# INVARIANCE PRINCIPLES FOR $G$-BROWNIAN-MOTION-DRIVEN STOCHASTIC DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS TO $G$-STOCHASTIC CONTROL

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**Abstract.** The $G$-Brownian-motion-driven stochastic differential equations ($G$-SDEs) as well as the $G$-expectation, which were seminally proposed by Peng and his colleagues, have been extensively applied to describing a particular kind of uncertainty arising in real-world systems modeling. Mathematically depicting long-time and limit behaviors of the solution produced by $G$-SDEs is beneficial to understanding the mechanisms of system's evolution. Here, we develop a new $G$-semimartingale convergence theorem and further establish a new invariance principle for investigating the long-time behaviors emergent in $G$-SDEs. We also validate the uniqueness and the global existence of the solution of $G$-SDEs whose vector fields are only locally Lipschitzian with a linear upper bound. To demonstrate the broad applicability of our analytically established results, we investigate its application to achieving $G$-stochastic control in a few representative dynamical systems.

**Key words.** $G$-stochastic differential equations, $G$-semimartingale convergence theorem, invariance principle, $G$-stochastic control

**AMS subject classifications.** 60G65, 60F17

## 1. Introduction.

Long-time and limit behaviors of the solutions generated by stochastic differential equations (SDEs) have received growing attention because such behaviors usually correspond to particular functions in real-world systems [10, 25, 31, 8, 3]. Interesting physical or/and biological phenomena have been systematically investigated, including asymptotic behaviors of random matrices in quantum physics [34], stochastic resonance [2], stochastic homogeneity [4], stochastic stabilization or synchronization [26, 32, 23, 20], and random-temporal-structure-induced emergence [11, 12, 14, 13]. Also developed were stochastic versions of invariance principle, which originated from LaSalle's invariance principle [17, 18] for deterministic

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systems and then has been extended successfully to study the SDEs [28, 40], the stochastic differential delayed equations (SDDEs) [27, 30], the stochastic functional differential equations (SFDEs) [29, 39] and even the discrete stochastic dynamical systems [41]. These versions of invariance principles are often used to elucidate the asymptotic behaviors, such as stability, boundedness, and invariance in some chaotic attractors, emergent in random systems.

In addition to the traditional frameworks of randomness and stochasticity, measuring uncertainties of randomness is another important issue in those areas replete with fluctuations and risks of high level, such as economics [16]. A seminal framework by means of sublinear expectation was fundamentally built by Peng and his colleagues to quantify such uncertainties [36] and then extended broadly in line with the modern probability theory. Indeed, the framework has been put forward to investigating the $G$-Brownian-motion-driven stochastic differential equations ($G$-SDEs), which thus provides a model to describe the randomness with uncertainties in evolutionary dynamics. Also systematically investigated was the well-posedness of $G$-SDEs [9, 36] and stochastic functional differential equations ($G$-SFDEs) [37, 7]. Furthermore, although the stability of $G$-SDEs has been widely investigated [21, 38], rigorously delicate descriptions of stability, boundedness, control and even invariance property in dynamical attractors using $G$-SDEs are still lacking.

In this article, we, therefore, intends to fill in this gap through novelly developing an invariance principle for $G$-SDEs and investigate its applicability to the stochastic control, especially in the case that the noise is uncertain. As such, this invariance principle can render the analytical investigations of dynamics produced by $G$-SDEs much clearer and more complete. In order to develop this new principle, we need to establish a new version of $G$-semimartingale convergence theorem, nontrivially generalizing the classical semimartingale convergence theorem developed in [24].

The remaining of this article is organized as follows. Section 2 introduces some basic concepts and provides some preliminary theorems of sublinear expectations. Section 3 rigorously proves the $G$-semimartingale convergence theorem as follows.

THEOREM 1.1. *Assume $A^1$ and $A^2$ are two non-decreasing process with initial value 0, $A^1(t)$ is a continuous process and $\hat{\mathbb{E}}[A^1(+\infty)] < +\infty$. Assume that $Z$ is a non-negative $G$-semimartingale satisfying $\hat{\mathbb{E}}[Z^+(0)] < \infty$ with the form as $Z(t) = Z(0)+A^1(t)-A^2(t)+M(t),\ t \geq 0$, where $M(t)$ is a continuous $G$-supermartingale with initial value 0. $M(t) \in L^1_G(\Omega_t)$ for every $t \geq 0$. Then, we have that $A^2(+\infty) < +\infty$, $\lim_{t\to+\infty} Z(t)$ finitely exists, and that $\lim_{t\to+\infty} M(t)$ finitely exists quasi-surely.*

Here, we sketch the proof of the above convergence theorem as follows. By extending the space of random variables, we generalize Fatou's Lemma on the $G$-conditional expectation. Combining with the uppercrossing inequality, we derive the $G$-martingale convergence theorem for a continuous process and then establish the essential $G$-semimartingale convergence theorem. Also in this section, we present the other more applicable versions of the $G$-semimartingale convergence theorem. With all these preparations, Section 4 presents our main result, the *invariance principle* for the $G$-SDEs, and validates it using the established $G$-semimartingale convergence theorem. Here, we show this principle as follows.

THEOREM 1.2. *With those conditions and assumptions listed in Section 4, we suppose that there exists a function* $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$*, a function* $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$ *and a continuous function* $\eta : \mathbb{R}^d \to \mathbb{R}_+$ *such that* $\lim_{|x|\to\infty} \inf_{0\le t<+\infty} V(\boldsymbol{x}, t) = \infty$ *and* $\mathcal{L}V(\boldsymbol{x}, t) \le \gamma(t) - \eta(\boldsymbol{x})$*, where the diffusive operator* $\mathcal{L}V = V_t + V_{x_i} f^i + G\Big((V_{x_k}(h^{kij} + h^{kji}) + V_{x_k x_l} g^{ki} g^{lj})^n_{i,j=1}\Big)$ *where Einstein's notations are applied here. Then, we have that* $\lim_{t\to+\infty} V(\boldsymbol{x}(t), t)$ *finitely exists quasi-surely and that* $\lim_{t\to+\infty} \eta(\boldsymbol{x}(t)) = 0$ *quasi-surely. Moreover, we have* $\lim_{t\to+\infty} d(\boldsymbol{x}(t), \mathrm{Ker}(\eta)) = 0$*. Here,* $\boldsymbol{x}(t)$ *is the solution of the G-SDEs which read*

$$\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t), t)\mathrm{d}t + \boldsymbol{g}(\boldsymbol{x}(t), t)\mathrm{d}\boldsymbol{B}(t) + \boldsymbol{h}(\boldsymbol{x}(t), t)\mathrm{d}\langle \boldsymbol{B}\rangle(t). \tag{1.1}$$

The proof of such theorem, though inspired by [28], is rather different. By $G$-Itô's formula, we write out the function in a form of the $G$-semimartingale and then apply the corresponding convergence theorem. By estimating the calculus of $\eta$ based on the uppercrossing stopping time, we show that all trajectories converge to the kernel of the function $\eta$ quasi-surely. Still in this section, we further present several generalized versions of invariance principle. All these build up a solid foundation for Section 5, where we use the $G$-stochastic control to stabilize representative complex dynamics, demonstrating the broad applicability of our analytically-established results. Finally, Section 6 provides some discussion and concluding remarks.

**2. Preliminaries.** In this section, we present some frequently used definitions and results of sublinear expectation theory, which will be useful for our following investigations. For more details, we refer to [5, 36, 35, 22].

To begin with, we let $\Omega$ be a given set, and $\mathcal{H}$ be the space of all real-valued functions defined on $\Omega$. Denote by $C_{l,\mathrm{Lip}}(\mathbb{R}^d)$ the space of all locally Lipschitz-continuous functions on $\mathbb{R}^d$. And, for any function $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^d)$, if $x_i(\omega) \in \mathcal{H}$ for all $i = 1, 2, \cdots, d$, then $\varphi(x_1(\omega), \cdots, x_d(\omega)) \in \mathcal{H}$.

Next, we provide some basic concepts on the sublinear expectation.

DEFINITION 2.1 (Sublinear Expectation [36]). *A functional* $\mathbb{E}[\cdot]$ *is said to be a sublinear expectation on* $\mathcal{H}$ *if it satisfies:* (1) $\mathbb{E}[c] = c$, for any $c \in \mathbb{R}$, (2) $\mathbb{E}[X] \le \mathbb{E}[Y]$, for any $X \le Y$, (3) $\mathbb{E}[X + Y] \le \mathbb{E}[X] + \mathbb{E}[Y]$, *and* (4) $\mathbb{E}[\lambda X] = \lambda\mathbb{E}[X]$, for any $\lambda \ge 0$.

DEFINITION 2.2 ($G$-Function [36]). *A function $G : \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ is said to be sublinear and monotone if it satisfies* (1) $G(\boldsymbol{p} + \bar{\boldsymbol{p}}, \boldsymbol{A} + \bar{\boldsymbol{A}}) \leq G(\boldsymbol{p}, \boldsymbol{A}) + G(\bar{\boldsymbol{p}}, \bar{\boldsymbol{A}})$, (2) $G(\boldsymbol{p}, \boldsymbol{A}) \leq G(\boldsymbol{p}, \bar{\boldsymbol{A}})$, *if* $\boldsymbol{A} \leq \bar{\boldsymbol{A}}$, *and* (3) $G(\lambda \boldsymbol{p}, \lambda \boldsymbol{A}) = \lambda G(\boldsymbol{p}, \boldsymbol{A}),\ \forall\ \lambda \geq 0$.

*Here, $\mathbb{S}^d$ denotes the space of $d \times d$ symmetric matrices. And $\boldsymbol{A} \leq \bar{\boldsymbol{A}}$ implies the nonnegativity of the symmetric matrix $\bar{\boldsymbol{A}} - \boldsymbol{A}$.*

In the following, we assume the function $G$ defined in Definition 2.2 is independent of the vector $p$. It is worthwhile to mention that, when $d = 1$, $G$ is reduced to the form $G(r) = \frac{1}{2}(r^+\overline{\sigma}^2 - r^-\underline{\sigma}^2)$ for some non-negative $\underline{\sigma} \leq \overline{\sigma}$. Here $r^+$ and $r^-$ correspond to the non-negative and the non-positive parts of $r$, respectively. Moreover, if a symmetric $G$-Brownian motion satisfies $\hat{\mathbb{E}}[\boldsymbol{A}\boldsymbol{B}(t), \boldsymbol{B}(t)] = 2G(\boldsymbol{A})t$ with $G(\boldsymbol{A}) = \frac{1}{2}\hat{\mathbb{E}}[\boldsymbol{A}\boldsymbol{B}(1), \boldsymbol{B}(1)]$, then $G$ is said to be a $G$-function related to the symmetric $G$-Brownian motion $\boldsymbol{B}$. Here, the definition of $G$-Brownian motion, as well as $G$-conditional expectation, can be found in [36].

Moreover, it is necessary to introduce some definitions on some spaces of functions and measures. Here, we denote, respectively, by

- $\mathscr{F}_t$ : The completion of $\sigma(\boldsymbol{B}(s) : s \leq t)$,
- $\mathscr{B}(\Omega)$ : The Borel $\sigma$-algebra on $\Omega$,
- $L^0(\Omega)$: The space of all $\mathscr{B}(\Omega)$-measurable functions,
- $L_G^p(\Omega)$ : The completion of the space $\mathrm{Lip}(\Omega)$ under the norm $\|\cdot\|_{L_G^p} := (\hat{\mathbb{E}}[|\cdot|^p])^{\frac{1}{p}}$,
- $\mathrm{Lip}\,(\Omega_t)$: $\{\varphi(\boldsymbol{B}(t_1), \boldsymbol{B}(t_2) - \boldsymbol{B}(t_1), \cdots, \boldsymbol{B}(t_k) - \boldsymbol{B}(t_{k-1})) : \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^{m\times k}),\ 0 \leq t_1 < \cdots < t_k \leq t\}$,
- $L_G^p(\Omega_t)$ : $L_G^p(\Omega) \cap \mathrm{Lip}\,(\Omega_t)$,
- $\mathcal{M}$ : The set of all probability measure defined on $\Omega$,
- $E_Q[\cdot]$: The expectation under the traditional probability measure $Q$,
- $\mathcal{P}(t, Q)$:= $\{R \in \mathcal{M} : E_Q[X] = E_R[X], \forall X \in \mathrm{Lip}\,(\Omega_t)\}$,
- $\mathcal{Q} := \{Q \in \mathcal{M} : E_Q[X] \leq \hat{\mathbb{E}}[X], \forall X \in L_G^1\,(\Omega)\}$, and
- $\mathcal{L}^0(\Omega)$:= $\{X \in L^0(\Omega)$: $E_Q[X]$ exists for any $Q \in \mathcal{Q}\}$.

From Theorem 1.2.1 in [35], it follows that the sublinear expectation satisfies $\hat{\mathbb{E}}[X] = \sup_{Q\in\mathcal{Q}} E_Q[X]$ for each $X \in \mathrm{Lip}(\Omega)$. Thus, the definition of $\hat{\mathbb{E}}[\cdot]$ can be extended to $\mathcal{L}^0(\Omega)$. In addition, for the $G$-conditional expectation defined above, it can be represented by means of the probability space.

THEOREM 2.3 ([15]). *For each $Q \in \mathcal{Q}$ and $X \in L_G^1(\Omega)$, $\hat{\mathbb{E}}_t[X] = \operatorname{ess\,sup}_{R\in\mathcal{P}(t,Q)}{}^{Q} E_R\left[X \mid \mathscr{F}_t\right]$, $Q$-a.s.. Here, if $Y = \operatorname{ess\,sup}_{R\in\mathcal{P}(t,Q)}{}^{Q} E_R\left[X \mid \mathscr{F}_t\right]$, it means that for every $R \in \mathcal{P}(t, Q)$, $E_R\left[X \mid \mathscr{F}_t\right] \leq Y, Q$-a.s.. Moreover, if $E_R\left[X \mid \mathscr{F}_t\right] \leq Z$ for each $R \in \mathcal{P}(t, Q)$, $Q$-a.s., then we must have $Y \leq Z, Q$-a.s..*

For introducing $G$-Itô's calculus, we define $M_G^p([0,T])$, a space of random process, and the $G$-Itô's calculus on it (refer to [36] for details). Moreover, the quadratic variation is defined in the same manner as that in normal stochastic analysis. However, the range of the quadratic variation here is much different.

LEMMA 2.4 ([36]). *For an $m$-dimensional $G$-Brownian motion $\boldsymbol{B}$, there exists a bounded, convex and closed set $\Gamma \in \mathbb{S}_+^m$ such that $\langle \boldsymbol{B} \rangle(t) \in t\Gamma := \{t\gamma : \gamma \in \Gamma\}$, where $\mathbb{S}_+^n$ represents the space of all positive symmetric matrices. Also, $\langle \boldsymbol{B} \rangle(t)$ and $\langle \boldsymbol{B} \rangle(t+s) - \langle \boldsymbol{B} \rangle(s)$ are identically distributed.*

*Remark* 2.5. In what follows, denote by $\bar{\gamma} := \max_{\gamma \in \Gamma}(|\gamma|_F \vee |\gamma|_2)$ where $|\cdot|_F$ and $|\cdot|_2$, respectively, are the Frobenius norm [1] and 2-norm for the matrix. Then, it follows from Lemma 2.4 that $|\langle \boldsymbol{B} \rangle(t)|_F \vee |\langle \boldsymbol{B} \rangle(t)|_2 \le \bar{\gamma} t$. Especially when $m = 1$, we have $\bar{\gamma} = \overline{\sigma}^2$. Also, the largest eigenvalue of a matrix is denoted by $\lambda_{\max}(\cdot)$.

There are some very useful inequalities for our investigation in this article. Combining the results of Sections 3.3-3.5 in [36], Lemma 2.4, and Remark 2.5, we give the conclusions as follows.

THEOREM 2.6. *For any $\eta(t),\ \gamma(t) \in M_G^2[0,T]$, we have $\hat{\mathbb{E}}\left(\int_0^T \eta(t)\ \mathrm{d}B_i(t)\right) = 0$ and $\hat{\mathbb{E}}\left(\int_0^T \eta(t)\ \mathrm{d}B_i(t) \int_0^T \gamma(t)\ \mathrm{d}B_j(t)\right) = \hat{\mathbb{E}}\left(\int_0^T \eta(t)\gamma(t)\ \mathrm{d}\langle B_i, B_j\rangle(t)\right) \le \bar{\gamma} \cdot \hat{\mathbb{E}}\left(\int_0^T |\eta(t)\gamma(t)|\ \mathrm{d}t\right)$.*

Now we introduce the Choquet capacity and some related propositions.

DEFINITION 2.7 (Choquet Capacity, [36]). *For $\mathcal{A} \in \mathscr{B}(\Omega)$, define by $c(\mathcal{A}) := \sup_{Q \in \mathcal{Q}} Q[\mathcal{A}] = \hat{\mathbb{E}}[1_{\mathcal{A}}]$. A property is called valid quasi-surely if this property is valid on the set $\Omega \backslash \mathcal{A}$ with $c(\mathcal{A}) = 0$.*

PROPOSITION 2.8 (Monotone Convergence Theorem, [5, 35]). *If $X(n) \uparrow X$, $\{X(n)\} \subset \mathcal{L}^0(\Omega)$, $X(n)$ is nonnegative, then $\hat{\mathbb{E}}[X(n)] \uparrow \hat{\mathbb{E}}[X]$.*

THEOREM 2.9 ([19]). *Assume that $\{M(n)\}$ is a $G$-supermartingale, satisfying $\sup_n \hat{\mathbb{E}}[M^-(n)] < +\infty$. Then, $\lim_{n\to\infty} M(n)$ exists, which is finite quasi-surely. Here, the definition of $G$-martingale can be found in [36].*

**3. $G$-Semimartingale Convergence Theorem.** In the literature, the semimartingale convergence theorem mainly describes the asymptotic property of the semimartingale, which is a random variable comprising a martingale and a process with bounded variation. Inspired by this well-established and broadly-applied convergence theorem, we are to establish a $G$-semimartingale convergence theorem and its variant. It will be shown that the $G$-semimartingale convergence theorem is based crucially on Doob's $G$-martingale convergence theorem. In fact, to our best knowledge, the continuous version of Doob's $G$-martingale convergence theorem has not yet been established until the result presented as follows.

PROPOSITION 3.1 ($G$-Martingale Convergence Theorem, A Continuous Version).

*Assume that* $\{M(t) : t \in [0, +\infty)\}$ *is a right- or left-continuous* $G$*-supermartingale, and* $M(t) \in L_G^1(\Omega_t)$*. Moreover, assume that* $\hat{\mathbb{E}}[\sup_{t\geq 0} M^-(t)] < +\infty$*. Then,* $M(t)$ *converges finitely to* $M(+\infty) \in L_G^{1^*_*}(\Omega)$ *quasi-surely. Moreover,* $\hat{\mathbb{E}}_t[M(+\infty)] \leq M(t)$*. Here, the definition of* $L_G^{1^*_*}(\Omega)$ *is provided in Definition 7.2 of Appendix 7.1.*

The proof of this proposition is tedious and tangential to the main focus of this article. To enhance the readability, we include the proof into Appendix 7.1. Now, with this preparation, we establish the following $G$-semimartingale convergence theorem.

THEOREM 3.2 ($G$-Semimartingale Convergence Theorem). *Assume that* $A^1$ *and* $A^2$ *are two non-decreasing processes with initial value* $0$*, and that* $A^1(t)$ *is a continuous process with* $\hat{\mathbb{E}}[A^1(+\infty)] < +\infty$*. Also, assume that* $Z$ *is a non-negative* $G$*-semimartingale satisfying* $\hat{\mathbb{E}}[Z^+(0)] < \infty$ *with the form* $Z(t) = Z(0) + A^1(t) - A^2(t) + M(t),\ t \geq 0$*, where* $M(t)$ *is a continuous* $G$*-supermartingale with initial value* $0$ *and* $M(t) \in L_G^1(\Omega_t)$ *for every* $t \geq 0$*. Then, we have that* $A^2(+\infty) < +\infty$*,* $\lim_{t\to+\infty} Z(t)$ *finitely exists and* $\lim_{t\to+\infty} M(t)$ *finitely exists quasi-surely.*

**Proof.** Notice that $M(t) = Z(t) - Z(0) - A^1(t) + A^2(t) \geq -Z(0) - A^1(+\infty)$. Then, $\sup_{t\geq 0} M^-(t) \leq Z^+(0) + A^1(+\infty)$. By Proposition 3.1, we have $\lim_{t\to\infty} M(t)$ finitely exists quasi-surely. Because $A^2(t) = Z(0)+A^1(t)+M(t)-Z(t) \leq Z(0)+A^1(t)+M(t)$ and $Z(t) = Z(0) + A^1(t) - A^2(t) + M(t)$, their limits also exist quasi-surely.

It is mentioned that this $G$-semimartingale convergence theorem can only deal with the case where the limit of $A^1(t)$ is supposed to be finite under the sublinear expectation. We now give its variant, the $G$-semimartingale convergence theorem with the $\mathbb{F}$-stopping time. It can deal with the case where the condition on the finite limit of $A^1(t)$ in Theorem 3.2 is removed. The tradeoff however requires more conditions for the $G$-martingale $M$.

THEOREM 3.3 ($G$-Semimartingale Convergence Theorem with Stopping Time). *Assume that* $A^1$ *and* $A^2$ *are two non-decreasing processes both with initial value* $0$*, and that* $A^1(t)$ *is a continuous adapted process. Also assume that* $Z$ *is a non-negative adapted process satisfying* $\hat{\mathbb{E}}[|Z(0)|] < \infty$ *with the form* $Z(t) = Z(0) + A^1(t) - A^2(t) + M(t),\quad t \geq 0$*, where* $M(t)$ *is a continuous process with initial value* $0$*. Furthermore, assume that there exists a series of* $\mathbb{F}$*-stopping times* $\tau_N$ *satisfying* $\{\tau_N \to +\infty\}$ *quasi-surely such that, for any* $Q \in \mathcal{Q}$*,* $E_Q[M(t \wedge \tau_N)|\mathscr{F}_s] = M(s \wedge \tau_N)$*. Then, we have quasi-surely*

$$\{\omega : A^1(+\infty) < +\infty\} \subset \left\{\omega : \lim_{t\to+\infty} Z(t) \text{ finitely exists}\right\}$$
$$\cap \{\omega : A^2(+\infty) < +\infty\} \cap \left\{\omega : \lim_{t\to+\infty} M(t) \text{ finitely exists}\right\}.$$

*Here,* $\mathcal{A} \subset \mathcal{B}$ *quasi-surely means that* $c(\mathcal{A}\backslash\mathcal{B}) = 0$*, where* $c$ *is the Choquet capacity provided in Definition 2.7.*

**Proof.** Denote by $\mathcal{A} = \Omega\backslash\Big(\{\omega : \lim_{t\to+\infty} Z(t) \text{ finitely exists}\} \cap \{\omega : A^2(+\infty) < +\infty\} \cap \{\omega : \lim_{t\to+\infty} M(t) \text{ finitely exists}\}\Big)$. For every $Q \in \mathcal{Q}$, we have $E_Q[|Z(0)|] \le \hat{\mathbb{E}}[|Z(0)|]$. By the $G$-semimartingale convergence theorem for the normal probability space [24], we have $Q(\mathcal{A}) = 0$. By the arbitrariness of the $Q$'s choice, we obtain that $c(\mathcal{A}) = \sup_{Q\in\mathcal{Q}} Q(\mathcal{A}) = 0$, which therefore completes the proof.

**4. Invariance Principle in Sublinear Expectation.** Now, we consider a $d$-dimensional $G$-stochastic differential equation which reads

$$\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t), t)\mathrm{d}t + \boldsymbol{g}(\boldsymbol{x}(t), t)\mathrm{d}\boldsymbol{B}(t) + \boldsymbol{h}(\boldsymbol{x}(t), t)\mathrm{d}\langle \boldsymbol{B}\rangle(t), \tag{4.1}$$

where the initial value $x(0) = x_0$. Furthermore, we denote, respectively, by $|\boldsymbol{A}|_2 := \sqrt{\mathrm{tr}(\boldsymbol{A}^\top \boldsymbol{A})}$ and $|\boldsymbol{A}| := |\boldsymbol{A}|_F = \sqrt{\sum_{i,j=1}^n a_{ij}^2}$ different norms of a given matrix $\boldsymbol{A}$. All functions $\boldsymbol{f} : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $\boldsymbol{g} : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$, and $\boldsymbol{h} : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times m\times m}$ are supposed to be continuous. In addition, $h^{kij} = h^{kji}$, and $f^i(\boldsymbol{x}, \cdot)$, $g^{ij}(\boldsymbol{x}, \cdot)$ and $h^{kij}(\boldsymbol{x}, \cdot) \in M_G^2[0, T]$ for every $T > 0$. We need the following assumptions.

*Assumption* 4.1. For any $N \in \mathbb{N}$, there exists a number $C_N$ such that $|\boldsymbol{f}(\boldsymbol{x}, t) - f(\boldsymbol{y}, t)| + |\boldsymbol{g}(\boldsymbol{x}, t) - \boldsymbol{g}(\boldsymbol{y}, t)| + |\boldsymbol{h}(\boldsymbol{x}, t) - \boldsymbol{h}(\boldsymbol{y}, t)| \le C_N|\boldsymbol{x} - \boldsymbol{y}|$ for all $|\boldsymbol{x}| \wedge |\boldsymbol{y}| \le N$. Here, $|\boldsymbol{h}|$ still represents the norm for $\boldsymbol{h}$ of $d \times m \times m$ dimensions.

*Assumption* 4.2. There exists a number $C_l$ such that $|\boldsymbol{f}(\boldsymbol{x}, t)| + |\boldsymbol{g}(\boldsymbol{x}, t)| + |\boldsymbol{h}(\boldsymbol{x}, t)| \le C_l(1 + |\boldsymbol{x}|)$, for all $(\boldsymbol{x}, t) \in \mathbb{R}^d \times \mathbb{R}_+$.

Underlying these assumptions as prerequisites, the solutions of Eq. (4.1) are well-posed from a certain perspective as follows.

PROPOSITION 4.3. *If Assumption 4.1 holds, there is a global unique solution in a quasi-sure sense on* $[0, \tau_\infty)$*, where* $\tau_\infty = \lim_{n\to+\infty} \tau_N$, $\tau_N := \inf\{t \ge 0 : |\boldsymbol{x}(t)| \ge N\}$*.For given* $N > 0$*, there exists* $\boldsymbol{x}^N \in M_G^2[0, T]$ *with* $T > 0$ *such that* $\boldsymbol{x} = \boldsymbol{x}^N$ *on* $[0, \tau_N)$*. Additionally, for* $\boldsymbol{A} = (a^{ij}) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$ *with* $a^{ij}(\boldsymbol{x}, \cdot) \in M_G^1[0, T]$ *and* $T > 0$*, we have* $\boldsymbol{M}(t) = \int_0^{t\wedge\tau_N} \boldsymbol{A}(\boldsymbol{x}(s), s)\mathrm{d}\boldsymbol{B}(s)$ *is* $Q$*-martingale for each* $Q \in \mathcal{Q}$*. If Assumption 4.2 holds, we have* $\tau_\infty = +\infty$ *quasi-surely.*

*Remark* 4.4. The proof of Proposition 4.3 is similar to those presented in Refs. [31, 21], which we omit here. It is worth mentioning that $\boldsymbol{x}(\cdot)$, the solution to Eq. (4.1), does not belong to $M_G^2([0, T]; \mathbb{R}^d)$. Actually, $\boldsymbol{x}(\cdot \wedge \tau_N) \in M_*^2([0, T]; \mathbb{R}^d)$ for each $N > 0$, which implies that our solution is locally integrable. In particular, if $\tau_\infty = +\infty$, we have $\boldsymbol{x}(\cdot) \in M_w^2([0, T]; \mathbb{R}^d)$ and it is globally integrable on $[0, +\infty)$ now. Here, both $M_*^2([0, T]; \mathbb{R}^d)$ and $M_w^2([0, T]; \mathbb{R}^d)$ are expanded integrand space defined in Chapter 8 of Ref. [36] satisfying $M_G^2([0, T]; \mathbb{R}^d) \subset M_*^2([0, T]; \mathbb{R}^d) \subset M_w^2([0, T]; \mathbb{R}^d)$.

Next, we introduce $G$-Itô's formula which is useful in the following discussions.

THEOREM 4.5 ($G$-Itô's formula [22]). *Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$. For the d-dimensional G-stochastic differential equations* $\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}(t)\mathrm{d}t + \boldsymbol{g}(t)\mathrm{d}\boldsymbol{B}(t) + \boldsymbol{h}(t)\mathrm{d}\langle \boldsymbol{B}\rangle(t)$ *with the initial value* $\boldsymbol{x}(0) = \boldsymbol{x}_0$. *Moreover,* $\boldsymbol{f} : \mathbb{R}_+ \to \mathbb{R}^d$, $\boldsymbol{g} : \mathbb{R}_+ \to \mathbb{R}^{d\times m}$, *and* $\boldsymbol{h} : \mathbb{R}_+ \to \mathbb{R}^{d\times m^2}$ *with* $f^i(\cdot)$, $g^{ij}(\cdot) \in M_G^1[0,T]$, $h^{kij}(\cdot) \in M_G^2[0,T]$ *for every* $T > 0$. *Then,* $V(\boldsymbol{x}(t),t) = V(\boldsymbol{x}_0,0) + \int_0^t V_t(\boldsymbol{x}(s),s)\mathrm{d}s + \int_0^t V_{x_i}(\boldsymbol{x}(s),s)f^i(s)\mathrm{d}s + \int_0^t V_{x_i}(\boldsymbol{x}(s),s)g^{ij}(s)\mathrm{d}B_j(s) + \int_0^t V_{x_k}(\boldsymbol{x}(s),s)h^{kij}(s)\mathrm{d}\langle B_i,B_j\rangle(s) + \int_0^t \frac{1}{2}V_{x_kx_l}(\boldsymbol{x}(s),s)g^{ki}(s)g^{lj}(s)\mathrm{d}\langle B_i,B_j\rangle(s)$.

Actually, $G$-Itô's formula presented above could be applicable to $M_*^2([0,T];\mathbb{R}^d)$ and $M_w^2([0,T];\mathbb{R}^d)$ according to Theorem 5.4 established in [22]. By virtue of $G$-Itô's formula, Assumption 4.2 used above can be replaced. To present this result, we introduce the notation as $\mathcal{L}V := V_t + V_{x_i}f^i + G\Big((V_{x_k}(h^{kij}+h^{kji}) + V_{x_kx_l}g^{ki}g^{lj})_{i,j=1}^n\Big)$, where the function $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$. As such, we obtain the following result.

PROPOSITION 4.6. *Suppose that Assumption 4.1 holds and that there exists a function* $\gamma \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ *such that* $\mathcal{L}V(\boldsymbol{x},t) \le \gamma(t)$. *Moreover,* $V$ *satisfies*

$$\lim_{|x|\to\infty} \inf_{0\le t<+\infty} V(\boldsymbol{x},t) = +\infty. \tag{4.2}$$

*Then,* $\tau_\infty$*, as defined in Proposition 4.3, satisfies* $\tau_\infty = +\infty$ *quasi-surely.*

For simplicity of expression, we still include the proof of Proposition 4.6 in Appendix 7.2, where the following proposition is needed.

PROPOSITION 4.7 ([21]). *Let* $M(t) = \int_0^t \kappa_{ij}(s)\mathrm{d}\langle B_i,B_j\rangle(s) - \int_0^t 2G(\boldsymbol{\kappa})\mathrm{d}s$, *where* $\boldsymbol{\kappa} \in M_G^1([0,T];\mathbb{S}^n)$. *Then, we have* $M(t) \le 0$ *quasi-surely. Particularly* $\hat{\mathbb{E}}[M(t)] \le 0$.

In addition, we present the following $G$-stochastic Barbalat's lemma that will be used later, and its proof is provided in Appendix 7.3.

LEMMA 4.8. *Suppose that Assumption 4.1 holds and* $\tau_\infty = +\infty$ *quasi-surely, where* $\tau_\infty$ *is defined in Proposition 4.3. Also suppose that the solution to Eq. (4.1) satisfies* $\sup_{t\in\mathbb{R}^+}|\boldsymbol{x}(t)| < +\infty$ *q.s.. Besides, there exists* $\eta \in C(\mathbb{R}^d;\mathbb{R}_+)$ *such that*

$$\int_0^{+\infty} \eta(\boldsymbol{x}(t))\mathrm{d}t < +\infty, \quad q.s.. \tag{4.3}$$

*Then, we have* $\lim_{t\to+\infty}\eta(\boldsymbol{x}(t)) = 0$ *quasi-surely.*

Now, with the following assumption, we state our main theorem.

*Assumption* 4.9. For each $N > 0$, $t \in \mathbb{R}_+$ and all $|\boldsymbol{x}| \le N$, there exists a number $K_N > 0$ such that $|\boldsymbol{f}(\boldsymbol{x},t)| + |\boldsymbol{g}(\boldsymbol{x},t)| + |\boldsymbol{h}(\boldsymbol{x},t)| \le K_N$.

THEOREM 4.10. *Suppose that Assumptions 4.1 and 4.9 hold. Also suppose that there exist three functions* $V \in C^{2,1}(\mathbb{R}^d\times\mathbb{R}_+;\mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ *and* $\eta \in C(\mathbb{R}^d;\mathbb{R}_+)$ *such that* (UB) $\lim_{|x|\to\infty}\inf_{0\le t<+\infty} V(\boldsymbol{x},t) = \infty$ *and* $\mathcal{L}V(\boldsymbol{x},t) \le \gamma(t) - \eta(\boldsymbol{x})$. *Then,* *we have that* $\lim_{t\to+\infty} V(\boldsymbol{x}(t),t)$ *finitely exists quasi-surely and that*

$$\lim_{t\to+\infty} \eta(\boldsymbol{x}(t)) = 0 \quad q.s.. \tag{4.4}$$

*Moreover,* $\lim_{t\to+\infty} d(\boldsymbol{x}, \mathrm{Ker}(\eta)) = 0$*, where* $d(\boldsymbol{x}, \mathrm{Ker}(\eta)) := \inf_{\boldsymbol{y}\in\mathrm{Ker}(\eta)} |\boldsymbol{x}-\boldsymbol{y}|$.

**Proof.** Using Proposition 4.6, the $G$-SDEs satisfying the conditions assumed in this theorem have a global solution on $[0,+\infty)$ with a property that $\mathcal{L}V(\boldsymbol{x},t) \le \gamma(t) - \eta(\boldsymbol{x}) \le \gamma(t)$. By $G$-Itô's formula in Theorem 4.5, Proposition 4.3 and Remark 4.4, we have

$$\begin{aligned}
&V(\boldsymbol{x}(t\wedge\tau_N), t\wedge\tau_N) = V(\boldsymbol{x}_0,0) + \int_0^{t\wedge\tau_N} V_t(\boldsymbol{x}(s),s)\mathrm{d}s \\
&+ \int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s),s) f^i(\boldsymbol{x}(s),s)\mathrm{d}s + \int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s),s) g^{ij}(\boldsymbol{x}(s),s)\mathrm{d}B_j(s) \\
&+ \int_0^{t\wedge\tau_N} V_{x_k}(\boldsymbol{x}(s),s) h^{kij}(\boldsymbol{x}(s),s)\mathrm{d}\langle B_i,B_j\rangle(s) + \int_0^{t\wedge\tau_N} \frac{1}{2} V_{x_k x_l}(\boldsymbol{x}(s),s) g^{ki}(\boldsymbol{x}(s),s) \\
&g^{lj}(\boldsymbol{x}(s),s)\mathrm{d}\langle B_i,B_j\rangle(s),
\end{aligned}$$

where $\tau_N := \inf\{t\ge 0 : |\boldsymbol{x}(t)| \ge N\}$. Letting $N\to+\infty$ and setting $\boldsymbol{\kappa} = (\kappa_{ij})_{i,j=1}^m$ for every $t\ge 0$ where $\kappa_{ij} = V_{x_k}(h^{kij} + h^{kji}) + V_{x_k x_l} g^{ki} g^{lj}$, we get that $\tau_N$ tends to $+\infty$ by Proposition 4.3 and

$$\begin{aligned}
V(\boldsymbol{x}(t),t) &= V(\boldsymbol{x}_0,0) + \int_0^t V_t(\boldsymbol{x}(s),s)\mathrm{d}s + \int_0^t V_{x_i}(\boldsymbol{x}(s),s) f^i(\boldsymbol{x}(s),s)\mathrm{d}s \\
&\quad + \int_0^t V_{x_i}(\boldsymbol{x}(s),s) g^{ij}(\boldsymbol{x}(s),s)\mathrm{d}B_j(s) + \int_0^t \frac{1}{2}\kappa_{ij}(\boldsymbol{x}(s),s)\mathrm{d}\langle B_i,B_j\rangle(s).
\end{aligned}$$

Thus, if we set

$$V(\boldsymbol{x}(t),t) = V(\boldsymbol{x}_0,0) + \int_0^t \gamma(s)\mathrm{d}s - A_2(t) + \int_0^t V_{x_i}(\boldsymbol{x}(s),s) g^{ij}(\boldsymbol{x}(s),s)\mathrm{d}B_j(s),$$

then $A_2(0) = 0$. Besides, according to Proposition 4.7, for every $0\le t_1 < t_2 < +\infty$, we have

$$\begin{aligned}
A_2(t_2) - A_2(t_1) &= \int_{t_1}^{t_2} \gamma(s)\mathrm{d}s - \int_{t_1}^{t_2} V_{x_i}(\boldsymbol{x}(s),s) f^i(\boldsymbol{x}(s),s)\mathrm{d}s \\
&\quad - \int_{t_1}^{t_2} V_t(\boldsymbol{x}(s),s)\mathrm{d}s - \int_{t_1}^{t_2} \frac{1}{2}\kappa_{ij}(\boldsymbol{x}(s),s)\mathrm{d}\langle B_i,B_j\rangle(s) \\
&\ge \int_{t_1}^{t_2} \gamma(s)\mathrm{d}s - \int_{t_1}^{t_2} V_t(\boldsymbol{x}(s),s)\mathrm{d}s \\
&\quad - \int_{t_1}^{t_2} V_{x_i}(\boldsymbol{x}(s),s) f^i(\boldsymbol{x}(s),s)\mathrm{d}s - \int_{t_1}^{t_2} G(\eta(\boldsymbol{x}(s),s))\mathrm{d}s \\
&= \int_{t_1}^{t_2} \gamma(s)\mathrm{d}s - \int_{t_1}^{t_2} \mathcal{L}V(\boldsymbol{x}(s),s)\mathrm{d}s \ge \int_{t_1}^{t_2} \eta(s)\mathrm{d}s \ge 0
\end{aligned}$$

which implies that $A_2(t)$ is a non-decreasing process. Using Proposition 4.3, we obtain that $\int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s),s)g^{ij}(\boldsymbol{x}(s),s)\mathrm{d}B_j(s)$ is a $Q$-martingale for every $Q \in \mathcal{Q}$. Noticing $\int_0^{+\infty}\gamma(s)\mathrm{d}s < +\infty$ and according to Proposition 3.3, we have a set $\Omega_0 \subset \Omega$ such that $c(\Omega\backslash\Omega_0) = 0$. Then, we have that, for all $\omega \in \Omega_0$, $\lim_{n\to+\infty} A_2(t)$ finitely exists and $\lim_{n\to+\infty} V(\boldsymbol{x}(t),t)$ finitely exists. Thus, on $\Omega_0$, $\int_0^{+\infty}\eta(\boldsymbol{x}(t))\mathrm{d}t < +\infty$. From the finite existence of the limit of $V$, we obtain that, on $\Omega_0$, $\sup_{t\geq 0} V(\boldsymbol{x}(t;\omega),t) < +\infty$. Hence, from the above-assumed condition (UB), it follows that there exists $K(\omega)$ such that $\sup_{t\geq 0}|\boldsymbol{x}(t;\omega)| \leq K(\omega)$. According to Lemma 4.8, we obtain $\lim_{t\to+\infty}\eta(\boldsymbol{x}(t)) = 0$ quasi-surely.

For every $\omega$ satisfying $\lim_{t\to+\infty}\eta(\boldsymbol{x}(t;\omega)) = 0$ and $\sup_{t\in\mathbb{R}_+}|\boldsymbol{x}(t;\omega)| < +\infty$, there exists $\boldsymbol{y}(\omega)$ and a sequence $\{t_i\}$ having $\lim_{i\to+\infty}\boldsymbol{x}(t_i;\omega) = \boldsymbol{y}(\omega)$. So, $\lim_{i\to+\infty}\eta(\boldsymbol{x}(t_i;\omega)) = \boldsymbol{\eta}(\boldsymbol{y}(\omega)) = 0$ and $\mathrm{Ker}(\eta) \neq \emptyset$. If $\limsup_{t\to+\infty} d(\boldsymbol{x}(t;\omega),\ker(\eta))$ is positive, there exist a sequence $\{t_i\}$ such that $d(\boldsymbol{x}(t_i;\omega),\ker(\eta)) \geq \epsilon$, for some $\epsilon > 0$. This implies $\eta(\boldsymbol{y}) > 0$, which is a contradiction.

*Remark* 4.11. Here, our conclusions nontrivially extend the corresponding results obtained for the traditional SDEs. Particularly, the significant differences do exist. First, in terms of the conclusions, we are able to induce relevant results even when the system randomness itself is uncertain, greatly surpassing the applicability scope of existing Brownian motion-driven stochastic systems. From a technical standpoint, our generalized stochastic differential equation (i.e., G-SDE) cannot measure the occurrence probability of events from the perspective of traditional probability measures, but the capacities instead. Second, the construction of the monotone functions in our semi-martingales differs significantly from the invariance principles in the traditional stochastic analysis.

Next, we present another version of invariance principle, where $\eta$ is a function with respect to the function $V$.

THEOREM 4.12. *Suppose that Assumption 4.1 holds, and that there exist three functions* $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+;\mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ *and* $\eta \in C(\mathbb{R}_+;\mathbb{R}_+)$ *such that* $\mathcal{L}V(\boldsymbol{x},t) \leq \gamma(t) - \eta(V(\boldsymbol{x},t))$ *for all* $(\boldsymbol{x},t) \in \mathbb{R}^d \times \mathbb{R}_+$. *Then, we obtain that* $\lim_{t\to+\infty} V(\boldsymbol{x}(t),t)$ *finitely exists quasi-surely and* $\lim_{t\to+\infty}\eta(V(\boldsymbol{x}(t),t)) = 0$ *q.s.. Moreover,* $\lim_{t\to+\infty} d(V(\boldsymbol{x}(t),t),\mathrm{Ker}(\eta)) = 0$.

**Proof.** Analogously, the $G$-SDEs have a global solution on $[0,+\infty)$ according to Proposition 4.6. By the arguments akin to those for validating Theorem 4.10, we obtain $V(\boldsymbol{x}(t),t) = V(\boldsymbol{x}_0,0) + \int_0^t\gamma(s)\mathrm{d}s - A_2(t) + \int_0^t V_{x_i}(\boldsymbol{x}(s),s)g^{ij}(\boldsymbol{x}(s),s)\mathrm{d}B_j(s)$, where $A_2(0) = 0$ and for every $0 \le t_1 < t_2 < +\infty$,

$$A_2(t_2) - A_2(t_1) = \int_{t_1}^{t_2} \gamma(s)\mathrm{d}s - \int_{t_1}^{t_2} V_{x_i}(\boldsymbol{x}(s), s) f^i(\boldsymbol{x}(s), s)\mathrm{d}s$$
$$- \int_{t_1}^{t_2} V_t(\boldsymbol{x}(s), s)\mathrm{d}s - \int_{t_1}^{t_2} \frac{1}{2}\kappa_{ij}(\boldsymbol{x}(s), s)\mathrm{d}\langle B_i, B_j\rangle(s)$$

$$\ge \int_{t_1}^{t_2} \gamma(s)\mathrm{d}s - \int_{t_1}^{t_2} V_{x_i}(\boldsymbol{x}(s), s) f^i(\boldsymbol{x}(s), s)\mathrm{d}s$$
$$- \int_{t_1}^{t_2} V_t(\boldsymbol{x}(s), s)\mathrm{d}s - \int_{t_1}^{t_2} G(\eta(\boldsymbol{x}(s), s))\mathrm{d}s$$
$$= \int_{t_1}^{t_2} \gamma(s)\mathrm{d}s - \int_{t_1}^{t_2} \mathcal{L}V(\boldsymbol{x}(s), s)\mathrm{d}s \ge \int_{t_1}^{t_2} \eta(V(\boldsymbol{x}(s), s))\mathrm{d}s \ge 0.$$

Hence, by the $G$-semimartingale Convergence Theorem 3.3, there exists $\bar{\Omega} \subset \Omega$ such that $c(\Omega\backslash\bar{\Omega}) = 0$. Furthermore, we have that, on $\bar{\Omega}$,

$$\int_0^{\infty} \eta(V(\boldsymbol{x}(t), t))\mathrm{d}t < +\infty \quad \text{and} \quad \lim_{n\to+\infty} V(\boldsymbol{x}(t), t) \text{ finitely exists.}$$

Now, we claim that, for every $\omega \in \bar{\Omega}$, we have $\lim_{t\to+\infty} \eta(V(\boldsymbol{x}(t;\omega), t)) = 0$. We validate the claim by contradiction. If this is not the case, then we have a sequence $\{t_k\}$ with $t_{k+1} - t_k > 1$ and $\epsilon > 0$, such that $\eta(V(\boldsymbol{x}(t_k;\omega), t_k)) > \epsilon$. Assume $\sup_{t\ge0} V(\boldsymbol{x}(t;\omega), t) \le K(\omega)$. Hence, there exists $\delta_1$ such that $|\eta(x) - \eta(y)| \le \frac{\epsilon}{2}$ for $0 \le x, y \le K(\omega)$ and $|x - y| \le \delta_1$. As $\lim_{t\to+\infty} V(\boldsymbol{x}(t;\omega), t)$ finitely exists and $V(\boldsymbol{x}(t;\omega), t)$ is continuous about $t$, we can easily check that it is uniformly continuous on $\mathbb{R}^+$. Thus, there exists $\delta_2 < 1$ such that $|V(\boldsymbol{x}(t;\omega), t) - V(\boldsymbol{x}(s;\omega), s)| < \delta_1, \ |t-s| < \delta_2$. Consequently, for $t_k \le t < t_k + \delta_2$, we have $\eta(V(\boldsymbol{x}(t;\omega), t)) \ge \eta(V(\boldsymbol{x}(t_k;\omega), t_k)) - |\eta(V(\boldsymbol{x}(t_k;\omega), t_k)) - \eta(V(\boldsymbol{x}(t;\omega), t))| \ge \frac{\epsilon}{2}$. Therefore, $+\infty > \int_0^\infty \eta(V(\boldsymbol{x}(t), t))\mathrm{d}t \ge \sum_{k=1}^{+\infty} \int_{t_k}^{t_k+\delta_2} \eta(V(\boldsymbol{x}(t), t))\mathrm{d}t \ge \sum_{k=1}^{+\infty} \frac{\epsilon\delta_2}{2} = +\infty$, which indicates a contradiction. Finally, the arguments for proving $\lim_{t\to+\infty} d(V(\boldsymbol{x}(t), t), \mathrm{Ker}(\eta)) = 0$ are the same as those for validating the last conclusion in Theorem 4.10.

*Remark* 4.13. A set $\mathcal{A} \in \mathscr{B}(\Omega)$ is said to be invariant if $c\big(\{\exists t \ge 0,\ x(t;\boldsymbol{x}_0) \notin \mathcal{A}\}\big) = 0$, for every $\boldsymbol{x}_0 \in \mathcal{A}$. Actually, if we suppose some conditions to be valid only in the invariant set $\mathcal{A}$ for Theorems 4.10 and 4.12, the conclusions in these theorems still sustain.

Finally, we present two corollaries which can be obtained directly form the invariance principles established above. These results are related to the stability or the exponential stability of the solution $\boldsymbol{x}(t)$.

COROLLARY 4.14. *Let Assumption 4.1 hold. Assume further that there exists a function $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$ such that*

$$\mu_1(|\boldsymbol{x}|) \le V(\boldsymbol{x}, t) \le \mu_2(|\boldsymbol{x}|), \quad \mathcal{L}V(\boldsymbol{x}, t) \le -\mu_3(|\boldsymbol{x}|), \tag{4.5}$$

*where* $\mu_1$, $\mu_2$ *and* $\mu_3$ *are three strictly increasing functions in* $[0,+\infty)$ *with the initial value* $0$ *and* $\mu_1(r), \mu_2(r) \to +\infty$ *as* $r \to +\infty$. *Then, we have* $\lim_{t\to+\infty} |\boldsymbol{x}(t)| = 0$ *q.s.*.

**Proof.** From the condition assumed in (4.5), it follows that $\mu_2^{-1}(V(\boldsymbol{x},t)) \le |\boldsymbol{x}|$, which implies $\mathcal{L}V(\boldsymbol{x},t) \le -\mu_3(\mu_2^{-1}(V(\boldsymbol{x},t)))$. According to Theorem 4.12, we have $\lim_{t\to\infty} \mu_3(\mu_2^{-1}(V(\boldsymbol{x}(t),t))) = 0$ *q.s.*, which implies $\lim_{t\to\infty} V(\boldsymbol{x}(t),t) = 0$ *q.s.*. Therefore, we have $\lim_{t\to\infty} \mu_1(|\boldsymbol{x}(t)|) = 0$ *q.s.*, which finally gives $\lim_{t\to\infty} |\boldsymbol{x}(t)| = 0$ *q.s.*.

COROLLARY 4.15. *Let Assumption 4.1 hold. Assume further that there exist two functions:* $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+;\mathbb{R}_+)$ *and* $\gamma \in L^1(\mathbb{R}_+;\mathbb{R}_+)$, *such that* $\mathrm{e}^{\lambda t}|\boldsymbol{x}|^p \le V(\boldsymbol{x}(t),t)$ and $\mathcal{L}V(\boldsymbol{x},t) \le \gamma(t)$, *where* $\lambda$ *and* $p$ *are positive numbers. Then, we have* $\overline{\lim}_{t\to+\infty} \frac{1}{t}\log|\boldsymbol{x}(t)| \le -\frac{\lambda}{p}$ *q.s.*.

**Proof.** Set $\eta = 0$ in Theorem 4.12. Then, $\lim_{t\to+\infty} V(\boldsymbol{x}(t),t)$ finitely exists quasi-surely. Further use the condition that $\mathrm{e}^{\lambda t}|\boldsymbol{x}|^p \le V(\boldsymbol{x}(t),t)$. The proof is therefore complete.

**5. Illustrative Examples: Applying $G$-invariance principle to achieving $G$-stochastic control.** In this section, we use several representative examples to illustrate the applicability of our analytical results to realizing $G$-stochastic control of the unstable dynamical systems.

*Example* 5.1. Consider a linear (complex network) system $\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t)\mathrm{d}t$. Here, $\boldsymbol{A} = [11,5,2;5,11,2;2,2,14]$. Then, it is easy to check that $\lambda_{\max}(\boldsymbol{A}) = 18$ and the system is unstable. Now, for a $G$-Brownian motion where $\underline{\sigma}^2 = 3.5$ and $\overline{\sigma}^2 = 4$, we choose $\boldsymbol{D} = \boldsymbol{I}_3$ and $\boldsymbol{C} = [-19,11,2;11,-19,2;2,2,-10]$ to $G$-stochastically control the linear system as $\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{x}(s)\mathrm{d}s + \int_0^t \boldsymbol{D}\boldsymbol{x}(s)\mathrm{d}\boldsymbol{B}(s) + \int_0^t \boldsymbol{C}\boldsymbol{x}(s)\mathrm{d}\langle\boldsymbol{B}\rangle(s)$. Choosing $V(\boldsymbol{x}) := |\boldsymbol{x}|^2$ yields: $\mathcal{L}V(\boldsymbol{x}) = 2\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x} + G(2\boldsymbol{x}^\top\boldsymbol{D}^\top\boldsymbol{D}\boldsymbol{x} + 4\boldsymbol{x}^\top\boldsymbol{C}\boldsymbol{x})$. As $\lambda_{\max}(\boldsymbol{C}) = -6$, we easily derive that $\mathcal{L}V(\boldsymbol{x}) \le -2.5|\boldsymbol{x}|^2$. This, according to Corollary 4.14, ensures the asymptotic stability of the controlled system in a quasi-sure sense.

Moreover, if we set $V(\boldsymbol{x},t) = \mathrm{e}^{\lambda t}|\boldsymbol{x}|^2$, we obtain that $\mathcal{L}V(\boldsymbol{x},t) = \mathcal{L}V(\boldsymbol{x}) = \left[\boldsymbol{x}^\top(2\boldsymbol{A}+\lambda\boldsymbol{I}_d)\boldsymbol{x} + G(2\boldsymbol{x}^\top\boldsymbol{D}^\top\boldsymbol{D}\boldsymbol{x} + 4\boldsymbol{x}^\top\boldsymbol{C}\boldsymbol{x})\right]\mathrm{e}^{\lambda t}$, which, using the parameters $\underline{\sigma}^2 = 3.5$ and $\overline{\sigma}^2 = 4$, yields $\mathcal{L}V(\boldsymbol{x},t) \le (\lambda - 1.5)|\boldsymbol{x}|^2$. If we set $\lambda \le 1.5$, using Corollary 4.15 gives $\overline{\lim}_{t\to+\infty} \frac{1}{t}\log|\boldsymbol{x}(t)| \le -0.75$ *q.s.*. This clearly illustrates the exponential stability of the controlled system.

*Example* 5.2. Consider an autonomous system, which reads $\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t))\mathrm{d}t$. Here, $\boldsymbol{f}$ satisfies Assumption 4.1 and $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. Moreover, $\boldsymbol{f}$ satisfies one-sided Lipschitz condition, i.e., there exists a number $L > 0$ such that $\langle \boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})\rangle \le L|\boldsymbol{x}|^2$. There are many systems, not globally Lipschitzian, only satisfying this one-sided Lipschitz condition. For instance, both $f(x) = x - x^3$ and the Lorenz system with $\boldsymbol{f}(\boldsymbol{x}) = [\sigma x_2 - \sigma x_1, \rho x_1 - x_3 x_1 - x_2, x_1 x_2 - \beta x_3]^\top$ satisfy the one-sided Lipschitz condition. Now, we apply the $G$-stochastic control to the original dynamics, which yields $\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t))\mathrm{d}t + k\sum_{j=1}^{m}\boldsymbol{x}(t)\mathrm{d}B_j(t)$ with $k > (-L/c_{-1})^{1/2}$ with $c_{-1} := G\left((-1)_{i,j=1}^{m}\right)$. Here, $(-1)_{i,j=1}^{m}$ corresponds to an $m \times m$ matrix with all elements are $-1$. Then, the controlled system becomes stochastically stable, whose proof is included in Appendix 7.4. Take the three-dimensional Lorenz system for example. We are able to use a one-dimensional $G$-Brownian motion to render the controlled system stable quasi-surely, if we set $m = 1$, $c_{-1} = G(-1) = -\frac{1}{2}\underline{\sigma}^2$, $L \le \frac{1}{2}(\sigma + \rho)$, and $k > (\sigma+\rho)^{1/2}\underline{\sigma}^{-1}$.

*Example* 5.3. Consider an oscillating system $\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{C}\boldsymbol{f}(\boldsymbol{x}(t))\mathrm{d}t$, where $\boldsymbol{C} = [1, 1, 4; 5, -1, 4; 8, 1, 0]$ and $\boldsymbol{f}(\boldsymbol{x}) = [-x_1, \arctan(x_2), \tanh(x_3)]^\top$. Now, we consider the $G$-stochastically controlled system as $\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{C}\boldsymbol{f}(\boldsymbol{x}(t))\mathrm{d}t + \boldsymbol{g}(\boldsymbol{x}(t))\mathrm{d}\boldsymbol{B}(t)$, where $\boldsymbol{B}$ is a two-dimensional, independent and identically distributed $G$-Brownian motion with $\bar{\sigma}^2 = 50$ and $\underline{\sigma}^2 = 40$, and $\boldsymbol{g}(\boldsymbol{x}) = [\boldsymbol{A}_1\boldsymbol{x}, \boldsymbol{A}_2\boldsymbol{x}]$ in which $\boldsymbol{A}_1 = [1, 0.5, 0; 0, 1, 0; 0, 0, 1]$ and $\boldsymbol{A}_2 = [1, 0, 0; 0, 1, 0.5; 0, 0, 1]$. Additionally, the $G$-function of $\boldsymbol{B}$ satisfies $\boldsymbol{G}(\boldsymbol{M}) = \sum_{j=1}^{2} G_j(a_{jj})$, where $\boldsymbol{M} = (m_{ij})_{i,j=1}^{2}$ is a two-dimensional matrix, and $G_j$ is the $G$-function related to the one-dimensional $G$-Brownian motion $B_j$. Set $V(\boldsymbol{x}) = |\boldsymbol{x}|^\alpha$ for some $\alpha > 0$. By Appendix 7.5, $\mathbb{R}^3\backslash\{\boldsymbol{0}\}$ is an invariant set of the system. It follows that, on $\mathbb{R}^3\backslash\{\boldsymbol{0}\}$,

$$\begin{aligned}
\mathcal{L}V(\boldsymbol{x}) = \alpha|\boldsymbol{x}|^{\alpha-2}\big[ & -x_1^2 + x_1\arctan(x_2) + 4x_1\tanh(x_3) - 5x_1x_2 - x_2\arctan(x_2) \\
& +4x_2\tanh(x_3) - 8x_3x_1 + x_3\arctan(x_2)\big] \\
& +\alpha|\boldsymbol{x}|^{\alpha-4}G\left(|\boldsymbol{x}|^2\boldsymbol{g}^\top\boldsymbol{g} + (\alpha-2)\boldsymbol{g}^\top\boldsymbol{x}\boldsymbol{x}^\top\boldsymbol{g}\right) \\
\le\ & \alpha|\boldsymbol{x}|^{\alpha-2}(-x_1^2 + 6|x_1x_2| + 12|x_1x_3| + 5|x_2x_3|) \\
& + \sum_{j=1}^{2}\alpha|\boldsymbol{x}|^{\alpha-4}G_j\left(|\boldsymbol{x}|^2|\boldsymbol{A}_j\boldsymbol{x}|^2 + (\alpha-2)(\boldsymbol{x}^\top\boldsymbol{A}_j\boldsymbol{x})^2\right).
\end{aligned}$$

Notice that $(\boldsymbol{x}^\top\boldsymbol{A}_j\boldsymbol{x})^2 \ge \frac{1}{2}|\boldsymbol{x}|^2|\boldsymbol{A}_j\boldsymbol{x}|^2 + \frac{1}{8}|\boldsymbol{x}|^4$ and $\boldsymbol{x}^\top\boldsymbol{A}_j\boldsymbol{x} \le \frac{5}{4}|\boldsymbol{x}|^2$ for $j = 1, 2$, and set $\alpha = \frac{2}{25}$. Then, we obtain $\mathcal{L}V(\boldsymbol{x}) \le \frac{17}{25}|\boldsymbol{x}|^{\frac{2}{25}} + \sum_{j=1}^{2}\frac{2}{25}|\boldsymbol{x}|^{-\frac{98}{25}}G_j\left(\frac{2}{25}(\boldsymbol{x}^\top\boldsymbol{A}_j\boldsymbol{x})^2 - \frac{1}{4}|\boldsymbol{x}|^4\right) \le -\frac{3}{25}|\boldsymbol{x}|^{\frac{2}{25}}$. Setting $\eta$ in Theorem 4.10 as $\eta(\boldsymbol{x}) = \frac{3}{25}|\boldsymbol{x}|^{\frac{2}{25}}$ guarantees the quasi-sure stability of the above controlled system.

In Appendix 7.6, we further provide a few numerical evidences for illustrating the above examples. It is emphasized that those numerically-presented results do not represent all the exact solution produced by the $G$-SDEs, but only provide some evidences partially supporting the analytical results obtained in the above examples. The numerical scheme used there is not complete, so it awaits further development for rigorously approximating the solution of $G$-SDEs.

**6. Conclusion.** In this article, we have developed several invariance principles for the stochastic differential equations driven by the $G$-Brownian motions. Our work is basically inspired by the seminal works from two directions: one is from the stability theory of the traditional SDEs [28] and the other is from the fundamentally-innovative works on the sublinear expectation [36]. Our contributions include not only the establishment of the $G$-semimartingale convergence theorem and its variants for the sublinear expectation, but also the establishment of several invariance principles and their applications in investigating the long-term behaviors of $G$-SDEs. Indeed, we anticipate that our analytical results can be beneficial to understanding and solving the problems associated with uncertain randomness in dynamical systems.

As for the future research directions, the assumption on the linear growth and the locally Lipschitz conditions can be further weakened through restricting the discussion for the operator $\mathcal{L}$ in some specific space. Also, further development of the invariance principles for the $G$-SDDEs and the $G$-SFDEs could be promoted. More practically, complete scheme for rigorously approximating the solution produced by the $G$-SFDEs deserves deep investigation.

## 7. Appendix.

**7.1. Proof of Proposition 3.1.** First, we establish Fatou's lemma for the $G$-conditional expectation, which is a prerequisite for our proposition to be demonstrated.

LEMMA 7.1 (Fatou's Lemma for $G$-conditional Expectation). *$\{X(n)\} \in L_G^1(\Omega)$ are a series of random vectors, and there exists a random variable $M$ such that $\hat{\mathbb{E}}[|M|] < +\infty$ and $X(n) \geq M$ for any $n > 0$. Then, $\hat{\mathbb{E}}_t\left[\underline{\lim}_{n\to\infty} X(n)\right] \leq \underline{\lim}_{n\to\infty} \hat{\mathbb{E}}_t[X(n)]$.*

In order to present the proof for this lemma, we need to extend the space of random variables and make some necessary preparations.

DEFINITION 7.2 ([15]). *Introduce some extended spaces of random variables as follows:*

$$\begin{aligned}
\mathcal{L}_G^{1^*}(\Omega) &:= \Big\{X \in L^0(\Omega) : \exists X(n) \in L_G^1(\Omega) \text{ such that } X(n) \downarrow X\Big\},\\
L_G^{1^*}(\Omega) &:= \Big\{X \in L^0(\Omega) : \hat{\mathbb{E}}[|X|] < +\infty, \quad X \in \mathcal{L}_G^{1^*}(\Omega)\Big\},\\
\mathcal{L}_G^{1^*_*}(\Omega) &:= \Big\{X \in L^0(\Omega) : \exists X(n) \in L_G^{1^*}(\Omega) \text{ such that } X(n) \uparrow X\Big\},\\
L_G^{1^*_*}(\Omega) &:= \Big\{X \in L^0(\Omega) : \hat{\mathbb{E}}[|X|] < +\infty, \quad X \in \mathcal{L}_G^{1^*_*}(\Omega)\Big\}.
\end{aligned}$$

*Then, we extend the $G$-conditional expectation on $\mathcal{L}_G^{1^*_*}(\Omega)$. Directly, we have $L_G^{1^*}(\Omega) \subset \mathcal{L}_G^{1^*}(\Omega) \subset \mathcal{L}_G^{1^*_*}(\Omega)$ and $L_G^{1^*}(\Omega) \subset L_G^{1^*_*}(\Omega) \subset \mathcal{L}_G^{1^*_*}(\Omega)$.*

LEMMA 7.3 ([15]). *Suppose that $\{X(n)\} \subset L_G^{1^*_*}(\Omega)$ is a series of non-decreasing random variables. Denote by $X := \lim_{n\to\infty} X(n)$. Then, we have quasi-surely $\lim_{n\to\infty} \hat{\mathbb{E}}_t[X(n)] = \hat{\mathbb{E}}_t[X]$.*

LEMMA 7.4. *If $X, Y \in L_G^1(\Omega)$, then $X \wedge Y \in L_G^1(\Omega)$ (resp. $X \vee Y \in L_G^1(\Omega)$).*

**Proof.** As $X, Y \in L_G^1(\Omega)$, there exists $\{X_n\}$ and $\{Y_n\}$ contained in $\mathrm{Lip}(\Omega)$ such that $\hat{\mathbb{E}}[|X(n)-X|] \to 0$ and $\hat{\mathbb{E}}[|Y(n)-Y|] \to 0$. For $\varphi,\ \psi \in C_{l,\mathrm{Lip}}(\Omega)$, we have $\varphi \wedge \psi = \frac{\varphi+\psi-|\varphi-\psi|}{2} \in C_{l,\mathrm{Lip}}(\Omega)$. Thus, $X(n)\wedge Y(n) \in \mathrm{Lip}(\Omega)$. So we derive $\hat{\mathbb{E}}[|X \wedge Y - X(n)\wedge Y(n)|] \le \hat{\mathbb{E}}[|X-X(n)|]+\hat{\mathbb{E}}[|Y-Y(n)|] \to 0$, which implies $X\wedge Y \in L_G^1(\Omega)$. The case that $X \vee Y \in L_G^1(\Omega)$ is analogous.

Lemma 7.5. *If $X(n) \in L_G^1(\Omega)$ and $X(n)$ converges to $X$, and there exists a random variable $M$ such that $\hat{\mathbb{E}}[|M|] < +\infty$ and $X(n) \ge M$ for any $n > 0$. Then, $X \in \mathcal{L}_G^{1^*}(\Omega)$.*

**Proof.** For any $m, n > 0$, by Lemma 7.4, we obtain that $\inf_{n\le k\le m} X(k) \in L_G^1(\Omega)$. Then, from Definition 7.2, it follows that $\inf_{k\ge n} X(k) \in \mathcal{L}_G^{1^*}(\Omega)$. Also, by the fact that $M \le \inf_{k\ge n} X(k) \le X(n)$, we have $|\inf_{k\ge n} X(k)| \le |X(n)| + |M|$. Thus, $\hat{\mathbb{E}}[|\inf_{k\ge n} X(k)|] \le +\infty$ and $\inf_{k\ge n} X(k) \in L_G^{1^*}(\Omega)$ using Definition 7.2. As $X = \lim_{n\to+\infty} \inf_{k\ge n} X(k)$, we immediately obtain the conclusion using Definition 7.2.

**Proof of Lemma 7.1.** Set $Y(n) := \inf_{k\ge n} \hat{\mathbb{E}}_t[X(k)]$. Using the arguments analogous to those performed in Lemma 7.5, we get $Y(n) \in L_G^{1^*}(\Omega)$. According to Lemma 7.3, we obtain $\lim_{n\to\infty} \hat{\mathbb{E}}_t[Y(n)] = \hat{\mathbb{E}}_t[\lim_{n\to\infty} Y(n)]$. Because of $Y(n) \le X(n)$, we derive $\hat{\mathbb{E}}_t[Y(n)] \le \hat{\mathbb{E}}_t[X(n)]$ and $\lim_{n\to\infty} \hat{\mathbb{E}}_t[Y(n)] \le \underline{\lim}_{n\to\infty} \hat{\mathbb{E}}_t[X(n)]$, which implies $\hat{\mathbb{E}}_t[\underline{\lim}_{n\to\infty} X(n)] \le \underline{\lim}_{n\to\infty} \hat{\mathbb{E}}_t[X(n)]$ we expect.

Now, we are in a position to prove the $G$-martingale convergence theorem step-by-step using the uppercrossing inequality.

Definition 7.6. *A random time $\tau : \Omega \to [0,+\infty)$ is called an $\mathbb{F}$-stopping time, if $\{\tau \le t\} \in \mathscr{F}_t$ for every $t \ge 0$.*

Definition 7.7. *For a finite subset $F \subset [0,+\infty)$, the interval $[\alpha,\beta]$ and the process $M = \{M(t)\}$ with $M(t) \in L_G^1(\Omega)$, we define the a series of $\mathbb{F}$-stopping times recursively by:*

$$\tau_1(\omega) = \min\{t \in F; M(t;\omega) \le \alpha\},\ \sigma_j(\omega) = \min\{t \in F; t \ge \tau_j(\omega),\quad M(t;\omega) \ge \beta\},$$
$$\tau_{j+1}(\omega) = \min\{t \in F; t \ge \sigma_j(\omega),\quad M(t;\omega) \le \alpha\}.$$

*And the minimum of an empty set is defined as $+\infty$. Let $U_F(\alpha,\beta;M(\omega))$ be the largest number $j$ such that $\sigma_j(\omega) < +\infty$. For any general set $I \subset [0,+\infty)$, we define $U_I(\alpha,\beta;M(\omega)) = \sup\{U_F(\alpha,\beta;M(\omega)); F \subseteq I,\ F \text{ is finite}\}$.*

Proposition 7.8 (Upcrossing Inequality, A Discrete Version, [19]). *Assume that $\{-M(n) : n = 1,2,\cdots,N\}$ is a $G$-supermartingale. If $M(n) \in L_G^1(\Omega_n)$, then we have $\hat{\mathbb{E}}[U_{\{1,2,\cdots,N\}}(\alpha,\beta;M(\omega))] \le \frac{\hat{\mathbb{E}}[(M(N)-\alpha)^+]}{\beta-\alpha}$.*

Lemma 7.9 (Uppercrossing Inequality, A Continuous Version). *Assume that $\{M(t) : t \in [0,+\infty)\}$ is a right- or left-continuous function and $\{-M(t) : t \in [0,+\infty)\}$ is a $G$-supermartingale. If $M(t) \in L_G^1(\Omega_t)$, then we have that, for any integer $n > 0$,*

$\hat{\mathbb{E}}[U_{[0,n]}(\alpha, \beta; M(\omega))] \leq \frac{\hat{\mathbb{E}}[(M(n)-\alpha)^+]}{\beta-\alpha}$.

**Proof.** Define $A_j := \cup_{1 \leq k \leq j}\{ni/k : i = 0, 1, \cdots, k\}$. Then, the monotone convergence theorem (Theorem 2.8), together with Definition 7.6 and Proposition 7.8, immediately yields: $\hat{\mathbb{E}}\left[U_{[0,n]\cap\mathbb{Q}}(\alpha, \beta; M(\omega))\right] = \lim_{j\to+\infty} \hat{\mathbb{E}}[U_{A_j}(\alpha, \beta; M(\omega))] \leq \frac{\hat{\mathbb{E}}[(M(n)-\alpha)^+]}{\beta-\alpha}$. Thus, for any sufficiently small $\epsilon > 0$, as $M$ is right- or left-continuous, $\hat{\mathbb{E}}\left[U_{[0,n]}(\alpha, \beta; M(\omega))\right] \leq \hat{\mathbb{E}}\left[U_{[0,n]\cap\mathbb{Q}}(\alpha+\epsilon, \beta-\epsilon; M(\omega))\right] \leq \frac{\hat{\mathbb{E}}[(M(n)-\alpha)^+]}{\beta-\alpha-2\epsilon}$, which validates the conclusion as required due to the arbitrariness of $\epsilon$'s selection.

**Proof of Proposition 3.1.** From Lemma 7.9 and Proposition 2.8, it follows that

$$\hat{\mathbb{E}}[U_{[0,+\infty)}(\alpha, \beta; -M(\omega))] = \lim_{n\to+\infty} \hat{\mathbb{E}}[U_{[0,n]}(\alpha, \beta; -M(\omega))] \leq \sup_{n\in\mathbb{N}} \frac{\hat{\mathbb{E}}[(-M(n)-\alpha)^+]}{\beta-\alpha} \leq$$

$$\frac{\sup_{t\geq 0} \hat{\mathbb{E}}[(-M)^+(t)] + |\alpha|}{\beta-\alpha} = \frac{\sup_{t\geq 0} \hat{\mathbb{E}}[M^-(t)] + |\alpha|}{\beta-\alpha} \leq \frac{\hat{\mathbb{E}}[\sup_{t\geq 0} M^-(t)] + |\alpha|}{\beta-\alpha} < +\infty.$$

So $U_{[0,+\infty)}(\alpha, \beta; -M(\omega)) < +\infty$ quasi-surely. Denote by $A_{\alpha,\beta} := \left\{U_{[0,+\infty)}(\alpha, \beta; -M(\omega)) = +\infty\right\}$. Since $\{\omega : -M(t;\omega) \text{ does not converge}\} \subset \cup_{\alpha,\beta\in\mathbb{Q}} A_{\alpha,\beta}$, $-M(t)$ converges quasi-surely to some $-M(+\infty)$. Here, $M(+\infty)$ can be $+\infty$ or $-\infty$. By the fact that $M(t) \geq \inf_{t\geq 0} -M^-(t) = -\sup_{t\geq 0} M^-(t)$ and Lemma 7.5, we have $M(+\infty) \in \mathcal{L}_G^{1^*_*}(\Omega)$. And by Lemma 7.1, we further have

$$\begin{aligned}
\hat{\mathbb{E}}[|M(+\infty)|] &\leq \varliminf_{n\to\infty} \hat{\mathbb{E}}\left[|M(n)|\right] < 2\hat{\mathbb{E}}\left[\sup_{n\in\mathbb{N}} M^-(n)\right] + \varliminf_{n\to\infty} \hat{\mathbb{E}}[M(n)] \\
&\leq 2\hat{\mathbb{E}}\left[\sup_{t\geq 0} M^-(t)\right] + \hat{\mathbb{E}}[M(1)] < \infty.
\end{aligned}$$

Thus, $M(+\infty)$, finite quasi-surely, belongs to $L_G^{1^*_*}(\Omega)$. Finally, by virtue of Lemma 7.1, we have $\hat{\mathbb{E}}_t[M(+\infty)] \leq \varliminf_{k\to+\infty} \hat{\mathbb{E}}_t[M(t_k)] \leq M(t)$, which completes the proof.

**7.2. Proof of Proposition 4.6.** From Propositions 4.5 and 4.3, it follows that

$$\begin{aligned}
&V(\boldsymbol{x}(t\wedge\tau_N), t\wedge\tau_N) = V(\boldsymbol{x}_0, 0) + \int_0^{t\wedge\tau_N} V_t(\boldsymbol{x}(s), s)\mathrm{d}s \\
&+ \int_0^{t\wedge\tau_N} V_{x_i}(x(s), s) f^i(\boldsymbol{x}(s), s)\mathrm{d}s + \int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s), s) g^{ij}(\boldsymbol{x}(s), s)\mathrm{d}B_j(s) \\
&+ \int_0^{t\wedge\tau_N} V_{x_k}(\boldsymbol{x}(s), s) h^{kij}(\boldsymbol{x}(s), s)\mathrm{d}\langle B_i, B_j\rangle(s) \\
&+ \int_0^{t\wedge\tau_N} \frac{1}{2} V_{x_k x_l}(\boldsymbol{x}(s), s) g^{ki}(\boldsymbol{x}(s), s) g^{lj}(\boldsymbol{x}(s), s)\mathrm{d}\langle B_i, B_j\rangle(s).
\end{aligned}$$

Set $\boldsymbol{\eta} = (\kappa_{ij}) \in M_G^1([0,T];\mathbb{S}^m)$, where $\eta_{ij} = V_{x_k}(h^{kij} + h^{kji}) + V_{x_k x_l} g^{ki} g^{lj}$. Using Proposition 4.7 leads us to the calculations as follows:

$$
\begin{aligned}
&V(\boldsymbol{x}(t\wedge\tau_N), t\wedge\tau_N) = V(\boldsymbol{x}_0, 0) + \int_0^{t\wedge\tau_N} V_t(\boldsymbol{x}(s), s)\mathrm{d}s \\
&\quad + \int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s), s) f^i(\boldsymbol{x}(s), s)\mathrm{d}s + \int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s), s) g^{ij}(\boldsymbol{x}(s), s)\mathrm{d}B_j(s) \\
&\quad + \int_0^{t\wedge\tau_N} \frac{1}{2}\kappa_{ij}(\boldsymbol{x}(s), s)\mathrm{d}\langle B_i, B_j\rangle \\
&\leq V(\boldsymbol{x}_0, 0) + \int_0^{t\wedge\tau_N} V_t(\boldsymbol{x}(s), s)\mathrm{d}s + \int_0^{t\wedge\tau_N} G(\boldsymbol{\eta})\mathrm{d}s \\
&\quad + \int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s), s) f^i(\boldsymbol{x}(s), s)\mathrm{d}s + \int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s), s) g^{ij}(\boldsymbol{x}(s), s)\mathrm{d}B_j(s) \\
&= V(\boldsymbol{x}_0, 0) + \int_0^{t\wedge\tau_N} \mathcal{L}V(\boldsymbol{x}(s), s)\mathrm{d}s + \int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s), s) g^{ij}(\boldsymbol{x}(s), s)\mathrm{d}B_j(s) \\
&\leq V(\boldsymbol{x}_0, 0) + \int_0^{+\infty} \gamma(t)\mathrm{d}t + \int_0^{t\wedge\tau_N} V_{x_i}(\boldsymbol{x}(s), s) g^{ij}(\boldsymbol{x}(s), s)\mathrm{d}B_j(s)
\end{aligned}
$$

Then, $\hat{\mathbb{E}}[|V(\boldsymbol{x}(t\wedge\tau_N), t\wedge\tau_N)|] \leq |V(\boldsymbol{x}_0, 0)| + \int_0^{+\infty}\gamma(t)\mathrm{d}t := K < +\infty$, which implies

$$
\begin{aligned}
&\infty > K \geq \hat{\mathbb{E}}[|V(\boldsymbol{x}(t\wedge\tau_N), t\wedge\tau_N)|] \geq \hat{\mathbb{E}}[\mu(|\boldsymbol{x}(t\wedge\tau_N)|)] \geq \\
&\geq \mu(N)c(\tau_N \leq t) \geq \mu(N)c(\tau_\infty \leq t) \qquad (7.1)
\end{aligned}
$$

where $\mu(r) := \inf_{|\boldsymbol{x}|\geq r, t\geq 0} V(\boldsymbol{x}, t)$ and $\lim_{r\to+\infty}\mu(r) = +\infty$ because of the condition assumed in (4.2). Now, letting $N \to +\infty$ in (7.1) yields $c(\tau_\infty \leq t) = 0$ for any $t$. Finally, further letting $t \to +\infty$ gives $c(\tau_\infty \leq +\infty) = 0$, which completes the proof.

**7.3. Proof of Lemma 4.8.** To prove Lemma 4.8, we first establish the inequality as follows.

LEMMA 7.10. *For $A_{ij}(t) \in M_G^2[0,T]$, denote by $\boldsymbol{A}(t) = (a_{ij}(t))_{d\times m}$. Then, we have* $\hat{\mathbb{E}}\left|\int_0^T \boldsymbol{A}(t)\ \mathrm{d}\boldsymbol{B}(t)\right|^2 \leq d\bar{\gamma}\ \hat{\mathbb{E}}\int_0^T |\boldsymbol{A}(t)|^2\ \mathrm{d}t$.

**Proof.** For simplicity of expression, we apply Einstein's notations [6] in the following arguments and throughout if they are necessary. From Theorem 2.6 and Remark 2.5, it follows that

$$
\begin{aligned}
&\hat{\mathbb{E}}\left|\int_0^T \boldsymbol{A}(t)\ \mathrm{d}\boldsymbol{B}(t)\right|^2 = \hat{\mathbb{E}}\left(\int_0^T a_{ij}(t)\ \mathrm{d}B_j(t)\int_0^T a_{ik}(t)\ \mathrm{d}B_k(t)\right) \\
&= \hat{\mathbb{E}}\int_0^T a_{ij}(t)a_{ik}(t)\ \mathrm{d}\langle B_j, B_k\rangle(t) = \hat{\mathbb{E}}\int_0^T \mathrm{trace}(\boldsymbol{A}(t)\ \mathrm{d}\langle\boldsymbol{B}\rangle(t)\boldsymbol{A}^\top(t)) \\
&\leq d\cdot\hat{\mathbb{E}}\int_0^T \lambda_{\max}(A(t)\ \mathrm{d}\langle B\rangle(t)A^\top(t)) = d\cdot\hat{\mathbb{E}}\int_0^T |\boldsymbol{A}(t)\ \mathrm{d}\langle\boldsymbol{B}\rangle(t)\boldsymbol{A}^\top(t)|_2 \\
&= d\cdot\hat{\mathbb{E}}\int_0^T |\boldsymbol{A}(t)|_2^2\ \mathrm{d}|\langle\boldsymbol{B}\rangle|_2(t) \leq d\bar{\gamma}\cdot\hat{\mathbb{E}}\int_0^T |\boldsymbol{A}(t)|_2^2\ \mathrm{d}t \leq d\bar{\gamma}\cdot\hat{\mathbb{E}}\int_0^T |\boldsymbol{A}(t)|^2\ \mathrm{d}t.
\end{aligned}
$$

The proof is therefore completed.

**Proof of Lemma 4.8.** Now, we need to prove the lemma using contradiction. If this is not true, then there exists $Q \in \mathcal{Q}$ such that $Q\left(\{\omega : \liminf_{t\to+\infty} \eta(\boldsymbol{x}(t;\omega)) > 0\}\right) > 0$. Thus, there exists $\epsilon > 0$ such that $Q(\Omega_1) \geq 2\epsilon$ with $\Omega_1 = \left\{\omega \in \Omega_0 : \liminf_{t\to+\infty} \eta(\boldsymbol{x}(t)) > 2\epsilon\right\}$. Since $\Omega_1 = \cup_{n=1}^{+\infty}\left(\Omega_1 \cap \left\{\omega : \sup_{t\geq 0} |\boldsymbol{x}(t;\omega)| < n\right\}\right)$, there exists a number $N > 0$ such that $Q(\Omega_2) \geq \epsilon$ in which $\Omega_2 = \Omega_1 \cap \left\{\omega : \sup_{t\geq 0} |\boldsymbol{x}(t;\omega)| < N\right\}$.

Now, we define the $\mathbb{F}$-stopping times as

$$\sigma_1(\omega) := \inf\{t : \eta(\boldsymbol{x}(t;\omega)) \geq 2\epsilon\}, \quad \sigma_{2i}(\omega) := \inf\{t : \eta(\boldsymbol{x}(t;\omega)) \leq \epsilon,\ t \geq \sigma_{2i-1}(\omega)\},$$
$$\sigma_{2i+1}(\omega) := \inf\{t : \eta(\boldsymbol{x}(t;\omega)) \geq 2\epsilon,\ t \geq \sigma_{2i}(\omega)\}, \quad \tau_N(\omega) := \inf\{t : |\boldsymbol{x}(t;\omega)| \geq N\}.$$

For all $\omega \in \Omega_2$, $\tau_N(\omega) = +\infty$ and $\sigma_i(\omega) < +\infty$ for all $i > 0$ using the formula (4.3) and the definition of $\Omega_1$. By virtue of Proposition 4.3, $\boldsymbol{M}(t) = \int_0^{t\wedge\tau_N} \boldsymbol{g}(\boldsymbol{x}(s), s)\mathrm{d}\boldsymbol{B}(s)$ is a $Q$-martingale for each $Q \in \mathcal{Q}$. Hence, using Assumption 4.1, Lemma 7.10, Hölder's inequality, and Doob's martingale inequality in traditional stochastic analysis, we obtain that for each $T > 0$,

$$
\begin{aligned}
&E_Q[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}} \sup_{0\leq t\leq T} |\boldsymbol{x}(\tau_N \wedge (\sigma_{2i-1}+t)) - \boldsymbol{x}(\tau_N\wedge\sigma_{2i-1})|^2] \\
&\leq 3E_Q\left[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}} \sup_{0\leq t\leq T}\left|\int_{\tau_N\wedge\sigma_{2i-1}}^{\tau_N\wedge(\sigma_{2i-1}+t)} \boldsymbol{f}(\boldsymbol{x}(s), s)\mathrm{d}s\right|^2\right] \\
&\quad +3E_Q\left[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}} \sup_{0\leq t\leq T}\left|\int_{\tau_N\wedge\sigma_{2i-1}}^{\tau_N\wedge(\sigma_{2i-1}+t)} \boldsymbol{g}(\boldsymbol{x}(s), s)\mathrm{d}\boldsymbol{B}(s)\right|^2\right] \\
&\quad +3E_Q\left[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}} \sup_{0\leq t\leq T}\left|\int_{\tau_N\wedge\sigma_{2i-1}}^{\tau_N\wedge(\sigma_{2i-1}+t)} \boldsymbol{h}(\boldsymbol{x}(s), s)\mathrm{d}\langle\boldsymbol{B}\rangle(s)\right|^2\right] \\
&\leq 3TE_Q\left[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}} \sup_{0\leq t\leq T}\int_{\tau_N\wedge\sigma_{2i-1}}^{\tau_N\wedge(\sigma_{2i-1}+t)} |\boldsymbol{f}(\boldsymbol{x}(s), s)|^2\mathrm{d}s\right] \\
&\quad +12E_Q\left[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}} \left|\int_{\tau_N\wedge\sigma_{2i-1}}^{\tau_N\wedge(\sigma_{2i-1}+T)} \boldsymbol{g}(\boldsymbol{x}(s), s)\mathrm{d}\boldsymbol{B}(s)\right|\right] \\
&\quad +3T\bar{\gamma}^2 m^2 E_Q\left[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}} \sup_{0\leq t\leq T}\int_{\tau_N\wedge\sigma_{2i-1}}^{\tau_N\wedge(\sigma_{2i-1}+t)} |\boldsymbol{h}(\boldsymbol{x}(s), s)|^2\mathrm{d}s\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le 3T\hat{\mathbb{E}}\left[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}}\int_{\tau_N\wedge\sigma_{2i-1}}^{\tau_N\wedge(\sigma_{2i-1}+T)}|\boldsymbol{f}(\boldsymbol{x}(s),s)|^2\mathrm{d}s\right]\\
&\quad+12d\bar{\gamma}\hat{\mathbb{E}}\left[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}}\int_{\tau_N\wedge\sigma_{2i-1}}^{\tau_N\wedge(\sigma_{2i-1}+T)}|\boldsymbol{g}(\boldsymbol{x}(s),s)|^2\mathrm{d}s\right]\\
&\quad+3T\bar{\gamma}^2m^2\hat{\mathbb{E}}\left[1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}}\int_{\tau_N\wedge\sigma_{2i-1}}^{\tau_N\wedge(\sigma_{2i-1}+T)}|\boldsymbol{h}(\boldsymbol{x}(s),s)|^2\mathrm{d}s\right]\\
&\le 3K_N^2T(T+4d\bar{\gamma}+T\bar{\gamma}^2m^2).
\end{aligned}
$$

As $\eta$ is continuous, there exists a number $\delta>0$ such that, for every $\boldsymbol{x},\boldsymbol{y}\in B(N)$ and $|\boldsymbol{x}-\boldsymbol{y}|\le\delta$, $|\eta(\boldsymbol{x})-\eta(\boldsymbol{y})|<\epsilon$. We select sufficiently small $T>0$ such that $3K_N^2T(T+4d\bar{\gamma}+T\bar{\gamma}^2m^2)/\delta^2<\frac{\epsilon}{2}$. Thus, we have $Q\left(1_{\{\tau_N\wedge\sigma_{2i-1}<+\infty\}}\sup_{0\le t\le T}|\boldsymbol{x}(\tau_N\wedge(\sigma_{2i-1}+t))-\boldsymbol{x}(\tau_N\wedge\sigma_{2i-1})|\ge\delta\right)\le 3K_N^2T(T+4d\bar{\gamma}+T\bar{\gamma}^2m^2)/\delta^2<\frac{\epsilon}{2}$. Hence, we have $Q(\{\sigma_{2i-1}<+\infty,\tau_N=+\infty\}\cap\{\sup_{0\le t\le T}|\boldsymbol{x}(\sigma_{2i-1}+t)-\boldsymbol{x}(\sigma_{2i-1})|\ge\delta\})\le\frac{\epsilon}{2}$. By the definition and the property of $\Omega_2$, we conclude that $Q\left(\{\sigma_{2i-1}<+\infty,\tau_N=+\infty\}\cap\left\{\sup_{0\le t\le T}|\boldsymbol{x}(\sigma_{2i-1}+t)-\boldsymbol{x}(\sigma_{2i-1})|<\delta\right\}\right)\ge\epsilon-\frac{\epsilon}{2}=\frac{\epsilon}{2}$, which further implies that

$$
\begin{aligned}
&Q\left(\{\sigma_{2i-1}<+\infty,\tau_N=+\infty\}\cap\left\{\sup_{0\le t\le T}|\eta(\boldsymbol{x}(\sigma_{2i-1}+t))-\eta(\boldsymbol{x}(\sigma_{2i-1}))|<\epsilon\right\}\right)\\
&\ge Q\left(\{\sigma_{2i-1}<+\infty,\tau_N=+\infty\}\cap\left\{\sup_{0\le t\le T}|\boldsymbol{x}(\sigma_{2i-1}+t)-\boldsymbol{x}(\sigma_{2i-1})|<\delta\right\}\right)\ge\frac{\epsilon}{2}.
\end{aligned}
$$

Define $\tilde{\Omega}_i:=\left\{\sup_{0\le t\le T}|\eta(\boldsymbol{x}(\sigma_{2i-1}+t))-\eta(\boldsymbol{x}(\sigma_{2i-1}))|<\epsilon\right\}$. Then, on $\tilde{\Omega}_i\cap\{\sigma_{2i-1}<+\infty\}$, we have $\sigma_{2i}-\sigma_{2i-1}\ge T$. By (4.4), if $\sigma_{2i-1}<+\infty$, then $\sigma_{2i}<+\infty$ quasi-surely. Thus,

$$
\begin{aligned}
+\infty>\hat{\mathbb{E}}\int_0^{+\infty}\eta(\boldsymbol{x}(t))\mathrm{d}t&\ge E_Q\int_0^{+\infty}\eta(\boldsymbol{x}(t))\mathrm{d}t\\
&\ge\sum_{i=1}^{+\infty}E_Q\left[1_{\{\tau_N=+\infty,\sigma_{2i-1}<+\infty,\sigma_{2i}<+\infty\}}\int_{\sigma_{2i-1}}^{\sigma_{2i}}\eta(\boldsymbol{x}(t))\mathrm{d}t\right]\\
&\ge\epsilon\sum_{i=1}^{+\infty}E_Q\left[1_{\{\tau_N=+\infty,\sigma_{2i-1}<+\infty\}}(\sigma_{2i}-\sigma_{2i-1})\right]\\
&\ge\epsilon\sum_{i=1}^{+\infty}E_Q\left[1_{\{\tau_N=+\infty,\sigma_{2i-1}<+\infty\}\cap\tilde{\Omega}_i}(\sigma_{2i}-\sigma_{2i-1})\right]\\
&\ge\epsilon T\sum_{i=1}^{+\infty}Q(\{\tau_N=+\infty,\sigma_{2i-1}<+\infty\}\cap\tilde{\Omega}_i)\ge\epsilon T\sum_{i=1}^{+\infty}\frac{\epsilon}{2}=+\infty,
\end{aligned}
$$

which indicates a contradiction. Consequently, we get $\lim_{t\to+\infty}\eta(\boldsymbol{x}(t))=0$ quasi-surely.

**7.4. Dynamic Stability in Example 5.2.** Here, we validate the quasi-sure stability of the considered equations in Example 5.2. To this end, we set $V(\boldsymbol{x}) := |\boldsymbol{x}|^\alpha$ for some given $0 < \alpha < 1$, which yields $\mathcal{L}V(\boldsymbol{x}) = \alpha|\boldsymbol{x}|^{\alpha-2}\langle \boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})\rangle + G\left(\left(k^2(\alpha-1)\alpha|\boldsymbol{x}|^\alpha\right)_{i,j=1}^m\right)$, where $\left(k^2(\alpha-1)\alpha|\boldsymbol{x}|^\alpha\right)_{i,j=1}^m$ stands for an $m \times m$ matrix such that all elements are $k^2(\alpha-1)\alpha|\boldsymbol{x}|^\alpha$. As $c_{-1} := (-1)_{i,j=1}^m$ is a non-positive symmetric matrix with eigenvalues 0 and $-m$, we have $c_{-1} < 0$. Set $0 < \alpha < 1 + \frac{L}{k^2 c_{-1}} < 1$, we obtain that $\mathcal{L}V(\boldsymbol{x}) = \alpha|\boldsymbol{x}|^{\alpha-2}\langle \boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})\rangle + k^2 c_{-1}(1-\alpha)\alpha|\boldsymbol{x}|^\alpha \le \alpha|\boldsymbol{x}|^\alpha(L + k^2 c_{-1}(1-\alpha))$. Set $\eta(\boldsymbol{x}) := \alpha|\boldsymbol{x}|^\alpha(L + k^2 c_{-1}(1-\alpha)) < 0$. Hence, in light of Proposition 4.6 and Theorem 4.10, if we could confirm a *statement* that the system in Example 5.2 does not reach $\boldsymbol{0}$ before it explodes, $V(\boldsymbol{x})$ with $\alpha < 1$ and along any trajectory apart from $\boldsymbol{0}$ is differentiable to the second order, so that the quasi-sure convergence of $\boldsymbol{x}$ is guaranteed to $\boldsymbol{0}$, the kernel of $\eta$. To make confirm the statement, we first introduce the following result.

PROPOSITION 7.11. *Let* $M(t) = \int_0^t \kappa_{ij}(s)\mathrm{d}\langle B_i, B_j\rangle(s) + \int_0^t 2G(-\boldsymbol{\eta})\mathrm{d}s$, *where* $\boldsymbol{\eta} \in M_G^1([0,T];\mathbb{S}^m)$. *Then, we have* $M(t) \ge 0$ *quasi-surely. Particularly,* $\hat{\mathbb{E}}[M(t)] \ge 0$.

The proof of the above proposition is akin to the proof for Proposition 4.7, which is omitted here.

Now, we make the final confirmation. We set $\tau_N := \inf\{t \ge 0 : |\boldsymbol{x}(t)| \ge N\}$ and $\xi_\epsilon = \inf\{t \ge 0 : |\boldsymbol{x}(t)| \le \epsilon\}$ for $\epsilon, N > 0$, and select $V(\boldsymbol{x}) = \log|\boldsymbol{x}|$. Then, using the formula presented in Theorem 4.5 and Proposition 4.3, we get

$$
\begin{aligned}
\log|\boldsymbol{x}(t\wedge\tau_N\wedge\xi_\epsilon)| &= \log|\boldsymbol{x}_0| + \int_0^{t\wedge\tau_N\wedge\xi_\epsilon} \frac{\langle \boldsymbol{x}(s), \boldsymbol{f}(\boldsymbol{x}(s))\rangle}{|\boldsymbol{x}|^2}\mathrm{d}s \\
&\quad + \sum_{j=1}^n \int_0^{t\wedge\tau_N\wedge\xi_\epsilon} k\mathrm{d}B_j(s) - \sum_{i,j=1}^n \int_0^{t\wedge\tau_N\wedge\xi_\epsilon} \frac{1}{2}k^2\langle B_i, B_j\rangle(s)
\end{aligned}
$$

Noticing the local Lipschitz property of $\boldsymbol{f}$ gives $|\langle \boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})\rangle| \le |\boldsymbol{x}||\boldsymbol{f}(\boldsymbol{x})| \le K_N|\boldsymbol{x}|^2$ on $[0, \tau_N)$. Set $c_1 := G((1)_{i,j=1}^m) > 0$. Then, by Proposition 7.11, we have $\hat{\mathbb{E}}[\log|\boldsymbol{x}(t\wedge\tau_N\wedge\xi_\epsilon)|] \ge \hat{\mathbb{E}}[\log|\boldsymbol{x}_0|] - \int_0^{t\wedge\tau_N\wedge\xi_\epsilon}(K_N + k^2c_1)\mathrm{d}s] \ge \hat{\mathbb{E}}[\log|\boldsymbol{x}_0|] - (K_N + k^2c_1)t$. On the other hand, $\hat{\mathbb{E}}[\log|\boldsymbol{x}(t\wedge\tau_N\wedge\xi_\epsilon)|] \le c(\xi_\epsilon < t\wedge\tau_N)\log\epsilon + c(\xi_\epsilon \ge t\wedge\tau_N)\log N \le c(\xi_\epsilon < t\wedge\tau_N)\log\epsilon + \log N$. Hence, we obtain $\hat{\mathbb{E}}[\log|\boldsymbol{x}_0|] - (K_N + k^2c_1)t \le c(\xi_\epsilon < t\wedge\tau_N)\log\epsilon + \log N$. First, letting $\epsilon \to 0$ results in $c(\xi_0 < t\wedge\tau_N) = 0$. Then, letting both $t$ and $N \to +\infty$ yields $c(\xi_0 < \tau_\infty) = 0$, which confirms the above statement and finally completes the proof.

**7.5. Invariant Set Associated with Autonomous $G$-SDEs.**

THEOREM 7.12. *We consider the following autonomous G-SDEs:*

$$
\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t))\mathrm{d}t + \boldsymbol{g}(\boldsymbol{x}(t))\mathrm{d}\boldsymbol{B}(t) + \boldsymbol{h}(\boldsymbol{x}(t))\mathrm{d}\langle\boldsymbol{B}\rangle(t), \tag{7.2}
$$

*where* $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^d$, $\boldsymbol{g} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$, $\boldsymbol{h} : \mathbb{R}^d \to \mathbb{R}^{d\times m^2}$, *and* $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{g}(\boldsymbol{a}) = \boldsymbol{h}(\boldsymbol{a}) = \boldsymbol{0}$.

*Clearly, $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$ are all globally Lipschitzian. Then, we have that, for all $\boldsymbol{x}_0 \neq a$, $c\big(\{\omega : \exists\ t > 0, \quad \boldsymbol{x}(t,\omega;\boldsymbol{x}_0) = \boldsymbol{a}\}\big) = 0$, which indicates that the trajectory does not approach $\boldsymbol{a}$ quasi-surely in a finite time.*

**Proof.** We know that the $G$-SDEs (7.2) have a unique solution on $M_G[0,T]$ for every $T > 0$ according to [36]. First, we need to perform the proof for the situation of $\boldsymbol{a} = \boldsymbol{0}$. Now set $\mathcal{A} := \{\omega : \boldsymbol{x}(t,\omega) = \boldsymbol{0}$ for some $t \in [0,+\infty)\}$. If $c(\mathcal{A}) > 0$, then there exists a number $T > 0$ such that $c(\mathcal{A}_T) > 0$ where $\mathcal{A}_T := \{\omega : \boldsymbol{x}(t,\omega) = \boldsymbol{0}$ for some $t \in [0,T]\}$, which is due to the fact that $\mathcal{A} = \cup_{T=1}^{+\infty}\mathcal{A}_T$. Next, introduce the stopping time $\tau_\epsilon := \inf\{t \in [0,+\infty) : |\boldsymbol{x}(t,\omega)| \leq \epsilon\}$. Set $V(\boldsymbol{x}) := 1/|\boldsymbol{x}| = (|\boldsymbol{x}|^2)^{-\frac{1}{2}}$. Then, we perform the calculations using $G$-Itô's formula, obtaining that

$$
\begin{aligned}
&V(\boldsymbol{x}(T\wedge\tau_\epsilon)) = V(\boldsymbol{x}_0) + \int_0^{T\wedge\tau_\epsilon} V_{x_i}(\boldsymbol{x}(s))f^i(\boldsymbol{x}(s))\mathrm{d}s \\
&+ \int_0^{T\wedge\tau_\epsilon} V_{x_i}(\boldsymbol{x}(s))g^{ij}(\boldsymbol{x}(s))\mathrm{d}B_j(s) + \int_0^{T\wedge\tau_\epsilon} \frac{1}{2}\kappa_{ij}(\boldsymbol{x}(s))\mathrm{d}\langle B_i,B_j\rangle(s) \\
&= V(\boldsymbol{x}_0) - \int_0^{T\wedge\tau_\epsilon} \frac{\langle \boldsymbol{x}(s), f(\boldsymbol{x}(s))\rangle}{|\boldsymbol{x}|^3}\mathrm{d}s - \int_0^{T\wedge\tau_\epsilon} \frac{x_i(s)g^{ij}(\boldsymbol{x}(s))}{|\boldsymbol{x}|^3}\mathrm{d}B_j(s) \\
&+ \int_0^{T\wedge\tau_\epsilon} \Bigg[ -\frac{g^{\mu i}(\boldsymbol{x}(s))g^{\mu j}(\boldsymbol{x}(s))}{2|\boldsymbol{x}|^3} + \frac{3}{2|\boldsymbol{x}|^5}x_\mu x_v g^{\mu i}(\boldsymbol{x}(s))g^{\nu j}(\boldsymbol{x}(s)) \\
&-\frac{x_v h^{vij}(\boldsymbol{x}(s))}{|\boldsymbol{x}|^3}\Bigg]\mathrm{d}\langle B_i,B_j\rangle(s) \leq V(\boldsymbol{x}_0) + \int_0^{T\wedge\tau_\epsilon}\Bigg[\frac{|\boldsymbol{f}(\boldsymbol{x})|}{|\boldsymbol{x}|^2} + \frac{d\bar{\gamma}|\boldsymbol{g}(\boldsymbol{x})|^2}{2|\boldsymbol{x}|^3} + \frac{3\bar{\gamma}|\boldsymbol{g}(\boldsymbol{x})|^2}{2|\boldsymbol{x}|^3} \\
&+\frac{|\boldsymbol{h}(\boldsymbol{x})|\bar{\gamma}}{|\boldsymbol{x}|^2}\Bigg]\mathrm{d}s + \int_0^{T\wedge\tau_\epsilon} V_{x_i}(\boldsymbol{x}(s))g^{ij}(\boldsymbol{x}(s))\mathrm{d}B_j(s),
\end{aligned}
$$

where $\kappa_{ij} = V_{x_k}(h^{kij} + h^{kji}) + V_{x_k x_l}g^{ki}g^{lj}$ and Einstein's notations are applied here. Let $\rho(\boldsymbol{x}) := \frac{|\boldsymbol{f}(\boldsymbol{x})|}{|\boldsymbol{x}|} + \frac{(d+3)\bar{\gamma}|\boldsymbol{g}(\boldsymbol{x})|^2}{2|\boldsymbol{x}|^2} + \frac{|\boldsymbol{h}(\boldsymbol{x})|\bar{\gamma}}{|\boldsymbol{x}|}$. Then, there exists a number $K > 0$ such that $\rho(\boldsymbol{x}) \leq K < +\infty$ because $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ are globally Lipschitzian as mentioned above. Hence, it follows that

$$
\begin{aligned}
&V(\boldsymbol{x}(T\wedge\tau_\epsilon)) \leq V(\boldsymbol{x}_0) + \int_0^{T\wedge\tau_\epsilon} V(\boldsymbol{x}(s))\rho(\boldsymbol{x}(s))\mathrm{d}s + \int_0^{T\wedge\tau_\epsilon} V_{x_i}(\boldsymbol{x}(s))g^{ij}(\boldsymbol{x}(s))\mathrm{d}B_j(s) \\
&= V(\boldsymbol{x}_0) + \int_0^{T} V(\boldsymbol{x}(s))\rho(\boldsymbol{x}(s))1_{[0,\tau_\epsilon]}\mathrm{d}s + \int_0^{T} V_{x_i}(\boldsymbol{x}(s))g^{ij}(\boldsymbol{x}(s))1_{[0,\tau_\epsilon]}\mathrm{d}B_j(s) \\
&\leq V(\boldsymbol{x}_0) + K\int_0^{T} V(\boldsymbol{x}(s))1_{[0,\tau_\epsilon]}\mathrm{d}s + \int_0^{T} V_{x_i}(\boldsymbol{x}(s))g^{ij}(\boldsymbol{x}(s))1_{[0,\tau_\epsilon]}\mathrm{d}B_j(s),
\end{aligned}
$$

which implies that $\hat{\mathbb{E}}[V(\boldsymbol{x}(T\wedge\tau_\epsilon))] \leq \hat{\mathbb{E}}[V(\boldsymbol{x}_0)] + K\hat{\mathbb{E}}\int_0^T V(\boldsymbol{x}(s))1_{[0,\tau_\epsilon]}\mathrm{d}s$ $\leq \hat{\mathbb{E}}[V(\boldsymbol{x}_0)] + K\int_0^T \hat{\mathbb{E}}[V(\boldsymbol{x}(s\wedge\tau_\epsilon))]\mathrm{d}s$. Now, using Gronwall's inequality, we have $\hat{\mathbb{E}}\left[\frac{1}{|\boldsymbol{x}(T\wedge\tau_\epsilon)|}\right] \leq \hat{\mathbb{E}}[V(\boldsymbol{x}_0)]\mathrm{e}^{KT}$. From the definition of $\tau_\epsilon$ and also from the continuity of $\boldsymbol{x}(t)$, it follows that $|\boldsymbol{x}(T\wedge\tau_\epsilon)| = \epsilon$ on $\mathcal{A}_T$. Thus, $c(\mathcal{A}_T) = \epsilon\hat{\mathbb{E}}\left[\frac{1}{|\boldsymbol{x}(T\wedge\tau_\epsilon)|}1_{\mathcal{A}_T}\right] \leq$

$\epsilon\hat{\mathbb{E}}[V(\boldsymbol{x}_0)]\mathrm{e}^{KT}$, which is valid for every $\epsilon > 0$. Therefore, we immediately obtain $c(\mathcal{A}_T) = 0$, which is a contradiction.

For the general situation of $\boldsymbol{a}$, we set $\boldsymbol{y}(t) := \boldsymbol{x}(t) - \boldsymbol{a}$. Then, $\boldsymbol{y}(t)$ satisfies the $G$-SDEs: $\mathrm{d}\boldsymbol{y}(t) = \boldsymbol{f}(\boldsymbol{y}(t)+\boldsymbol{a})\mathrm{d}t + \boldsymbol{g}(\boldsymbol{y}(t)+\boldsymbol{a})\mathrm{d}\boldsymbol{B}(t) + \boldsymbol{h}(\boldsymbol{y}(t)+\boldsymbol{a})\mathrm{d}\langle\boldsymbol{B}\rangle(t)$. Consequently, we know that $\boldsymbol{y}(t)$ never approaches $\boldsymbol{0}$ quasi-surely, i.e., $\boldsymbol{x}(t)$ never approaches $\boldsymbol{a}$ quasi-surely. Therefore, the proof is complete.

**7.6. Numerical evidences.** Here, we describe the numerical scheme that we use for partially illustrating the analytical results obtained in the main text. Actually, we do not provide a complete simulation for the solutions of $G$-SDEs but only simulate the corresponding SDEs under a group of probability measures. A rigorous and complete scheme for simulating the solution of $G$-SDEs still awaits further investigations.

To this end, we first suppose $\boldsymbol{W}(t)$ to be a standard $m$-dimensional Brownian motion on the probability space $(\Omega, \mathcal{B}(\Omega), P)$. Also suppose that $\Theta$ is a bounded, closed and convex subset of $\mathbb{R}^{m\times m}$, where $\Theta = [\underline{\sigma}, \overline{\sigma}]$ for $m = 1$. In addition, $\tilde{\mathcal{Q}} := \left\{P_{\boldsymbol{\theta}} \in \mathcal{M} : P_{\boldsymbol{\theta}} \text{ is the law of process } \int_0^t \boldsymbol{\theta}(s)\mathrm{d}\boldsymbol{W}(s) \text{ for } \forall\ t \geq 0, \boldsymbol{\theta} \in \mathscr{A}_{0,\infty}^{\Theta}\right\} \subset \mathcal{Q}$, where $\mathscr{A}_{0,\infty}^{\Theta}$ denotes the collection of all $\Theta$-valued $\mathscr{F}$ adapted function in $[0, +\infty)$. According to Remark 15 in Ref. [15], the capacity satisfies $c(\mathcal{A}) = \sup_{Q\in\tilde{\mathcal{Q}}} P[\mathcal{A}]$ for any $\mathcal{A} \in \mathscr{B}(\Omega)$, so we can check whether an event is correct quasi-surely on the probability measures space $\tilde{\mathcal{Q}}$. Thus, we make our numerical simulations on a finite subset of $\tilde{\mathcal{Q}}$ repeatedly as follows and use the case where $\langle B_i, B_j\rangle = 0$ for each $i \neq j$ and all $B_i$ are identically distributed.

For the time interval $[0, T]$, we introduce a uniform time partition $0 = t_0 < t_1 < \cdots < t_N = T$ with $\Delta t := t_{n+1} - t_n = T/N$. We use the following Euler-Maruyama scheme, as proposed in [33], to investigate the solution of the SDEs correspondingly from the $G$-SDEs in (4.1):

(7.3)
$$\boldsymbol{X}(n+1) = \boldsymbol{X}(n) + \boldsymbol{f}(\boldsymbol{X}(n), t_n)\Delta t + \boldsymbol{g}(\boldsymbol{X}(n), t_n)\Delta\boldsymbol{B}(t_n) + \boldsymbol{h}(\boldsymbol{X}(n), t_n)\Delta\langle\boldsymbol{B}\rangle(t_n)$$

with $\boldsymbol{X}(0) = \boldsymbol{x}_0$ and $n = 0, 1, \cdots, N-1$. Here, $\Delta B_i(t_n) \sim \mathcal{N}(0, \sigma_{i,n}^2\Delta t)$ and $\Delta\langle B_i\rangle(t_n) = \sigma_{i,n}^2\Delta t$ with $\sigma_{i,n} \in [\underline{\sigma}, \overline{\sigma}]$ and $i = 0, 1, \cdots, m$.

In order to investigate the dynamics of the corresponding SDEs on the probability measures space $\tilde{\mathcal{Q}}$, the covariance $\{\sigma_{i,n}\}_{1\leq i\leq m, 1\leq n\leq N}$ should be taken from all the element of the set $[\underline{\sigma}, \overline{\sigma}]^{m\times N}$. To do this numerically, we introduce a uniform interval partition $\underline{\sigma} = \sigma_0 < \sigma_1 < \cdots < \sigma_k = \overline{\sigma}$ with $\Delta\sigma = \sigma_{i+1} - \sigma_i = (\overline{\sigma} - \underline{\sigma})/k$. Denote by $\Sigma_{jl} := \{\sigma_i | j \leq i \leq l\}$, where $1 \leq j \leq l \leq k$. For any given tuple $(j, l)$, we choose an element $(\mu_{in})_{1\leq i\leq m, 1\leq n\leq N} \in \Sigma_{jl}^{m\times N}$, set $\sigma_{i,n} = \mu_{in}$ for all $1 \leq i \leq m, 1 \leq n \leq N$, and then approximate the dynamics of the SDEs correspondingly from (4.1) using the scheme specified in (7.3), which enables us to numerically produce a large number of simulating trials.

In Figure 1, we show the numerical results, respectively, for Examples 5.1-5.3.

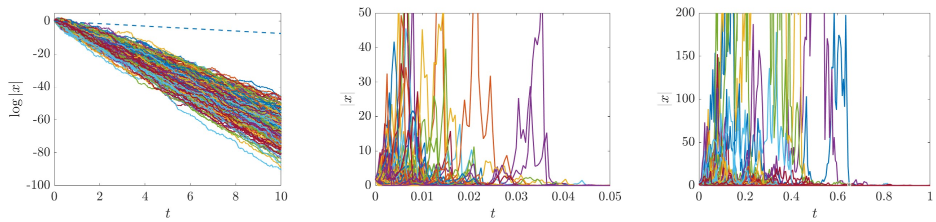


Fig. 1. *(a)The dynamics of* $\log|\boldsymbol{x}|$ *change with* $t$ *for a group of SDEs correspondingly from the G-SDEs in Example 5.1. Here, simulated are the 400 trials using the settings,* $\underline{\sigma}^2 = 3.5$, $\overline{\sigma}^2 = 4$. *(b)The dynamics of* $|\boldsymbol{x}|$ *change with* $t$ *for a group of SDEs correspondingly from the G-SDEs in Example 5.2. Here, simulated are the 400 trials using the settings:* $\underline{\sigma}^2 = 40$, $\overline{\sigma}^2 = 50$, $\sigma = 10$, $\rho = 10$, $\beta = 8/3$, *and* $k = 5$. *(c)The dynamics of* $|\boldsymbol{x}|$ *change with* $t$ *for a group of SDEs correspondingly from the G-SDEs in Example 5.3. Here, simulated are the 400 trials using the settings:* $\underline{\sigma}^2 = 40$ *and* $\overline{\sigma}^2 = 50$.

## REFERENCES


[1] G. Belitskii, *Matrix norms and their applications*, vol. 36, Birkhäuser, Basel, 2013.

[2] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, *A theory of stochastic resonance in climatic change*, SIAM Journal on applied mathematics, 43 (1983), pp. 565–578.

[3] A. Brockwell, K. Borovkov, and R. Evans, *Stability of an adaptive regulator for partially known nonlinear stochastic systems*, SIAM Journal on Control and Optimization, 37 (1999), pp. 1553–1567.

[4] N. Dabrock, M. Hofmanová, and M. Röger, *Existence of martingale solutions and large-time behavior for a stochastic mean curvature flow of graphs*, Probability Theory and Related Fields, 179 (2021), pp. 407–449.

[5] L. Denis, M. Hu, and S. Peng, *Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths*, Potential Analysis, 34 (2011), pp. 139–161.

[6] A. Einstein, *The general theory of relativity*, Springer, Berlin, 1922.

[7] F. Faizullah, *Existence of solutions for G-SFDEs with Cauchy-Maruyama approximation scheme*, Abstract and Applied Analysis,2014,(2014-9-11), 2014 (2014), pp. 1–8.

[8] P. Florchinger, *Lyapunov-like techniques for stochastic stability*, SIAM Journal on Control and Optimization, 33 (1995), pp. 1151–1169.

[9] F. Gao, *Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion*, Stochastic Processes & Their Applications, 119 (2009), pp. 3356–3382.

[10] M. Gevers, G. Goodwin, and V. Wertz, *Continuous-time stochastic adaptive control*, SIAM Journal on Control and Optimization, 29 (1991), pp. 264–282.

[11] Y. Guo, W. Lin, and G. Chen, *Stability of switched systems on randomly switching durations with random interaction matrices*, IEEE Transactions on Automatic Control, 63 (2017), pp. 21–36.

[12] Y. Guo, W. Lin, Y. Chen, and J. Wu, *Instability in time-delayed switched systems induced by fast and random switching*, Journal of Differential Equations, 263 (2017), pp. 880–909.

[13] Y. Guo, W. Lin, and D. W. Ho, *Discrete-time systems with random switches: From systems stability to networks synchronization*, Chaos: An Interdisciplinary Journal of Nonlinear Science, 26 (2016), p. 033113.

[14] Y. Guo, W. Lin, and M. A. Sanjuan, *The efficiency of a random and fast switch in complex dynamical systems*, New Journal of Physics, 14 (2012), p. 083022.

[15] M. Hu and S. Peng, *Extended conditional g-expectations and related stopping times*, Probability, Uncertainty and Quantitative Risk, 6 (2021), pp. 369–390.

[16] F. H. Knight, *Risk, Uncertainty and Profit*, Houghton Mifflin, Boston, 1921.

[17] J. P. LaSalle, *Stability theory for ordinary differential equations*, Journal of Differential Equations, 4 (1968), pp. 57–65.

[18] J. P. LaSalle, *Stability theory and invariance principles*, Dynamical Systems, 1 (1976), pp. 211–222.

[19] H. Li, *Martingale inequalities under G-expectation and their applications*, Acta Mathematica Scientia, 41 (2021), pp. 349–360.

[20] W. Li, X.-J. Xie, and S. Zhang, *Output-feedback stabilization of stochastic high-order nonlinear systems under weaker conditions*, SIAM Journal on Control and Optimization, 49 (2011), pp. 1262–1282.

[21] X. Li, X. Lin, and Y. Lin, *Lyapunov-type conditions and stochastic differential equations driven by G-Brownian motion*, Journal of Mathematical Analysis and Applications, 439 (2016), pp. 235–255.

[22] X. Li and S. Peng, *Stopping times and related Itô's calculus with G-Brownian motion*, Stochastic Processes & Their Applications, 121 (2009), pp. 1492–1508.

[23] Z. Li and J. Chen, *Robust consensus for multi-agent systems communicating over stochastic uncertain networks*, SIAM Journal on Control and Optimization, 57 (2019), pp. 3553–3570.

[24] R. Liptser and A. N. Shiryayev, *Theory of martingales*, vol. 49, Springer Science & Business Media, Berlin, 2012.

[25] X. Mao, *Exponential stability of stochastic differential equations*, Marcel Dekker, New York, 1994.

[26] X. Mao, *Stochastic stabilization and destabilization*, Systems & control letters, 23 (1994), pp. 279–290.

[27] X. Mao, *LaSalle-type theorems for stochastic differential delay equations*, Journal of Mathematical Analysis & Applications, 236 (1999), pp. 350–369.

[28] X. Mao, *Stochastic versions of the LaSalle theorem*, Journal of Differential Equations, 153 (1999), pp. 175–195.

[29] X. Mao, *The LaSalle-type theorems for stochastic functional differential equations*, Nonlinear Studies, 7 (2000), pp. 307–328.

[30] X. Mao, *A note on the LaSalle-type theorems for stochastic differential delay equations*, Journal of Mathematical Analysis and Applications, 268 (2002), pp. 125–142.

[31] X. Mao, *Stochastic differential equations and applications*, Elsevier, Amsterdam, 2007.

[32] X. Mao, G. G. Yin, and C. Yuan, *Stabilization and destabilization of hybrid systems of stochastic differential equations*, Automatica, 43 (2007), pp. 264–273.

[33] G. Maruyama, *On the transition probability functions of the markov process*, Natural Science Reports of the Ochanomizu University, 5 (1954), pp. 10–20.

[34] S. Mendelson and G. Paouris, *On the singular values of random matrices*, Journal of the European Mathematical Society, 16 (2014), pp. 823–834.

[35] S. Peng, *G-expectation, G-Brownian motion and related stochastic calculus of Ito's type*, in Stochastic Analysis and Applications, Berlin, Heidelberg, 2007, Springer Berlin Heidelberg, pp. 541–567.

[36] S. Peng, *Nonlinear Expectations and Stochastic Calculus under Uncertainty*, vol. 95, Springer, Berlin, 2019.

[37] Y. Ren, Q. Bi, and S. Rathinasamy, *Stochastic functional differential equations with infinite delay driven by G-Brownian motion*, Mathematical Methods in the Applied Sciences, 36 (2013), pp. 1746–1759.

[38] Y. Ren, W. Yin, and R. Sakthivel, *Stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation*, Automatica, 95 (2018), pp. 146–151.

[39] S. Yi, L. Qi, and X. Mao, *The improved LaSalle-type theorems for stochastic functional differential equations*, Journal of Mathematical Analysis and Applications, 318 (2006), pp. 134–154.

[40] S. Zhou, W. Lin, X. Mao, and J. Wu, *Generalized invariance principles for stochastic dynamical systems and their applications*, IEEE Transactions on Automatic Control, (2023).

[41] S. Zhou, W. Lin, and J. Wu, *Generalized invariance principles for discrete-time stochastic dynamical systems*, Automatica, 143 (2022), p. 110436.