This is a peer-reviewed, accepted author manuscript of the following research article: Xu, H & Mao, X 2024, 'Razumikhin technique for stabilisation of highly nonlinear hybrid systems by bounded discrete-time state feedback control working intermittently', Numerical Algebra, Control and Optimization. https://doi.org/10.3934/naco.2024003

RAZUMIKHIN TECHNIQUE FOR STABILISATION OF HIGHLY NONLINEAR HYBRID SYSTEMS BY BOUNDED DISCRETE-TIME STATE FEEDBACK CONTROL WORKING INTERMITTENTLY

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Dedicated to Professor George Yin on the Occasion of his 70th Birthday

(Communicated by Handling Editor)

Abstract. This paper applies the Razumikhin idea to study the stabilisation of hybrid stochastic systems by discrete-time state feedback control, which works intermittently and is designed boundedly. Theoretically, the Razumikhin method is generalised in view of time-varying functions, rather than constants, where the time-inhomogeneous property of intermittent control could be fully made use of. In practice, the control cost could be reduced significantly since the controller is bounded, not observed continuously and having rest time. Moreover, there will be a wider range of applications especially for models that do not satisfy the linear growth condition (say highly nonlinear). An example of the coupled Van der Pol–Duffing oscillator system is hence provided to show the practicability of the developed theory.

1. **Introduction.** Mao in [1] initialed the study of using a state feedback control control $u(x(t_\tau), t, r(t))$, where $t_\tau = [t/\tau]\tau$, which is based on the state observations at discrete times, $0, \tau, 2\tau, \cdots$, to stabilise an unstable hybrid stochastic differential equation (SDE)

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t),$$

where $x(t) \in \mathbb{R}^d$ is the system state, $r(t)$ is a Markov chain taking values in a finite space, $W(t)$ is a Brownian motion, $[t/\tau]$ is the integer part of t/τ. Compared with the classical feedback control $u(x(t), t, r(t))$, for which we need to observe the system states continuously, discrete-time state feedback control $u(x(t_\tau), t, r(t))$ is more practical and less costly. After several years' development, there have been many results on this stabilisation problem (see, e.g. [2, 3, 4, 5, 6, 7, 8, 9, 10]).

Although the discrete-time state feedback control is more advanced than before, there are still two issues deserved our attention that could help improve the control design. Firstly, it should be pointed out that the control function $u(x, t, i)$ in many papers such as [1, 3, 6, 7] is usually designed on every observable discrete-time state, such as the linear form $\mu_i x$ with $\mu_1 = -3$ and $\mu_2 = 0$ in the first example of [7].

2020 *Mathematics Subject Classification.* Primary: 93D15, 93D20; Secondary: 93E03.
Key words and phrases. Razumikhin technique, intermittent control, bounded feedback control, discrete-time observations, highly nonlinear systems.

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But this sometimes would lead to some unnecessary control cost since the cost is in general proportional to the system state value $|x(t_\tau)|$. In particular, if the initial data is given large, the cost on the beginning stage will be relatively high. This then begs a question: is it really necessary to impose control on every discrete-time state? The answer at least in this paper is negative, and we will propose a scheme where the control function is designed in a bounded state area.

The second one is intermittent control strategy. In fact, most of the discrete-state-feedback stabilisation results (see, e.g. [2, 5, 6, 7, 8]) are based on the controller imposed to the system for all the time without any rest. This undoubtedly will shorten the life of our controller. Therefore, a more practical technique is to let the controller working intermittently, where we divide the whole time periodically, and each period is consisted of working time and rest time. Then, the controller becomes $u(x(t_\tau), t, r(t))I(t)$ with $I(t) = \sum_{k=0}^{\infty} 1_{[kT, kT+\delta T)}(t)$, where $T > 0$ is the control period, δT is the working width with strength $\delta \in (0, 1)$.

Due to its efficiency in reducing the control cost, intermittent control recently has drawn abundant interest (see, e.g. [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). But unfortunately, due to the difficulty in dealing with two discrete strategies, $x(t_\tau)$ and $I(t)$, at the same time, there are only a few results (e.g. [9, 10, 16, 17, 18]) considering them together. Until now, the comparison idea has been proven to be the most helpful method to study the intermittent discrete-time state feedback control. However, the observation duration τ obtained by using this method is always awesome. Even worse, it only works well when the underlying hybrid SDEs are globally Lipschitz continuous (see Appendix in [19]), which might exclude many important practical systems such as stochastic volatility model [20, 21], biological system [22, 23], and oscillator network [15]. It is hence necessary to develop new techniques to deal with this stabilisation problem to improve the value of τ and cover more general models.

Among others, in the classical discrete-state-feedback stabilisation theory, Razumikhin technique has received much attention for these two purposes. For example, by the Razumikhin technique, a better τ was obtained in [6] than that in [1] (using the comparison idea). This method was first applied in [7] to the highly nonlinear hybrid SDEs, which did not satisfy the linear growth condition. Could we then generalise the Razumikhin method to the stabilization problem of intermittent discrete-state feedback control? We will give a positive answer to this question. But it should be pointed out that most of the existing Razumikhin results (e.g. [7, 24]) might not be used to the intermittent control problem directly. This is because the following fundamental assumption cannot be met

$$E\mathcal{L}U(x(t), x(t_\tau), t, r(t)) \leq -\lambda_1 EU(x(t), t, r(t))$$

if for some t, $\sup_{-\tau \leq \theta \leq 0} EU(x(t+\theta), t+\theta, r(t+\theta)) \leq qEU(x(t), t, r(t))$. The detailed explanation of this condition will be given later. On the one hand, $I(t)$ is a piece-wise constant function, which could not be considered into the construction of continuous Lyapunov function $U(x, t, i)$. On the other hand, λ_1 is time-homogeneous, which would let the time-varying property of $I(t)$ be ignored. Therefore, it is wiser to establish the Razumikhin theory based on the function $\lambda_1(t)$ rather than the constant λ_1. This change will make our stability analysis more technical than before.

In conclusion, this paper will be devoted to the stabilisation problem of highly nonlinear hybrid SDEs by discrete-time state feedback control, which is designed intermittently and boundedly, based on [1, 7, 9]. The main method we will use is the Razumikhin technique developed in [7], but will be further generalised to take advantage of the time-inhomogeneous property of the systems. We will also give an application example to demonstrate the availability of our proposed results.

2. Model description.

2.1. Preliminary.

We firstly list the notations widely used in this paper. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space with Euclidean norm $|\cdot|$, and $\mathbb{R}_+$ be the collection of all non-negative numbers. For any real constant a, b, denote by $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. If A is a vector or matrix, A^{T} is its transpose. If A is a matrix, $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$ is its trace norm. If A is a symmetric real-valued matrix, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. Let $\mathcal{I}_d$ be the d-dimensioanl identity matrix. For a subset F_1 included in some universal set F, 1_{F_1} denotes its indicator function, that is, $1_{F_1}(a) = 1$ if $a \in F_1$, otherwise, 0. There are also some positive constants whose specific forms are useless, which for simplicity are denoted by C, regardless of their values.

We will work on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing, right-continuous and $\mathcal{F}_0$ contains all P-null sets). Let $W(t) = (W_1(t), \cdots, W_m(t))^{\mathrm{T}}$ be an m-dimensional Brownian motion, and $r(t)$ a right-continuous Markov chain taking values in a finite state space $\mathbb{S} = \{1, \cdots, S\}$ with transition rate matrix $Q = (q_{ij})_{S\times S}$ given by

$$P\left(r(t+\Delta) = j | r(t) = i\right) = \begin{cases} 1 + q_{ij}\Delta + o(\Delta), & \text{if } i = j, \\ q_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \end{cases}$$

as $\Delta \downarrow 0$. Here $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $q_{ii} = -\sum_{j\neq i} q_{ij}$. Assume that the Brownian motion $W(t)$ and the Markov chain $r(t)$ are independent under the probability measure P.

2.2. Standing hypotheses.

Consider the hybrid SDE

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t) \tag{1}$$

on $t \in \mathbb{R}_+$ with initial data $x(0) = x_0 \in \mathbb{R}^d$ and $r(0) = r_0 \in \mathbb{S}$. The drift coefficient $f : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^d$ and the diffusion coefficient $g : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{d\times m}$ should be locally Lipschitz continuous (see Theorem 3.15, [24]), and satisfty the following polynomial growth condition.

Assumption 2.1. Assume that there are non-negative constants L_1, L_2, and $p > 1$ such that

$$|f(x, t, i)| \leq L_1|x| + L_2|x|^p, \ \forall (x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}. \tag{2}$$

Assumption 2 is clearly more advanced than the classical linear growth condition, which is used to restrict the growth of f. But it cannot guarantee the hybrid SDE (1) has a unique global solution. For this aim, the following Khasminskii-type condition is always needed.

Assumption 2.2. Assume that there are a pair of constants $\hat{\alpha} > 0$ and $q \geq 3p - 1$ such that for any $(x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$

$$x^{\mathrm{T}} f(x, t, i) + \frac{q-1}{2}|g(x, t, i)|^2 \leq \hat{\alpha}|x|^2. \tag{3}$$

Remark 2.3. Here the reader might wonder whether it is possible for g to go very rapidly since we only impose growth condition on f. This worry in fact could be denied if we combine Assumptions 2.1 with 2.2. It is easy to calculate for any $(x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$

$$|g(x,t,i)|^2 \leq \frac{2}{q-1}\left(\hat{\alpha}|x|^2 + |x|\left(L_1|x| + L_2|x|^p\right)\right) \leq C\left(|x|^2 + |x|^{p+1}\right). \tag{4}$$

This observation means that $g(x, t, i)$ is also controlled by a polynomial.

According to Theorem 3.19 in [24], we show that under Assumptions 2.1 and 2.2, the hybrid SDE (1) with initial data x_0 and r_0 has a unique solution $x(t)$, satisfying the property that for any $t \in \mathbb{R}_+$

$$\sup_{0 \leq s \leq t} E\left(|x(s)|^q\right) < \infty. \tag{5}$$

Before closing this part, we introduce an additional assumption on the underlying SDE (1), which will play a vital role for our control design later.

Assumption 2.4. For each $i \in \mathbb{S}$, assume that there are non-negative constants α_i, $\bar{\alpha}_i$, and positive constants β_i, $\bar{\beta}_i$ such that for any $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$

$$\begin{cases} x^{\mathrm{T}} f(x,t,i) + \dfrac{1}{2}|g(x,t,i)|^2 \leq \alpha_i|x|^2 - \beta_i|x|^{p+1}, \\ x^{\mathrm{T}} f(x,t,i) + \dfrac{p}{2}|g(x,t,i)|^2 \leq \bar{\alpha}_i|x|^2 - \bar{\beta}_i|x|^{p+1}. \end{cases} \tag{6}$$

The reader might find Assumptions 2.2 and 2.4 look very similar. Is it necessary to give them at the same time? The following remark may be helpful.

Remark 2.5. It should be pointed out that Assumptions 2.2 and 2.4 are both needed. At first, they play different roles. Assumption 2.2 is used to ensure the existence of global solution with certain moment properties, while Assumption 2.4 is for stabilisation and control design. Secondly, these two conditions are quite different, and any one cannot be deduced from the other. Let us use a scalar example operating just in one mode to explain this.

Case 1: $f(x, t, 1) = x - x^3 \sin^2(x)$, $g(x, t, 1) = x$. Then Assumption 2.2 is satisfied with $p = 3$, $q = 8$, $\hat{\alpha} = 3.5$. But we could not find a $\beta_1 > 0$ so that for all $x \in \mathbb{R}$

$$x^{\mathrm{T}} f(x,t,1) + \frac{1}{2}|g(x,t,1)|^2 = 1.5|x|^2 - |x|^4 \sin^2(x) \leq 1.5|x|^2 - \beta_1|x|^4.$$

Case 2: $f(x, t, 1) = x - 2x^3$, $g(x, t, 1) = x^2$. It is easy to verify that Assumption 2.4 holds with $\alpha_1 = 1$, $\beta_1 = 1.5$, $\bar{\alpha} = 1$, $\hat{\beta} = 0.5$. But Assumption 2.2 is not satisfied since

$$x^{\mathrm{T}} f(x,t,1) + \frac{q}{2}|g(x,t,1)|^2 = |x|^2 + \left(\frac{q-1}{2} - 2\right)|x|^4,$$

where $\frac{q-1}{2} - 2 \geq 1.5$, as $q \geq 3p - 1 = 8$.

2.3. **Control design.** But (5) does not indicate the underlying SDE (1) is stable. If not, we would like to design a state feedback control based on discrete-time observations working intermittently, $u(x(t_\tau), t, r(t))I(t)$, to make the controlled SDE

$$\mathrm{d}x(t) = \Big(f(x(t), t, r(t)) + u(x(t_\tau), t, r(t))I(t)\Big)\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t) \tag{7}$$

become stable.

Next, we will explain how to design the bounded control function $u(x,t,i)$. For convenience, we denote by $B_a = \{x \in \mathbb{R}^d : |x| \le a\}$, $B_a^c = \{x \in \mathbb{R}^d : |x| > a\}$, $B_b - B_a = \{x \in \mathbb{R}^d : a < |x| \le b\}$ for any $0 < a < b$.

Rule 2.6. Choose non-negative constants $\gamma_i (i \in \mathbb{S})$ such that

$$\begin{cases} \mathcal{A} := -2\mathrm{diag}\,(\alpha_1 - \gamma_1, \cdots, \alpha_N - \gamma_S) - Q \\ \bar{\mathcal{A}} := -(p+1)\mathrm{diag}\,(\bar{\alpha}_1 - \gamma_1, \cdots, \bar{\alpha}_S - \gamma_S) - Q \end{cases}$$

are non-singular M-matrices. Then for each $i \in \mathbb{S}$, set $R_i = \left(\frac{2\gamma_i}{\beta_i \wedge \beta_i}\right)^{\frac{1}{p-1}}$. The control function can be designed as:

- when $x \in B_{R_i}$, design $u(x,t,i)$ such that we can find a non-negative constant K_i to let for any $(x,y,t) \in B_{R_i} \times B_{R_i} \times \mathbb{R}_+$
$$|u(x,t,i) - u(y,t,i)| \le K_i |x-y| \tag{8}$$
and
$$x^{\mathrm{T}} u(x,t,i) \le -\gamma_i |x|^2 \tag{9}$$
hold, and moreover $u(0,t,i) = 0$ for all $t \in \mathbb{R}_+$;
- when $x \in B_{2R_i} - B_{R_i}$, let $u(x,t,i) = u\left(\left(\frac{2R_i}{|x|} - 1\right)x, t, i\right)$ for all $t \in \mathbb{R}_+$;
- when $x \in B^c_{2R_i}$, let $u(x,t,i) = 0$ for all $t \in \mathbb{R}_+$.

Remark 2.7. From Rule 2.6, we see that the control is only imposed in a bounded state area B_{2R_i}. Such a control is much smaller since R_i was set to infinity before (e.g., [1, 7]), so that the control cost could be reduced significantly, especially for large system states. In fact, we could let $u(x,t,i) = 0$ for $x \in B^c_{R_i}$. But the additional connect area $B_{2R_i} - B_{R_i}$ is required for the purpose of continuity of $u(x,t,i)$ in x to guarantee the existence of unique global solution of the controlled system (1), which is stated as the following lemma.

Lemma 2.8. *Let Rule 2.6 hold. Let $K_M = \max_{i \in \mathbb{S}} K_i$. Then for all $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,*
$$|u(x,t,i) - u(y,t,i)| \le K_M |x-y|. \tag{10}$$
Furthermore, $u(0,t,i) = 0$ for all $(t,i) \in \mathbb{R}_+ \times \mathbb{S}$.

The proof of Lemma 2.8 can be found in the Appendix. It also yields the following linear growth condition
$$|u(x,t,i)| \le K_M |x|, \ \forall (x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}. \tag{11}$$
In addition, (2) and (4) tells us that $f(0,t,i) \equiv 0$ and $g(0,t,i) \equiv 0$. As a result, we could see that the controlled SDE (7) admits a zero solution.

Theorem 2.9. *Let Assumptions 2.1, 2.2, 2.4 hold. Design the control function to meet Rule 2.6. Then for any initial data x_0 and r_0, there exists a unique global solution $x(t)$ of the controlled SDE (7). Moreover it satisfies that for any $t \in \mathbb{R}_+$*
$$\sup_{0 \le s \le t} E\left(|x(s)|^q\right) < \infty \tag{12}$$
and
$$E\left(\sup_{0 \le s \le t} |x(s)|^{2p}\right) < \infty. \tag{13}$$

The existence and uniqueness of $x(t)$ as well as property (12) can be shown in the similar way of proofing Theorem 7.13 in [24]. Property (13) can be proved using the same procedure of proofing Lemma 3.1 in [7]. So we omit them.

Before closing this section, we give some comments on the availability of our control scheme.

Remark 2.10. According to Rule 2.6, the control design for $x \in B^c_{R_i}$ is very clear. The remaining question is whether we could find appropriate γ_i and control function $u(x,t,i)$ in B_{R_i}. There are actually many choices. For example, it is easy to find a positive constant γ large enough such that $-2\text{diag}\,(\alpha_1-\gamma,\cdots,\alpha_S-\gamma)-Q$ and $-(p+1)\text{diag}\,(\bar\alpha_1-\gamma,\cdots,\bar\alpha_S-\gamma)-Q$ are non-singular M-matrices. Therefore, we can let $\gamma_i=\gamma$ for each $i\in\mathbb{S}$. Next, use control function in the linear form $u(x,t,i)=-\mathcal{B}_i x$ with $\mathcal{B}_i$ being symmetric and positive-definite such that $\lambda_{\min}(\mathcal{B}_i)\ge\gamma$. Then we see that conditions (8) and (9) are satisfied.

3. Stabilisation results.

3.1. Lyapunov function.

Define the Lyapunov function $U:\mathbb{R}^d\times\mathbb{S}\to\mathbb{R}_+$ by

$$U(x,i)=\eta_i|x|^2+\bar\eta_i|x|^{p+1},$$

where $(\eta_1,\cdots,\eta_S)^{\mathrm{T}}=\mathcal{A}^{-1}(1,\cdots,1)^{\mathrm{T}}$ and $(\bar\eta_1,\cdots,\bar\eta_S)^{\mathrm{T}}=\bar{\mathcal{A}}^{-1}(1,\cdots,1)^{\mathrm{T}}$. Since $\mathcal{A}$, $\bar{\mathcal{A}}$ are non-singular M-matrices, all η_i, $\bar\eta_i$ are positive. Then define the operator $\mathcal{L}U(x,y,t,i):\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{S}\to\mathbb{R}$ with respect to the controlled SDE (7) by

$$\mathcal{L}U(x,y,t,i)=LU(x,t,i)+\bar U(x,y,t,i)$$

where

$$\begin{aligned}LU(x,t,i)=&U_x(x,i)(f(x,t,i)+u(x,t,i)I(t))\\&+\frac{1}{2}\text{trace}\left(g^{\mathrm{T}}(x,t,i)U_{xx}(x,i)g(x,t,i)\right)+\sum_{j=1}^S q_{ij}U(x,j)\end{aligned}$$

and

$$\bar U(x,y,t,i)=U_x(x,i)(u(y,t,i)-u(x,t,i))I(t)$$

with

$$\begin{aligned}U_x(x,i)=&2\eta_i x^{\mathrm{T}}+(p+1)\bar\eta_i|x|^{p-1}x^{\mathrm{T}},\\U_{xx}(x,i)=&2\eta_i\mathcal{I}_d+(p+1)\bar\eta_i\left(|x|^{p-1}\mathcal{I}_d+(p-1)|x|^{p-3}xx^{\mathrm{T}}\right).\end{aligned}$$

In Lyapunov stability analysis, we always want the operator $\mathcal{L}U$ to be negative. This is hence very significant to estimate $\mathcal{L}U$. The first estimation is about LU.

Lemma 3.1. *Let all the conditions in Theorem 2.9 hold. Then*

$$LU(x,t,i)\le(-\mu_1 I(t)+\mu_2(1-I(t)))U(x,i)-\frac{p+1}{2}\bar\eta_i\bar\beta_i|x|^{2p}\tag{14}$$

holds for any $(x,t,i)\in\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{S}$. Here μ_1, μ_2 are positive constants given by

$$\begin{aligned}\mu_1=&\frac{1}{\eta_M}\wedge\min_{i\in\mathbb{S}}\left(\frac{1+\eta_i\beta_i}{\bar\eta_i}\right),\ \eta_M=\max_{i\in\mathbb{S}}\eta_i,\\\mu_2=&\max_{i\in\mathbb{S}}\left(\frac{1}{\eta_i}\left(2\eta_i\alpha_i+\sum_{j=1}^S q_{ij}\eta_j\right)\vee\frac{1}{\bar\eta_i}\left(-\eta_i\beta_i+(p+1)\bar\eta_i\bar\alpha_i+\sum_{j=1}^S q_{ij}\bar\eta_j\right)\right).\end{aligned}$$

Next, we state the estimation of $\bar{U}$. For convenience, we extend $x(t)$, $r(t)$ to $[-\tau, 0)$ by defining $x(\theta) = x_0$, $r(\theta) = r_0$ for $\theta \in [-\tau, 0)$.

Lemma 3.2. *Let all the conditions in Theorem 2.9 hold. Then the solution of the controlled SDE (7) satisfies that for any $t \in \mathbb{R}_+$*

$$\begin{aligned} &E\bar{U}(x(t), x(t_\tau), t, r(t)) \\ &\leq \left(K_M EU(x(t), r(t)) + H_1(\tau) \sup_{-\tau\leq\theta\leq 0} EU(x(t+\theta), r(t+\theta)) \right) \sqrt{\tau} I(t) \\ &+ \frac{p+1}{2} E\left(\bar{\eta}_{r(t)} \bar{\beta}_{r(t)} |x(t)|^{2p} \right), \end{aligned} \tag{15}$$

where

$$H_1(\tau) = \left(\frac{2\alpha_M + 4K_M + 2L_1}{\eta_m} + \frac{2L_2}{\bar{\eta}_m} \right) \left(K_M \eta_M + \frac{(p+1)K_M^2 \bar{\eta}_M}{2\bar{\beta}_m} \sqrt{\tau} \right).$$

The detailed proof of Lemmas 3.1 and 3.2 can be found in the Appendix. The main technique to study stability in this paper is Razumikhin method, which requires the continuity of U and $\mathcal{L}U$ strongly. Thus we need to prepare a useful lemma.

Lemma 3.3. *Let all the conditions in Theorem 2.9 hold. Then as functions of t, $EU(x(t), r(t))$ is continuous, and $E\mathcal{L}U(x(t), x(t_\tau), t, r(t))$ is right-continuous.*

Proof. For any $t \in \mathbb{R}_+$, applying the generalized Itô formula to $U(x, i)$, we see that

$$\begin{aligned} U(x(t), r(t)) =& U(x_0, r_0) + \int_0^t \mathcal{L}U(x(s), x(s_\tau), s, r(s)) \mathrm{d}s \\ &+ \int_0^t U_x(x(s), r(s)) g(x(s), s, r(s)) \mathrm{d}W(s) + M(t), \end{aligned} \tag{16}$$

where

$$M(t) = \int_0^t \int_{\mathbb{R}} \Big(U(x(s), r_0 + \hat{h}(r(s), l)) - U(x(s), r(s)) \Big) \hat{\mu}(\mathrm{d}s, \mathrm{d}l).$$

The detailed explanation of the function $\hat{h}$ and martingale measure $\hat{\mu}$ is given in Theorem 1.45 in [24], but is of no use in this paper so we omit it here. It is easy to see that $M(t)$ is a continuous martingale vanishing at $t = 0$. Letting $\bar{\eta}_M = \max_{i\in\mathbb{S}} \bar{\eta}_i$ and using (2), (4), (11), we obtain that for any $(x, y, t, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$

$$\begin{aligned} |\mathcal{L}U(x, y, t, i)| \leq& (2\eta_M|x| + (p+1)\bar{\eta}_M|x|^p) (L_1|x| + L_2|x|^p + K_M|y|) \\ &+ \left(\eta_M + \frac{p(p+1)}{2} \bar{\eta}_M |x|^{p-1} \right) C \left(|x|^2 + |x|^{p+1} \right) \\ &+ S \max_{1\leq i,j\leq S} |q_{ij}| \left(\eta_M |x|^2 + \bar{\eta}_M |x|^{p+1} \right) \\ \leq& C \left(|x|^2 + |x|^{p+1} + |x|^{2p} + |y|^2 + |y|^{p+1} \right) \\ \leq& C \left(1 + |x|^{2p} + |y|^{2p} \right) \end{aligned} \tag{17}$$

and

$$|U_x(x, i) g(x, t, i)| \leq (2\eta_M|x| + (p+1)\bar{\eta}_M|x|^p) \sqrt{C\left(|x|^2 + |x|^{p+1}\right)} \leq C(1 + |x|^{2p}).$$

Recalling (12), we know that for $s \in [0, t]$, $E|x(s)|^{2p} < \infty$, which implies that

$$E|\mathcal{L}U(x(s), x(s_\tau), s, r(s))| \leq C\left(1 + E|x(s)|^{2p} + E|x(s_\tau)|^{2p}\right) < \infty$$

and

$$E|U_x(x(s),r(s))g(x(s),s,r(s))| \le C\left(1+E|x(s)|^{2p}\right) < \infty.$$

Let k_0 be a sufficiently large integer for $k > |x_0|$. For each interger $k \ge k_0$, define the stopping time $\sigma_k = \inf\{t \in \mathbb{R}_+ \mid |x(t)| \ge k\}$. Clearly, $\sigma_k \uparrow \infty$ a.s. It then follows from (16) that

$$EU(x(t\wedge\sigma_k),r(t\wedge\sigma_k)) = U(x_0,r_0) + E\int_0^{t\wedge\sigma_k} \mathcal{L}U(x(s),x(s_\tau),s,r(s))\mathrm{d}s. \quad (18)$$

For each $k \ge k_0$, we have

$$\left|\int_0^{t\wedge\sigma_k} \mathcal{L}U(x(s),x(s_\tau),s,r(s))\mathrm{d}s\right| \le \int_0^t |\mathcal{L}U(x(s),x(s_\tau),s,r(s))|\mathrm{d}s$$

and

$$E\int_0^t |\mathcal{L}U(x(s),x(s_\tau),s,r(s))|\,\mathrm{d}s = \int_0^t E\,|\mathcal{L}U(x(s),x(s_\tau),s,r(s))|\,\mathrm{d}s < \infty.$$

On the other hand,

$$U(x(t\wedge\sigma_k),r(t\wedge\sigma_k)) \le \eta_M|x(t\wedge\sigma_k)|^2+\bar{\eta}_M|x(t\wedge\sigma_k)|^{p+1} \le C\left(1+\sup_{0\le s\le t}|x(s)|^{p+1}\right).$$

By (13), we know that $E\left(\sup_{0\le s\le t}|x(s)|^{p+1}\right) < \infty$. Thus, we can let $k \to \infty$ on both sides of (18) and use the dominated convergence theorem to get that

$$EU(x(t),r(t)) = U(x_0,r_0) + \int_0^t E\mathcal{L}U(x(s),x(s_\tau),s,r(s))\mathrm{d}s. \quad (19)$$

Clearly, $EU(x(t),r(t))$ is continuous at time t.

Next, let $\Delta > 0$ be a sufficiently small number. Observing from (17) again gives that for any $s \in [0, t+\Delta]$,

$$|\mathcal{L}U(x(s),x(s_\tau),s,r(s))| \le C\left(1+\sup_{0\le s\le t+\Delta}|x(s)|^{2p}\right).$$

Therefore,

$$\sup_{0\le s\le t+\Delta}|\mathcal{L}U(x(s),x(s_\tau),s,r(s))| \le C\left(1+\sup_{0\le s\le t+\Delta}|x(s)|^{2p}\right).$$

From (13), we see that the expectation of the right-hand side exists. By the right-continuity of $\mathcal{L}U(x(t),x(t_\tau),t,r(t))$, we derive from the dominated convergence theorem that

$$\begin{aligned}\lim_{s\to t^+} E\mathcal{L}U(x(s),x(s_\tau),s,r(s)) =& E\left(\lim_{s\to t^+}\mathcal{L}U(x(s),x(s_\tau),s,r(s))\right)\\ =& E\mathcal{L}U(x(t),x(s_\tau),t,r(t)),\end{aligned}$$

which implies the right-continuity of $E\mathcal{L}U(x(t),x(t_\tau),t,r(t))$. The proof is complete. $\square$

Remark 3.4. From the proof of the continuity of $EU(x(t),r(t))$, we find that only

$$\sup_{0\le s\le t} E|x(s)|^{2p} < \infty,\ E\left(\sup_{0\le s\le t}|x(s)|^{p+1}\right) < \infty,\ \forall t \ge 0$$

are used. In other words, we might relax $q \ge 3p-1$ in Assumption 2.2 to $q \ge 2p$. Moreover, to obtain (19), there is no need to require the boundedness of $E|\mathcal{L}U|$ or more specifically, the boundedness of $E|x(t)|^{2p}$, such as [5] (Theorem 3.1) and [8]

(Theorem 3.3). In this case, why do we impose $q \geq 3p-1$? It is actually given to guarantee (13), by which the dominated convergence theorem could be used to prove the right-continuity of $E\mathcal{L}U$.

3.2. **Exponential stability.** We will mainly focus on the exponential stabilisation in the sense of L^{p+1} and almost surely. Before giving the stability results, we list the notations that will be used later. Let $K_M = \max_{i\in\mathbb{S}} K_i$, $\eta_m = \min_{i\in\mathbb{S}} \eta_i$, $\bar{\eta}_m = \min_{i\in\mathbb{S}} \bar{\eta}_i$, $\eta_M = \max_{i\in\mathbb{S}} \eta_i$, $\bar{\eta}_M = \max_{i\in\mathbb{S}} \bar{\eta}_i$, $\alpha_M = \max_{i\in\mathbb{S}} \alpha_i$, $\bar{\beta}_m = \min_{i\in\mathbb{S}} \bar{\beta}_i$.

Theorem 3.5. *Let all the conditions in Theorem 2.9 hold. Choose* $\delta \in \left(\frac{\mu_2}{\mu_1+\mu_2}, 1\right)$ *and let* τ^* *be the unique root of* $\varphi(\tau) = 0$ *on* $(0, \hat{\tau}]$, *where*

$$\varphi(\tau) := \exp\left(\left(\frac{\mu_2}{\delta} - \mu_2\right)\tau\right) - \frac{1}{H_1(\tau)}\left(\frac{\mu_1 + \mu_2 - \frac{\mu_2}{\delta}}{\sqrt{\tau}} - K_M\right),$$

and $\hat{\tau} = \left(\frac{\mu_1+\mu_2-\frac{\mu_2}{\delta}}{K_M}\right)^2$. *Then given* $\tau < \tau^*$, *the solution of the controlled SDE (7) satisfies that*

$$\limsup_{t\to\infty} \frac{1}{t}\log\left(E|x(t)|^{p+1}\right) < 0 \tag{20}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log\left(|x(t)|\right) < 0 \ \ a.s. \tag{21}$$

Proof. We divide the proof into four steps.

Step 1. Let $\delta \in \left(\frac{\mu_2}{\mu_1+\mu_2}, 1\right)$ be chosen. It is easy to see that $\varphi(\cdot)$ is an increasing continuous function on $(0, \hat{\tau}]$. Moreover, $\lim_{\tau\to 0^+}\varphi(\tau) = -\infty$, and $\varphi(\hat{\tau}) > 0$. Therefore, there is a unique zero solution of φ, and so the definition of τ^* is clear. For fixed $\tau < \tau^*$, write $H_1 = H_1(\tau)$ for simplicity. Since φ is increasing, we naturally have

$$\exp\left(\left(\frac{\mu_2}{\delta} - \mu_2\right)\tau\right) < \frac{1}{H_1}\left(\frac{\mu_1 + \mu_2 - \frac{\mu_2}{\delta}}{\sqrt{\tau}} - K_M\right).$$

Therefore, we could choose a constant q such that

$$\exp\left(\left(\frac{\mu_2}{\delta} - \mu_2\right)\tau\right) < q < \frac{1}{H_1}\left(\frac{\mu_1 + \mu_2 - \frac{\mu_2}{\delta}}{\sqrt{\tau}} - K_M\right). \tag{22}$$

Step 2. If $x_0 = 0$, the result is obvious. Thus we always assume that $|x_0| > 0$. If for some t, the solution satisfying that

$$\sup_{-\tau\leq\theta\leq 0} EU(x(t+\theta), r(t+\theta)) \leq qEU(x(t), r(t)), \tag{23}$$

we then derive from (14) and Lemma 3.2 that

$$\begin{aligned}
&E\mathcal{L}U(x(t), x(t_\tau), t, r(t)) \\
\leq&(-\mu_1 I(t) + \mu_2(1-I(t)))EU(x(t), r(t)) - \frac{p+1}{2}E\left(\bar{\eta}_{r(t)}\bar{\beta}_r(t)|x(t)|^{2p}\right) \\
&+\left(K_M EU(x(t), r(t)) + H_1 \sup_{-\tau\leq\theta\leq 0} EU(x(t+\theta), r(t+\theta))\right)\sqrt{\tau}I(t) \\
&+\frac{p+1}{2}E\left(\bar{\eta}_{r(t)}\bar{\beta}_{r(t)}|x(t)|^{2p}\right) \\
\leq&-\Big((\mu_1 - (K_M + H_1 q)\sqrt{\tau})I(t) - \mu_2(1-I(t))\Big)EU(x(t), r(t)).
\end{aligned}$$

Letting $H_2 = \mu_1 - (K_M + H_1 q)\sqrt{\tau}$ and $\lambda_1(t) = H_2 I(t) - \mu_2(1 - I(t))$, we have

$$E\mathcal{L}U(x(t), x(t_\tau), t, r(t)) \le -\lambda_1(t) EU(x(t), r(t)). \tag{24}$$

Step 3. For $t \ge 0$, define $\lambda_2(t) = \lambda_1(t) \wedge \frac{\log(q)}{\tau}$, and

$$V(t) = \exp\left(\int_0^t \lambda_2(s)\mathrm{d}s\right) EU(x(t), r(t)).$$

It is easy to see that $V(t)$ is well-defined, and further is a continuous function. Next, we claim that

$$V(t) \le V(0), \ \ \forall t \ge 0, \tag{25}$$

where $V(0) = U(x_0, r_0)$ is a positive constant. If (25) is true, it then follows that

$$E|x(t)|^{p+1} \le \frac{1}{\bar{\eta}_M} EU(x(t), r(t)) \le \frac{V(0)}{\bar{\eta}_M} \exp\left(-\int_0^t \lambda_2(s)\mathrm{d}s\right).$$

Since $\delta < 1$, we can easily observe that

$$q < \frac{1}{H_1}\left(\frac{\mu_1 + \mu_2 - \frac{\mu_2}{\delta}}{\sqrt{\tau}} - K_M\right) < \frac{1}{H_1}\left(\frac{\mu_1}{\sqrt{\tau}} - K_M\right),$$

which implies that $H_2 > 0$. As a consequence,

$$\begin{aligned}\int_0^t \lambda_2(s)\mathrm{d}s &= \left(H_2 \wedge \frac{\log(q)}{\tau}\right)\int_0^t I(s)\mathrm{d}s - \mu_2 \int_0^t (1 - I(s))\mathrm{d}s \\ &= -\mu_2 t + \left(\mu_2 + H_2 \wedge \frac{\log(q)}{\tau}\right)\int_0^t I(s)\mathrm{d}s.\end{aligned}$$

For any fixed $t \ge 0$, we can find a non-negative integer k such that $kT \le t < (k+1)T$. If $t \in [kT, kT + \delta T)$, we obtain that

$$\begin{aligned}\int_0^t \lambda_2(s)\mathrm{d}s &= -\mu_2 t + \left(\mu_2 + H_2 \wedge \frac{\log(q)}{\tau}\right)(\delta kT + t - kT) \\ &= \left(-\mu_2 + \mu_2\delta + H_2\delta \wedge \frac{\log(q)}{\tau}\delta\right) kT \\ &\quad + \left(-\mu_2 + \mu_2 + H_2 \wedge \frac{\log(q)}{\tau}\right)(t - kT) \\ &\ge \left(-\mu_2 + \mu_2\delta + H_2\delta \wedge \frac{\log(q)}{\tau}\delta\right) kT.\end{aligned}$$

From the first inequality of (22), we derive that $\frac{\log(q)}{\tau} > \frac{\mu_2}{\delta} - \mu_2$. The other side of (22) implies that $H_2 > \frac{\mu_2}{\delta} - \mu_2$. In other words, $\bar{\lambda} := -\mu_2 + \mu_2\delta + H_2\delta \wedge \frac{\log(q)}{\tau}\delta$ is positive, and so

$$\int_0^t \lambda_2(s)\mathrm{d}s \ge \bar{\lambda}(t - T).$$

If $t \in [kT + \delta T, (k+1)T)$, we have

$$\begin{aligned}\int_0^t \lambda_2(s)\mathrm{d}s &= -\mu_2 t + \left(\mu_2 + H_2 \wedge \frac{\log(q)}{\tau}\right)\delta(k+1)T \\ &\ge -\mu_2 t + \left(\mu_2 + H_2 \wedge \frac{\log(q)}{\tau}\right)\delta t \ge \bar{\lambda}(t - T).\end{aligned}$$

In conclusion, we have shown that for any $t \geq 0$

$$E|x(t)|^{p+1} \leq \frac{V(0)}{\bar{\eta}_M} \exp\left(-\int_0^t \lambda_2(s)\mathrm{d}s\right) \leq \frac{V(0)}{\bar{\eta}_M} \exp\left(\bar{\lambda}T\right) \exp\left(-\bar{\lambda}t\right),$$

which yields that

$$\frac{1}{t}\log\left(E|x(t)|^{p+1}\right) \leq \frac{1}{t}\log\left(\frac{V(0)}{\bar{\eta}_M}\exp\left(\bar{\lambda}T\right)\right) - \bar{\lambda}.$$

Letting $t \to \infty$ gives the assertion (20). After achieving the moment exponential stability, we can use the same analysis as in the proof of Theorem 5.4 in [5] to prove the assertion (21).

Step 4. The remaining work is to prove claim (25). Supposing not, there will be some $t > 0$ such that $V(t) > V(0)$. We can set $\hat{t} = \inf\{t > 0 | V(t) > V(0)\}$. Because of the continuity of $V(t)$, we see that for $0 \leq t < \hat{t}$, $V(t) \leq V(0)$; for $t = \hat{t}$, $V(\hat{t}) = V(0)$; and there is a sequence $\{t_n\}_{n\geq 1}$ such that $t_n > \hat{t}$, $t_n \downarrow \hat{t}$, and $V(t_n) > V(0)$.

On the other hand, for any $\theta \in [-\tau, 0]$, if $\hat{t} + \theta > 0$, we obtain that

$$EU(x(\hat{t}+\theta), r(\hat{t}+\theta)) \leq \exp\left(\int_{\hat{t}+\theta}^{\hat{t}} \lambda_2(s)\mathrm{d}s\right) EU(x(\hat{t}), r(\hat{t})) \leq qEU(x(\hat{t}), r(\hat{t}))$$

since $V(\hat{t}+\theta) \leq V(0) = V(\hat{t})$ and $\int_{\hat{t}+\theta}^{\hat{t}} \lambda_2(s)\mathrm{d}s \leq \frac{\log(q)}{\tau}(-\theta) \leq \log(q)$. Otherwise, as $\hat{t} \leq \tau$, we have

$$\begin{aligned} EU(x(\hat{t}+\theta), r(\hat{t}+\theta)) =& U(x_0, r_0) = V(\hat{t}) = \exp\left(\int_0^{\hat{t}} \lambda_2(s)\mathrm{d}s\right) EU(x(\hat{t}), r(\hat{t})) \\ \leq& qEU(x(\hat{t}), r(\hat{t})). \end{aligned}$$

In other words, we have shown that for any $-\tau \leq \theta \leq 0$,

$$EU(x(\hat{t}+\theta), r(\hat{t}+\theta)) \leq qEU(x(\hat{t}), r(\hat{t})),$$

which is exactly condition (23). Using the results in Step 2, we have

$$\begin{aligned} E\mathcal{L}U(x(\hat{t}), x(\hat{t}_\tau), \hat{t}, r(\hat{t})) \leq& -\lambda_1(\hat{t})EU(x(\hat{t}), r(\hat{t})) \\ \leq& -\lambda_2(\hat{t})EU(x(\hat{t}), r(\hat{t})) < -\lambda_2(\hat{t})EU(x(\hat{t}), r(\hat{t})) + \varepsilon, \end{aligned}$$

where ε is an arbitrary positive constant. We can also find a non-negative integer K such that $KT \leq \hat{t} < (K+1)T$. Let $\Delta > 0$ be small enough so that $\Delta < (KT + \delta T - \hat{t})1_{\{KT+\delta T > \hat{t}\}} + ((K+1)T - \hat{t})1_{\{KT+\delta T \leq \hat{t}\}}$ and $\Delta < \tau$. We then see from the right-continuity of $E\mathcal{L}U$ that

$$E\mathcal{L}U(x(t), x(t_\tau), t, r(t)) < -\lambda_2(t)EU(x(t), r(t)) + \varepsilon, \ \forall t \in [\hat{t}, \hat{t}+\Delta].$$

It is easy to see that the interval $[\hat{t}, \hat{t}+\Delta]$ is either in $[KT, KT+\delta T)$ or $[KT+\delta T, (K+1)T)$. Thus applying the generalised Itô formula to $\exp\left(\int_0^t \lambda_2(s)\mathrm{d}s\right) U(x(t), r(t))$ gives that

$$\begin{aligned} & V(\hat{t}+\Delta) - V(\hat{t}) \\ =& \exp\left(\int_0^{\hat{t}+\Delta} \lambda_2(s)\mathrm{d}s\right) EU(x(\hat{t}+\Delta), r(\hat{t}+\Delta)) - \exp\left(\int_0^{\hat{t}} \lambda_2(s)\mathrm{d}s\right) EU(x(\hat{t}), r(\hat{t})) \\ =& \int_{\hat{t}}^{\hat{t}+\Delta} \exp\left(\int_0^s \lambda_2(v)\mathrm{d}v\right) \Big(E\mathcal{L}U(x(s), x(s_\tau), s, r(s)) + \lambda_2(s)EU(x(s), r(s))\Big)\mathrm{d}s \end{aligned}$$

$$<\varepsilon \int_{\hat{t}}^{\hat{t}+\Delta} \exp\left(\int_0^s \lambda_2(v)\mathrm{d}v\right)\mathrm{d}s \leq \varepsilon \int_{\hat{t}}^{\hat{t}+\tau} \exp\left(\frac{\log(q)}{\tau}s\right)\mathrm{d}s.$$

Since $\int_{\hat{t}}^{\hat{t}+\tau} \exp\left(\frac{\log(q)}{\tau}s\right)\mathrm{d}s$ is a positive constant and ε is chosen arbitrarily, $V(\hat{t}+\Delta) - V(\hat{t}) \leq 0$. For sufficiently large n with $t_n - \hat{t} \leq \Delta$, we obtain that

$$V(t_n) \leq V(\hat{t}) = V(0),$$

which is a contradiction with the fact that $V(t_n) > V(0)$ derived before. Therefore claim (25) must be true. The proof is therefore complete. □

4. Application.

4.1. **Coupled oscillators.** Consider the coupled Van der Pol–Duffing oscillator system investigated in [15], which is consisted of N oscillators and the n-th oscillator is described as

$$\begin{cases} \mathrm{d}x_n(t) = \Big(- \big(a_n(r(t)) + b_n(r(t))\big)x_n(t) + B_n(r(t))(y_n(t) - x_n(t))^3 \\ \qquad\qquad + b_n(r(t))y_n(t) - A_n(r(t))x_n^3(t)\Big)\mathrm{d}t + \nu_n(r(t))x_n(t)\mathrm{d}w_n^{(1)}(t), \\ \mathrm{d}y_n(t) = \Big(b_n(r(t))x_n(t) - z_n(t) - B_n(r(t))(y_n(t) - x_n(t))^3 \\ \qquad\qquad - (b_n(r(t)) + 1)y_n(t) - C_n(r(t))y_n^3(t))\Big)\mathrm{d}t + \nu_n(r(t))y_n(t)\mathrm{d}w_n^{(2)}(t), \\ \mathrm{d}z_n(t) = \left(y_n(t) + \displaystyle\sum_{j=1}^{S} e_{nj}(r(t))\Pi_{nj}(z_n(t), z_j(t), r(t)) + P_n(z_n(t), r(t))\right)\mathrm{d}t \\ \qquad\qquad + \nu_n(r(t))z_n(t)\mathrm{d}w_n^{(3)}(t), \end{cases} \tag{26}$$

where $x_n, y_n, z_n \in \mathbb{R}$, $a_n(i), b_n(i), A_n(i), B_n(i), C_n(i), \nu_n(i)$ are positive constants, $e_{nj}(i)$ stands for connection weight from oscillator j to oscillator n, $\Pi_{nj}(z_n, z_j, i)$ and $P_n(z_n, i)$ are locally Lipscitz continuous functions in the i-th mode. Here, we need to impose the following conditions on these functions.

Assumption 4.1. For every $i \in \mathbb{S}$ and $n, j = 1, \cdots, N$, assume that there are positive constants $\Lambda_{nj}(i)$, $J_n^{(1)}(i)$, $J_n^{(2)}(i)$, $D_n^{(1)}(i)$, $D_n^{(2)}(i)$ so that for all $x, y \in \mathbb{R}$

$$|\Pi_{n,j}(x, y, i)| \leq \Lambda_{nj}(i)(|x| + |y|) \tag{27}$$

and

$$|P_n(x, i)| \leq J_n^{(1)}(i)|x| + J_n^{(2)}(i)|x|^3, \; xP_n(x, i) \leq D_n^{(1)}(i)|x|^2 - D_n^{(2)}(i)|x|^4. \tag{28}$$

Let $X_n = (x_n, y_n, z_n)^{\mathrm{T}}$, $X = \left(X_1^{\mathrm{T}}, \cdots, X_N^{\mathrm{T}}\right)^{\mathrm{T}}$, $W_n = \left(w_n^{(1)}, w_n^{(2)}, w_n^{(3)}\right)^{\mathrm{T}}$, $W = \left(W_1^{\mathrm{T}}, \cdots, W_N^{\mathrm{T}}\right)^{\mathrm{T}}$. Then the oscillator system can be written as

$$\mathrm{d}X(t) = F(X(t), r(t))\mathrm{d}t + G(X(t), r(t))\mathrm{d}W(t), \tag{29}$$

where

$$F(X, i) = \left(F_1^{\mathrm{T}}(X_1, i), \cdots, F_N^{\mathrm{T}}(X_N, i)\right)^{\mathrm{T}},$$

$$G(X, i) = \begin{pmatrix} G_1(X_1, i) & & \\ & \ddots & \\ & & G_N(X_N, i) \end{pmatrix},$$

with $G_n(X_n, i) = \nu_n(i)\mathrm{diag}(x_n, y_n, z_n)$ and

$$F_n(X_n, i) = \begin{pmatrix} -(a_n(i)+b_n(i))x_n + B_n(i)(y_n - x_n)^3 + b_n(i)y_n - A_n(i)x_n^3 \\ b_n(i)x_n - z_n - B_n(i)(y_n - x_n)^3 - (b_n(i)+1)y_n - C_n(i)y_n^3 \\ y_n + \sum_{j=1}^S e_{nj}(i)\Pi_{nj}(z_n, z_j, i) + P_n(z_n, i) \end{pmatrix}.$$

With the detailed calculation, we derive that for each $i \in \mathbb{S}$, $|F(X,i)|^2 \le L_{1i}|X|^2 + L_{2i}|X|^6$, where

$$\begin{aligned} L_{1i} = & \max_{1\le n\le N}\left(4(a_n(i)+b_n(i))^2 + 5b_n^2(i)\right) \vee \max_{1\le n\le N}\left(4b_n^2(i) + 5(b_n(i)+1)^2 + 3\right) \\ & \vee \left(\max_{1\le n\le N}\left(5 + 3(J_n^{(1)})^2 + 6N\sum_{j=1}^S(|e_{nj}(i)|\Lambda_{nj}(i))^2\right)\right. \\ & \left. + 6N \max_{1\le n,j\le N}(|e_{nj}(i)|\Lambda_{nj}(i))^2 \right) \end{aligned}$$

and

$$L_{2i} = \max_{1\le n\le N}\left(4A_n^2(i) + 288B_n^2(i)\right) \vee \max_{1\le n\le N}\left(5C_n^2(i) + 288B_n^2(i)\right) \vee \max_{1\le n\le N} 3(J_n^{(2)})^2.$$

Therefore, Assumption 2.1 is satisfied with $p = 3$, $L_1 = \max_{i\in\mathbb{S}}\sqrt{L_{1i}}$, $L_2 = \max_{i\in\mathbb{S}}\sqrt{L_{2i}}$. Next, compute

$$\begin{aligned} X^{\mathrm{T}}F(X,i) \le & \sum_{n=1}^N \left(-a_n(i)x_n^2 - y_n^2 + \sum_{j=1}^S |e_{nj}(i)\Pi_{nj}(z_n, z_j, i)z_n| - A_n(i)x_n^4 \right. \\ & \left. - C_n(i)y_n^4 + z_nP_n(z_n, i) \right) \\ \le & \sum_{n=1}^N \left(\sum_{j=1}^S |e_{nj}(i)\Lambda_{nj}(i)(z_n^2 + |z_nz_j|)| + D_n^{(1)}(i)z_n^2 - A_n(i)x_n^4 \right. \\ & \left. - C_n(i)y_n^4 - D_n^{(2)}(i)z_n^4 \right). \end{aligned}$$

Since $|X|^4 \le 3N\sum_{n=1}^N(x_n^4 + y_n^4 + z_n^4)$, we further have

$$X^{\mathrm{T}}F(X,i) \le h_i|X|^2 - \frac{1}{3N}\min_{1\le n\le N}\left(A_n(i)\wedge C_n(i)\wedge D_n^{(2)}(i)\right)|X|^4,$$

where

$$h_i = \max_{1\le n\le N}\left(\frac{3}{2}\sum_{j=1}^S|e_{nj}(i)|\Lambda_{nj}(i) + D_n^{(1)}(i)\right) + \frac{1}{2}\max_{1\le n,j\le N}(|e_{nj}(i)|\Lambda_{nj}(i)).$$

It is easy to see that

$$|G(X,i)|^2 \le \sum_{n=1}^N \nu_n^2(i)(x_n^2 + y_n^2 + z_n^2) \le \max_{1\le n\le N}\nu_n^2(i)|X|^2.$$

As a result, Assumption 2.2 holds with any number $q \ge 8$ and

$$\hat{\alpha} = \max_{i\in\mathbb{S}}\left(h_i + \frac{q-1}{2}\max_{1\le n\le N}\nu_n^2(i)\right).$$

Assumption 2.4 is also satisfied with

$$\begin{aligned}\alpha_i =& h_i + \frac{1}{2}\max_{1\le n\le N}\nu_n^2(i),\ \bar{\alpha}_i = h_i + \max_{1\le n\le N}\nu_n^2(i),\\ \beta_i =& \bar{\beta}_i = \frac{1}{3N}\min_{1\le n\le N}\left(A_n(i)\wedge C_n(i)\wedge D_n^{(2)}(i)\right).\end{aligned}$$

But the oscillator system (29) might not be stable (e.g., see the simulation in Fig. 2). It is hence necessary to design controller according to the results above to achieve stabilisation. At first, the control function $\mathcal{U}(X,i)$ can be designed as follows.

Rule 4.2. Choose non-negative constants $\gamma_i (i\in\mathbb{S})$ such that $\mathcal{A}$ and $\bar{\mathcal{A}}$ are non-singular M-matrices. Then for each $i\in\mathbb{S}$, letting $R_i=\sqrt{\frac{2\gamma_i}{\beta_i}}$, we can design the control function as follows

$$\mathcal{U}(X,i)=\begin{cases}-\gamma_i X, & X\in B_{R_i},\\ -\gamma_i\left(\dfrac{2R_i}{|X|}-1\right)X, & X\in B_{2R_i}-B_{R_i},\\ 0, & X\in B^c_{R_i}.\end{cases}\tag{30}$$

It is easy to verify that $\mathcal{U}(X,i)$ designed in Rule 4.2 meets with Rule 2.6 with $K_i=\gamma_i$. Next, we let the feedback control $\mathcal{U}(X(t),r(t))$ working imminently with strength δ and being observed at discrete times $0,\tau,2\tau,\cdots$ In other words, the controlled oscillator system is given as

$$\mathrm{d}X(t)=(F(X(t),r(t))+\mathcal{U}(X(t_\tau),r(t))I(t))\mathrm{d}t+G(X(t),r(t))\mathrm{d}W(t),\tag{31}$$

where $t_\tau=[t/\tau]\tau$ and $I(t)=\sum_{k=0}^{\infty}1_{[kT,kT+\delta T)}(t)$ are the same as before. By Theorem 3.5, we can make the following assertion.

Theorem 4.3. *Let Assumption 4.1 hold and the control function $\mathcal{U}(X,i)$ be given in Rule 4.2. Then choosing $\delta\in\left(\frac{\mu_2}{\mu_1+\mu_2},1\right)$ and letting $\tau<\tau^*$, the controlled oscillator system (31) is exponential stable in the sense of L^4 and almost surely. Here, τ^* can be determined by using the method in Theorem 3.5, where the required values of L_1, L_2, α_i, $\bar{\alpha}_i$, β_i, $\bar{\beta}_i$, K_i are all given above.*

4.2. **Numerical simulations.** For the sake of showing the viability of our results, a numerical example is provided in this part. We let the Markov chain $r(t)$ taking values in $\mathbb{S}=\{1,2\}$ with $Q=\begin{pmatrix}-10 & 10\\ 10 & -10\end{pmatrix}$. We consider the oscillator system (29) with 25 oscillators. The parameters are given as

$$\begin{aligned}&a_n(1)=0.2,\quad b_n(1)=0.3,\quad A_n(1)=1.6,\quad B_n(1)=0.05,\quad C_n(1)=1.7,\quad \nu_n(1)=0.5,\\ &a_n(2)=0.5,\quad b_n(2)=0.4,\quad A_n(2)=2,\quad B_n(2)=0.03,\quad C_n(2)=1.9,\quad \nu_n(2)=0.8,\end{aligned}$$

and the functions are given as $\Pi_{n,j}(x,y,1)=0.01(x-y)$, $\Pi_{n,j}(x,y,2)=0.005(x-y)$, $P_n(x,1)=0.5x-1.5x^3$, $P_n(x,2)=0.3x-1.8x^3$ for all $n,j=1,\cdots,25$. The connection weight $(e_{n,j}(i))_{20\times 20}$ can be obtained from the connection graphs in Fig. 1. Here for both two modes, node n stands for the n-th oscillator, directed edge (n,j) means the output of the j-th oscillator is connected with the input of the n-th oscillator, the number on the edge (n,j) is the value of $e_{n,j}(i)$. It is then easy to verify that Assumption 4.1 is satisfied with $\Gamma_{n,j}(1)=0.01$, $\Gamma_{n,j}(2)=0.005$, $J_n^{(1)}(1)=D_n^{(1)}(1)=0.5$, $J_n^{(2)}(1)=D_n^{(2)}(1)=1.5$, $J_n^{(1)}(2)=D_n^{(1)}(2)=0.3$, $J_n^{(2)}(2)=D_n^{(2)}(2)=1.8$.

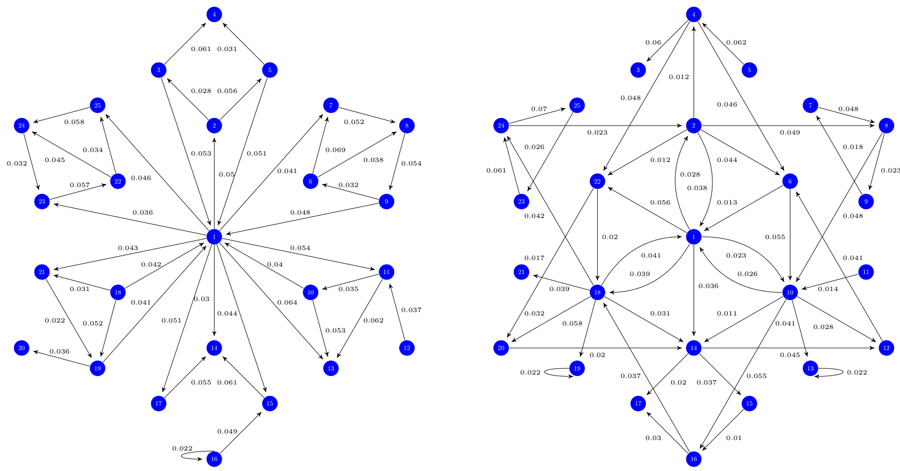

FIGURE 1. The oscillator connection graphs at mode 1 (left) and mode 2 (right).

Through computer simulations (see Fig. 2), we find the oscillator system (29) is indeed unstable. Therefore, we want to use the controller $\mathcal{U}(X(t_\tau), r(t))I(t)$ to realise stabilisation. Before that, we can easily get $L_1(1) = 11.81$, $L_1(2) = 13.44$, $L_2(1) = 15.17$, $L_2(2) = 18.3092$, $h(1) = 0.5045$, $h(2) = 0.3012$, $\alpha(1) = 0.6295$, $\alpha(2) = 0.6212$, $\bar{\alpha}(1) = 0.7545$, $\bar{\alpha}(2) = 0.9412$, $\beta(1) = \bar{\beta}(1) = 0.02$, $\beta(2) = \bar{\beta}(2) = 0.024$. We choose $\gamma_1 = 6$ and $\gamma_2 = 5$, as a result of which $\mathcal{A} = \begin{pmatrix} 20.7411 & -10 \\ -10 & 18.7566 \end{pmatrix}$ and $\bar{\mathcal{A}} = \begin{pmatrix} 30.9821 & -10 \\ -10 & 26.2351 \end{pmatrix}$ are non-singular M-matrices. The bounds of control area are given as $R_1 = 24.4949$ and $R_2 = 20.4124$. Then Rule 4.2 is fulfilled. With detailed calculation, we derive that $\mu_1 = 1.0391$, $\mu_2 = 4.2888$. Thus we can take the control rate $\delta = 0.9$ to get the value of τ^* as 2.17×10^{-6}. By Theorem 4.3, we can conclude that the controlled oscillator system (31) is exponential stable in the sense of L^4 and almost surely if $\delta = 0.9$ and $\tau < 2.17 \times 10^{-6}$. The simulation results support our theory clearly (see Fig. 2).

5. **Conclusion.** In this paper, we have designed the discrete-time state intermittent feedback control in a bounded state area to stabilise a kind of hybrid SDEs, in the sense of moment and almost surely exponential stability. Not only more general stochastic systems could be covered especially for those who are not globally Lipschitz continuous, but also the control cost could be saved significantly. The Razumikhin theory has been further developed, that is, condition (24) was given in a function $\lambda_1(t)$ rather than a constant as before. But the structure of the function we considered was quite simple, which was piece-wise continuous with only two values. In the future, we will generalise the Razumikhin theory in terms of more general functions.

Appendix. In this Appendix, we will give the proofs of several results in the previous parts.

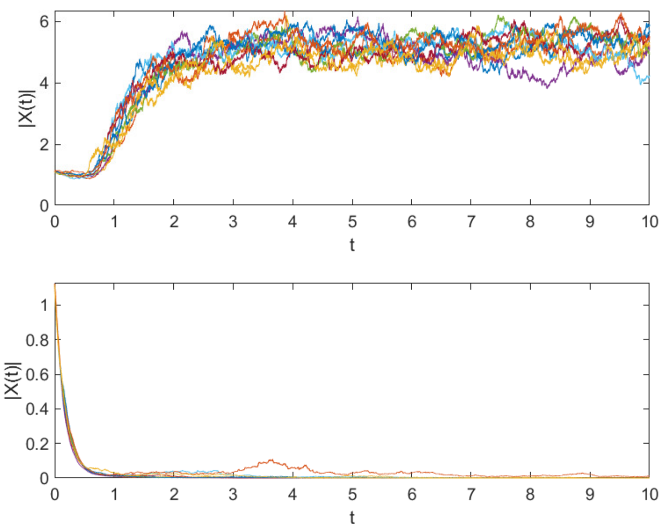

FIGURE 2. Ten sample paths of oscillator system (29) (top), controlled oscillator system (31) with $\delta = 0.9$ and $\tau = 1 \times 10^{-6}$ (bottom). Here the initial data is fiexed as $x_n(0) = 0.2$, $y_n(0) = 0.1$, $z_n(0) = 0$ for each $n = 1, \cdots, 25$.

Proof of Lemma 2.8. For any $(x, y, t, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$, we always assume that $|x| \le |y|$ without loss of generality. To show the desired assertion, let us consider the following five possible cases. For convenience, denote by $u_{xy} = |u(x, t, i) - u(y, t, i)|$.

- For $x, y \in B_{R_i}$ or $x, y \in B^c_{2R_i}$, the results are clear.
- For $x, y \in B_{2R_i} - B_{R_i}$, we see that $\left(\frac{2R_i}{|x|} - 1\right) x$ and $\left(\frac{2R_i}{|y|} - 1\right) y$ are both in B_{R_i}. Thus by condition (8), we have $u_{xy} \le K_M \left|\left(\frac{2R_i}{|x|} - 1\right) x - \left(\frac{2R_i}{|y|} - 1\right) y\right|$. Then compute
$$\left|\left(\frac{2R_i}{|x|} - 1\right) x - \left(\frac{2R_i}{|y|} - 1\right) y\right|^2 \le \frac{4R_i}{|x||y|}(2R_i - |x| - |y|)(|x||y| - x^{\mathrm{T}} y) + |x - y|^2,$$
which yields that $|u_{xy}| \le K_M |x - y|$, since $|x|, |y| > R_i$.
- For $x \in B_{R_i}, y \in B_{2R_i} - B_{R_i}$, we have $u_{xy} \le K_M \left|x - \left(\frac{2R_i}{|y|} - 1\right) y\right|$. Then the required assertion follows since
$$\left|x - \left(\frac{2R_i}{|y|} - 1\right) y\right|^2 - |x - y|^2 = 4(|y| - R_i)\left(\frac{x^{\mathrm{T}} y}{|y|} - R_i\right) \le 0.$$
- For $x \in B_{R_i}, y \in B^c_{2R_i}$, it is easy to derive that
$$u_{xy} = |u(x, t, i)| \le K_M |x| \le K_M R_i \le K_M \big||y| - |x|\big| \le K_M |x - y|.$$
- For $x \in B_{2R_i} - B_{R_i}, y \in B^c_{2R_i}$, we derive that
$$u_{xy} = \left|u(x, t, i) - u\left(\frac{2R_i}{|y|} y, t, i\right)\right| \le K_M \left|x - \frac{2R_i}{|y|} y\right|.$$

Here we use the fact that $u(y,t,i) = u\left(\frac{2R_i}{|y|}y,t,i\right) = 0$ and the result in the second case. Next compute

$$\left|x - \frac{2R_i}{|y|}y\right|^2 - |x-y|^2 = (|y| - 2R_i)\left(\frac{2x^{\mathrm{T}}y}{|y|} - (2R_i + |y|)\right) \le 0.$$

The required assertion then follows.

The proof is therefore complete. □

Proof of Lemma 3.1. At first, we show that for any $(x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$

$$\begin{cases} x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)I(t)) + \dfrac{1}{2}|g(x,t,i)|^2 \le (\alpha_i - \gamma_i I(t))|x|^2 - \dfrac{\beta_i}{2}|x|^{p+1}, \\ x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)I(t)) + \dfrac{p}{2}|g(x,t,i)|^2 \le (\bar{\alpha}_i - \gamma_i I(t))|x|^2 - \dfrac{\bar{\beta}_i}{2}|x|^{p+1}. \end{cases} \tag{32}$$

Fix $(t,i) \in \mathbb{R}_+ \times \mathbb{S}$ arbitrarily. For $x \in B_{R_i}$, by (6) and (11), it is easy to see that

$$\begin{aligned} x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)I(t)) + \frac{1}{2}|g(x,t,i)|^2 \le& (\alpha_i - \gamma_i I(t))|x|^2 - \beta_i|x|^{p+1} \\ \le& (\alpha_i - \gamma_i I(t))|x|^2 - \frac{\beta_i}{2}|x|^{p+1}. \end{aligned}$$

On the other hand, we have that for $x \in B_{2R_i} - B_{R_i}$, $x^{\mathrm{T}}u(x,t,i) = \left(\frac{2R_i}{|x|} - 1\right)|x|^2 \le 0$, and for $x \in B^c_{2R_i}$, $x^{\mathrm{T}}u(x,t,i) = 0$. Therefore, for $x \in B^c_{R_i}$,

$$\begin{aligned} & x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)I(t)) + \frac{1}{2}|g(x,t,i)|^2 \\ \le& \alpha_i|x|^2 - \beta_i|x|^{p+1} \\ =& (\alpha_i - \gamma_i I(t))|x|^2 - \frac{\beta_i}{2}|x|^{p+1} + \left(\gamma_i I(t)|x|^2 - \frac{\beta_i}{2}|x|^{p+1}\right) \\ \le& (\alpha_i - \gamma_i I(t))|x|^2 - \frac{\beta_i}{2}|x|^{p+1} \end{aligned}$$

since $\gamma_i I(t)|x|^2 - \frac{\beta_i}{2}|x|^{p+1} \le \gamma_i|x|^2 - \frac{\beta_i}{2}|x|^{p+1} \le 0$ when $|x| > R_i$. The second inequality of (32) can be proven in the similar way.

Next, making use of (32), compute

$$\begin{aligned} & LU(x,t,i) \\ \le& 2\eta_i\left((\alpha_i - \gamma_i I(t))|x|^2 - \frac{\beta_i}{2}|x|^{p+1}\right) + \sum_{j=1}^{S} q_{ij}\eta_j|x|^2 \\ & + (p+1)\bar{\eta}_i|x|^{p-1}\left((\bar{\alpha}_i - \gamma_i I(t))|x|^2 - \frac{\bar{\beta}_i}{2}|x|^{p+1}\right) + \sum_{j=1}^{S} q_{ij}\bar{\eta}_j|x|^{p+1} \\ =& \left(2\eta_i(\alpha_i - \gamma_i) + \sum_{j=1}^{S} q_{ij}\eta_j\right)I(t)|x|^2 + \left(2\eta_i\alpha_i + \sum_{j=1}^{S} q_{ij}\eta_j\right)(1 - I(t))|x|^2 \\ & + \left(-\eta_i\beta_i + (p+1)\bar{\eta}_i(\bar{\alpha}_i - \gamma_i) + \sum_{j=1}^{S} q_{ij}\bar{\eta}_j\right)I(t)|x|^{p+1} \end{aligned}$$

$$
\begin{aligned}
&+\left(-\eta_i\beta_i+(p+1)\bar{\eta}_i\bar{\alpha}_i+\sum_{j=1}^{S}q_{ij}\bar{\eta}_j\right)(1-I(t))|x|^{p+1}-\frac{p+1}{2}\bar{\eta}_i\bar{\beta}_i|x|^{2p}\\
=&-I(t)|x|^2-(\eta_i\beta_i+1)I(t)|x|^{p+1}+\left(2\eta_i\alpha_i+\sum_{j=1}^{S}q_{ij}\eta_j\right)(1-I(t))|x|^2\\
&+\left(-\eta_i\beta_i+(p+1)\bar{\eta}_i\bar{\alpha}_i+\sum_{j=1}^{S}q_{ij}\bar{\eta}_j\right)(1-I(t))|x|^{p+1}-\frac{p+1}{2}\bar{\eta}_i\bar{\beta}_i|x|^{2p}.
\end{aligned}
$$

This completes the proof. $\square$

Proof of Lemma 3.2. Let t be fixed. It is easy to derive from (8) that

$$
\begin{aligned}
&E\bar{U}(x(t),x(t_\tau),t,r(t))\\
\le&E\left(\left(2\eta_{r(t)}|x(t)|+(p+1)\bar{\eta}_{r(t)}|x(t)|^p\right)K_M|x(t)-x(t_\tau)|\right)I(t)\\
\le&E\bigg(K_M\eta_{r(t)}\sqrt{\tau}|x(t)|^2+K_M\eta_{r(t)}\frac{1}{\sqrt{\tau}}|x(t)-x(t_\tau)|^2\\
&+\frac{p+1}{2}\bar{\eta}_{r(t)}\bar{\beta}_{r(t)}|x(t)|^{2p}+\frac{(p+1)K_M^2\bar{\eta}_{r(t)}}{2\bar{\beta}_{r(t)}}|x(t)-x(t_\tau)|^2\bigg)I(t).
\end{aligned}
\tag{33}
$$

We can find a non-negative integer n such that $n\tau\le t<(n+1)\tau$. Then we have $t_\tau=n\tau$, and $s_\tau=n\tau$ for any $s\in[n\tau,t]$. Applying the Itô formula and using (6), (11) yields that

$$
\begin{aligned}
E|x(t)-x(n\tau)|^2\le&E\int_{n\tau}^{t}\left(2(x(s)-x(n\tau))^{\mathrm{T}}(f(x(s),s,r(s))+u(x(n\tau),s,r(s))I(s))\right.\\
&\left.+|g(x(s),s,r(s))|^2\right)\mathrm{d}s\\
\le&(2\alpha_M+K_M+L_1)\int_{n\tau}^{t}E|x(s)|^2\mathrm{d}s+\frac{2L_2p}{p+1}\int_{n\tau}^{t}E|x(s)|^{p+1}\mathrm{d}s\\
&+(3K_M+L_1)\int_{k\tau}^{t}E|x(n\tau)|^2\mathrm{d}s+\frac{2L_2}{p+1}\int_{n\tau}^{t}E|x(n\tau)|^{p+1}\mathrm{d}s\\
\le&\left(\frac{2\alpha_M+4K_M+2L_1}{\eta_m}+\frac{2L_2}{\bar{\eta}_m}\right)\tau\sup_{-\tau\le\theta\le0}EU(x(t+\theta),r(t+\theta)).
\end{aligned}
$$

Substituting this into (33), we obtain the assertion (15). This ends the proof. $\square$

Acknowledgments. The authors would like to give sincere gratitude to the reviewers for their professional comments and suggestions. The authors want to thank the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society of Edinburgh (RSE1832), Shanghai Administration of Foreign Experts Affairs (21WZ2503700, the Foreign Expert Program), Chinese Scholarship Council and Strathclyde University (PhD studentship) for their financial support.

REFERENCES

[1] X. Mao, Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, *Automatica*, **49** (2013), 3677-3681.

[2] Y. Li and C. Kou, Robust stabilization of hybrid uncertain stochastic systems with time-varying delay by discrete-time feedback control, *Adv. Differ. Equ.*, **2017** (2017), 1-17.

[3] Y. Ren, W. Yin and R. Sakthivel, Stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation, *Automatica*, **95** (2018), 146-151.

[4] Y. Wu, S. Yan, M. Fan and W. Li, Stabilization of stochastic coupled systems with Markovian switching via feedback control based on discrete-time state observations, *Automatica*, **28** (2018), 247-265.

[5] C. Fei, W. Fei, X. Mao, D. Xia and L. Yan, Stabilization of highly nonlinear hybrid systems by feedback control based on discrete-time state observations, *IEEE Trans. Autom. Control*, **65** (2020), 2899-2912.

[6] Y. Li, J. Lu, C. Kou, X. Mao and J. Pan, Robust discrete-state-feedback stabilization of hybrid stochastic systems with time-varying delay based on Razumikhin technique, *Stat. Probab. Lett.*, **139** (2018), 152-161.

[7] H. Xu and X. Mao, Advances in discrete-state-feedback stabilization of highly nonlinear hybrid systems by Razumikhin technique, *IEEE Trans. Autom. Control*, **68** (2023), 6098-6113.

[8] B. Shi, X. Mao and F. Wu, Stabilisation of hybrid system with different structures by feedback control based on discrete-time state observations, *Nonlinear Anal.-Hybrid Syst.*, **45** (2022), 101198.

[9] Y. Jiang, L. Hu, J. Lu, W. Mao and X. Mao, Stabilization of hybrid systems by intermittent feedback controls based on discrete-time observations with a time delay, *IET Contr. Theory Appl.*, **15** (2021), 2039-2052.

[10] W. Mao, S. You, Y. Jiang and X. Mao, Stochastic stabilization of hybrid neural networks by periodically intermittent control based on discrete-time state observations, *Nonlinear Anal.-Hybrid Syst.*, **48** (2023), 101331.

[11] C. Li, G. Feng and X. Liao, Stabilization of nonlinear systems via periodically intermittent control, *IEEE Trans. Circuits Syst. II-Express Briefs*, **54** (2007), 1019-1023.

[12] W. Xia and J. Cao, Pinning synchronization of delayed dynamical networks via periodically intermittent control, *Chaos*, **19** (2009), 013120.

[13] B. Zhang, F. Deng, S. Peng and S. Xie, Stabilization and destabilization of nonlinear systems via intermittent stochastic noise with application to memristor-based system, *J. Frankl. Inst.-Eng. Appl. Math.*, **355** (2018), 3829-3852.

[14] P. Wang, Y. Hong and H. Su, Stabilization of stochastic complex-valued coupled delayed systems with Markovian switching via periodically intermittent control, *Nonlinear Anal.-Hybrid Syst.*, **29** (2018), 395-413.

[15] Y. Liu, J. Liu and W. Li, Stabilization of highly nonlinear stochastic coupled systems via periodically intermittent control, *IEEE Trans. Autom. Control*, **66** (2020), 4799-4806.

[16] L. Liu and Z. Wu, Intermittent stochastic stabilization based on discrete-time observation with time delay, *Syst. Control Lett.*, **137** (2020), 104626.

[17] T. Chen, W. Wang and Y. Wu, Quasi-synchronization of fuzzy heterogeneous complex networks via intermittent discrete-time state observations control, *IEEE Trans. Fuzzy Syst.*, **30** (2021), 3085-3097.

[18] X. He, C. Ahn and P. Shi, Periodically intermittent stabilization of neural networks based on discrete-time observations, *IEEE Trans. Circuits Syst. II-Express Briefs*, **67** (2020), 3497-3501.

[19] J. Hu, W. Liu, F. Deng and X. Mao, Advances in stabilization of hybrid stochastic differential equations by delay feedback control, *SIAM J. Control Optim.*, **58** (2020), 735-754.

[20] S. Heston, *A simple new formula for options with stochastic volatility*, Course notes of Washington University in St. Louis, Missouri, 1997.

[21] A. Lewis, *Option Valuation under Stochastic Volatility: with Mathematica Code*, Finance Press, 2020.

[22] L. Zhang and Z. Teng, N-species non-autonomous Lotka–Volterra competitive systems with delays and impulsive perturbations, *Nonlinear Anal.-Real World Appl.*, **12** (2011), 3152-3169.

[23] Z. Jin, M. Qiu, K. Tran and G. Yin, A survey of numerical solutions for stochastic control problems: Some recent progress, *Numer. Algebr. Control Optim.*, **12** (2022), 213-253.

[24] X. Mao and C. Yuan, *Stochastic differential equations with Markovian switching*, Imperial college press, 2006.