Methodology and Computing in Applied Probability (2024) 26:6
https://doi.org/10.1007/s11009-024-10077-y

RESEARCH

On Survival of Coherent Systems Subject to Random Shocks

Dheeraj Goyal[1] · **Nil Kamal Hazra**[1,2] · **Maxim Finkelstein**[3,4]

Received: 6 December 2022 / Revised: 27 April 2023 / Accepted: 24 January 2024

Abstract
We consider coherent systems subject to random shocks that can damage a random number of components of a system. Based on the distribution of the number of failed components, we discuss three models, namely, (i) a shock can damage any number of components (including zero) with the same probability, (ii) each shock damages, at least, one component, and (iii) a shock can damage, at most, one component. Shocks arrival times are modeled using three important counting processes, namely, the Poisson generalized gamma process, the Poisson phase-type process and the renewal process with matrix Mittag-Leffler distributed inter-arrival times. For the defined shock models, we discuss relevant reliability properties of coherent systems. An optimal replacement policy for repairable systems is considered as an application of the proposed modeling.

Keywords Coherent system · Shock models · Poisson generalized gamma process · Poisson phase-type process · Renewal process of the matrix Mittag-Leffler type

Mathematics Subject Classification 60E15 · 60K10

1 Introduction

Most of the real-world systems operate in random environments and hence, are often subject to random external shocks. In a broad sense, the term "shock" is used to represent a potentially harmful event of a relatively short duration (e.g., voltage surges in power generation systems, wind gusts for wind turbines, earthquakes for various structures (for example, bridges), failures of cooling systems that result in a sharp rise of temperature of the main system, etc.). In the literature, various shock models have been introduced based on different failure mecha-

✉ Maxim Finkelstein
FinkelM@ufs.ac.za

1 Department of Mathematics, Indian Institute of Technology Jodhpur, 342037 Karwar, Rajasthan, India

2 School of AI & DS, Indian Institute of Technology Jodhpur, 342037 Karwar, Rajasthan, India

3 Department of Mathematical Statistics and Actuarial Science, University of the Free State, 339 Bloemfontein 9300, South Africa

4 Department of Management Science, University of Strathclyde, Glasgow, UK

Published online: 19 February 2024

nisms in systems (see, e.g., Gut and Hüsler 1999; Cha and Finkelstein 2016; Gut 1990; Gong et al. 2018, 2020; Eryilmaz 2017; Goyal et al. 2022a; Mallor and Omey 2001 to name a few).

Shock models were studied not only for single unit systems but also for multi-component systems, namely, k-out-of-n systems, series–parallel system, parallel-series system and general coherent systems (see Barlow and Proschan 1975 for the definitions). Some follow-up articles review a few of these studies. For instance, Sheu and Liou (1992) have considered the optimal replacement policy for a k-out-of-n system subject to random shocks occurred according to the non-homogeneous Poisson process (NHPP). Skoulakis (2000) have discussed reliability characteristics of systems subject to random shocks described by the renewal process. Sheu and Chang (2001) have studied the optimal replacement policy for a k-out-of-n system based on the NHPP. Levitin and Finkelstein (2017a) have studied the optimal backup in non-repairable 1-out-of-N heterogeneous warm standby systems by considering two different causes of failures of components, namely, internal failures and external shocks. Levitin and Finkelstein (2017b) developed a new general approach to obtain the performance characteristics of complex non-repairable systems in the presence of shocks affecting individual element as well as groups of elements. Eryilmaz and Devrim (2019) have considered reliability of a k-out-of-n system subject to random shocks that occur at random times and cause the failure of random number of components. Moreover, it was assumed in their model that shocks occur according to the phase-type (PH) renewal process. Huang et al. (2019) have performed the reliability analysis of coherent systems subject to internal failures and external shocks. Bian et al. (2021) have studied multi-component systems subject to dependent competing failure processes under the extreme and the cumulative shock models. Wang et al. (2022) have introduced a novel mixed shock model for a multi-state weighted k-out-of-n system by considering system's resistance against shocks. Lorvand and Kelkinnama (2023) have studied some reliability properties of k-out-of-n systems subject to random shocks under the δ-shock model.

A brief literature review of multi-component systems subject to random shocks shows that the NHPP and the renewal processes are mostly used to model the process of shocks that affect these systems. However, the NHPP has some basic limitations, e.g., the independent increment property which is a very restrictive in many real-life applications. For example, if there is a larger number of shocks in the past, we may expect the same in the future (positive dependence). There are many more general counting processes that a free from this limitation. For instance, the Poisson generalized gamma process (PGGP) and the Poisson phase-type process (PPHP) are two rather general counting processes that possess the dependent increment property. The PGGP was introduced and studied by Cha and Mercier (2021). This process is mathematically tractable and contains many well-known processes (namely, the HPP), the NHPP, the Pólya process, and the generalized Pólya process (GPP)) as the particular cases. The PPHP was defined and studied by Goyal et al. (2022b). Due to the denseness property of the mixing distribution (namely, the PH distribution), the PPHP can be used to approximate any mixed Poisson process. Moreover, this process is also mathematically tractable due to its matrix-based expressions. Both the PGGP and the PPHP are mixed Poisson processes and hence, have the positive dependence property (see Theorem 4 in Cha and Mercier 2021).

It should be noted that the renewal processes, considered so far in the literature, are not tailored to describe the heavy tail property for inter-arrival times. However, there are many real-life scenarios (e.g., shocks generated by earthquakes, etc.) where the inter-arrival time between two shocks may have the heavy-tailed behavior. These cases can be modeled by using the well-known heavy-tailed distributions (e.g.,, Pareto, log-normal, heavy-tailed Weibull distribution, etc.). In this paper, we propose to use the renewal process of the matrix-Mittag Leffler type (RPMML) as a more general model. Inter-arrival times of the RPMML follow the matrix Mittag-Leffler (MML) distribution which has a heavy-tailed behavior. Moreover, the MML distribution contains many popular distributions (with heavy-tailed behavior) as the particular cases, namely, Mittag-Leffler distribution, fractional Erlang distribution, etc. Moreover, the set of MML distributions is dense in the set of all distributions with non-negative support. It is important to note that the fractional homogeneous Poisson process (FHPP), one of the popular counting processes, is a 'very' special case of the RPMML. The contribution of our paper can be summarized as follows:

We suggest and describe three new models for the effect of shocks on performance of coherent systems. Specifically,

(i) A shock can damage any number of components (including zero) with the same probability;
(ii) Each shock damages at least one component;
(iii) A shock can damage at most one component.

For the defined models, we obtain survival probabilities for coherent systems. These results hold for any counting process. Furthermore, we discuss in detail some special cases by considering three important counting processes, namely, the PPHP, the PGGP and the RPMML. The specific case of the k-out-of-n system is also considered. Our results are presented in the closed form that allow for straightforward computations.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries that include definitions, notation, acronyms and descriptions of some important counting processes. In Section 3, we formulate our model. In Section 4, we discuss some general results for the defined models. In Section 5, we derive some results for the defined models for three specific counting processes, namely, the PGGP, the PPHP and the RPMML. In Section 6, as an application, we discuss the optimal replacement policy. Lastly, concluding remarks are given in Section 7.

To enhance the readability of the paper, all proofs of theorems, lemmas, and corollaries, wherever given, are deferred to Appendix A. Further, a more general model is discussed in Appendix B.

2 Preliminaries

For any random variable U, we denote the cumulative distribution function (cdf) by $F_U(\cdot)$, the survival/reliability function by $\bar{F}_U(\cdot)$, and the probability density function (pdf) by $f_U(\cdot)$; here $\bar{F}_U(\cdot) \equiv 1 - F_U(\cdot)$. Further, we denote the set of natural numbers by $\mathbb{N}$, and the set of real numbers by $\mathbb{R}$. For any set A, $|A|$ represents the cardinality of A. We define $\sum_{i=1}^{0}(\cdot) \equiv 0$. Further, we denote the identity matrix by I. In what follows, we give a list of acronyms to be used throughout the paper:

Acronyms:
PH: phase-type
pdf: probability density function
pmf: probablity mass function
cdf: cumulative distribution function
i.i.d.: independent and identically distributed
HPP: homogeneous Poisson process
MML: matrix Mittag-Leffler
NHPP: non-homogeneous Poisson process
PGGP: Poisson generalized gamma process

PPHP: Poisson phase-type process
FHPP: fractional homogeneous Poisson process
RPMML: renewal process of matrix Mittag-Leffler type

Next, we briefly discuss some important counting processes.

2.1 PGGP

The PGGP is a mixed Poisson process with the generalized gamma mixing distribution. It was introduced by Cha and Mercier (2021). This process could be viewed as a Poisson process having a random intensity function which is a product of a deterministic intensity function and a random variable. Below we give the formal definition of the PGGP. First, we recall the definition of the generalized gamma distribution (see Agarwal and Kalla 1996).

Definition 2.1 A random variable Q is said to have a generalized gamma distribution (GGD) with the set of parameters $\{\nu, \mu, \alpha, l\}$, $\nu \geq 0,\ \mu, \alpha, l > 0$, denoted by $Q \sim GG(\nu, \mu, \alpha, l)$, if its pdf is given by

$$f_Q(q) = \frac{\alpha^{\mu-\nu}}{\Gamma_\nu(\mu, \alpha l)} \frac{q^{\mu-1}\exp\{-\alpha q\}}{(q+l)^\nu}, \quad q > 0,$$

where

$$\Gamma_\nu(\mu, \alpha l) = \int_0^\infty \frac{\alpha^{\mu-\nu} y^{\mu-1}\exp\{-\alpha y\}}{(y+l)^\nu} dy. \tag{2.1}$$

Definition 2.2 A counting process $\{N(t), t \geq 0\}$ is said to be the PGGP with the set of parameters $\{\lambda(t), \nu, \mu, \alpha, l\}$, $\lambda(t) > 0$, for all $t \geq 0$, $\nu \geq 0$, $\mu, \alpha, l > 0$, denoted by $PGGP(\lambda(t), \nu, \mu, \alpha, l)$, if

(a) $\{N(t), t \geq 0 | Q = q\} \sim NHPP(q\lambda(t))$;
(b) $Q \sim GG(\nu, \mu, \alpha, l)$. □

Remark 2.1 The following statements are true (see Goyal et al. 2022a).

(a) The $PGGP(\lambda(t), \nu, \mu, \alpha, l)$, where $\lambda(t) = \lambda\ (> 0)$, $\nu = 0, \alpha = \mu$ and $\mu \to \infty$, is the HPP with the intensity λ, regardless of l;
(b) The $PGGP(\lambda(t), \nu, \mu, \alpha, l)$, where $\nu = 0, \alpha = \mu$ and $\mu \to \infty$, is the NHPP with the intensity function $\lambda(\cdot)$, regardless of l;
(c) The $PGGP(\lambda(t), \nu, \mu, \alpha, l)$, where $\lambda(t) = 1/b\ (> 0)$, $\nu = 0, \mu = \beta, \alpha = 1$, is the Pólya process with the set of parameters $\{\beta, b\}$, regardless of l;
(d) The $PGGP(\lambda(t), \nu, \mu, \alpha, l)$, where $\nu = 0, \mu = \tau/\zeta, \alpha = 1/\zeta$ and $\lambda(t) = \eta(t)\exp\{\zeta \int_0^t \eta(x)dx\}$, is the GPP with the set of parameters $(\eta(t), \zeta, \tau)$, regardless of l.

2.2 PPHP

The PPHP, introduced by Goyal et al. (2022b), is the mixed NHPP with the continuous phase type (PH) mixing distribution. It is well known that "The set of PH distributions is dense in the set of probability distributions on the non-negative real half-line". Thus, the PH distribution can be used to approximate any lifetime distribution and consequently, any mixed Poisson process can also be approximated by the PPHP. This important property of the PPHP makes it distinct from other counting processes. Below, we first define the PH distribution and then give the definition of the PPHP.

Definition 2.3 A non-negative random variable X is said to have a PH distribution, denoted by $X \sim PH(\boldsymbol{\pi}, T)$, if

$$F_X(x) = 1 - \boldsymbol{\pi}\exp\{Tx\}\boldsymbol{e} = 1 - \boldsymbol{\pi}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}T^n\right)\boldsymbol{e}, \quad x \geq 0,$$

where

(i) $\boldsymbol{e}$ is the column vector with all elements being one;
(ii) $\boldsymbol{\pi}$ is a substochastic vector of order m, i.e., $\boldsymbol{\pi}$ is a row vector, all elements of $\boldsymbol{\pi}$ are nonnegative, and $\boldsymbol{\pi}\boldsymbol{e} \leq 1$, where m is a positive integer; and
(iii) T is a subgenerator of order m, i.e., T is an $m \times m$ matrix such that: (a) all diagonal elements are negative; (b) all off-diagonal elements are nonnegative; (c) all row sums are non-positive; and (d) T is invertible. □

The pair $(\boldsymbol{\pi}, T)$ is called a PH representation of X. Without loss of generality, we assume $\boldsymbol{\pi}\boldsymbol{e} = 1$ throughout the paper.

Definition 2.4 A counting process $\{N(t) : t \geq 0\}$ is said to be the PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\pi}, T\}$, denoted by $PPHP(\lambda(t), \boldsymbol{\pi}, T)$, if

(i) $\{N(t) : t \geq 0\}|(X = x) \sim$ NHPP $(x\lambda(t))$;
(ii) $X \sim PH(\boldsymbol{\pi}, T)$,

where $\lambda(t) > 0$ and $(\boldsymbol{\pi}, T)$ is a PH representation of X. □

In the following lemma we give the pmf of the random variable $N(t)$ following the PPHP (see Goyal et al. 2022b).

Lemma 2.1 *Let $\{N(t) : t \geq 0\}$ be the $PPHP(\lambda(t), \boldsymbol{\pi}, T)$. Then pmf of $N(t)$ is given by*

$$P(N(t) = n) = (\Lambda(t))^n\boldsymbol{\pi}(\Lambda(t)I - T)^{-(n+1)}\boldsymbol{T^0}, \quad t > 0, n \in \mathbb{N} \cup \{0\}.$$

2.3 RPMML

The class of MML distributions was defined and studied by Albrecher et al. (2020). These distributions have heavier tails. Below we give the definition of a MML distribution (see Albrecher et al. 2020).

Definition 2.5 Let $(\boldsymbol{\pi}, T)$ be a PH reperesentation and $0 < \gamma \leq 1$. A random variable X is said to have a MML distribution with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$, if its Laplace transform is given by

$$L_X(u) = \boldsymbol{\pi}(u^\gamma I - T)^{-1}\boldsymbol{T^0},$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$. □

We write $X \sim MML(\gamma, \boldsymbol{\pi}, T)$ to indicate that X has a MML distribution with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$. Further, the the cdf of X is given by

$$F_X(x) = 1 - \boldsymbol{\pi}E_{\gamma,1}(Tx^\gamma)\boldsymbol{e}, \tag{2.2}$$

where

$$E_{\gamma,1}(Tx^\gamma) = \sum_{j=0}^{\infty}\frac{T^j x^{\gamma j}}{\Gamma(\gamma j + 1)}.$$

Clearly, when $\gamma = 1$, $X \sim PH(\boldsymbol{\pi}, T)$. Further, if $\boldsymbol{\pi} = 1$ and $T = -\lambda$, then X has a Mittag-Leffler distribution with parameters γ and λ (see Kataria and Vellaisamy 2019).

Next, we define the RPMML. Before that we give the definition of the FHPP.

Definition 2.6 The FHPP with the set of parameters $\{\gamma, \lambda\}$, denoted by FHPP(γ, λ), is a renewal process with inter-arrival times following the Mittag-Leffler distribution with parameters γ and λ.

Definition 2.7 A renewal process with inter-arrival times following the MML distribution with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$ is called the RPMML with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$. □

For convenience, we denote the RPMML with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$ by $RPMML(\gamma, \boldsymbol{\pi}, T)$.

Remark 2.2 The following observations can be made:

(i) The $RPMML(1, \boldsymbol{\pi}, T)$ is the PH renewal process;
(ii) The $RPMML(\gamma, 1, -\lambda)$ is the FHPP (γ, λ). □

In the following lemma we give the pmf of the random variable $N(t)$ following RPMML$(\gamma, \boldsymbol{\pi}, T)$.

Lemma 2.2 *Let $\{N(t) : t \geq 0\}$ be the RPMML$(\gamma, \boldsymbol{\pi}, T)$. Then the pmf of $N(t)$ is given by*

$$P(N(t) = n) = \boldsymbol{\pi}_{\boldsymbol{n+1}} E_{\gamma,1}(T_{n+1} t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{n}} E_{\gamma,1}(T_n t^{\gamma})\boldsymbol{e}, \quad t > 0, \ n = 0, 1, 2, \dots,$$

where $\boldsymbol{\pi}_{\boldsymbol{n}} = (\boldsymbol{\pi}, 0, \dots, 0)$ and

$$T_n = \begin{pmatrix} T & -T\boldsymbol{e}\boldsymbol{\pi}_{\boldsymbol{n-1}} \\ 0 & T_{n-1} \end{pmatrix},$$

for $n \geq 2$, with $\boldsymbol{\pi}_{\boldsymbol{1}} = \boldsymbol{\pi}$, $T_1 = T$ and $\boldsymbol{\pi}_{\boldsymbol{0}} = T_0 = 0$.

3 Model Description

Let L be a random variable representing the lifetime of a coherent system with n i.i.d. components that started operating time $t = 0$. Assume that the system is subject to external shocks that arrive at random times being the only cause of system's failure. Let $\{N(t) : t \geq 0\}$ be an orderly counting process where $N(t)$ represents the number of shocks arrived by the time t. Assume that each shock can affect a random number of components. Let Z_i be the random variable representing the number of components failed due to the i-th shock, $i \in \mathbb{N}$. In what follows, we give a list of model assumptions.

Assumptions:

- Each shock is harmless with probability p_0 and harmful with probability $1 - p_0$, i.e., $P(Z_i = 0) = p_0$ and $P(Z_i \neq 0) = 1 - p_0$, for $i \in \mathbb{N}$.
- If the i-th shock is harmful then it can damage j components with probability p_j, i.e., $P(Z_i = j) = p_j$, for all $j = 1, 2, \dots, n$, $i \in \mathbb{N}$. Clearly, $p_1 + p_2 + \cdots + p_n = 1 - p_0$.
- $\{Z_i : i \in \mathbb{N}\}$ is a sequence of i.i.d. random variables. This assumption is made for mathematical convenience. Appendix B contains a more general setup where Z_i's are dependent with a specific dependency structure.
- The shock process $\{N(t) : t \geq 0\}$ and $\{Z_i : i \in \mathbb{N}\}$ are independent.

In what follows in this paper, we consider three special cases (with respect to distribution of Z_i):

- **M1**: $P(Z_i = j) = 1/(n+1)$, $j = 0, 1, 2, \ldots, n$, i.e., Z_i follows the uniform distribution over $\{0, 1, 2, \ldots, n\}$. In other words, a shock can damage any number of components (including zero) with the same probability. This model may be appropriate to use in a situation where the distribution of Z_i's is in question.
- **M2**: Each shock is assumed to be harmful, and at least one component is affected by the shock i.e., $p_0 = 0$. Consequently, the range of Z_i is given by $\{1, 2, \ldots, n\}$, for all $i \in \mathbb{N}$.
- **M3**: Each shock can harm at most one component i.e., $p_2 = p_3 = \cdots = p_n = 0$. Consequently, the range of Z_i is given by $\{0, 1\}$, for all $i \in \mathbb{N}$.

Some real-world settings that comply with the described models are given below:

- Consider an apartment lighting system where n lamps are used to illuminate the apartment. The apartment's lighting is suitable if at least k ($1 \leq k \leq n$) lamps are on. If the magnitude of electricity is above a fixed threshold, then some of these lamps may be damaged. Consequently, the brightness of the apartment will be reduced. If each lamp is considered a component and electric impulses with high magnitudes are considered as shocks, then one can associate this problem with the model introduced in this paper. This example is borrowed from Lorvand and Kelkinnama (2023).
- The Proton Exchange Membrane Fuel Cells power systems, described in detail in Eryilmaz and Devrim (2019), is another example of the proposed model.
- In a cricket match, the batting team can be considered as a 2-out-of-11 system. Each ball to the batsman can be considered a shock for the team (system). This setting can be described by the M_3 model.

4 General Model

In this section, we obtain some general results. We derive the expression for the survival function of a coherent system for the defined model that is valid for an arbitrary orderly counting process $\{N(t) : t \geq 0\}$. We first consider the k-out-of-n systems, and then generalize the result to coherent systems.

Let $M(t)$ be a random variable representing the number of failed components of a system by time t. Then

$$M(t) = \sum_{i=1}^{N(t)} Z_i. \tag{4.1}$$

Further, let $L_{k:n}$ be a random variable representing the lifetime of a k-out-of-n system with n i.i.d. components. Note that this system functions until the $(n-k+1)$-th component fails. Then

$$\bar{F}_{L_{k:n}}(t) = P(M(t) < n-k+1) = \sum_{i=0}^{n-k} P(M(t) = i). \tag{4.2}$$

Further, let L be a random variable representing the lifetime of a coherent system with n i.i.d. components, described by the system's signature vector $(s_1, s_2, \ldots, s_n)$ (see Samaniego 2007). Then

$$\bar{F}_L(t) = \sum_{k=1}^{n} P(L > t | L = L_{k:n}) P(L = L_{k:n}) = \sum_{k=1}^{n} s_k \bar{F}_{L_{k:n}}(t), \tag{4.3}$$

where $s_k = P(L = L_{k:n})$, for $k = 1, 2, \ldots, n$.

To obtain the survival functions of L and $L_{k:n}$, we first need to find the probabilities $P(M(t) = i)$, for $i = 0, 1, 2, \ldots, n-k$. In the following lemma, we derive these probabilities.

Lemma 4.1 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, $i \in \mathbb{N}$, and $\{N(t) : t \geq 0\}$ be a counting process. Then*

$$P(M(t) = 0) = \sum_{j=0}^{\infty} p_0^j P(N(t) = j)$$

and

$$P(M(t) = i) = \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t) = j), \quad i = 1, 2, \ldots, n-1,$$

where $\Omega_{m,i} = \{z = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \leq z_l \leq n, 1 \leq l \leq m\}$ and $1 \leq m \leq i < n$. □

The next theorem immediately follows from Lemma 4.1 and (4.2).

Theorem 4.1 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, $i \in \mathbb{N}$. Assume that shocks occur according to a counting process $\{N(t) : t \geq 0\}$. Then*

$$\bar{F}_{L_{k:n}}(t) = \sum_{j=0}^{\infty} p_0^j P(N(t) = j) + \sum_{i=1}^{n-k} \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t) = j),$$

where $\Omega_{m,i} = \{z = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \leq z_l \leq n, 1 \leq l \leq m\}$ and $1 \leq m \leq i \leq n-k$. □

The succeeding corollary follows from Theorem 4.1.

Corollary 4.1 *The following results hold true.*

(i) For model M_1:

$$\bar{F}_{L_{k:n}}(t) = \sum_{j=0}^{\infty} \left(\frac{1}{n+1} \right)^j P(N(t) = j) + \sum_{i=1}^{n-k} \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \binom{i-1}{m-1} \left(\frac{1}{n+1} \right)^j P(N(t) = j);$$

(ii) For model M_2:

$$\bar{F}_{L_{k:n}}(t) = P(N(t) = 0) + \sum_{i=1}^{n-k} \sum_{m=1}^{i} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) P(N(t) = m);$$

(iii) For model M_3:

$$\bar{F}_{L_{k:n}}(t) = \sum_{j=0}^{\infty} p_0^j P(N(t)=j) + \sum_{i=1}^{n-k} \sum_{j=i}^{\infty} \binom{j}{i} p_1^i p_0^{j-i} P(N(t)=j).$$

□

In the next theorem we derive the survival function of a coherent system for the defined model. The proof is straightforward by using (4.3) and Theortem 4.1.

Theorem 4.2 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, $i \in \mathbb{N}$. Assume that shocks occur according to a counting process $\{N(t) : t \geq 0\}$. Then*

$$\bar{F}_L(t) = \sum_{j=0}^{\infty} p_0^j P(N(t)=j) + \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \sum_{m=1}^{i} \sum_{j=m}^{\infty} s_k \binom{j}{m} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t)=j),$$

where $\Omega_{m,i} = \{\mathbf{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \leq z_l \leq n, 1 \leq l \leq m\}$, $s_k = P(L = L_{k:n})$, $1 \leq k \leq n-1$ and $1 \leq m \leq i \leq n-k$.

Remark 4.1 By proceeding in the same line as in Corollary 4.1, the survival function of a coherent system for models M_1, M_2 and M_3 can be obtained.

5 Some Special Cases

In this section, we derive survival functions for the defined model for some specific shock processes, namely, the PGGP, the PPHP and the RPMML. As these processes are quite 'sophisticated', each case needs specific technique and reasoning.

5.1 PGGP

Theorem 5.1 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, $i \in \mathbb{N}$. Assume that shocks occur according to the $PGGP(\lambda(t), \nu, \mu, \alpha, l)$. Then*

$$\bar{F}_{L_{k:n}}(t) = \left(\frac{\alpha}{\alpha + (1-p_0)\Lambda(t)} \right)^{\mu-\nu} \left[\frac{\Gamma_\nu(\mu, (\alpha + (1-p_0)\Lambda(t))l)}{\Gamma_\nu(\mu, \alpha l)} + \sum_{i=1}^{n-k} \sum_{m=1}^{i} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) \left(\frac{\Lambda(t)}{\alpha + (1-p_0)\Lambda(t)} \right)^m \frac{\Gamma_\nu(m+\mu, (\alpha + (1-p_0)\Lambda(t))l)}{m! \Gamma_\nu(\mu, \alpha l)} \right],$$

where $\Omega_{m,i} = \{\mathbf{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \leq z_l \leq n, 1 \leq l \leq m\}$ and $1 \leq m \leq i \leq n-k$. □

The next corollary immediately follows from Theorem 5.1. Here, we obtain the survival function of the k-out-of-n system for models M_1, M_2 and M_3.

Corollary 5.1 *The following results hold true.*

(i) For model M_1:

$$\bar{F}_{L_{k:n}}(t) = \left(\frac{\alpha}{\alpha + (n/(n+1))\Lambda(t)}\right)^{\mu-\nu} \left[\frac{\Gamma_\nu(\mu, (\alpha + (n/(n+1))\Lambda(t))l)}{\Gamma_\nu(\mu, \alpha l)} + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\binom{i-1}{m-1} \times \left(\frac{1}{n+1}\right)^m \left(\frac{\Lambda(t)}{\alpha + (n/(n+1))\Lambda(t)}\right)^m \frac{\Gamma_\nu(m+\mu, (\alpha + (n/(n+1))\Lambda(t))l)}{m!\Gamma_\nu(\mu, \alpha l)}\right];$$

(ii) For model M_2:

$$\bar{F}_{L_{k:n}}(t) = \left(\frac{\alpha}{\alpha + \Lambda(t)}\right)^{\mu-\nu} \left[\frac{\Gamma_\nu(\mu, (\alpha + \Lambda(t))l)}{\Gamma_\nu(\mu, \alpha l)} + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\left(\frac{\Lambda(t)}{\alpha + \Lambda(t)}\right)^m \frac{\Gamma_\nu(m+\mu, (\alpha + \Lambda(t))l)}{m!\Gamma_\nu(\mu, \alpha l)}\right];$$

(iii) For model M_3:

$$\bar{F}_{L_{k:n}}(t) = \left(\frac{\alpha}{\alpha + p_1\Lambda(t)}\right)^{\mu-\nu} \left[\frac{\Gamma_\nu(\mu, (\alpha + p_1\Lambda(t))l)}{\Gamma_\nu(\mu, \alpha l)} + \sum_{i=1}^{n-k}\left(\frac{p_1\Lambda(t)}{\alpha + p_1\Lambda(t)}\right)^i \frac{\Gamma_\nu(i+\mu, (\alpha + p_1\Lambda(t))l)}{i!\Gamma_\nu(\mu, \alpha l)}\right].$$

□

The next corollary immediately follows from Theorem 5.1, and Remark 2.1 (b) and (c).

Corollary 5.2 *For the general model, the following results hold true for the k-out-of-n system.*

(i) Assume that shocks occur according to the NHPP with intensity $\lambda(t)$. Then

$$\bar{F}_{L_{k:n}}(t) = \exp\{-(1-p_0)\Lambda(t)\}\left[1 + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Lambda(t)^m}{m!}\right].$$

(ii) Assume that shocks occur according to the Pólya with the set of parameters $\{\beta, b\}$. Then

$$\bar{F}_{L_{k:n}}(t) = \left(\frac{b}{(1-p_0)t + b}\right)^{\beta}\left[1 + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Gamma(\beta+m)}{\Gamma(\beta)m!}\left(\frac{t}{(1-p_0)t + b}\right)^m\right].$$

Remark 5.1 From the above corollary, one can obtain the survival functions of the system for models M_1, M_2 and M_3 for the NHPP and the Pólya shock processes. □

In the succeeding theorem, we derive the survival function of a coherent system for the defined model. The proof follows from (4.3) and Theorem 5.1.

Theorem 5.2 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, $i \in \mathbb{N}$. Assume that shocks occur according to the $PGGP(\lambda(t), \nu, \mu, \alpha, l)$. Then*

$$\bar{F}_L(t) = \left(\frac{\alpha}{\alpha + (1-p_0)\Lambda(t)}\right)^{\mu-\nu}\left[\frac{\Gamma_\nu(\mu, (\alpha + (1-p_0)\Lambda(t))l)}{\Gamma_\nu(\mu, \alpha l)} + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k \left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right) \times \left(\frac{\Lambda(t)}{\alpha + (1-p_0)\Lambda(t)}\right)^m \frac{\Gamma_\nu(m+\mu, (\alpha + (1-p_0)\Lambda(t))l)}{m!\Gamma_\nu(\mu, \alpha l)}\right],$$

where $\Omega_{m,i} = \{z = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$, $s_k = P(L = L_{k:n})$, and $1 \le m \le i \le n-k$. □

The next corollary follows from Theorem 5.2, and Remark 2.1 (b) and (c).

Corollary 5.3 *For the general model, the following results hold true.*

(i) Assume that shocks occur according to the NHPP with intensity $\lambda(t)$. Then

$$\bar{F}_L(t) = \exp\{-(1-p_0)\Lambda(t)\}\left[1 + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k \left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Lambda(t)^m}{m!}\right].$$

(ii) Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$. Then

$$\bar{F}_L(t) = \left(\frac{b}{(1-p_0)t + b}\right)^{\beta}\left[1 + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k \left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Gamma(\beta+m)}{\Gamma(\beta)m!}\left(\frac{t}{(1-p_0)t + b}\right)^m\right].$$

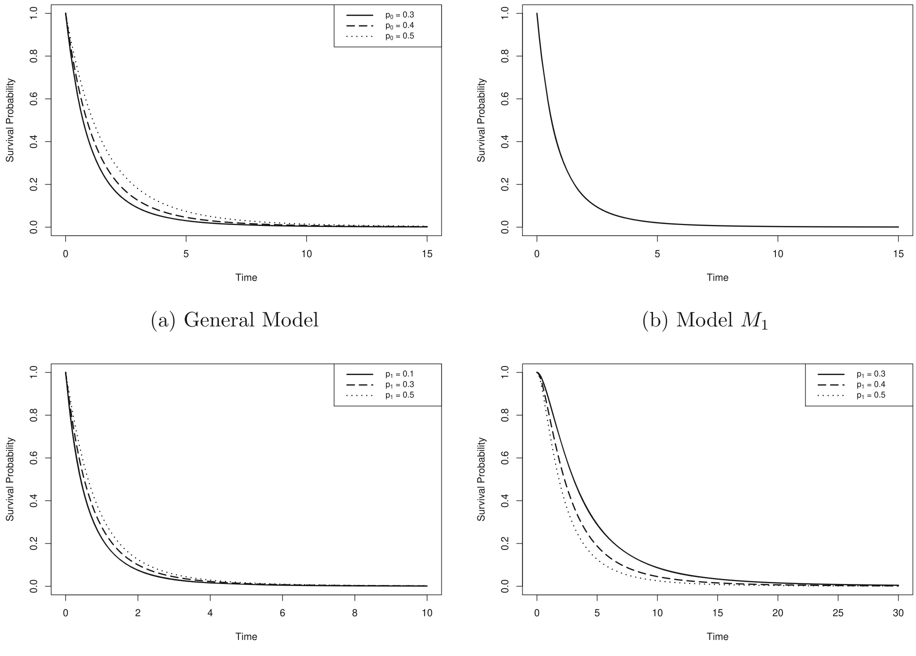
□

In the succeeding example, we illustrate the result given in Corollary 5.2 (ii).

Example 5.1 Consider a 3-out-of-4 system, i.e., $k = 3$ and $n = 4$. Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$. Then, from Corollary 5.2 (ii), we get

$$\bar{F}_{L_{3:4}}(t) = \left(\frac{b}{(1-p_0)t + b}\right)^{\beta}\left[1 + \beta p_1 \left(\frac{t}{(1-p_0)t + b}\right)\right].$$

Let $\beta = 4$ and $b = 2$. In Fig. 1a, we plot the above survival function against $t \in [0, 15]$, for different values of p_0 and for fixed $p_1 = 0.2$. This figure shows that the system's survival probability increases as p_0 increases. In Fig. 1b, we plot the system's survival function against $t \in [0, 15]$, for model M_1, with $p_0 = 1/5$ and $p_2 = 1/5$. In Fig. 1c, the system's survival function against $t \in [0, 10]$, for different values of p_1, is plotted for model M_2. From this figure, we conclude that an increment in p_1 increases the system's survivability. In Fig. 1d, we plot the system's survival function against $t \in [0, 30]$, for different values of p_1, for model M_3. This figure shows that an increment in p_1 decreases the system's survivability. □



(a) General Model (b) Model M_1

(c) Model M_2 (d) Model M_3

Fig. 1 Plot of the survival function of a 3-out-of-4 system against time

In the next example, we illustrate the result given in Corollary 5.3 (ii).

Example 5.2 Consider a coherent system with the structure function (see Barlow and Proschan 1975) given by

$$\phi(x_1, x_2, x_3) = min(x_1, max(x_2, x_3)).$$

Now, it can easily be shown that $(s_1, s_2, s_3) = (1/3, 2/3, 0)$. Then, from Corollary 5.3 (ii), we get

$$\begin{aligned}\bar{F}_L(t) = \left(\frac{b}{(1-p_0)t+b}\right)^{\beta} \Bigg[1 &+ p_1 \left(\frac{\beta t}{(1-p_0)t+b}\right) \\ &+ \frac{1}{3} p_2 \left(\frac{\beta t}{(1-p_0)t+b}\right) \\ &+ \frac{1}{3}\left(\frac{\beta(\beta+1)}{2}\right)\left(\frac{p_1 t}{(1-p_0)t+b}\right)^2 \Bigg].\end{aligned}$$

For fixed $b = 2$ and $\beta = 4$, we plot the survival function of the above coherent system for different failure probability vectors (p_0, p_1, p_2, p_3) in Fig. 2.

In the succeeding theorem, the mean lifetime of a k-out-of-n system is derived under the assumption that shocks occur according to the Pólya process or the HPP.

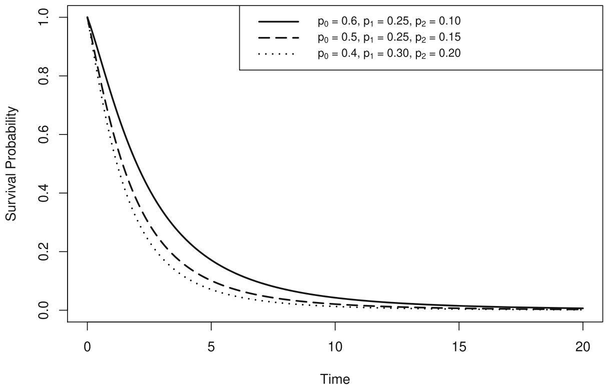


Fig. 2 Plot of the survival function of a coherent system with the signature vector $(s_1, s_2, s_3) = (1/3, 2/3, 0)$ against time

Theorem 5.3 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$.*

(i) Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta > 1$. Then the mean lifetime of the k-out-of-n system is given by

$$E(L_{k:n}) = \left(\frac{b}{\beta-1}\right)\left(\frac{1}{1-p_0}\right)\left[1+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\left(\frac{1}{1-p_0}\right)^m\right];$$

(ii) Assume that shocks occur according to the HPP with the intensity λ. Then the mean lifetime of the k-out-of-n system is given by

$$E(L_{k:n}) = \frac{1}{\lambda}\left(\frac{1}{1-p_0}\right)\left[1+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\left(\frac{1}{1-p_0}\right)^m\right];$$

here $\Omega_{m,i} = \{\mathbf{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$ and $1 \le m \le i \le n-k$.

The next corollary immediately follows from the above theorem. Here we discuss the results for models M_1, M_2 and M_3.

Corollary 5.4 *The following results hold true.*

(i) For model M_1: If the shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta > 1$, then

$$E(L_{k:n}) = \left(\frac{b}{\beta-1}\right)\left(1+\frac{1}{n}\right)^{n-k+1},$$

and if shocks occur according to the HPP with the intensity λ, then

$$E(L_{k:n}) = \frac{1}{\lambda}\left(1+\frac{1}{n}\right)^{n-k+1}.$$

(ii) For model M_2: If shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta > 1$, then

$$E(L_{k:n}) = \left(\frac{b}{\beta-1}\right)\left[1+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\right],$$

and if shocks occur according to the HPP with the intensity λ*, then*

$$E(L_{k:n}) = \frac{1}{\lambda}\left[1 + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\right].$$

(iii) For model M_3*: If shocks occur according to the Pólya process with the set of parameters* $\{\beta, b\}$*,* $\beta > 1$*, then*

$$E(L_{k:n}) = \left(\frac{b}{\beta - 1}\right)\left(\frac{1+n-k}{p_1}\right),$$

and if shocks occur according to the HPP with the intensity λ*, then*

$$E(L_{k:n}) = \frac{1}{\lambda}\left(\frac{1+n-k}{p_1}\right).$$

□

The following example illustrates the results given in Theorem 5.3 (ii) and Corollary 5.4.

Example 5.3 Consider a 3-out-of-4 system. Assume that shocks occur according to the HPP with the intensity λ. In Table 1, we calculate the mean lifetime of the system for different models and different intensity λ of the HPP. Table 1 shows that an increment in parameter λ decreases the system's mean lifetime for each model.

In the following theorem we derive the mean lifetime of a coherent system for the general model.

Theorem 5.4 *Let* $P(Z_i = j) = p_j$*, for all* $j = 0, 1, 2, \ldots, n$*, and* $i \in \mathbb{N}$*.*

(i) Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$*,* $\beta > 1$*. Then the mean lifetime of the coherent system is given by*

$$E(L) = \left(\frac{b}{\beta - 1}\right)\left(\frac{1}{1-p_0}\right)\left[1 + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\left(\frac{1}{1-p_0}\right)^k\right];$$

Table 1 The mean lifetime of the system for HPP with the intensity λ

Models	(p_0, p_1, p_2, p_3)	Intensity (λ)	Mean lifetime ($E(L_{3:4})$)
General Model	$(p_0, p_1, p_2, p_3) =$	$\lambda = 1$	2.50
	(0.4,0.3,0.2,0.1)	$\lambda = 2$	1.25
		$\lambda = 3$	0.83
M_1	$(p_0, p_1, p_2, p_3) =$	$\lambda = 1$	1.56
	(0.4,0.3,0.2,0.1)	$\lambda = 2$	0.78
		$\lambda = 3$	0.52
M_2	$(p_0, p_1, p_2, p_3) =$	$\lambda = 1$	1.30
	(0.4,0.3,0.2,0.1)	$\lambda = 2$	0.65
		$\lambda = 3$	0.43
M_3	$(p_0, p_1, p_2, p_3) =$	$\lambda = 1$	6.67
	(0.7,0.3,0,0)	$\lambda = 2$	3.33
		$\lambda = 3$	2.22

(ii) Assume that shocks occur according to the HPP with the intensity λ. *Then the mean lifetime of the coherent system is given by*

$$E(L) = \left(\frac{1}{\lambda}\right)\left(\frac{1}{1-p_0}\right)\left[1+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\left(\frac{1}{1-p_0}\right)^k\right];$$

here $\Omega_{m,i} = \{z = (z_1, z_2, \ldots, z_m)|z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$, $s_k = P(L = L_{k:n})$, $1 \le k \le n-1$ *and* $1 \le m \le i \le n-k$. □

The following example illustrates the results given in Theorem 5.4 (ii).

Example 5.4 Consider the coherent system mentioned in Example 5.2. Assume that shocks occur according to the HPP with the intensity λ. Then, from Theorem 5.4 (ii), we have

$$E(L) = \left(\frac{1}{\lambda}\right)\left(\frac{1}{1-p_0}\right)\left[1+\left(\frac{1}{3}\right)\left(\frac{p_1+p_2+p_1^2}{1-p_0}\right)+\left(\frac{2p_1}{3}\right)\left(\frac{1}{1-p_0}\right)^2\right].$$

In Table 2, we calculate the mean lifetime of the system for different intensity λ of the HPP by considering parameters $p_0 = 0.4$, $p_1 = 0.3$, $p_2 = 0.2$ and $p_3 = 0.1$. This table shows that an increment in intensity (λ) of the shock process (HPP) decreases the system lifetime.

5.2 PPHP

Theorem 5.5 *Let* $P(Z_i = j) = p_j$, *for all* $j = 0, 1, 2, \ldots, n$, *and* $i \in \mathbb{N}$. *Assume that shocks occur according to the* $PPHP(\lambda(t), \boldsymbol{\pi}, T)$. *Then*

$$\begin{aligned}\bar{F}_{L_{k:n}}(t) =& \boldsymbol{\pi}(\Lambda(t)(1-p_0)I-T)^{-1}\boldsymbol{T^0}\\ &+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\Lambda(t)^m\boldsymbol{\pi}(\Lambda(t)(1-p_0)I-T)^{-(m+1)}\boldsymbol{T^0},\end{aligned}$$

where $\Omega_{m,i} = \{z = (z_1, z_2, \ldots, z_m)|z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$ *and* $1 \le m \le i \le n-k$. □

The following corollary immediately follows from Theorem 5.5. Here we discuss the results for specific models M_1, M_2 and M_3.

Corollary 5.5 *The following results hold true.*

(i) For model M_1:

$$\begin{aligned}\bar{F}_{L_{k:n}}(t) = & \boldsymbol{\pi}\left(\Lambda(t)\left(\frac{n}{n+1}\right)I-T\right)^{-1}\boldsymbol{T^0}\\ &+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\binom{i-1}{m-1}\left(\frac{\Lambda(t)}{n+1}\right)^m\boldsymbol{\pi}\left(\Lambda(t)\left(\frac{n}{n+1}\right)I-T\right)^{-(m+1)}\boldsymbol{T^0};\end{aligned}$$

Table 2 The mean lifetime of a coherent system for HPP with the intensity λ

Intensity (λ)	Mean lifetime ($E(L)$)
$\lambda = 1$	3.13
$\lambda = 2$	1.57
$\lambda = 3$	1.04

(ii) For model M_2:

$$\bar{F}_{L_{k:n}}(t) = \boldsymbol{\pi}(\Lambda(t)I - T)^{-1}\boldsymbol{T^0} + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\Lambda(t)^m\boldsymbol{\pi}(\Lambda(t)I - T)^{-(m+1)}\boldsymbol{T^0};$$

(iii) For model M_3:

$$\bar{F}_{L_{k:n}}(t) = \boldsymbol{\pi}(\Lambda(t)p_1 I - T)^{-1}\boldsymbol{T^0} + \sum_{i=1}^{n-k} p_1^i\Lambda(t)^i\boldsymbol{\pi}(\Lambda(t)p_1 I - T)^{-(i+1)}\boldsymbol{T^0}.$$

In the next theorem, we derive the survival function of a coherent system for the general model.

Theorem 5.6 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$. Assume that shocks occur according to the $PPHP(\lambda(t), \boldsymbol{\pi}, T)$. Then*

$$\bar{F}_L(t) = \boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-1}\boldsymbol{T^0} + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\Lambda(t)^m\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-(m+1)}\boldsymbol{T^0},$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$, $s_k = P(L = L_{k:n})$, $1 \le k \le n-1$ and $1 \le m \le i \le n-k$. □

The results given in above theorems and corollary are illustrated by the following two examples.

Example 5.5 Consider a 3-out-of-4 system. Then, from Theorem 5.5, we have

$$\bar{F}_{L_{3:4}}(t) = \boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-1}\boldsymbol{T^0} + p_1\Lambda(t)\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-2}\boldsymbol{T^0}.$$

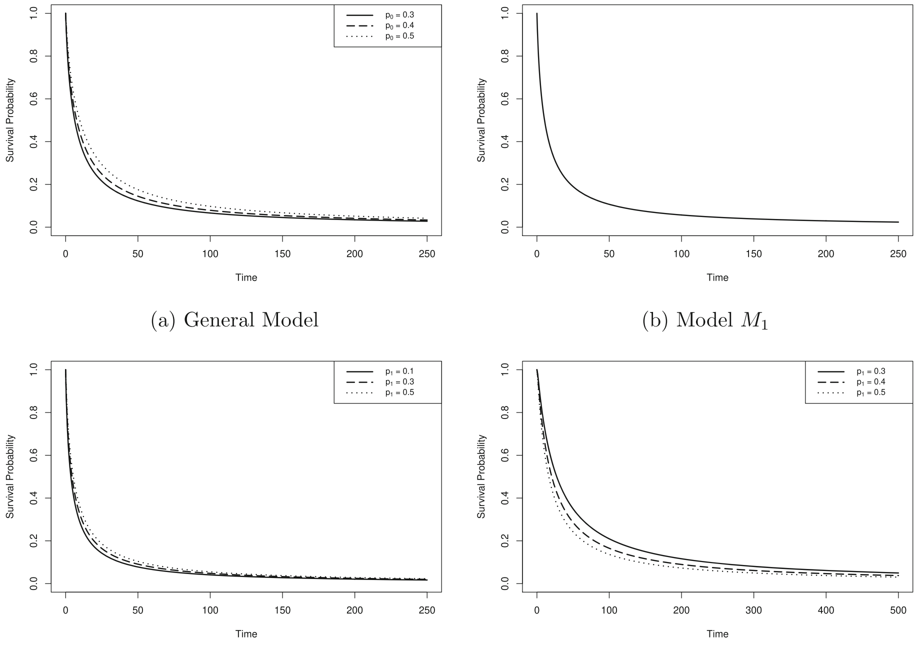
Let the parameters be

$$\boldsymbol{\pi} = (0.2, 0.8), \quad T = \begin{pmatrix} -2 & 1 \\ 0.5 & -10 \end{pmatrix} \text{ and } \lambda = 2.$$

In Fig. 3a, we plot the survival function of the system (given in Theorem 5.5) against $t \in [0, 250]$, for different values of p_0 and for fixed $p_1 = 0.2$. In Fig. 3b, we plot the system's survival function (given in Corollary 5.5 (i)) against $t \in [0, 250]$, for fixed $p_0 = 1/5$ and $p_2 = 1/5$. In Fig. 3c, we plot the system's survival function (given in Corollary 5.5 (ii)) against $t \in [0, 250]$, for different values of p_1. Lastly, in Fig. 3d, we plot the system's survival function (given in Corollary 5.5 (iii)) against $t \in [0, 500]$, for different values of p_1. From all these figures, we can make the same observations as in Example 5.1.

Example 5.6 Consider the coherent system mentioned in Example 5.2. Then, from Theorem 5.6, we have

$$\begin{aligned}\bar{F}_L(t) = {} & \boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-1}\boldsymbol{T^0} \\ & + p_1\Lambda(t)\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-2}\boldsymbol{T^0} \\ & + s_1 p_2\Lambda(t)\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-2}\boldsymbol{T^0} \\ & + s_1 p_1^2\Lambda(t)^2\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-3}\boldsymbol{T^0}.\end{aligned}$$



(a) General Model (b) Model M_1

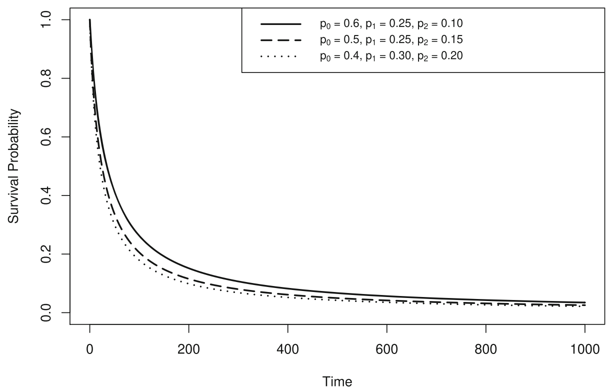
(c) Model M_2 (d) Model M_3

Fig. 3 Plot of the survival function of a 3-out-of-4 system against time, for different models

In Fig. 4, we plot this against $t \in [0, 1000]$, for different failure probability vectors (p_0, p_1, p_2, p_3) and for fixed parameters values given by

$$\boldsymbol{\pi} = (0.2, 0.8),\ T = \begin{pmatrix} -2 & 1 \\ 0.5 & -10 \end{pmatrix}, \lambda = 1.$$

Fig. 4 Plot of $\bar{F}_L(t)$ against $t \in [0, 1000]$



5.3 RPMML

In the following theorem we derive the survival function of a k-out-of-n system for the defined model by assuming the shock process as the RPMML. The proof immediately follows from Theorem 4.1 and Lemma 2.2 and hence, omitted.

Theorem 5.7 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$. Assume that shocks occur according to the $RPMML(\gamma, \boldsymbol{\pi}, T)$. Then*

$$\bar{F}_{L_{k:n}}(t) = \boldsymbol{\pi} E_{\gamma,1}(Tt^{\gamma})\boldsymbol{e} + \sum_{j=1}^{\infty} p_0^j (\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e})$$
$$+ \sum_{i=1}^{n-k} \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} (\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e}$$
$$- \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}),$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$, $1 \le m \le i \le n - k$, and $\boldsymbol{\pi}_{\boldsymbol{j}}$'s and T_j's are the same as in Lemma 2.2. □

The next corollary immediately follows from Theorem 5.7.

Corollary 5.6 *The following results hold true.*

(i) For model M_1:

$$\bar{F}_{L_{k:n}}(t) = \sum_{j=0}^{\infty} (1/(n+1))^j \, (\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e})$$
$$+ \sum_{i=1}^{n-k} \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \binom{i-1}{m-1} (1/(n+1))^j$$
$$(\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e});$$

(ii) For model M_2:

$$\bar{F}_{L_{k:n}}(t) = \boldsymbol{\pi} E_{\gamma,1}(Tt^{\gamma})\boldsymbol{e}$$
$$+ \sum_{i=1}^{n-k} \sum_{m=1}^{i} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) (\boldsymbol{\pi}_{\boldsymbol{m+1}} E_{\gamma,1}(T_{m+1}t^{\gamma})\boldsymbol{e}$$
$$- \boldsymbol{\pi}_{\boldsymbol{m}} E_{\gamma,1}(T_m t^{\gamma})\boldsymbol{e});$$

(iii) For model M_3:

$$\bar{F}_{L_{k:n}}(t) = \sum_{j=0}^{\infty} p_0^j (\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e})$$
$$+ \sum_{i=1}^{n-k} \sum_{j=i}^{\infty} \binom{j}{i} p_1^i p_0^{j-i} (\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}).$$

The survival function of a coherent system for the defined model is given in the following theorem. The proof is obvious and hence, omitted.

Theorem 5.8 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$. Assume that shocks occur according to the $RPMML(\gamma, \boldsymbol{\pi}, T)$. Then*

$$\bar{F}_L(t) = \boldsymbol{\pi} E_{\gamma,1}(Tt^\gamma)\boldsymbol{e} + \sum_{j=1}^{\infty} p_0^j (\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^\gamma)\boldsymbol{e})$$
$$+ \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i}\sum_{j=m}^{\infty} s_k \binom{j}{m} \left(\sum_{z\in\Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m}$$
$$(\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^\gamma)\boldsymbol{e}),$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$, $s_k = P(L = L_{k:n})$, $1 \le k \le n-1$, $1 \le m \le i \le n-k$, and $\boldsymbol{\pi}_{\boldsymbol{j}}$'s and T_j's are the same as in Lemma 2.2. □

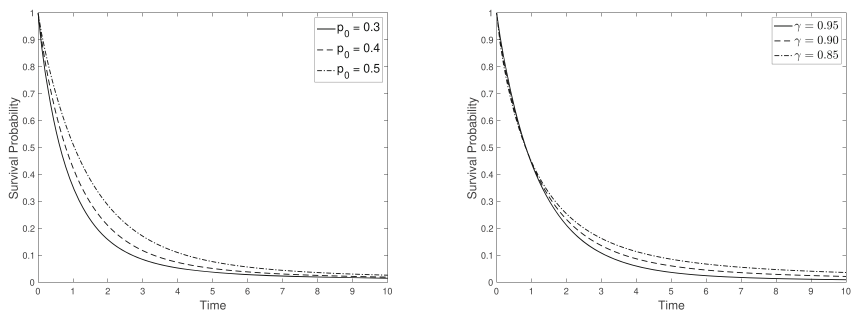
Remark 5.2 Since $\int_0^\infty \boldsymbol{\pi} E_{\gamma,1}(Tt^\gamma)\boldsymbol{e}dt = \infty$ for $\gamma < 1$, one can conclude from Theorem 5.8 that $E(L) = \infty$, for $0 < \gamma < 1$.

In the next two examples, we illustrate the foregoing results.

Example 5.7 Consider a 3-out-of-4 system. Then, from Theorem 5.7, we have

$$\bar{F}_{L_{3:4}}(t) = \boldsymbol{\pi} E_{\gamma,1}(Tt^\gamma)\boldsymbol{e} + \sum_{j=1}^{\infty}(p_0^j + jp_0^{j-1}p_1)(\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^\gamma)\boldsymbol{e}).$$

Let the parameters be $\boldsymbol{\pi}_{\boldsymbol{1}} = 1$, $T = -\lambda$ and $\lambda = 2$, i.e., the shock process be the FHPP with the set of parameters $\{\gamma, 2\}$. In Fig. 5a, we plot the survival function of the system (given in Theorem 5.7) against $t \in [0, 10]$, for different values of p_0, and for fixed $\gamma = 0.9$ and $p_1 = 0.2$. This figure show that the system survivability increases as p_0 increases. In Fig. 5b, we plot the survival function of the system against $t \in [0, 10]$ for the same model, for different values of γ, and for fixed $p_0 = 0.5$ and $p_1 = 0.1$.



(a) Plot of system's survival function against time, for different values of p_0.

(b) Plot of system's survival function against time, for different values of γ.

Fig. 5 Plot of the survival function of a 3-out-of-4 system against time

Example 5.8 Consider the coherent system given in Example 5.2. Then, from Theorem 5.8, we have

$$\bar{F}_L(t) = \boldsymbol{\pi} E_{\gamma,1}(Tt^{\gamma})\boldsymbol{e} + \sum_{j=1}^{\infty}(p_1 + s_1 p_2) j p_0^{j-1}(\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e})$$
$$+ s_1 \sum_{j=2}^{\infty} p_1^2 \binom{j}{2} p_0^{j-2}(\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}).$$

Let the shock process be the FHPP with the set of parameters $\{\gamma, \lambda\}$, where $\gamma = 0.9$ and $\lambda = 2$. In Fig. 6, we plot $\bar{F}_L(t)$ against $t \in [0, 10]$, for different failure probability vectors (p_0, p_1, p_2, p_3).

6 Application: Optimal Replacement Time

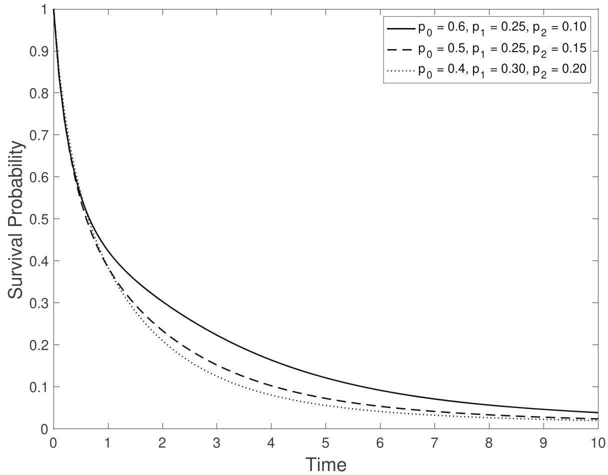
The study of the optimal replacement policies is one of the important applications of reliability theory. In this section, we consider the problem of finding the optimal replacement time for a coherent system that minimizes the long-run average cost per unit time. According to the classical age replacement policy, a system should be replaced upon failure or reaching the predetermined age t, whichever occurs first.

Below we give a list of assumptions which are similar to those given in Eryilmaz and Devrim (2019).

Assumptions:

(a) A new coherent system with the lifetime L described by the defined shock model is incepted into operation at time zero.
(b) The external shocks are the only reason for the system's failure, and occur according to the $RPMML(\gamma, \boldsymbol{\pi}, T)$
(c) The system is replaced by a new one either upon failure or after reaching the predetermined age t, whichever occurs first.

Fig. 6 Plot of $\bar{F}_L(t)$ against $t \in [0, 10]$



(d) Let c_1 and c_2 denote the cost of replacement of the system before failure and the cost of replacement of the system after failure, respectively.
(e) If the system is replaced before failure, then all failed components are replaced by new components and all non-failed components are repaired.
(f) A component is as good as new after repair and hence, the system after replacement becomes new.
(g) Let c_a denote the acquisition cost of one component and c_r be the repair cost for a non-failed component. Further, let c_f denote the additional cost of the system failure.

Note that the expected number of components to be replaced before failure is equal to $E(M(t)|L > t)$, and the expected number of components to be repaired before failure is equal to $E(n - M(t)|L > t)$. Then

$$c_1 = c_a E(M(t)|L > t) + c_r E(n - M(t)|L > t) = nc_r + (c_a - c_r)E(M(t)|L > t).$$

On the other hand, the cost of replacement of the system after failure is given by $c_2 = nc_a + c_f$. Consequently, the total long-run average cost per unit time is given by

$$\begin{aligned} C_L(t) &= \frac{[nc_r + (c_a - c_r)E(M(t)|L > t)]\bar{F}_L(t) + (nc_a + c_f)F_L(t)}{E(min(L,t))} \\ &= \frac{nc_a + c_f + [nc_r + (c_a - c_r)E(M(t)|L > t) - nc_a - c_f]\bar{F}_L(t)}{\int_0^t \bar{F}_L(u)du}. \end{aligned} \tag{6.1}$$

Note that the denominator of the above equation tends to ∞ as $t \to \infty$ (see Remark 5.2) and hence, the finite optimal replacement time cannot be determined on the infinite horizon. In real life, due to specifications and internal degradation process, an item might be replaced at some large time t^u (say) even if it is not an optimal decision. Thus, we study the optimal replacement policy under the updated Assumption (c) as given below.

Updated Assumption (c): The system is replaced by a new one either upon its failure or after reaching its age to a predetermined threshold value $t \leq t^u$, whichever occurs first.

Consequently, our goal is to find the optimal t^* $(\leq t^u)$ such that $C_L(t^*) < C_L(t^u)$ and $C_L(t^*) = min_{t\in(0,t^u]} C_L(t)$. Now, to evaluate $C_L(t)$, we need the expressions of $E(M(t)|L > t)$ and $\bar{F}_L(t)$. The first expression is given by

$$E(M(t)|L > t) = \frac{1}{\bar{F}_L(t)} \sum_{k=1}^{n-1} s_k \sum_{i=1}^{n-k} i P(M(t) = i),$$

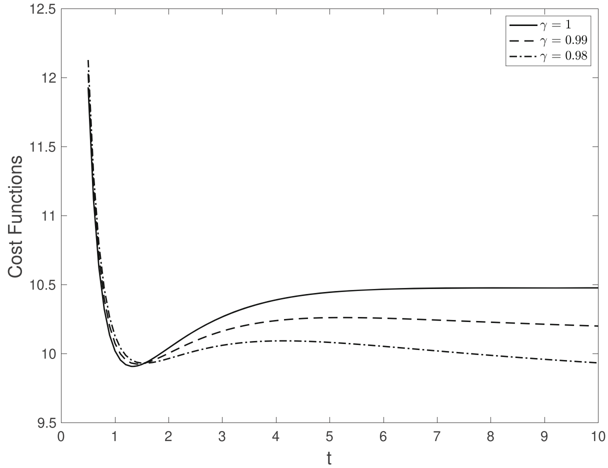
where the pmf of $M(t)$ is the same as in Lemma 4.1. Further, the expression of $\bar{F}_L(t)$ is obtained from Theorem 5.8.

Due to mathematical complexity, the aforementioned problem cannot be analytically solved. Thus, we consider a numerical example as given below.

Example 6.1 Consider the coherent system mentioned in Example 5.2. Assume that shocks occur according to the RPMML with the parameter set $\{\gamma, \boldsymbol{\pi}, T\}$, where $\boldsymbol{\pi} = (1, 0)$, $T = \begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix}$ and $\lambda = 1.2$. Then

$$E(M(t)|L > t) = \frac{1}{\bar{F}_L(t)}(P(M(t) = 1) + 2s_1 P(M(t) = 2)), \tag{6.2}$$

Fig. 7 Plot of $C_L(t)$ against time



where $P(M(t)=1)$ and $P(M(t)=2)$ are the same as in Lemma 4.1, and

$$\bar{F}_L(t) = \boldsymbol{\pi} E_{\gamma,1}(Tt^{\gamma})\boldsymbol{e} + \sum_{j=1}^{\infty}(p_1+s_1p_2)jp_0^{j-1}(\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}) + s_1\sum_{j=2}^{\infty} p_1^2\binom{j}{2}p_0^{j-2}(\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}). \tag{6.3}$$

Note that $\int_0^t \boldsymbol{\pi} E_{\gamma,1}(Tu^{\gamma})\boldsymbol{e}du = t\boldsymbol{\pi} E_{\gamma,2}(Tt^{\gamma})\boldsymbol{e}$. Consequently,

$$\int_0^t \bar{F}_L(u)du = t\Bigg[\boldsymbol{\pi} E_{\gamma,2}(Tt^{\gamma})\boldsymbol{e} + \sum_{j=1}^{\infty}(p_1+s_1p_2)jp_0^{j-1}(\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,2}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,2}(T_j t^{\gamma})\boldsymbol{e}) + s_1\sum_{j=2}^{\infty} p_1^2\binom{j}{2}p_0^{j-2}(\boldsymbol{\pi}_{\boldsymbol{j+1}} E_{\gamma,2}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi}_{\boldsymbol{j}} E_{\gamma,2}(T_j t^{\gamma})\boldsymbol{e})\Bigg]. \tag{6.4}$$

By using (6.2), (6.3), and (6.4) in (6.1), we get the expression of $C_L(t)$. Now, in Fig. 7, we plot this $C_L(t)$ against $t \in [0, 10]$, for different values of γ, and for fixed $c_a = 3$, $c_f = 10$, $c_r = 1$, $p_0 = 0$, $p_1 = 0.2$ and $p_2 = 0.1$. From this figure, we see that the the cost function $C_L(t)$ is in U-shaped in initial time period. Let $t^u = 10$. Then, for $\gamma = 1$, $t^* = 1.3$ and $C_L(t^*) = 9.9075$; for $\gamma = 0.99$, $t^* = 1.4$ and $C_L(t^*) = 9.9243$, and, for $\gamma = 0.98$, $t^* = 1.5$ and $C_L(t^*) = 9.9308$. Moreover, $C_L(t)$ tends to 0 when $t \to \infty$ because $E(L) = \infty$ for $\gamma < 1$.

7 Concluding remarks

The study of shock models for multi-component systems is an important area in modern reliability theory. In this paper, we consider three rather general counting processes for modeling shock occurrences, namely, the PGGP, the PPHP and the RPMML. All these processes are mathematically tractable.

It is important to note that the PPHP can be used to approximate any mixed Poisson process. On the other hand, the RPMML is very useful in modeling the arrivals of shocks with heavy-tailed inter-arrival times. All considered processes have not been used in modeling lifetimes of multi-component systems subject to external shocks. Therefore, our paper is the first to obtain relationships for the survival functions and expected lifetimes for different specific models of the effect of shocks on the components of a coherent system.

We expect that the obtained in this paper innovative results can be used in various applications. As an example, the optimal replacement policy under the RPMML process of shocks was considered.

The δ-shock model is one of the popular shock models used in many applications. The study of the δ-shock model for a multi-component system based on the PGGP/PPHP/RPMML can be considered as a potential problem yet to be explored. Further, in this paper, we mostly derive the results for coherent systems with i.i.d. or exchangeable components. The study of coherent systems with dependent components can be considered in future as well.

Appendix A

Proof of Lemma 2.2 Let $S_1, S_2, \ldots$ be a sequence of arrival times. Then

$$P(N(t) = n) = P(S_{n+1} > t) - P(S_n > t). \tag{A1}$$

Note that $S_n = X_1 + X_2 + \cdots + X_n \sim MML(\gamma, \boldsymbol{\pi_n}, T_n)$, where X_i's are inter-arrival time. Then, from (2.2), we have $P(S_n > t) = \boldsymbol{\pi_n} E_{\gamma,1}(T_n t^{\gamma})\boldsymbol{e}$. Consequently, the result follows from (A1). □

Proof of Lemma 4.1 Note that the event "$M(t) = 0$" happens if and only if each shock, occurred till time t, is harmless for the system. Then,

$$\begin{aligned} P(M(t) = 0) &= P(N(t) = 0) + \sum_{j=1}^{\infty} P(N(t) = j, Z_1 = 0, Z_2 = 0, Z_3 = 0, \ldots, Z_j = 0) \\ &= \sum_{j=0}^{\infty} p_0^j P(N(t) = j). \end{aligned}$$

Now, for $1 \leq i < n$, the event "$M(t) = i$" means that exactly i components of the system failed till time t. Thus, if $N(t) = j$, then there are exactly m harmful shocks out of j shocks, where $j \geq m$, $1 \leq m \leq i$. Consequently, we can write

$$\begin{aligned} P(M(t) = i) &= \sum_{m=1}^{i} \sum_{j=m}^{\infty} \text{P}(i \text{ components fail due to exactly } m \text{ harmful shocks when } N(t) = j) \\ &= \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \sum_{z \in \Omega_{m,i}} P(N(t) = j, Z_1 = z_1, \ldots, Z_m = z_m, Z_{m+1} = 0, \ldots, Z_j = 0) \\ &= \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t) = j), \end{aligned}$$

where the last equality holds because Z_i's are i.i.d, and the process $\{N(t) : t \geq 0\}$ and $\{Z_i : i \in \mathbb{N}\}$ are independent. Hence the result is proved. □

Proof of Corollary 4.1

Proof (i): If $p_j = 1/(n+1)$, for all $j = 1, 2, \ldots, n$, then

$$\sum_{z\in\Omega_{m,i}} \prod_{r=1}^{m} P_{z_r} = \left(\frac{1}{n+1}\right)^m \sum_{z\in\Omega_{m,i}} 1 = \left(\frac{1}{n+1}\right)^m |\Omega_{m,i}|.$$

By using Bose-Einstien value, one can show that $|\Omega_{m,i}| = \binom{i-1}{m-1}$. Thus

$$\sum_{z\in\Omega_{m,i}} \prod_{r=1}^{m} P_{z_r} = \left(\frac{1}{n+1}\right)^m \binom{i-1}{m-1}.$$

By using the above equality in Theorem 4.1, we get the required result.

Proof (ii): The result immediately follows from Theorem 4.1 by substituting $p_0 = 0$.

Proof (iii): Note that

$$\sum_{z\in\Omega_{m,i}} \prod_{r=1}^{m} P_{z_r} = \begin{cases} 0 & if\ m < i \\ p_1^m & if\ m = i. \end{cases}$$

By using the above equality in Theorem 4.1, we get the required result. □

Proof of Theorem 5.1 From Theorem 4.1, we have

$$\begin{aligned} \bar{F}_{L_{k:n}}(t) =& \sum_{j=0}^{\infty} p_0^j P(N(t)=j) \\ &+ \sum_{i=1}^{n-k} \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{z\in\Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t)=j). \end{aligned} \tag{A2}$$

Let Q be the mixing distribution of the PGGP. Then, for $m \geq 0$, we have

$$\begin{aligned} \sum_{j=m}^{\infty} \binom{j}{m} p_0^{j-m} P(N(t)=j) &= \sum_{j=m}^{\infty} \binom{j}{m} p_0^{j-m} \int_0^{\infty} \exp\{-q\Lambda(t)\} \frac{(q\Lambda(t))^j}{j!} dF_Q(q)dq \\ &= \int_0^{\infty} \exp\{-q\Lambda(t)\} \frac{(q\Lambda(t))^m}{m!} \left(\sum_{j=m}^{\infty} \frac{(qp_0\Lambda(t))^{j-m}}{(j-m)!} \right) dF_Q(q) \\ &= \int_0^{\infty} \exp\{-q(1-p_0)\Lambda(t)\} \frac{(q\Lambda(t))^m}{m!} dF_Q(q) \\ &= \left(\frac{\alpha}{\alpha + (1-p_0)\Lambda(t)} \right)^{\mu-\nu} \left(\frac{\Lambda(t)}{\alpha + (1-p_0)\Lambda(t)} \right)^m \\ &\quad \times \frac{\Gamma_\nu(m+\mu, (\alpha + (1-p_0)\Lambda(t))l)}{m!\Gamma_\nu(\mu, \alpha l)}, \end{aligned} \tag{A3}$$

where the first equality follows from Definition 2.2; the second equality holds due to the *Dominated Convergence Theorem*, and the last equality follows from (2.1). Now, by using (A3) in (A2), we get the required result. □

Proof of Theorem 5.3 We only prove part (i). Part (ii) can be proved in the same line. Now, from Corollary 5.2 (ii), we have

$$\begin{aligned} E(L_{k:n}) &= \int_0^\infty \bar{F}_{L_{k:n}}(t)dt \\ &= \int_0^\infty \left(\frac{b}{(1-p_0)t+b}\right)^\beta dt + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Gamma(\beta+m)}{\Gamma(\beta)m!} \\ &\quad \times \int_0^\infty \left(\frac{b}{(1-p_0)t+b}\right)^\beta \left(\frac{t}{(1-p_0)t+b}\right)^m dt \\ &= \left(\frac{b}{\beta-1}\right)\left(\frac{1}{1-p_0}\right) + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Gamma(\beta+m)}{\Gamma(\beta)m!} \\ &\quad \times \left(\frac{1}{1-p_0}\right)^{m+1}\int_0^\infty \left(\frac{b}{z+b}\right)^\beta \left(\frac{z}{z+b}\right)^m dz. \end{aligned} \tag{A4}$$

Again, we have

$$\int_0^\infty \left(\frac{b}{z+b}\right)^\beta \left(\frac{z}{z+b}\right)^m = \frac{b\Gamma(m+1)\Gamma(\beta-1)}{\Gamma(m+\beta)}.$$

By using the above equality in (A4), we get the required result. □

Proof of Corollary 5.4 We only prove part (i). Other parts can be proved in same line. Now, from Theorem 5.3 (i), we have, for model M_1,

$$\begin{aligned} E(L_{k:n}) &= \left(\frac{b}{\beta-1}\right)\left(\frac{n+1}{n}\right)\left[1+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\binom{i-1}{m-1}\left(\frac{1}{n}\right)^m\right] \\ &= \left(\frac{b}{\beta-1}\right)\left(\frac{n+1}{n}\right)\left[1+\left(\frac{1}{n}\right)\sum_{i=1}^{n-k}\left(1+\frac{1}{n}\right)^{i-1}\right] \\ &= \left(\frac{b}{\beta-1}\right)\left(\frac{n+1}{n}\right)\left(1+\frac{1}{n}\right)^{n-k} = \left(\frac{b}{\beta-1}\right)\left(1+\frac{1}{n}\right)^{n-k+1}. \end{aligned}$$

Hence part (i) is proved. □

Proof of Theorem 5.4 We have $P(L>t) = \sum_{i=1}^n s_i P(L_{i:n}>t)$, which implies $E(L) = \sum_{i=1}^n s_i E(L_{i:n})$. Consequently, the result follows from Theorem 5.3. □

Proof of Theorem 5.5 From Theorem 4.1 and Lemma 2.1, we get

$$\begin{aligned} \bar{F}_{L_{k:n}}(t) &= \sum_{j=0}^{\infty} p_0^j \Lambda(t)^j \boldsymbol{\pi}(\Lambda(t)I-T)^{-(j+1)}\boldsymbol{T^0} \\ &\quad + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\sum_{j=m}^{\infty}\binom{j}{m}\left(\sum_{z\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right) p_0^{j-m}\Lambda(t)^j\boldsymbol{\pi}(\Lambda(t)I-T)^{-(j+1)}\boldsymbol{T^0}. \end{aligned} \tag{A5}$$

Further, we can write

$$\Lambda(t)^j \boldsymbol{\pi}(\Lambda(t)I - T)^{-(j+1)}\boldsymbol{T^0} = \Lambda(t)^{-1}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(j+1)}\boldsymbol{T^0}, \;\; j \in \mathbb{N} \cup \{0\}. \tag{A6}$$

Now, for $m \geq 0$, we have

$$\begin{aligned}
&\sum_{j=m}^{\infty} \binom{j}{m} p_0^{j-m} \Lambda(t)^j \boldsymbol{\pi}(\Lambda(t)I - T)^{-(j+1)}\boldsymbol{T^0} \\
&= \Lambda(t)^{-1} \sum_{j=m}^{\infty} \binom{j}{m} p_0^{j-m} \boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(j+1)}\boldsymbol{T^0} \\
&= \Lambda(t)^{-1}\boldsymbol{\pi} \left(\sum_{j=m}^{\infty} \binom{j}{m} p_0^{j-m} (I - \Lambda(t)^{-1}T)^{-(j+1)} \right) \boldsymbol{T^0} \\
&= \Lambda(t)^{-1}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(m+1)} \left(\sum_{j=m}^{\infty} \binom{j}{m} p_0^{j-m} (I - \Lambda(t)^{-1}T)^{-(j-m)} \right) \boldsymbol{T^0} \\
&= \Lambda(t)^{-1}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(m+1)} \left(\sum_{l=0}^{\infty} \binom{m+l}{m} p_0^{l} (I - \Lambda(t)^{-1}T)^{-l} \right) \boldsymbol{T^0} \\
&= \Lambda(t)^{-1}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(m+1)} \left(\sum_{l=0}^{\infty} \binom{m+l}{l} p_0^{l} (I - \Lambda(t)^{-1}T)^{-l} \right) \boldsymbol{T^0} \\
&= \Lambda(t)^{-1}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(m+1)}(I - p_0(I - \Lambda(t)^{-1}T)^{-1})^{-(m+1)}\boldsymbol{T^0} \\
&= \Lambda(t)^{-1}\boldsymbol{\pi}((1 - p_0)I - \Lambda(t)^{-1}T)^{-(m+1)}\boldsymbol{T^0} \\
&= \Lambda(t)^{m}\boldsymbol{\pi}((1 - p_0)\Lambda(t)I - T)^{-(m+1)}\boldsymbol{T^0},
\end{aligned} \tag{A7}$$

where the first equality follows from (A6); the fifth equality holds because $\binom{m+l}{m} = \binom{m+l}{l}$, and the sixth equality holds because $\sum_{l=0}^{\infty} \binom{m+l}{l} T^l = (I - T)^{-(m+1)}$. Now, by using (A7), we can write

$$\begin{aligned}
&\sum_{i=1}^{n-k} \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} \Lambda(t)^j \boldsymbol{\pi}(\Lambda(t)I - T)^{-(j+1)}\boldsymbol{T^0} \\
&= \sum_{i=1}^{n-k} \sum_{m=1}^{i} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) \Lambda(t)^m \boldsymbol{\pi}((1 - p_0)\Lambda(t)I - T)^{-(m+1)}\boldsymbol{T^0}.
\end{aligned} \tag{A8}$$

Finally, by using (A8) in (A5), we get the required result. □

Appendix B

Here we discuss the case: Z_i's are dependant random variables with a specific dependency structure. We consider the following assumptions:

Assumptions:

- Each shock is harmless with probability p_0 and harmful with probability $1 - p_0$, i.e., $P(Z_i = 0) = p_0$ and $P(Z_i \neq 0) = 1 - p_0$, for $i \in \mathbb{N}$.

- If the first shock is harmful then it can damage j components with probability p_j, i.e., $P(Z_1 = j) = p_j$ for all $j = 1, 2, \ldots, n$. Clearly, $p_1 + p_2 + \cdots + p_n = 1 - p_0$.
- The random variables $Z_1, Z_2, \ldots, Z_m, Z_{m+1}$ are dependent with the dependancy structure given by

$$P(Z_{m+1} = z_{m+1} | Z_1 = z_1, \ldots, Z_m = z_m)$$
$$= \begin{cases} p_{z_{m+1}}, & \text{if } z_{m+1} < n - z_1 - z_2 - \cdots - z_m \\ p_{z_{m+1}} + \cdots + p_n, & \text{if } z_{m+1} = n - z_1 - z_2 - \cdots - z_m, \end{cases}$$

  where $z_1 + z_2 + \cdots + z_m < n$ and $m \geq 1$.
- The random variables $Z_1, Z_2, Z_3, \ldots$ are exchangeable.
- $\{N(t) : t \geq 0\}$ and $\{Z_i : i \in \mathbb{N}\}$ are independent.

Based on the above assumptions, we derive the expression for $P(M(t) = i)$, for $i = 0, 1, 2, \ldots, n-1$, in the following lemma.

Lemma 7.1 Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, $i \in \mathbb{N}$, and let $\{N(t) : t \geq 0\}$ be a counting process. Then

$$P(M(t) = 0) = \sum_{j=0}^{\infty} p_0^j P(N(t) = j)$$

and

$$P(M(t) = i) = \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t) = j),$$

where $\Omega_{m,i} = \{z = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \leq z_l \leq n, 1 \leq l \leq m\}$ and $1 \leq m \leq i < n - 1$.

Proof: By proceeding in the same line as in the proof of Lemma 4.1, we get

$$P(M(t) = 0) = P(N(t) = 0) + \sum_{j=1}^{\infty} P(N(t) = j, Z_1 = 0, Z_2 = 0, Z_3 = 0, \ldots, Z_j = 0)$$
$$= \sum_{j=0}^{\infty} p_0^j P(N(t) = j),$$

and, for $1 \leq i \leq n - 1$,

$$\begin{aligned} P(M(t) = i) &= \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \sum_{z \in \Omega_{m,i}} P(N(t) = j, Z_1 = z_1, \ldots, Z_m = z_m, \\ &\qquad Z_{m+1} = 0, \ldots, Z_j = 0) \\ &= \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \sum_{z \in \Omega_{m,i}} P(N(t) = j) P(Z_1 = z_1, \ldots, Z_m = z_m, \\ &\qquad Z_{m+1} = 0, \ldots, Z_j = 0), \end{aligned} \tag{B1}$$

where the last equality holds because $\{N(t) : t \geq 0\}$ and $\{Z_i : i \in \mathbb{N}\}$ are independent. Now, consider

$$\begin{aligned} &P(Z_1 = z_1, \ldots, Z_m = z_m, Z_{m+1} = 0, \ldots, Z_j = 0) \\ &\quad = P(Z_j = 0 | Z_1 = z_1, \ldots, Z_m = z_m, Z_{m+1} = 0, \ldots, Z_{j-1} = 0) \\ &\qquad \times P(Z_1 = z_1, \ldots, Z_m = z_m, Z_{m+1} = 0, \ldots, Z_{j-1} = 0) \\ &\quad = p_0 P(Z_1 = z_1, \ldots, Z_m = z_m, Z_{m+1} = 0, \ldots, Z_{j-1} = 0). \end{aligned}$$

By using the above equality recursively, we get

$$P(Z_1 = z_1, \ldots, Z_m = z_m, Z_{m+1} = 0, \ldots, Z_j = 0) = p_0^{j-m} P(Z_1 = z_1, \ldots, Z_m = z_m).$$

Since $z_1 + z_2 + \cdots + z_m = i < n$, we get

$$\begin{aligned} P(Z_1 = z_1, \ldots, Z_m = z_m) \quad & P(Z_m = z_m | Z_1 = z_1, \ldots, Z_{m-1} = z_{m-1}) \\ & P(Z_1 = z_1, \ldots, Z_{m-1} = z_{m-1}) \\ &= p_{z_m} P(Z_1 = z_1, \ldots, Z_{m-1} = z_{m-1}) \\ &= p_{z_m} p_{z_{m-1}} P(Z_1 = z_1, \ldots, Z_{m-1} = z_{m-2}) \\ &= \prod_{r=1}^{m} p_{z_r}. \end{aligned}$$

Consequently,

$$P(Z_1 = z_1, \ldots, Z_m = z_m, Z_{m+1} = 0, \ldots, Z_j = 0) = p_0^{j-m} \prod_{r=1}^{m} p_{z_r}.$$

By using above equality in (B1), we get

$$P(M(t) = i) = \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{z \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t) = j)$$

and hence, the required result is proved. □

One may note that, even if a specific dependency structure between Z_i's is assumed in Lemma 7.1, the results given in Lemmas 7.1 and 4.1 are the same. Thus, all results developed for the case of independent Z_i's also hold for the aforementioned dependant Z_i's setup.

Author Contributions All others discussed the setting, the methods and derivations. The first author made the major writing and derivations. The second author supervised the work and checked the derivation. The third author participated in writing and discussions on all stages.

Funding The first author sincerely acknowledges the financial support received from UGC, Govt. of India. The work of the second author was supported by IIT. Jodhpur, India.

Data Availability No data was used in this study.

Declarations

Competing Interests The authors declare no competing interests.

References

Agarwal SK, Kalla SL (1996) A generalized gamma distribution and its application in reliability. Commun Stat Theory Methods 25:201–210

Albrecher H, Bladt M, Bladt M (2020) Matrix Mittag-Leffler distributions and modeling heavy-tailed risks. Extremes 23:425–450

Barlow RE, Proschan F (1975) Statistical theory of reliability and life testing: Probability models. Holt, Rinehart and Winston, New York

Bian L, Wang G, Liu P (2021) Reliability analysis for multi-component systems with interdependent competing failure processes. Appl Math Model 94:446–459

Cha JH, Finkelstein M (2016) New shock models based on the generalized Polya process. Eur J Oper Res 251:135–141

Cha JH, Mercier S (2021) Poisson generalized gamma process and its properties. Stochastics 93:1123–1140

Eryilmaz S (2017) δ-shock model based on Polya process and its optimal replacement policy. Eur J Oper Res 263:690–697

Eryilmaz S, Devrim Y (2019) Reliability and optimal replacement policy for a k-out-of-n system subject to shocks. Reliab Eng Syst Saf 188:393–397

Gong M, Xie M, Yang Y (2018) Reliability assessment of system under a generalized run shock model. J Appl Probab 55:1249–1260

Gong M, Eryilmaz S, Xie M (2020) Reliability assessment of system under a generalized cumulative shock model. Proc Inst Mech Eng Part O: J Risk Reliab 234:129–137

Goyal D, Hazra NK, Finkelstein M (2022a) On the general δ-shock model. TEST 31:994–1029. https://doi.org/10.1007/s11749-022-00810-5

Goyal D, Hazra NK, Finkelstein M (2022b) On properties of the phase-type mixed Poisson process and its applications to reliability shock modeling. Methodol Comput Appl Probab 24:2933–2960. https://doi.org/10.1007/s11009-022-09961-2

Gut A (1990) Cumulative shock models. Adv Appl Probab 22:504–507

Gut A, Hüsler J (1999) Extreme shock models. Extremes 2:295–307

Huang X, Jin S, He X, He D (2019) Reliability analysis of coherent systems subject to internal failures and external shocks. Reliab Eng Syst Saf 181:75–83

Kataria KK, Vellaisamy P (2019) On the convolution of Mittag-Leffler distributions and its applications to fractional point processes. Stoch Anal Appl 37:115–122

Levitin G, Finkelstein M (2017a) Optimal backup in heterogeneous standby systems exposed to shocks. Reliab Eng Syst Saf 165:336–344

Levitin G, Finkelstein M (2017b) Effect of element separation in series-parallel systems exposed to random shocks. Eur J Oper Res 260:305–315

Lorvand H, Kelkinnama M (2023) Reliability analysis and optimal replacement for a k-out-of-n system under a δ-shock model. Proc Inst Mech Eng Part O: J Risk Reliab 237:98–111. https://doi.org/10.1177/1748006X221082762

Mallor F, Omey E (2001) Shocks, runs and random sums. J Appl Probab 38:438–448

Samaniego FJ (2007) System signatures and their applications in engineering reliability. Wiley, New York

Sheu SH, Chang TH (2001) An optimal replacement period for a k-out-of-n: F system subject to shocks. Int J Syst Sci 32:565–573

Sheu SH, Liou CT (1992) Optimal replacement of a k-out-of-n system subject to shocks. Microelectron Reliab 32:649–655

Skoulakis G (2000) A general shock model for a reliability system. J Appl Probab 37:925–935

Wang X, Zhao X, Wu C, Wang S (2022) Mixed shock model for multi-state weighted k-out-of-n: F systems with degraded resistance against shocks. Reliab Eng Syst Saf 217:108098