymbol{Z}_2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \tilde{\mathscr{B}} \boldsymbol{M}_{\mathrm{srp}} \mathrm{d}\psi_g \mathrm{d}\psi_l,$$

   and $\boldsymbol{W}_{1_{\mathrm{H}}} = \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \chi_1$, with χ_1 fulfilling

$$-n_{\psi_l} \frac{\partial \chi_1}{\partial \psi_l} - n_{\psi_l} \frac{\partial \chi_1}{\partial \psi_g} + \varepsilon^2 \left(\mathcal{V}_{\mathrm{g}} + \mathcal{V}_{\mathrm{m}} \right) = \varepsilon^2 \mathcal{Z}_2, \quad \mathcal{Z}_2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left(\mathcal{V}_{\mathrm{g}} + \mathcal{V}_{\mathrm{m}} \right) \mathrm{d}\psi_g \mathrm{d}\psi_l.$$

   The vector field in Eq. 40 is transformed into

$$\frac{\mathrm{d}\tilde{\boldsymbol{s}}^{(1)}}{\mathrm{d}t} = \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \mathcal{H}_{\mathrm{S}} + \varepsilon \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \left(n J_M \right) + \varepsilon^2 \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \mathcal{Z}_2 + \varepsilon^2 \boldsymbol{Z}_2 + \mathcal{O}(\varepsilon^3). \tag{41}$$

2. Let $Z_{2,i}$, $i = 1..8$, be the components of $\boldsymbol{Z}_2$, and let $\overline{Z}_{2,i}$ be their average over M. The second Lie transformation has generator $\boldsymbol{W}_2$ equal to

$$\boldsymbol{W}_2 = \frac{\varepsilon^2}{n_{\psi_g}} \begin{bmatrix} 2(Z_{2,1} - \overline{Z}_{2,1}) \arctan\left(\tan \frac{\psi_g}{2} \right), \\ 2(Z_{2,2} - \overline{Z}_{2,2}) \arctan\left(\tan \frac{\psi_g}{2} \right) \\ 0 \\ \frac{2}{n_{\psi_g}} \left(\frac{\partial n_{\psi_l}}{\partial \zeta} (Z_{2,1} - \overline{Z}_{2,1}) + \frac{\partial n_{\psi_l}}{\partial J_g} (Z_{2,2} - \overline{Z}_{2,2}) \right) \arctan^2\left(\tan \frac{\psi_g}{2} \right) \\ \frac{2}{n_{\psi_g}} \left(\frac{\partial n_{\psi_g}}{\partial \zeta} (Z_{2,1} - \overline{Z}_{2,1}) + \frac{\partial n_{\psi_g}}{\partial J_g} (Z_{2,2} - \overline{Z}_{2,2}) \right) \arctan^2\left(\tan \frac{\psi_g}{2} \right) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

   leading to

$$\frac{\mathrm{d}\tilde{\boldsymbol{s}}^{(2)}}{\mathrm{d}t} = \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \mathcal{H}_{\mathrm{S}} + \varepsilon \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \left(n J_M \right) + \varepsilon^2 \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \mathcal{Z}_2 + \varepsilon^2 \tilde{\boldsymbol{Z}}_2 + \mathcal{O}(\varepsilon^3),$$

   with $\tilde{\boldsymbol{Z}}_2 = \left(\overline{Z}_{2,1}, \overline{Z}_{2,2}, Z_{2,3}, Z_{2,4}, Z_{2,5}, Z_{2,6}, Z_{2,7}, Z_{2,8} \right)^{\mathrm{T}}$.

3. The last transformation has generator $\boldsymbol{W}_3 = \boldsymbol{W}_{3_{\mathrm{NH}}} + \boldsymbol{W}_{3_{\mathrm{H}}} - \overline{\boldsymbol{W}_3}$, where $\boldsymbol{W}_{3_{\mathrm{NH}}}$ fulfils

$$\tilde{\mathscr{L}}_{\boldsymbol{W}_{3_{\mathrm{NH}}}} \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} (n J_M) + \varepsilon^2 \tilde{\boldsymbol{Z}}_2 = \varepsilon^2 \overline{\boldsymbol{Z}}_2,$$

   with $\overline{\boldsymbol{Z}}_2$ a vector of components $\overline{Z}_{2,i}$, $i = 1..8$, $\boldsymbol{W}_{3_{\mathrm{H}}} = \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \chi_3$, with χ_3 fulfilling

$$-n \frac{\partial \chi_3}{\partial M} \varepsilon - + \varepsilon^2 \mathcal{Z}_2 = \varepsilon^2 \overline{\mathcal{Z}}_2, \quad \overline{\mathcal{Z}}_2 = \frac{1}{2\pi} \int_0^{2\pi} \overline{\mathcal{Z}}_2 \mathrm{d}M,$$

   and $\overline{W_3}$ the average of $\boldsymbol{W}_{3_{\mathrm{NH}}} + \boldsymbol{W}_{3_{\mathrm{H}}}$ over M. The vector field is then transformed into

$$\frac{\mathrm{d}\tilde{\boldsymbol{s}}^{(3)}}{\mathrm{d}t} = \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \mathcal{H}_{\mathrm{S}} + \varepsilon \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \left(n J_M \right) + \varepsilon^2 \tilde{\mathscr{A}} \nabla_{\tilde{\boldsymbol{s}}} \overline{\mathcal{Z}}_2 + \varepsilon^2 \overline{\boldsymbol{Z}}_2 + \mathcal{O}(\varepsilon^3). \tag{42}$$

Neglecting the terms above the second order in the Lie transformation (39), the mean variables can be computed as

$$\overline{\tilde{\boldsymbol{s}}} = \tilde{\boldsymbol{s}} - \boldsymbol{W}_1(\tilde{\boldsymbol{s}}) - \boldsymbol{W}_2\left(\tilde{\boldsymbol{s}} - \boldsymbol{W}_1(\tilde{\boldsymbol{s}})\right) - \boldsymbol{W}_3\left(\tilde{\boldsymbol{s}} - \boldsymbol{W}_1(\tilde{\boldsymbol{s}}) - \boldsymbol{W}_2\left(\tilde{\boldsymbol{s}} - \boldsymbol{W}_1(\tilde{\boldsymbol{s}})\right)\right).$$

The approach described is useful also to estimate the remainder, i.e. the difference between the original vector field and the semi-analytical model. This information could be used to determine and add higher-order terms in the semi-analytical model, increasing its accuracy.

For satellites on LEOs, it is necessary to consider the perturbation by the aerodynamic torque and the above procedure is not suitable. Indeed, the average of the corresponding term in the equations of motion is partially performed numerically. In this case or if the hypothesis of a fast-rotating satellite is not appropriate, the initial values of the mean variables can be estimated through the numerical procedure described in Ref. 21.

Singularities

The equations of motion in Eq. (24) have singularities. In particular, the time derivatives of ψ_l and ψ_g have a singularity when either $m = 0$ or $m = 1$; the time derivatives of ψ_g and ψ_g have a singularity if $\sin\tilde{\delta} = 0$.

The first singularity $m = 0$ is peculiar as it disappears from the equation of motions of the semi-analytical model. These equations still present terms with m at the denominator, but their limit for m tending to zero is finite. For example, in the semi-analytical model, the component of the time derivative of ψ_l due to the magnetic torques is

$$\left[\frac{\mathrm{d}\psi_l}{\mathrm{d}t}\right]_{\mathrm{m}} = \Psi\left(J_g, J_h, \psi_h, a, P_1, P_2, Q_1, Q_2; \mu_m, \boldsymbol{I}_{\mathrm{m}}\right) \frac{\kappa\,(m-1)\,K(m) + E(m)}{K(m)^3 m};$$

and, by means of L'Hôpital's rule, it results

$$\lim_{m\to 0} \frac{\kappa\,(m-1)\,K(m) + E(m)}{K(m)^3 m} = \frac{2\kappa + 4}{\pi^2}.$$

The singularity $m = 1$ occurs when $J_d = B$. In the free-torque problem, relation $J_d = B$ holds along the separatrix between the SAM and the LAM. Consider the free-torque problem expressed in the Andoyer-Serret variables with Hamiltonian (1). Fig. 1 shows the phase portrait represented in the (l, L) plane. There exist two elliptic fixed points at $(l, L) = (\pi/2, 0)$ and $(l, L) = (3\pi/2, 0)$, corresponding to stable periodic orbits. The hyperbolic fixed point at $(l, L) = (0, 0)$ and $(l, L) = (\pi, 0)$ correspond to unstable periodic orbits. The stable and unstable invariant manifolds of the unstable periodic orbits coincide and form the separatrix between LAM and SAM. The LAM corresponds to the region with the librational tori (closed curves) and the SAM corresponds to the region with the rotational tori. When perturbations are introduced, the region of the phase space in the neighborhood of the separatrix becomes chaotic. For initial conditions in this region, the semi-analytical model is quantitatively inaccurate. Since it is not suitable anymore, the singularity $m = 1$ does not present an issue.

The last singularity, $\sin\tilde{\delta} = 0$, can be handled by introducing alternative variables obtained by a suitable combination of the Sadov variables. A possible set is $(\zeta, J_g, J_h, \psi_l, J_5, J_6, J_7)$ with

$$J_5 = \psi_g + \frac{J_h}{J_g}\psi_h, \quad J_6 = \sqrt{J_g^2 - J_h^2}\cos\psi_h, \quad J_7 = \sqrt{J_g^2 - J_h^2}\sin\psi_h. \tag{43}$$

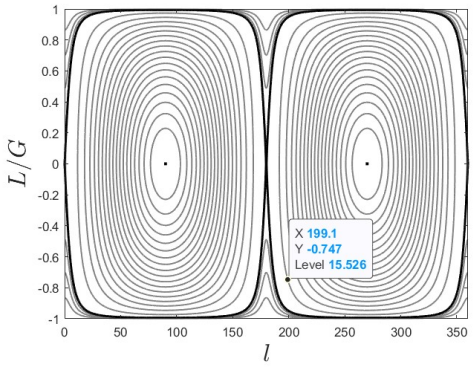

Figure 1: Phase portrait of the free-torque problem

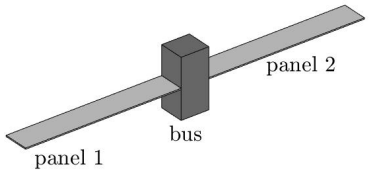

Figure 2: Fictitious satellite employed in the numerical simulations

Let us remark that the use of these variables is necessary only when the initial conditions are such that the value of $\sin\tilde{\delta}$ is close to zero. Indeed, this singularity is not associated with any attractive equilibrium state.

NUMERICAL SIMULATIONS

For the numerical tests, the fictitious satellite shown in Fig. 2 is employed. Its mass, moments of inertia and intrinsic magnetic moments are

$$m = 733 \text{ kg}, \quad A = 385.716 \text{ kg m}^2, \quad B = 2769.143 \text{ kg m}^2, \quad C = 3007.037 \text{ kg m}^2,$$

$$\boldsymbol{I}_{\text{m}} = [0.1,\ 1,\ 1]\text{A m}^2.$$

Its bus has a total reflectivity $\rho_i = 0.6$ and the fraction s_i of ρ_i which is specular is $s_i = 1$; the front of panels 1 and 2 has $\rho_i = 0.27$ and $s_i = 1$; the back of panels 1 and 2 has $\rho_i = 0.07$ and $s_i = 0$. In the first test, we consider the satellite along a Keplerian orbit with equinoctial elements

$$a = 20000\,\text{km},\ P_1 = 8.910{\cdot}10^{-2},\ P_2 = 4.540{\cdot}10^{-2},\ Q_1 = 4.605{\cdot}10^{-1},\ Q_2 = 2.659{\cdot}10^{-1}, \quad (44)$$

perturbed by only the gravity and magnetic torques. In the second test, we consider the satellite along a Keplerian orbit with equinoctial elements

$$a = 29600\,\text{km},\ P_1 = 8.910{\cdot}10^{-4},\ P_2 = 4.540{\cdot}10^{-4},\ Q_1 = 4.605{\cdot}10^{-1},\ Q_2 = 2.659{\cdot}10^{-1}, \quad (45)$$

perturbed by the gravity, magnetic, and solar radiation pressure torques. In the third test, the satellite is along a Keplerian orbit with equinoctial elements

$$a = 7500\,\text{km},\ P_1 = 2.670{\cdot}10^{-2},\ P_2 = 1.360{\cdot}10^{-2},\ Q_1 = 4.605{\cdot}10^{-1},\ Q_2 = 2.659{\cdot}10^{-1}, \quad (46)$$

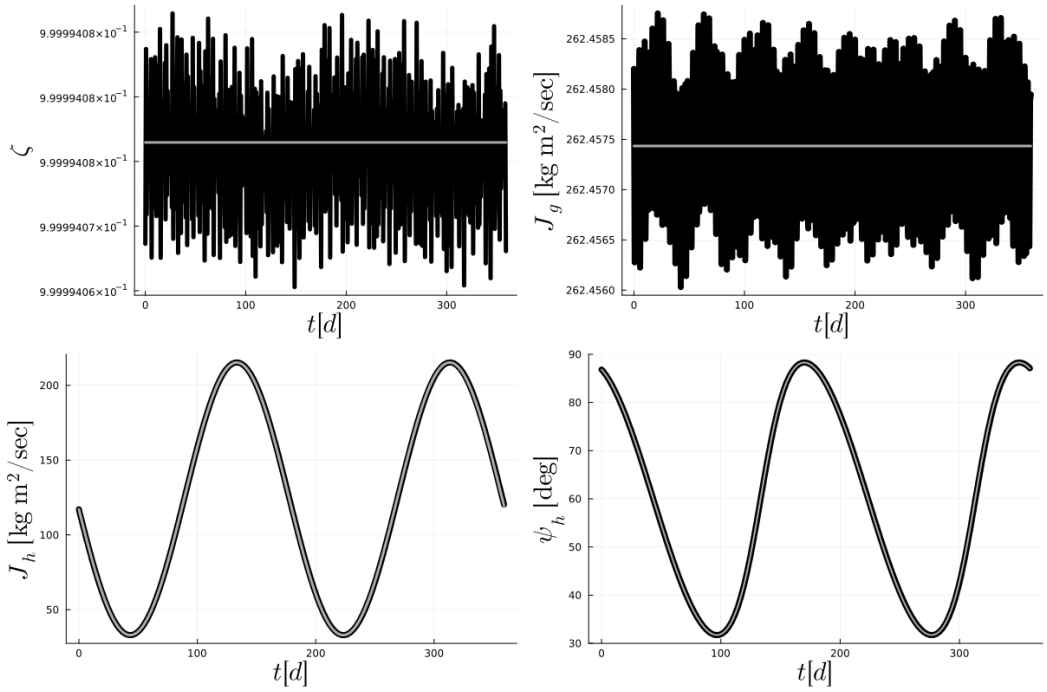

Figure 3: Evolution of the slow attitude variables computed by means of an osculating propagator (black line) and the semi-analytical propagator (grey line) for the problem with perturbations by the gravity and magnetic torques with initial conditions (44) and (47).

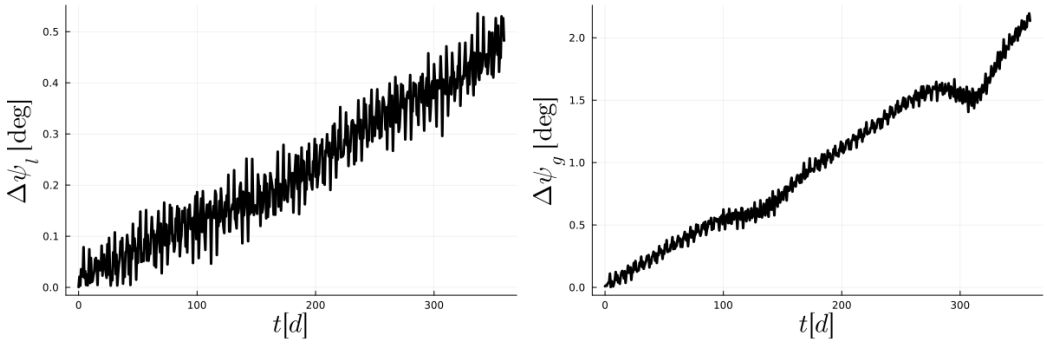

Figure 4: Evolution of the error of the fast Sadov angle for the same test in Fig. 3.

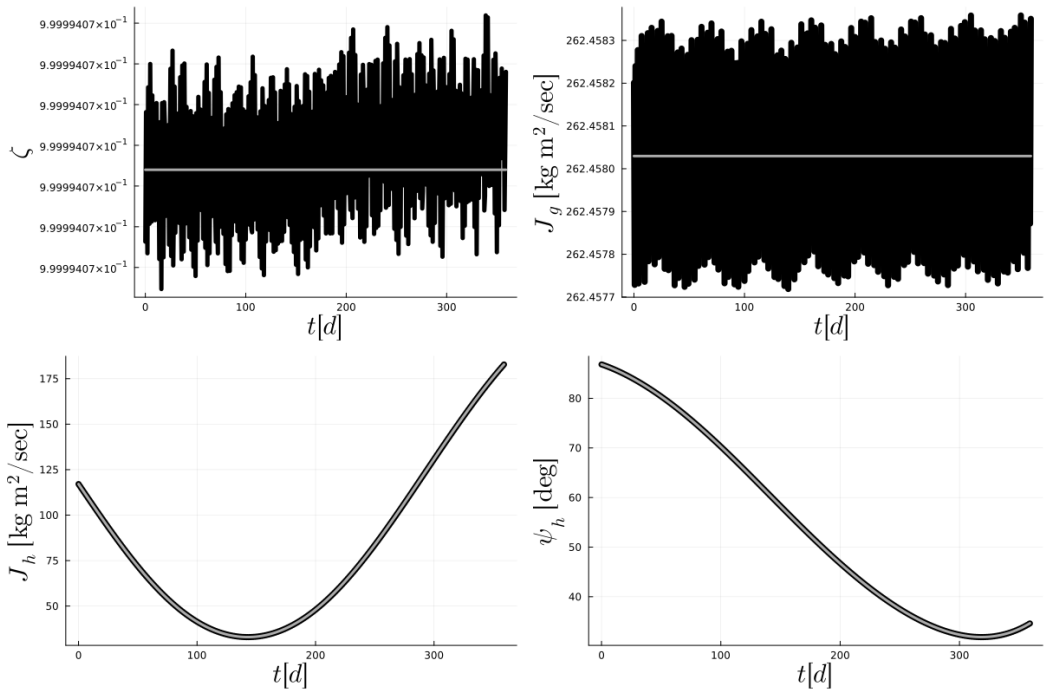

Figure 5: Evolution of the slow attitude variables computed by means of an osculating propagator (black line) and the semi-analytical propagator (grey line) for the problem with perturbations by the gravity, magnetic and solar radiation pressure torques with initial conditions (45) and (47).

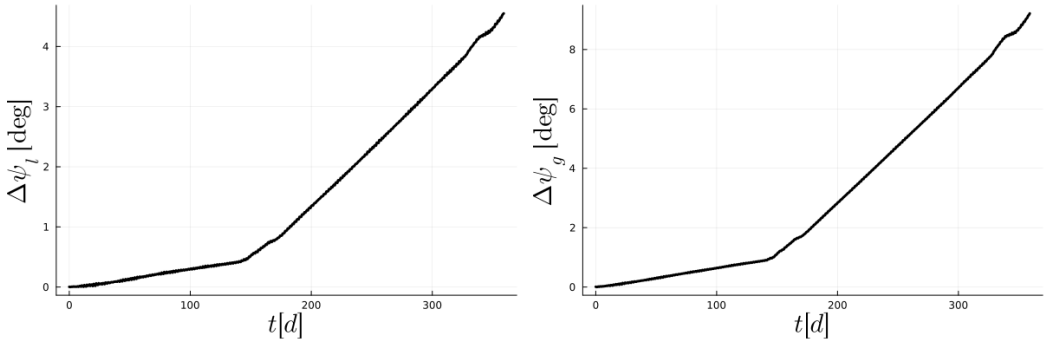

Figure 6: Evolution of the error of the fast Sadov angle for the same test in Fig. 5.

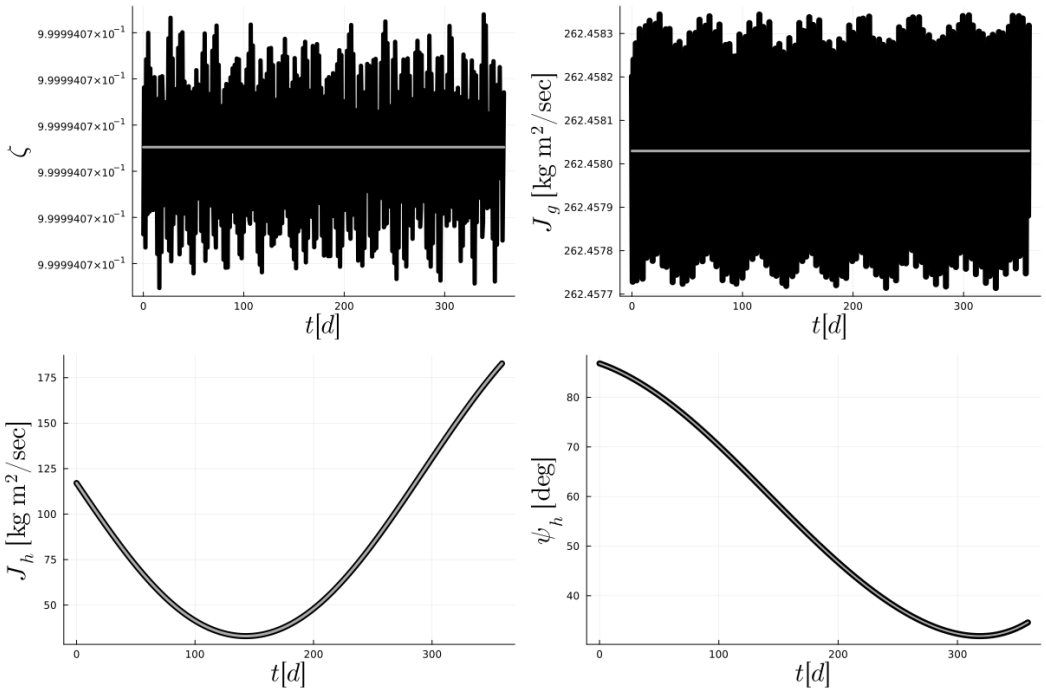

Figure 7: Evolution of the slow attitude variables computed by means of an osculating propagator (black line) and the semi-analytical propagator (grey line) for the problem with perturbations by the gravity, magnetic and solar radiation pressure torques with initial conditions (45) and (47), not taking into account the shadowing effects.

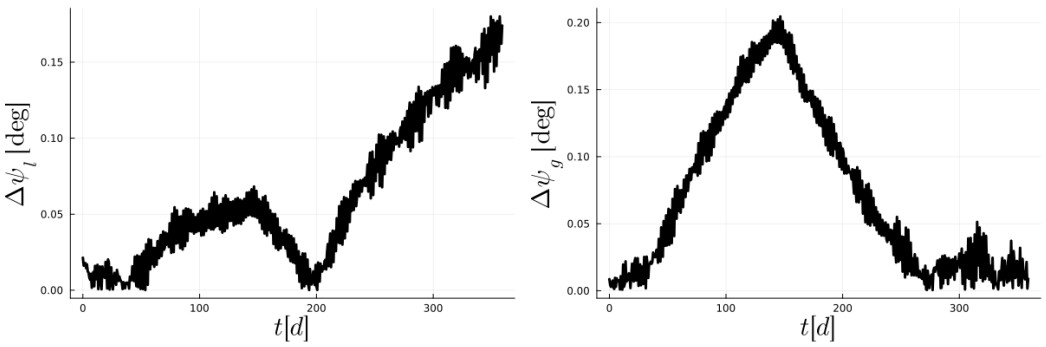

Figure 8: Evolution of the error of the fast Sadov angle for the same test in Fig. 7.

perturbed by the gravity, magnetic, and aerodynamic torques. In all three tests, the satellite has randomly selected initial attitude variables equal to

$$\begin{aligned} \zeta = 0.999994, \quad J_g = 262.458\,\mathrm{kg\,m^2/sec}, \quad J_h = 117.085\,\mathrm{kg\,m^2/sec}, \\ \psi_l = 301.779\,\mathrm{deg}, \quad \psi_g = 294.2137\,\mathrm{deg}, \quad \psi_h = 86.8244\,\mathrm{deg}. \end{aligned} \tag{47}$$

The outcomes of the semi-analytical propagator are compared with those of an osculating propagator built by using the same models of the external torques. As done in Ref. 2, the osculating propagation is performed by employing Euler's equations and quaternions. The simulated period of time is 360 days.

The outcomes of the first test are shown in Fig. 3 and 4. Those of the second test are shown in Fig. 5 and 6. Those of the third test are shown in Fig. 9 and 10. Looking at the evolution of the slow variables $(\zeta, J_g, J_h, \psi_h)$, the accuracy is good over the whole time of propagation. For the first test, this holds true also concerning the variables ψ_l and ψ_g. Instead, in the case of the second and third tests, the evolution of ψ_l and ψ_g is affected by a fast-increasing error. In the second test, after almost one year, the phase shift is still acceptable, but it becomes significant in the third test. Probably, this is due to the largest values of the perturbations at lower altitudes.

The error in the second test is mainly caused by the Earth's shadow effects. Indeed, neglecting the effects of the Earth's shadow, i.e. not multiplying the solar radiation pressure torque by the shadow function $v_s(\Upsilon)$ in Eq. (30), the results significantly improve, as shown in Fig. 7 and 8.

In order to increase the accuracy in the evolution of the fast angles, higher-order corrections can be included in the model. As proof, we determine higher-order corrections related to the perturbations by the gravity and the magnetic torques and we applied them in the third test. For this purpose, we employ the Lie approach to estimate the remainder of book-keeping order equal to 3 in the transformed vector field (42). Then, the average of this remainder term over the fast angles (ψ_l, ψ_g, M) is computed and included in the semi-analytical model. Fig. 11 and 12 show the improved results obtained by adding the higher-order corrections. One can notice, considering the evolution of the fast angles error, that the accuracy is significantly increased.

In terms of computational time, the osculating propagator takes about 15 minutes for the first test, 3 hours for the second test, and 1.75 hours for the third test. On the contrary, the semi-analytical propagator takes 21 seconds for the first test and $96.8\,\mathrm{sec}$ for the second test. Concerning the third test, the computational time is $146\,\mathrm{sec}$ without the higher-order corrections and $210\,\mathrm{sec}$ with the higher-order corrections.

CONCLUSIONS AND FUTURE WORK

A semi-analytical model for the attitude dynamics of Earth satellites is developed by employing the attitude Sadov variables and exploiting a Lie series approach, whenever it is possible. The tests performed show the advantage of using the semi-analytical model in terms of computational cost. Qualitatively, the attitude dynamics is well reproduced over long periods of time. Quantitatively, the evolution of the slow attitude variables is typically well reproduced. The error grows faster and may become significant when considering the evolution of the Sadov fast angles. We proved that this trend can be slowed down by introducing higher-order corrections in the semi-analytical model. These corrections are already computed and implemented for the perturbations by the gravity and magnetic torques. In the future, we plan to compute similar corrections also for the perturbations by the solar radiation pressure and the aerodynamic torques. The current semi-analytical model is

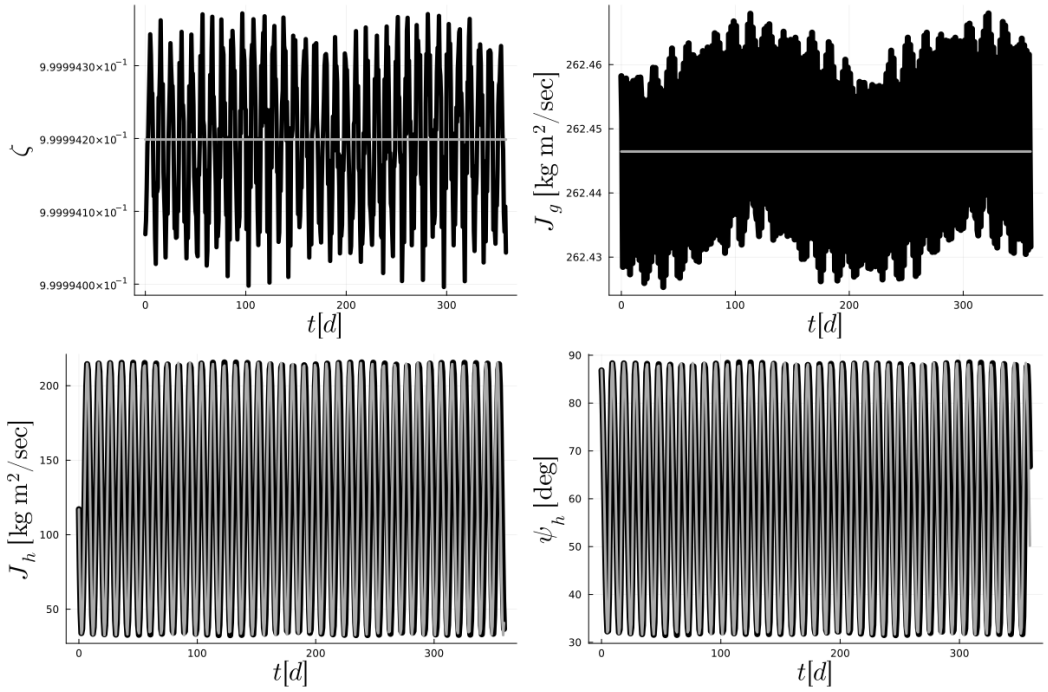

Figure 9: Evolution of the slow attitude variables computed by means of an osculating propagator (black line) and the semi-analytical propagator (grey line) for the problem with perturbations by the gravity, magnetic and aerodynamic torques with initial conditions (46) and (47).

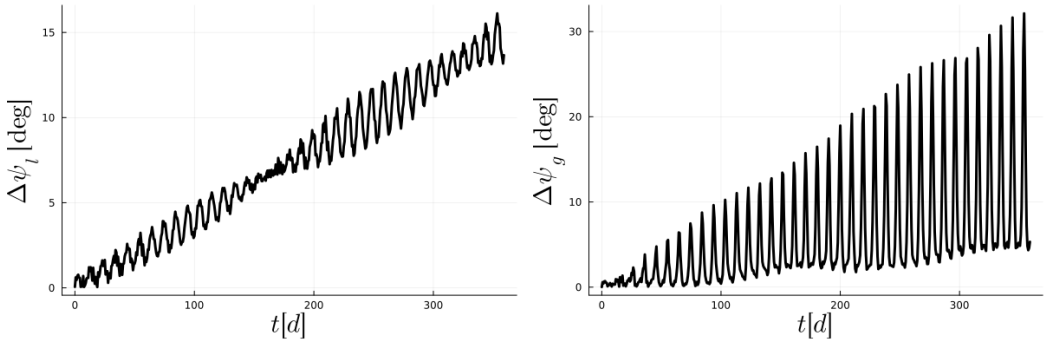

Figure 10: Evolution of the error of the fast Sadov angle for the same test in Fig. 9.

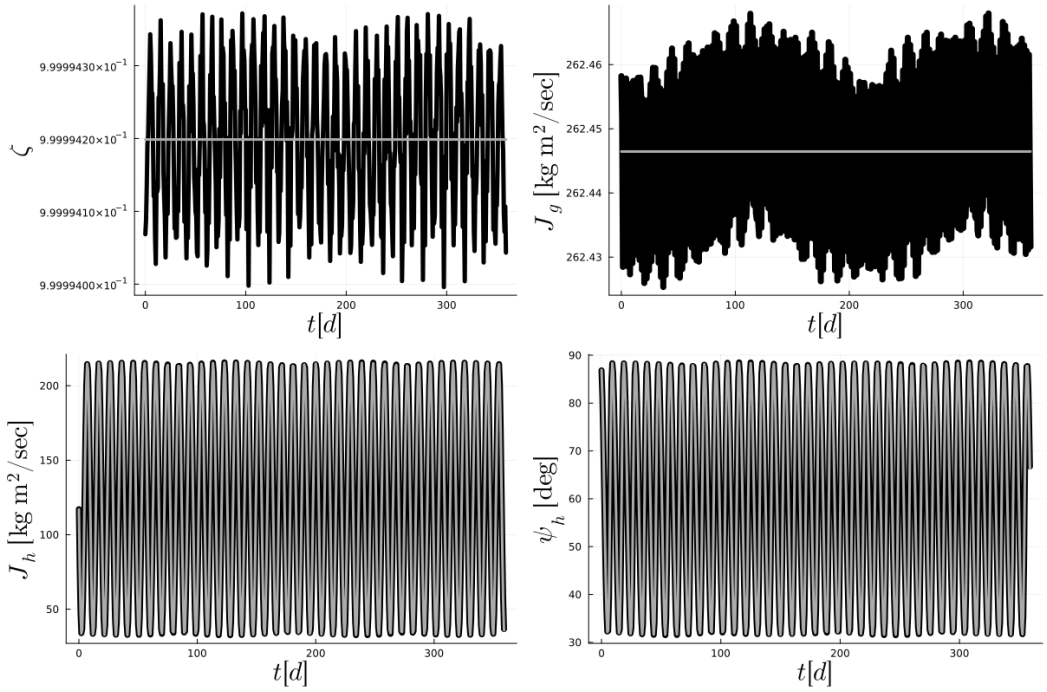

Figure 11: Evolution of the slow attitude variables computed by means of an osculating propagator (black line) and the semi-analytical propagator with higher-order corrections (grey line) for the same problem of Fig. 9.

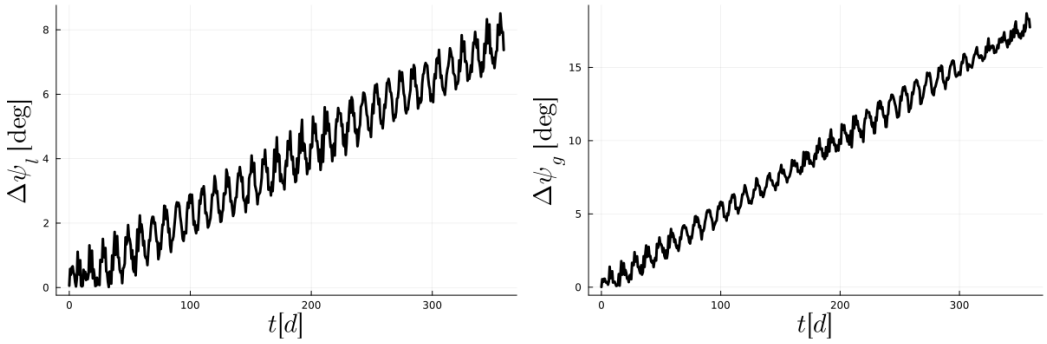

Figure 12: Evolution of the error of the fast Sadov angle for the same test in Fig. 11.

developed by assuming the space object along Keplerian orbits. The next step is to integrate also the effects of the main perturbations affecting the satellite's orbit.

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