

Generating multiparticle entangled states by self-organization of driven ultracold atoms

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(Dated: December 15, 2022)

We study a methodology for guiding the dynamical evolution of ultracold atomic motional degrees of freedom towards multiparticle entangled Dicke-like states, via nonlinear self-organization under external driving. Two examples of nonlinear many-body models are investigated. In the first model the external drive is a temporally oscillating magnetic field, leading to self-organization by interatomic scattering. In the second model the drive is a pump laser, leading to self-organization by photon-atom scattering in a ring cavity. We demonstrate highly efficient generation of multiparticle entangled states of atomic motion and discuss prospective experimental realizations of the models. Our results highlight the potential for using self-organization of atomic motion in quantum technological applications.

The study of self-organization of ultracold atoms is a well-established research direction, with many notable experimental and theoretical results [1, 2]. Following the pioneering works on self-organization of cold [3, 4] and ultracold [5, 6] atoms coupled to a single longitudinal mode of a Fabry-Perot cavity, the multimode aspects of optomechanical self-organization in cold and ultracold atoms have recently started to generate significant interest [7–26]. In parallel to the work on optomechanical self-organization, there has been great progress in studying the atomic self-organization arising due to atom-atom interactions being modulated by an external B-field [27–31].

Although the majority of these works have studied the nonequilibrium phase diagrams in the mean field limit, where quantum correlations can be neglected, a number of works have shown that the quantum nature of light and matter can play an important role for self-organization [6, 32–41].

Quantum correlated squeezed and entangled states can be used for quantum enhanced measurements, which go beyond classical metrology [42, 43]. In this context, squeezing and entanglement of the internal atomic degrees of freedom [44–53], but recently also the external ones [54–58], have been recognized as attractive tools for metrological applications.

Here, we focus on external degrees of freedom and demonstrate numerically the spontaneous generation of multiparticle entangled (also called Dicke squeezed) states in the atomic motion [52, 59–64], by transverse self-organization of an externally driven Bose-Einstein condensate (BEC). The mechanism of this phenomenon is four-wave mixing [65–71] of zero-order modes with the spontaneously generated transverse sidebands, and we demonstrate the effect numerically for two different models. In the first model, the self-organization arises due to an external periodic modulation in interatomic scattering, driven by a temporally oscillating B-field. In the second model, it arises from laser driving in a ring cavity. A strong confinement along the y and z axes allows us to restrict the analysis to 1D structures in an elongated cloud.

Driving by B-field modulation. In the first model, shown in Fig. 1(a), a spatially homogeneous and temporally oscillating B-field in the z -direction is modulated near a Feshbach resonance of the atoms. The B-field sinusoidally modulates the atomic

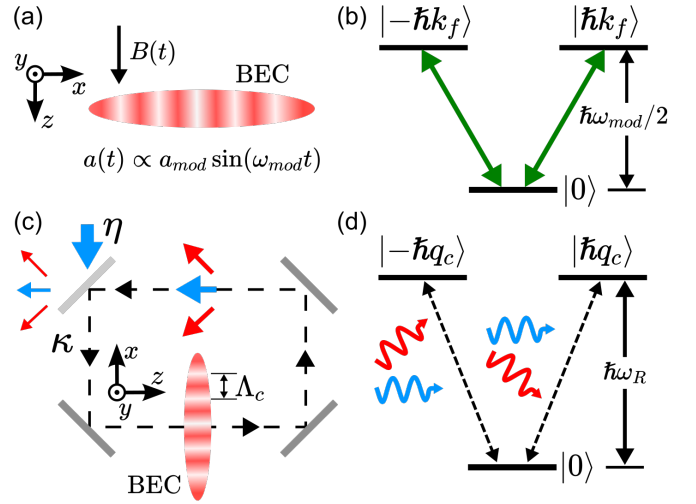


FIG. 1. Nonlinear self-organization via four-wave mixing in a driven elongated BEC. (a) The self-organization along the x axis can be driven by applying a B-field $B(t)$ oscillating near a Feshbach resonance, leading to an oscillating scattering length $a(t)$ [31]. (b) Absorption of a quantum of energy $\hbar\omega_{mod}$ leads here to a momentum-conserving scattering of two atoms with zero transverse momentum into modes with transverse momenta $\pm\hbar k_f$, where $k_f = \sqrt{m\omega_{mod}/\hbar}$. (c) Transverse self-organization with spatial period Λ_c for laser pumped atoms in a ring cavity (η - pump rate) with photon leakage rate κ . (d) In this system, an atomic momentum sideband state with transverse momentum $\pm\hbar q_c$ ($q_c = 2\pi/\Lambda_c$) is excited by scattering of an on-axis photon (blue) into a mode with transverse wavenumber $\mp q_c$ (red) and absorbing a quantum of energy $\hbar\omega_R$. The momentum is here conserved for the combined photon-atom processes.

s -wave scattering length with frequency ω_{mod} and amplitude a_{mod} , thus driving the pattern formation [31]. This four-wave mixing process is described by the Hamiltonian:

$$H_B = i\hbar g_{mod} b_+^\dagger b_-^\dagger b_0 b_0 + \text{H.c.}, \quad (1)$$

where $g_{mod} = 2\pi\hbar a_{mod}/mV$, m is the atom mass, V is the volume of the condensate, and b_0 , b_\pm are the bosonic momentum annihilation operators for the transverse modes of momenta 0 , $\pm\hbar k_f$, where $k_f = \sqrt{m\omega_{mod}/\hbar}$. The mechanism of pattern

formation in this system is illustrated in Fig. 1b). When a pair of atoms with zero transverse momenta absorbs a quantum of energy $\hbar\omega_{mod}$, they scatter into modes with opposite momenta and kinetic energy $\hbar\omega_{mod}/2$. This momentum conserving process leads in the semiclassical picture to a formation of stripe patterns in the atomic density.

The nonlinear Hamiltonian H_B is a simple model for describing pattern formation in the quantum regime. Indeed, it can readily be shown that H_B describes the somewhat idealized situation for which the excitation of higher order transverse momentum modes is precluded (see below). A similar model was initially studied in laser driven materials with cavity feedback, where the medium's degrees of freedom were eliminated and only the photonic modes were relevant [32–34].

Laser driving with ring cavity feedback. The second model we study is based on laser driving of a BEC in a ring cavity, as illustrated in Fig. 1c). This novel setup is inspired by earlier theoretical [32–34, 72–74] and experimental [75, 76] work in nonlinear optics. Here, in addition to the three atomic motional modes with annihilation operators b_0, b_{\pm} for transverse momenta $0, \pm\hbar q_c$ (with $q_c = 2\pi/\Lambda_c$ for the pattern length-scale Λ_c), we have also the corresponding intracavity photonic modes with annihilation operators a_0, a_{\pm} . In this system, a continuous-wave (cw) laser of frequency ω drives the zero-order cavity mode of frequency ω_0 with pump rate η . Above some critical pump level η_c , this leads to spontaneous generation of sideband modes with frequency ω'_0 , concurrently with atomic momentum sideband modes. The effective Hamiltonian of the problem is given by $H_{cav} = H_0 + H_{FWM}^{(1)} + H_{FWM}^{(2)}$, with:

$$H_0 = -\hbar\bar{\Delta}_c n_0 - \hbar\bar{\Delta}'_c (n_+ + n_-) + \hbar\omega_R (N_+ + N_-) + i\hbar\eta (a_0^\dagger - a_0), \quad (2)$$

and the four-wave mixing terms:

$$H_{FWM}^{(1)} = \hbar U_0 [(a_+^\dagger b_-^\dagger + a_-^\dagger b_+^\dagger) a_0 b_0 + \text{H.c.}] + \hbar U_0 [a_0^\dagger (b_+^\dagger a_+ + b_-^\dagger a_-) b_0 + \text{H.c.}], \quad (3)$$

$$H_{FWM}^{(2)} = \hbar U_0 (a_+^\dagger a_- b_-^\dagger b_+ + \text{H.c.}), \quad (4)$$

where $\bar{\Delta}_c = \omega - \omega_0 - NU_0$, $\bar{\Delta}'_c = \omega - \omega'_0 - NU_0$ are the effective pump detunings from the on-axis and sideband cavity modes, respectively, $n_0 = a_0^\dagger a_0$, $n_{\pm} = a_{\pm}^\dagger a_{\pm}$, $N_0 = b_0^\dagger b_0$, $N_{\pm} = b_{\pm}^\dagger b_{\pm}$, $U_0 = g_0^2/\Delta_a$ is the single atom light shift, $\Delta_a = \omega - \omega_a$ is the laser-atom detuning, g_0 is the atom-cavity coupling strength, and $\hbar\omega_R = (\hbar q_c)^2/2m$ is the transverse recoil energy. Following [5, 6], we here concentrate on the system dynamics for $\bar{\Delta}'_c < 0$, where ω'_0 (i.e. Λ_c) is tunable via Fourier filtering of the intracavity light [77].

Generation of transverse sidebands via the four-wave mixing terms of H_{cav} can be explained by the momentum conserving processes illustrated in Fig. 1d). Two opposite atomic momentum sidebands are created by the term $a_+^\dagger a_0 b_-^\dagger b_0$ in $H_{FWM}^{(1)}$ by scattering an on-axis photon into the $+q_c$ mode, which excites an atomic state with transverse momentum $-\hbar q_c$, and the term $a_-^\dagger a_0 b_+^\dagger b_0$, which scatters an on-axis photon into the $-q_c$ mode and excites an atomic state with transverse momentum $\hbar q_c$.

The spontaneously generated transverse lattice potential attracts atoms towards the maxima of the light intensity for $U_0 < 0$, or their minima for $U_0 > 0$. The ordering of atoms in turn increases the light diffraction and thus also the depth of the lattice, which turns into a runaway process when the pump rate η is sufficiently strong to compensate for the kinetic energy cost and dissipation.

Dicke squeezed states in transverse atomic momentum. Our aim is to study the phenomenon of Dicke squeezing and the associated multiparticle (many-atom) entanglement for the Hamiltonians H_B and H_{cav} . To this end we define the sideband operators: $\delta n = n_+ - n_-$, $\delta N = N_+ - N_-$, along with: $J_x = (b_+^\dagger b_- + b_-^\dagger b_+)/2$, $J_y = (b_+^\dagger b_- - b_-^\dagger b_+)/2i$, $J_z = \delta N/2$, which are analogous to Schwinger's angular momentum operators [78], with $J_{eff}^2 = J_x^2 + J_y^2$.

Dicke squeezed states for the transverse momentum sidebands, depicted on the Bloch sphere in Fig. 2b), are characterized by a low $\langle(\Delta J_z)^2\rangle$, large $\langle J_{eff}^2\rangle$ and $\langle J_x\rangle = \langle J_y\rangle = \langle J_z\rangle = 0$ [52, 60, 61, 63, 64]. For the two transverse modes, the criterion for Dicke state multiparticle entanglement of identical atoms is given by [64, 79]:

$$\xi_{gen}^2 = (N-1) \frac{\langle(\Delta J_z)^2\rangle}{\langle J_{eff}^2\rangle - N/2} < 1, \quad (5)$$

where $N = N_0 + N_+ + N_-$ is the total number of atoms, taken as constant in our simulations.

One can easily show that $J_{eff}^2 = N_+ N_- + (N_+ + N_-)/2$, such that $\langle J_{eff}^2\rangle = \langle N_+ N_- \rangle + \langle N_+ + N_- \rangle/2$. For the maximally entangled state, we have $\langle(\Delta J_z)^2\rangle = 0$ and $\langle J_{eff}^2\rangle = N(N+2)/4$. In the atomic many-body basis, this maximally multiparticle entangled state is the ideal Dicke state, for even N given by $|0\rangle_0 |N/2\rangle_+ |N/2\rangle_-$. An alternative measure of Dicke squeezing at large $\langle J_{eff}^2\rangle$, practical in the large N limit since it scales in this ideal case as $1/(N+2)$, is the quantity $\xi_D = N[\langle(\Delta J_z)^2\rangle + 0.25]/\langle J_{eff}^2\rangle$ [63].

Continuous translational symmetry of H_B and H_{cav} . Both H_B and H_{cav} are symmetric to continuous translations along the x axis, which leads to $[\delta N, H_B] = 0$ and $[\delta n + \delta N, H_{cav}] = 0$, where $\langle J_x\rangle = \langle J_y\rangle = \langle J_z\rangle = 0$ for both unitary evolution with H_B and dissipative evolution with H_{cav} [80]. The continuous translational symmetry of H_B and H_{cav} is preserved by the density matrix during temporal evolution. These translationally symmetric self-organized states are analogous to maximally amplitude squeezed photonic states, for which the phase of the electric field is undetermined [81, 82].

Simulations of unitary evolution under H_B . We now discuss the solutions of the Schrödinger equation for H_B . Due to $[J_z, H_B] = 0$, for a system starting in the state $|N\rangle_0 |0\rangle_+ |0\rangle_-$, all moments of J_z will be equal to zero at all t , leading to $\xi_{gen}^2 = 0$. This vanishing of the J_z moments arises from the fact that the transverse sideband states are populated by the atom momentum-conserving process, creating pairs of sideband excitations.

The temporal evolution of $\langle J_{eff}^2\rangle$ is plotted in Fig. 2c). The $\langle J_{eff}^2\rangle$, which is a measure of the atomic self-organization,

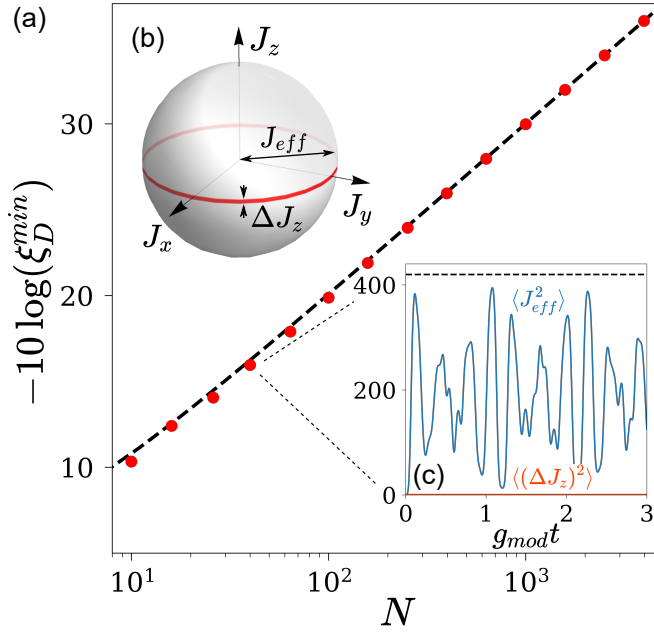


FIG. 2. Populating the sideband Dicke-like states by self-organization via the Hamiltonian H_B . (a) Scan of minimal ξ_D reached during temporal evolution, denoted by ξ_D^{\min} (red dots), against the total atom number N , with the Heisenberg limit of $1/(N+2)$ denoted by the dashed line. (b) The Dicke squeezed state forms a band around the equator of the Bloch sphere, characterized by a large radius $\langle J_{eff}^2 \rangle = \langle J_x^2 + J_y^2 \rangle$ and a vanishing variance of the spin distribution along the z axis, $\langle (\Delta J_z)^2 \rangle$. (c) For unitary evolution with H_B , $\langle (\Delta J_z)^2 \rangle = 0$ (orange line), and $\langle J_{eff}^2 \rangle$ performs sloshing oscillations in time (blue line), nearly reaching the limit value of $N(N+2)/4 = 420$ (dashed line) at $N = 40$.

initially rises, and then starts to oscillate in time, indicating sloshing dynamics, with continuous oscillations between the bunched and nearly homogeneous structures in the self-ordered atomic lattice (see also [6, 83]). During this time, the atoms scatter in and out of the transverse sidebands by absorbing and emitting energy quanta from and to the driving magnetic field. The value of $\langle J_{eff}^2 \rangle$ oscillates with a large amplitude, nearly reaching 0 at its trough and $N(N+2)/4$ at its peak. Such large values of $\langle J_{eff}^2 \rangle$ lead to Dicke state entanglement very close to the Heisenberg limit for the Hamiltonian H_B .

Due to momentum conservation in the interatomic scattering processes, the available Hilbert space for this initial condition is significantly reduced, as states keep zero transverse atomic momentum during unitary evolution, allowing for simulations with relatively large N . Plotting the minimal ξ_D reached during temporal evolution, denoted by ξ_D^{\min} , versus N , reveals that it closely follows the Heisenberg limit of $1/(N+2)$ over nearly three orders of magnitude, up to the maximum atom number tractable by the computational resources available, see Fig. 2(a).

Simulations of dissipative evolution under H_{cav} . We now look at the full photon-atom dynamics for both the coherent unitary evolution and for including the cavity photon decay in

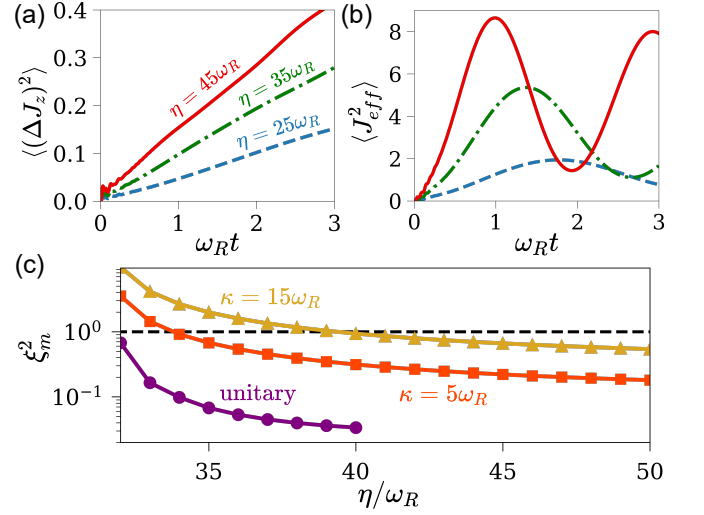


FIG. 3. Generating multiparticle entanglement in the Dicke-like states via transverse self-organization for the Hamiltonian H_{cav} . The irreversible dynamical evolution of (a) $\langle (\Delta J_z)^2 \rangle$, (b) $\langle J_{eff}^2 \rangle$ for η values of $45\omega_R$ (red, solid), $35\omega_R$ (green, dot-dashed) and $25\omega_R$ (blue, dashed). (c) The η scan of ξ_m^2 , given by the lowest ξ_{gen}^2 reached during temporal evolution, for the unitary case (purple, dots), and $\kappa = 5\omega_R$ (red, squares), $\kappa = 15\omega_R$ (yellow, triangles). Crossing of the dashed line ($\xi_{gen}^2 = 1$) indicates the existence of many particle entanglement [79]. The lines are guide to the eyes. Simulation parameters: (a), (b) $(\bar{\Delta}_c, \bar{\Delta}'_c, U_0, \kappa) = (110, -45, 10, 5)\omega_R$, and (c) $(\bar{\Delta}_c, \bar{\Delta}'_c, U_0) = (110, -45, 10)\omega_R$, with (a)-(c) $N = 8$.

the Lindblad master equation. Due to conservation of momentum in the combined photon-atom four-wave mixing, we have $[\delta n + \delta N, H_{cav}] = 0$, but $[J_z, H_{cav}] \neq 0$. The Hilbert space is in this problem considerably larger than for H_B , which limits the tractable atom number to $N = 8$ for the calculations of full quantum dynamics. Parameter values for this case are taken such that maximal Dicke squeezing is observed, and experimentally accessible parameters are discussed in [80].

Due to $[\delta n + \delta N, H_{cav}] = 0$, the variance of $2J_z$ is equal to the variance of δn for the unitarily evolving system. This indicates that a reduction of ξ_D will benefit from lower δn variances, which in general occur when $\langle n_0 \rangle$ and $\langle n_{\pm} \rangle$ are lower, e.g. at larger $|\bar{\Delta}_c|$, $|\bar{\Delta}'_c|$, as long as $\langle J_{eff}^2 \rangle$ is large.

In Figs. 3(a,b) we plot the temporal evolution of the relevant atomic observables for varying the pump strength η in the case with dissipation of photons out of the cavity. For the dissipative case, the equality of δN and δn variances no longer holds and the dissipation of photons out of the cavity makes $\langle (\Delta J_z)^2 \rangle$ increase almost linearly with time, with the slope increasing with the pump amplitude η . This indicates that there is still some correlation between δN and δn even in the dissipative case, i.e. the low δn variances obtained for lower η and $\langle n_{\pm} \rangle$ will lead to lower δN variances.

The $\langle J_{eff}^2 \rangle$ initially increases and oscillates in time, again indicating sloshing dynamics (see Fig. 2). Increasing the pump η , both the growth rate and maximum value of $\langle J_{eff}^2 \rangle$ are increased. The growth rate of $\langle J_{eff}^2 \rangle$ increases faster with dis-

tance to threshold than the growth rate of $\langle(\Delta J_z)^2\rangle$, which, from Eq. (5), leads to a decrease in ξ_{gen}^2 for larger η . Increasing the η values several times above the semiclassically calculated threshold $\eta_c = [-\omega_R(\bar{\Delta}_c^2 + \kappa^2)(\bar{\Delta}_c'^2 + \kappa^2)/(4NU_0^2\bar{\Delta}_c')^{1/2}] \approx 13\omega_R$ [80], the ξ_{gen}^2 reaches values below 1, see Fig. 3c). The lowest ξ_{gen}^2 values for these results are $\xi_{gen}^2 = 0.03$ for the unitary and $\xi_{gen}^2 = 0.18$ for the dissipative case. The maximal η available for these simulations is limited by the size of the numerically tractable Hilbert space.

Conclusion. Each of the two discussed models for producing Dicke squeezed states has its relative advantages and disadvantages regarding prospective experimental implementations. For H_B , the squeezing appears more pronounced, and the system is technically simpler. However, it can be shown that the Hamiltonian H_B is a good description of the atom-atom scattering only when the medium is collisionally thin, such that $\langle N_0 \rangle \gg \langle N_{\pm} \rangle$ [31]. Indeed, adding the higher order terms to H_B decreases the achieved value of multiparticle entanglement for the first order sidebands, by increasing $\langle(\Delta J_z)^2\rangle$ and decreasing $\langle J_{eff}^2 \rangle$ [80]. In prospective experiments, this can however be amended by postselecting the output state to the total number of atoms in the first order sidebands [52, 64]. The maximum value of $\langle J_{eff}^2 \rangle \approx 4 \times 10^6$, attained in the results of Fig. 2, should thus not be achievable for the experiment of Ref. [31] at $N = 3982$ atoms, although it is possible that the same value of $\langle J_{eff}^2 \rangle$ may be reachable for a larger total number of atoms, for the case when $\langle N_0 \rangle \gg \langle N_{\pm} \rangle$ is satisfied. Researching the experimental protocols for realizing the Hamiltonian H_B in the $\langle N_{\pm} \rangle \gg \langle N_0 \rangle$ regime is a highly intriguing topic for future theoretical and numerical work, given the extremely efficient multiparticle entanglement generation shown in Fig. 2.

For the Hamiltonian H_{cav} , the excitation of higher order sidebands is precluded for shallow self-organized optical lattices, which is the case for the results shown in Fig. 3 [80]. However, as the experimental system is more complex, there is a greater number of potential noise sources. For instance, as discussed in [33], if experimental imperfections break the x -axis translational symmetry of the system, e.g. via misalignment in the optical setup, the variance of δn will be increased due to the number-phase uncertainty relation for the photonic sidebands [84], which may reduce the multiparticle entanglement in the atomic momentum sidebands. Other potential drawbacks stem from the limitations in the experimentally available parameter space.

We note that steady state Dicke squeezing can be achieved for both H_B and H_{cav} by rapidly switching off the driving at the moment of optimal squeezing, providing states for prospective metrology applications [80]. Details of the procedure for using the generated states in quantum metrology, which would rely on measuring the quantities $\langle(\Delta J_z)^2\rangle$ and $\langle J_{eff}^2 \rangle$ by quantum non-demolition measurements of the atomic momentum distribution [85], e.g. by time-of-flight imaging [60, 64], are currently under investigation.

To summarize, we have theoretically studied and numerically demonstrated a method for generating momentum state

Dicke squeezing via nonlinear self-organization of ultracold atoms driven by an external field. The four-wave mixing mechanism generating the quantum correlations between ultracold atoms was studied in two distinct Hamiltonians, H_B for which the drive is an oscillating magnetic field, related to the experiment of Ref. [31] and a novel model H_{cav} , where driving is provided by a laser with ring cavity feedback. In both of the considered systems, the quantum correlations appear in the atomic motional states of a BEC, which are more robust to noise from the environment as compared to internal atomic degrees of freedom, used commonly in previous ultracold atom based entanglement schemes [52]. Dicke squeezing near the Heisenberg limit, achieved for the case of H_B , highlights the great potential of applying nonlinear self-organization in emerging quantum technologies. Note that both H_B and H_{cav} have continuous symmetry, which is also the case for some prominent recent models describing self-organization of ultracold atoms [17, 23]. It may thus be possible to observe multiparticle entanglement generation in experimental setups of Refs. [17, 23], along with the setup of Ref. [31].

Acknowledgements. I.K. thanks Helmut Ritsch for useful comments on the early version of the manuscript and Oliver Diekmann for helpful discussions. The work of I. K. was funded by the Austrian Science Fund (FWF) Lise Meitner Postdoctoral Fellowship M3011. The dynamical evolution equations were solved numerically by using the open-source framework QuantumOptics.jl [86]. The computational results presented here have been achieved using the Vienna Scientific Cluster (VSC).

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Supplemental Material - Generating multiparticle entangled states by self-organization of driven ultracold atoms

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(Dated: December 15, 2022)

DERIVATION OF H_{cav}

The derivation of H_B is given for 2D clouds in the Methods section of Ref. [S1], and can be readily simplified to the 1D case (see also below). Hence, we provide here only the details of the derivation of H_{cav} . A cigar-shaped zero-temperature Bose-Einstein condensate (BEC) is placed in an effectively plano-planar ring cavity of effective diffractive length L with one lossy and three perfectly reflecting mirrors (κ - cavity photon decay rate), which is pumped by a coherent electric field with pump strength η at frequency ω (see Fig. S1a) and section ‘‘Experimental design of the ring cavity setup’’ for details). A strong confinement along the y and z axes allows to restrict the analysis to 1D structures. The pump drive excites on-axis running waves with spatial profile e^{ik_0z} , with the spontaneously generated sidebands having the profile $e^{ik_0z}e^{\pm iq_c x}$, where $k_0 = 2\pi/\lambda_0$ is the cavity longitudinal mode (also called the on-axis or zero-order mode) wavenumber, $q_c = 2\pi/\Lambda_c$ and Λ_c is the pattern lengthscale, tunable via Fourier filtering of intracavity light [S2].

Following Refs. [S3–S5], we take the electric field modes as:

$$E(\mathbf{r}) = a_0 e^{ik_0 z} + a_+ e^{ik_0 z} e^{iq_c x} + a_- e^{ik_0 z} e^{-iq_c x}, \quad (\text{S1})$$

where a_j are the photonic annihilation operators in the j -th mode. The atomic field operator is given by:

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{V}} (b_0 + b_+ e^{iq_c x} + b_- e^{-iq_c x}), \quad (\text{S2})$$

where b_j is the bosonic annihilation operator of the j -th transverse atomic momentum mode. The effective many-body Hamiltonian for the photons and atomic motional states can be derived from the Jaynes-Cummings model (see e.g. [S6–S8]). For two-level atoms, using the dipole and rotating wave approximations in the low saturation (far-detuned) limit, we therefore have:

$$H_{cav} = -\hbar\Delta_c n_0 - \hbar\Delta'_c (n_+ + n_-) + i\hbar(\eta a_0^\dagger - \eta^* a_0) + \int_V d^3r \psi^\dagger(\mathbf{r}) \left[\frac{\mathbf{p}^2}{2m} + \hbar U_0 E^\dagger(\mathbf{r}) E(\mathbf{r}) \right] \psi(\mathbf{r}), \quad (\text{S3})$$

where $\Delta_c = \omega - \omega_0$, $\Delta'_c = \omega - \omega'_0$ are the pump detunings from the on-axis and sideband cavity modes, respectively, $n_0 = a_0^\dagger a_0$, $n_\pm = a_\pm^\dagger a_\pm$, $U_0 = g_0^2/\Delta_a$ is the single atom light shift, $\Delta_a = \omega - \omega_a$ is the laser-atom detuning, and g_0 is the atom-cavity coupling strength. In writing Eq. (S3) we have neglected the random collisions between the atoms in the dilute BEC cloud, as we are here interested on highlighting the consequences of light-matter interaction. The number of atoms is fixed and given by $N = N_0 + N_+ + N_-$, where $N_0 = b_0^\dagger b_0$, $N_\pm = b_\pm^\dagger b_\pm$.

We insert Eqs. (S1) and (S2) into Eq. (S3) for real-valued η and perform the integration over the BEC cloud volume V to get the effective total Hamiltonian $H_{cav} = H_0 + H_{FWM}^{(1)} + H_{FWM}^{(2)}$, where:

$$H_0 = -\hbar\bar{\Delta}_c n_0 - \hbar\bar{\Delta}'_c (n_+ + n_-) + \hbar\omega_R (N_+ + N_-) + i\hbar\eta (a_0^\dagger - a_0), \quad (\text{S4})$$

and the four wave mixing terms are:

$$H_{FWM}^{(1)} = \hbar U_0 [(a_+^\dagger b_-^\dagger + a_-^\dagger b_+^\dagger) a_0 b_0 + \text{H.c.}] + \hbar U_0 [a_0^\dagger (b_+^\dagger a_+ + b_-^\dagger a_-) b_0 + \text{H.c.}], \quad (\text{S5})$$

$$H_{FWM}^{(2)} = \hbar U_0 (a_+^\dagger a_- b_-^\dagger b_+ + \text{H.c.}), \quad (\text{S6})$$

where $\bar{\Delta}_c = \Delta_c - NU_0$, $\bar{\Delta}'_c = \Delta'_c - NU_0$, $n = n_0 + n_+ + n_-$ and $\hbar\omega_R = (\hbar q_c)^2/2m$ is the transverse recoil energy.

Generation of transverse sidebands via the four wave mixing term $a_+^\dagger a_0 b_-^\dagger b_0$ in $H_{FWM}^{(1)}$ can be explained by the momentum conserving processes illustrated in Fig. 1d) of the main text. The $a_0^\dagger (b_+^\dagger a_+ + b_-^\dagger a_-) b_0$ term of $H_{FWM}^{(1)}$ scatters a photon with $\pm q_c$ into the on-axis mode and excites an atomic sideband with transverse momentum $\pm \hbar q_c$. The $H_{FWM}^{(2)}$ describes the secondary wave mixing process for stripe patterns, in which a scattering of a photon sideband with $\pm q_c$ into the mode with $\mp q_c$ leads to a transition of an atom from the state with transverse momentum $\mp \hbar q_c$ into the state with transverse momentum $\pm \hbar q_c$. This process leads to saturation of the sideband mode population far above threshold [S9].

MEAN FIELD EVOLUTION EQUATIONS FOR H_{cav}

We first look at the temporal evolution of the field operators [S10]. The Heisenberg equation for an operator $O(t)$ has the form:

$$\frac{dO}{dt} = \frac{i}{\hbar} [H, O]. \quad (\text{S7})$$

while the nonunitary evolution, including the dissipation of photons from the cavity, can be described by the Lindblad-type evolution via the equation:

$$\frac{dO}{dt} = \frac{i}{\hbar}[H, O] + \kappa \sum_{j=0, \pm} (2a_j^\dagger O a_j - a_j^\dagger a_j O - O a_j^\dagger a_j). \quad (\text{S8})$$

We here use the ring cavity Hamiltonian $H = H_{cav} = H_0 + H_{FWM}^{(1)} + H_{FWM}^{(2)}$, with the three parts given by:

$$H_0 = -\hbar\bar{\Delta}_c n_0 - \hbar\bar{\Delta}'_c (n_+ + n_-) + \hbar\omega_R (N_+ + N_-) + i\hbar\eta (a_0^\dagger - a_0), \quad (\text{S9})$$

$$H_{FWM}^{(1)} = \hbar U_0 [(a_+^\dagger b_+^\dagger + a_-^\dagger b_+^\dagger) a_0 b_0 + \text{H.c.}] + \hbar U_0 [a_0^\dagger (b_+^\dagger a_+ + b_-^\dagger a_-) b_0 + \text{H.c.}], \quad (\text{S10})$$

$$H_{FWM}^{(2)} = \hbar U_0 (a_+^\dagger a_- b_-^\dagger b_+ + \text{H.c.}), \quad (\text{S11})$$

and the commutation relations of the bosonic modes for photons and atoms: $[a_j, a_k^\dagger] = \delta_{j,k}$, $[a_j^\dagger, a_k^\dagger] = [a_j, a_k] = 0$, and $[b_j, b_k^\dagger] = \delta_{j,k}$, $[b_j^\dagger, b_k^\dagger] = [b_j, b_k] = 0$, respectively, where $j, k = 0, +, -$. For the photonic modes, the Eq. (S8) now gives:

$$\dot{a}_0 = (i\bar{\Delta}_c - \kappa) a_0 - iU_0 [(b_+^\dagger a_+ + b_-^\dagger a_-) b_0 + b_0^\dagger (a_+ b_- + a_- b_+)] + \eta^*, \quad (\text{S12})$$

$$\dot{a}_+ = (i\bar{\Delta}'_c - \kappa) a_+ - iU_0 [(b_-^\dagger b_0 + b_0^\dagger b_+) a_0 + a_- b_-^\dagger b_+], \quad (\text{S13})$$

$$\dot{a}_- = (i\bar{\Delta}'_c - \kappa) a_- - iU_0 [(b_+^\dagger b_0 + b_0^\dagger b_-) a_0 + a_+ b_+^\dagger b_-], \quad (\text{S14})$$

while for the atomic momentum modes, the Eq. (S7) gives:

$$\dot{b}_0 = -iU_0 [a_0^\dagger (a_+ b_- + a_- b_+) + (a_+^\dagger b_+ + a_-^\dagger b_-) a_0], \quad (\text{S15})$$

$$\dot{b}_+ = -i\omega_R b_+ - iU_0 [(a_-^\dagger a_0 + a_0^\dagger a_+) b_0 + a_-^\dagger a_+ b_-], \quad (\text{S16})$$

$$\dot{b}_- = -i\omega_R b_- - iU_0 [(a_+^\dagger a_0 + a_0^\dagger a_-) b_0 + a_+^\dagger a_- b_+]. \quad (\text{S17})$$

Taking now the expectation values of the right- and left-hand sides, writing $\langle O_1 O_2 O_3 \rangle \rightarrow \langle O_1 \rangle \langle O_2 \rangle \langle O_3 \rangle$, and using $a_j \rightarrow \langle a_j \rangle = \sqrt{N} \alpha_j(t)$, $b_j \rightarrow \langle b_j \rangle = \sqrt{N} \beta_j(t)$, we get the mean field dynamical equations:

$$\dot{\alpha}_0 = (i\bar{\Delta}_c - \kappa) \alpha_0 - iu_0 [(\beta_+^* \alpha_+ + \beta_-^* \alpha_-) \beta_0 + \beta_0^* (\alpha_+ \beta_- + \alpha_- \beta_+)] + y^*, \quad (\text{S18})$$

$$\dot{\alpha}_+ = (i\bar{\Delta}'_c - \kappa) \alpha_+ - iu_0 [(\beta_-^* \beta_0 + \beta_0^* \beta_+) \alpha_0 + \alpha_- \beta_-^* \beta_+], \quad (\text{S19})$$

$$\dot{\alpha}_- = (i\bar{\Delta}'_c - \kappa) \alpha_- - iu_0 [(\beta_+^* \beta_0 + \beta_0^* \beta_-) \alpha_0 + \alpha_+ \beta_+^* \beta_-], \quad (\text{S20})$$

where $u_0 = NU_0$, $y = \eta/\sqrt{N}$, and

$$\dot{\beta}_0 = -iu_0 [\alpha_0^* (\alpha_+ \beta_- + \alpha_- \beta_+) + (\alpha_+^* \beta_+ + \alpha_-^* \beta_-) \alpha_0], \quad (\text{S21})$$

$$\dot{\beta}_+ = -i\omega_R \beta_+ - iu_0 [(\alpha_-^* \alpha_0 + \alpha_0^* \alpha_+) \beta_0 + \alpha_-^* \alpha_+ \beta_-], \quad (\text{S22})$$

$$\dot{\beta}_- = -i\omega_R \beta_- - iu_0 [(\alpha_+^* \alpha_0 + \alpha_0^* \alpha_-) \beta_0 + \alpha_+^* \alpha_- \beta_+]. \quad (\text{S23})$$

MEAN FIELD THRESHOLD FOR H_{cav}

We now look at the steady state limit of the above mean field dynamical equations. Writing now $\alpha_j(t) \rightarrow \alpha_j$ and $\beta_j(t) \rightarrow \beta_j$, we get the equations:

$$0 = (i\bar{\Delta}_c - \kappa) \alpha_0 - iu_0 [(\beta_+^* \alpha_+ + \beta_-^* \alpha_-) \beta_0 + \beta_0^* (\alpha_+ \beta_- + \alpha_- \beta_+)] + y^*, \quad (\text{S24})$$

$$0 = (i\bar{\Delta}'_c - \kappa) \alpha_+ - iu_0 [(\beta_-^* \beta_0 + \beta_0^* \beta_+) \alpha_0 + \alpha_- \beta_-^* \beta_+], \quad (\text{S25})$$

$$0 = (i\bar{\Delta}'_c - \kappa) \alpha_- - iu_0 [(\beta_+^* \beta_0 + \beta_0^* \beta_-) \alpha_0 + \alpha_+ \beta_+^* \beta_-], \quad (\text{S26})$$

and

$$0 = u_0 [\alpha_0^* (\alpha_+ \beta_- + \alpha_- \beta_+) + (\alpha_+^* \beta_+ + \alpha_-^* \beta_-) \alpha_0], \quad (\text{S27})$$

$$0 = \omega_R \beta_+ + u_0 [(\alpha_-^* \alpha_0 + \alpha_0^* \alpha_+) \beta_0 + \alpha_-^* \alpha_+ \beta_-], \quad (\text{S28})$$

$$0 = \omega_R \beta_- + u_0 [(\alpha_+^* \alpha_0 + \alpha_0^* \alpha_-) \beta_0 + \alpha_+^* \alpha_- \beta_+]. \quad (\text{S29})$$

At threshold, we neglect all terms square or higher order in the sidebands. We choose a real-valued $\beta_0 = \sqrt{1 - |\beta_+|^2 - |\beta_-|^2} \approx 1$ and find the homogeneous field amplitude of the on-axis mode to be

$$\alpha_0 = \frac{y^*}{\sqrt{\bar{\Delta}_c^2 + \kappa^2}} e^{i \arctan(\bar{\Delta}_c/\kappa)}. \quad (\text{S30})$$

One can then easily calculate that in this approximation:

$$\beta_{\pm} = -\frac{u_0}{\omega_R} (\alpha_{\pm}^* \alpha_0 + \alpha_{\mp} \alpha_0^*). \quad (\text{S31})$$

Inserting this relation in the equations for the fields, we get:

$$0 = (i\bar{\Delta}'_c - \kappa) \alpha_+ + 2i \frac{u_0^2}{\omega_R} (\alpha_+ \alpha_0^* + \alpha_-^* \alpha_0) \alpha_0, \quad (\text{S32})$$

$$0 = (i\bar{\Delta}'_c - \kappa) \alpha_- + 2i \frac{u_0^2}{\omega_R} (\alpha_+^* \alpha_0 + \alpha_- \alpha_0^*) \alpha_0. \quad (\text{S33})$$

By inserting the complex conjugate of

$$\alpha_+ \alpha_0^* + \alpha_-^* \alpha_0 = -\frac{(i\bar{\Delta}'_c - \kappa) \alpha_+}{2i \frac{u_0^2}{\omega_R} \alpha_0} \quad (\text{S34})$$

into the second equation of the above, we get:

$$\alpha_- = \frac{i\bar{\Delta}'_c + \kappa}{i\bar{\Delta}'_c - \kappa} \frac{\alpha_0 \alpha_+^*}{\alpha_0^*}. \quad (\text{S35})$$

As the sidebands have equal amplitudes, we can write:

$$\alpha_{\pm} = A e^{i\chi_{\pm}}, \quad \beta_{\pm} = B e^{i\epsilon_{\pm}}. \quad (\text{S36})$$

From (S31), we then get that at threshold $\bar{\epsilon} = \epsilon_+ + \epsilon_- = 0$, while Eq. (S35) gives $\bar{\chi} = \chi_+ + \chi_- = 2[\arctan(\bar{\Delta}_c/\kappa) + \arctan(\bar{\Delta}'_c/\kappa) - \arg y]$. Inserting the complex conjugate of Eq. (S35) into the Eq. (S32), we get for the critical intracavity field:

$$|\alpha_0^c|^2 = -\frac{\omega_R(\bar{\Delta}_c^2 + \kappa^2)}{4u_0^2 \bar{\Delta}'_c} \quad (\text{S37})$$

which gives the threshold for the real-valued critical input electric field amplitude y_c :

$$y_c^2 = -\frac{\omega_R(\bar{\Delta}_c^2 + \kappa^2)(\bar{\Delta}'_c^2 + \kappa^2)}{4u_0^2 \bar{\Delta}'_c}, \quad (\text{S38})$$

which leads to

$$\eta_c = \sqrt{\frac{-\omega_R(\bar{\Delta}_c^2 + \kappa^2)(\bar{\Delta}'_c^2 + \kappa^2)}{4NU_0^2 \bar{\Delta}'_c}}. \quad (\text{S39})$$

The structure of the expression (S39) for η_c bears similarity to the threshold for the transversely pumped single mode cavity, given by $\sqrt{-\omega_R(\bar{\Delta}_c^2 + \kappa^2)/(2N\bar{\Delta}_c)}$ [S11].

We now numerically investigate the system dynamics in the mean-field limit of the dissipative evolution for the Hamiltonian H_{cav} . In Fig. S1b) we plot the square root of the diffracted photon number $|\alpha_{\pm}^S| = (\langle n_{\pm} \rangle_S / N)^{1/2}$ and the transverse atomic sideband population $|\beta_{\pm}^S| = (\langle N_{\pm} \rangle_S / N)^{1/2}$ vs. the pump η . For the results in Fig. S1b), the equations (S18)-(S23) are solved numerically and the steady state values are plotted for different η 's. In the inset we plot the temporal evolution of the roll patterns in the atomic density, given by $n(x,t) = |\beta_0(t) + \beta_+(t) \exp(iq_c x) + \beta_-(t) \exp(-iq_c x)|^2$, and in the electric field intensity, given by $I(x,t) = |\alpha_0(t) + \alpha_+(t) \exp(iq_c x) + \alpha_-(t) \exp(-iq_c x)|^2$, normalized to the steady state $|\alpha_0^S|^2$, denoted by I_0 .

The mean field patterns appear for the pump rate $\eta > \eta_c$. The initial sharp increase in the transverse excitations, seen in Fig. S1b) for $\eta \gtrsim \eta_c$, gives way to saturation for larger η 's. In the inset we plot the temporal evolution of the semiclassical stripe patterns in the atomic density $n(x,t)$ and the electric field intensity $I(x,t)$, normalized to the steady state $|\alpha_0^S|^2$, denoted by I_0 .

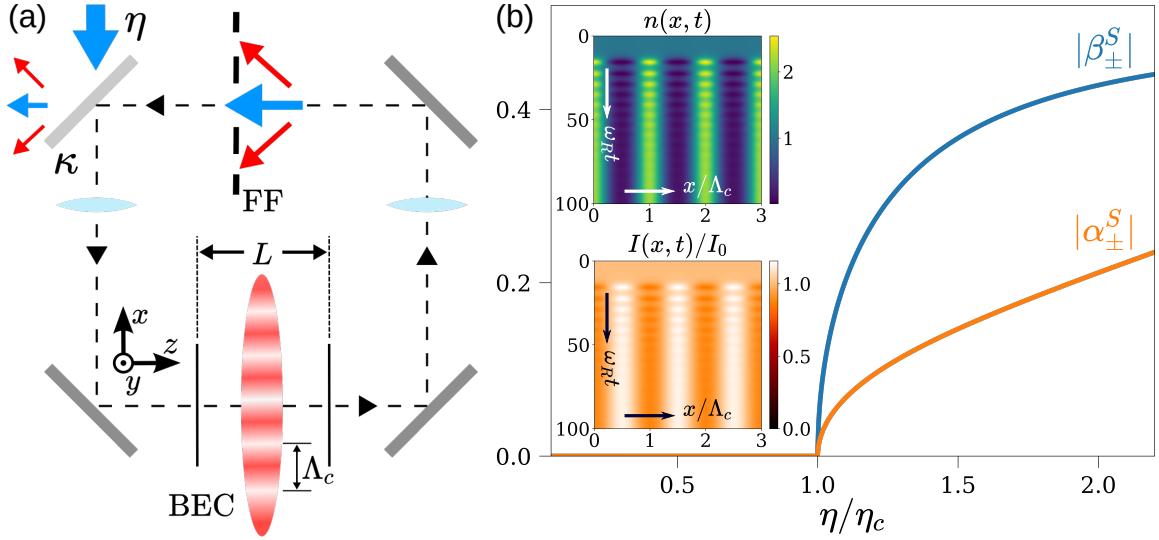


FIG. S1. (a) Schematics of the proposed experimental setup for observing Dicke state entanglement via self-organization of ultracold atoms in a laser pumped ring cavity. The ultracold BEC gas, strongly confined along the y and z axes, is placed in a ring cavity with linewidth κ , which is pumped by coherent on-axis light with drive amplitude η (blue arrow). Effective cavity length L can be controlled by adjustment of intracavity lenses (light blue) around the afocal telescopic condition, while pattern lengthscale Λ_c is tuned via Fourier filtering (FF) of the photonic sidebands (red arrows). (b) Mean-field transverse optomechanical self-ordering in a leaky ring cavity. The square root of the steady state diffracted photon number $|\alpha_{\pm}^S|$ (orange) and transverse atomic sideband population $|\beta_{\pm}^S|$ (blue) for varying the input beam pump rate η (see text). Inset: temporal evolution of the semiclassical stripe patterns in the atomic density $n(x, t)$ (upper) and the normalized electric field $I(x, t)/I_0$ (lower) at $\eta = 1.2\eta_c$ (see text). Simulation parameters: $N = 10^4$ and $(\bar{\Delta}_c, \bar{\Delta}'_c, U_0, \kappa) = (8.8, -10, 1.2 \times 10^{-4}, 10)\omega_R$, with $\hbar = 1$.

The system starts with $\alpha_{\pm}(0) = \alpha_{\pm}(0) = 0$, with $\alpha_{\pm}(t)$ rising rapidly on the scale $1/\kappa$, which is not detectable on the plot since the t range is too large. This homogeneous state becomes unstable after around $10/\omega_R$, a time determined by the initial fluctuations in the atomic density, which were made artificially small in the plots shown (note that the steady state in this self-organizing mean field model is the same for all reasonable values of initial fluctuations). Before reaching the steady state, the $|\alpha_{\pm}(t)|$ and $|\beta_{\pm}(t)|$ oscillate at a frequency of a few ω_R . This oscillation of the sideband populations is a signature of the sloshing dynamics, i.e. the continuous oscillation between the bunched and homogeneous atomic structure in the optical lattice (studied e.g. in [S12, S13]). The stripes in $n(x, t)$ and $I(x, t)$ are complementary, which is a consequence of the optical dipole potential repulsing the atoms away from the intensity peaks for $U_0 > 0$ [S14, S15].

CONTINUOUS TRANSLATIONAL SYMMETRY OF H_B AND H_{cav}

We start by writing again the operators for the photon and atomic momentum sidebands, which have the form $\delta n = n_+ - n_-$, $\delta N = N_+ - N_-$, with $n_{\pm} = a_{\pm}^{\dagger} a_{\pm}$ and $N_{\pm} = b_{\pm}^{\dagger} b_{\pm}$. The Hamiltonian H_B is symmetric to translations by a real-valued distance parameter d along the x axis, which transforms the wavefunction as $\psi(x) \rightarrow \psi(x+d)$, meaning it is symmetric under simultaneous transformations: $b_0 \rightarrow b_0$, $b_{\pm} \rightarrow e^{\pm idk_f} b_{\pm}$. The generator of this symmetry is the x component of the atomic momentum operator, in the many-body formalism given simply by $p_x = \hbar k_f \delta N$. The corresponding translation operation is performed by the unitary operator $T_B(d) = e^{idp_x/\hbar}$. This continuous symmetry of H_B leads to:

$$T_B^{\dagger}(d)H_B T_B(d) = H_B \rightarrow [\delta N, H_B] = 0. \quad (\text{S40})$$

Note that the same conclusion can be reached from the opposite direction, by explicitly calculating $[\delta N, H_B] = 0$ and finding the corresponding unitary symmetry operator. By using the Baker-Campbell-Hausdorff formula and the commutator $[\delta N, b_{\pm}] = \mp b_{\pm}$, it can then be readily shown that $T_B^{\dagger}(d)b_{\pm} T_B(d) = e^{\pm idk_f} b_{\pm}$.

The Hamiltonian H_{cav} is also symmetric to continuous translations by d along the x axis, which transforms the wavefunction and the electric field as $\psi(x) \rightarrow \psi(x+d)$, $E(x) \rightarrow E(x+d)$, meaning it is symmetric under simultaneous transformations: $a_0 \rightarrow a_0$, $a_{\pm} \rightarrow e^{\pm iq_c d} a_{\pm}$, $b_0 \rightarrow b_0$, $b_{\pm} \rightarrow e^{\pm iq_c d} b_{\pm}$. The generator of this symmetry is the x component of the combined photonic and atomic momentum operator, given by $P_x = \hbar q_c (\delta n + \delta N)$. The corresponding translation operation is performed by the

unitary operator $T_{cav}(d) = e^{idP_x/\hbar}$. This continuous symmetry of H_{cav} leads to:

$$T_{cav}^\dagger(d)H_{cav}T_{cav}(d) = H_{cav} \rightarrow [\delta n + \delta N, H_{cav}] = 0. \quad (\text{S41})$$

Again, the same conclusion can be reached from the opposite direction, by explicitly calculating $[\delta n + \delta N, H_{cav}] = 0$ and finding the corresponding unitary symmetry operator. Following the same procedure as above, it can readily be shown that now $T_{cav}^\dagger(d)a_\pm T_{cav}(d) = e^{\pm idq_c}a_\pm$ and $T_{cav}^\dagger(d)b_\pm T_{cav}(d) = e^{\pm idq_c}b_\pm$.

To illustrate the meaning of preservation of translational symmetry during temporal evolution, we calculate the atomic spatial probability distribution $\langle \psi^\dagger(x)\psi(x) \rangle$ for $V = 1$ as:

$$\langle \psi^\dagger(x)\psi(x) \rangle = \langle N_0 \rangle + \langle N_+ \rangle + \langle N_- \rangle + \langle b_0^\dagger b_+ + b_-^\dagger b_0 \rangle e^{ikx} + \langle b_0^\dagger b_- + b_+^\dagger b_0 \rangle e^{-ikx} + \langle b_-^\dagger b_+ \rangle e^{2ikx} + \langle b_+^\dagger b_- \rangle e^{-2ikx}, \quad (\text{S42})$$

where $k = k_f$ for H_B and $k = q_c$ for H_{cav} . It can readily be shown that for a translationally invariant density matrix $\rho(t) = T^\dagger(d)\rho(t)T(d)$, the inhomogeneous terms of Eq. (S42) vanish (see below), and $\langle \psi^\dagger(x)\psi(x) \rangle = N$, in contrast to the results for the semiclassical (mean-field) framework shown in Fig. S1. Such situation where atomic spatial probability distribution is homogeneous, even though self-organization takes place, occurs because of the Heisenberg uncertainty relation for position and momentum measurements. This means that, as the atoms are in a continuously translationally invariant state, a pattern realization with any displacement from $x = 0$ (i.e. pattern phase) is equally probable.

Performing a measurement of the atomic position will collapse the system onto a state with an undetermined (i.e. one with maximum variance in) total momentum, which means the inhomogeneous terms of Eq. (S42) will be nonzero. The spatial probability distribution will then be sinusoidal, with a fixed spatial phase of the patterns. As the measured states have undetermined atomic positions, the measurement of the atomic distributions should yield random values of the pattern phases. This was indeed seen in the experiments of Ref. [S1], where pattern realizations with random pattern phases (displacements) and orientations, in a 2D system, were reported.

We note that the above described translationally invariant states are analogous to the maximally amplitude squeezed photonic states, for which the phase of the electric field is undetermined [S16]. Similar conclusions were previously also reached for the photonic self-organized patterns, as reported in [S4].

In ultracold atom experiments, the two most common complementary measurements of density (position) and momentum distributions of atoms in a cloud are near-field microscopy and time-of-flight imaging, respectively. To efficiently generate multiparticle entangled states, the experiments should gather information by measuring the momentum distribution via quantum nondemolition measurements [S17], and completely avoid measuring the density distribution, as done e.g. in the experiment of Ref. [S18].

VANISHING OF $\langle \mathbf{J} \rangle$ DURING DYNAMICAL EVOLUTION

It can readily be shown that the expectation value of the total (vector) ‘‘angular momentum’’ operator of the atomic momentum sidebands vanishes, i.e. $\langle \mathbf{J} \rangle = 0$, for both unitary evolution under H_B , along with unitary and dissipative evolution under H_{cav} . To do this, we first write the atomic momentum ladder operators $J_\pm = b_\pm^\dagger b_\mp$, where the J_x, J_y are now $J_x = (J_+ + J_-)/2$, $J_y = (J_+ - J_-)/2i$.

For H_B , in our simulations the system evolves from the initial state $|\psi_0\rangle = |N\rangle_0|0\rangle_+|0\rangle_-$ to the state $|\psi(t)\rangle$, by unitary evolution $|\psi(t)\rangle = U_B(t)|\psi_0\rangle$, where $U_B(t) = e^{-iH_B t/\hbar}$. The expectation values of J_\pm at time t are thus given by $\langle J_\pm \rangle = \langle \psi(t)|J_\pm|\psi(t)\rangle$. Now, the temporal evolution of the state $|\psi_0\rangle$ via H_B leaves the state $|\psi(t)\rangle$ on a part of the Hilbert space spanned by states with zero transverse momentum, since H_B conserves the transverse momentum. In contrast, the operators J_\pm act to transfer the state $|\psi(t)\rangle$ to an orthogonal part of the Hilbert space, since they increase/decrease the transverse momentum by $2\hbar k_f$. The orthogonality of $J_\pm|\psi(t)\rangle$ and $|\psi(t)\rangle$ thus leads to $\langle J_\pm \rangle = 0$, and correspondingly to $\langle J_x \rangle = \langle J_y \rangle = 0$. Likewise, $\langle J_z \rangle = 0$ follows from the fact that the total transverse momentum of the state $|\psi(t)\rangle$ remains zero for all time.

For unitary evolution under H_{cav} , the initial state is in our simulations given by $|\psi_0\rangle = |0, 0, 0\rangle_{ph}|N\rangle_0|0\rangle_+|0\rangle_-$. The dynamics leads now to the state $|\psi(t)\rangle = U_{cav}(t)|\psi_0\rangle$, where $U_{cav}(t) = e^{-iH_{cav}t/\hbar}$, for which the total photonic and atomic transverse momentum is equal to zero. As the operators J_\pm do not conserve the transverse momentum, they take the state $|\psi(t)\rangle$ to an orthogonal part of the Hilbert space, and we again have $\langle J_x \rangle = \langle J_y \rangle = 0$.

For H_{cav} , $\langle J_z \rangle = 0$ follows from parity ($+$ \leftrightarrow $-$) symmetry of H_{cav} , i.e. $P^{-1}H_{cav}P = H_{cav}$, where the operator P inverts the x axis, switching between the $+$ and $-$ modes. The parity symmetry of H_{cav} and $|\psi_0\rangle$ leads to a parity symmetry of the state $|\psi(t)\rangle$, which means $\langle J_z \rangle = \langle \psi(t)|J_z|\psi(t)\rangle = \langle \psi(t)|P^{-1}J_zP|\psi(t)\rangle = -\langle J_z \rangle$, leading to $\langle J_z \rangle = 0$.

To show that $\langle \mathbf{J} \rangle = 0$ during dissipative evolution for H_{cav} , we write the Lindblad master equation for the density matrix $\rho(t)$ as

[S10]:

$$\frac{d\rho}{dt} = \mathcal{L}[\rho], \text{ where } \mathcal{L}[\rho] = -\frac{i}{\hbar}[H, \rho] + \kappa \sum_{j=0, \pm} (2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j). \quad (\text{S43})$$

The above equation can be formally solved as [S19]:

$$\rho(t) = e^{\mathcal{L}t}[\rho(0)] = \rho(0) + t\mathcal{L}[\rho(0)] + \frac{t^2}{2}\mathcal{L}[\mathcal{L}[\rho(0)]] + \dots \quad (\text{S44})$$

By noting that $T_{cav}(d)\rho(0)T_{cav}^\dagger(d) = \rho(0)$, it can readily be shown that $T_{cav}(d)\rho(t)T_{cav}^\dagger(d) = \rho(t)$. To demonstrate this, we write for the first order term of (S44):

$$T_{cav}(d)\mathcal{L}[\rho(0)]T_{cav}^\dagger(d) = T_{cav}(d)\left[-\frac{i}{\hbar}[H, \rho(0)] + \kappa \sum_{j=0, \pm} (2a_j \rho(0) a_j^\dagger - a_j^\dagger a_j \rho(0) - \rho(0) a_j^\dagger a_j)\right]T_{cav}^\dagger(d). \quad (\text{S45})$$

Now, the relations $T_{cav}(d)H_{cav}T_{cav}^\dagger(d) = H_{cav}$, $T_{cav}(d)a_0T_{cav}^\dagger(d) = a_0$ and $T_{cav}(d)a_\pm T_{cav}^\dagger(d) = e^{\mp idq_c} a_\pm$ lead to:

$$T_{cav}(d)\mathcal{L}[\rho(0)]T_{cav}^\dagger(d) = \mathcal{L}[\rho(0)]. \quad (\text{S46})$$

Similarly, we write the second order term of (S44) as:

$$T_{cav}(d)\mathcal{L}[\mathcal{L}[\rho(0)]]T_{cav}^\dagger(d) = T_{cav}(d)\left[-\frac{i}{\hbar}[H, \mathcal{L}[\rho(0)]] + \kappa \sum_{j=0, \pm} (2a_j \mathcal{L}[\rho(0)] a_j^\dagger - a_j^\dagger a_j \mathcal{L}[\rho(0)] - \mathcal{L}[\rho(0)] a_j^\dagger a_j)\right]T_{cav}^\dagger(d). \quad (\text{S47})$$

Using now the same relations as above and Eq. (S46) leads to:

$$T_{cav}(d)\mathcal{L}[\mathcal{L}[\rho(0)]]T_{cav}^\dagger(d) = \mathcal{L}[\mathcal{L}[\rho(0)]]. \quad (\text{S48})$$

Repeating the same procedure for all the higher order terms leads to:

$$T_{cav}(d)\rho(t)T_{cav}^\dagger(d) = \rho(t), \quad (\text{S49})$$

which can also be shown by using Eq. (2.4) of Ref. [S20].

One can then show that $\langle T_{cav}^\dagger(d)J_\pm T_{cav}(d) \rangle = e^{\mp 2idq_c} \langle J_\pm \rangle = \text{Tr}(T_{cav}^\dagger(d)J_\pm T_{cav}(d)\rho(t)) = \text{Tr}(J_\pm T_{cav}(d)\rho(t)T_{cav}^\dagger(d)) = \text{Tr}(J_\pm \rho(t)) = \langle J_\pm \rangle$, where we have used the invariance of trace under cyclic permutations, and the relations above. Since $e^{\mp 2idq_c} \langle J_\pm \rangle = \langle J_\pm \rangle$ for any d in the 1D space of the problem, we have $\langle J_\pm \rangle = \langle J_x \rangle = \langle J_y \rangle = 0$ during dynamical evolution.

To show that $\langle J_z \rangle = 0$, we first note that the same procedure as above leads from $P\rho(0)P^{-1} = \rho(0)$ to $P\rho(t)P^{-1} = \rho(t)$. We then have $\langle P^{-1}J_z P \rangle = -\langle J_z \rangle = \text{Tr}(P^{-1}J_z P \rho(t)) = \text{Tr}(J_z P \rho(t)P^{-1}) = \text{Tr}(J_z \rho(t)) = \langle J_z \rangle$, which leads to $\langle J_z \rangle = 0$ during dynamical evolution.

Finally, we note that for the dissipative evolution, in contrast to the unitary case, $d(\delta n + \delta N)/dt \neq 0$ (see Eq. (S51)), meaning the total momentum is not conserved, even though $\langle \delta n \rangle = \langle \delta N \rangle = 0$. However, the continuous translational symmetry of the state is still preserved during temporal evolution, as $\rho(t) = T_{cav}^\dagger(d)\rho(t)T_{cav}(d)$, because $T_{cav}(d)H_{cav}T_{cav}^\dagger(d) = H_{cav}$ and $T_{cav}(d)a_j T_{cav}^\dagger(d) = e^{i\phi_j} a_j$, for $j = 0, \pm$. Continuous symmetry operators of this type were discussed also in Ref. [S20].

TEMPORAL EVOLUTION OF THE VARIANCES FOR H_{cav}

We now discuss qualitatively the consequence of $[H_{cav}, \delta n + \delta N] = 0$ on the behavior of the variances of the $\delta n + \delta N$, δn and δN operators by looking at their temporal evolution, described by Eqs. (S7) and (S8). The operator $\delta n + \delta N$ is given by a zero matrix at $t = 0$. In the unitary case, $\langle \delta n + \delta N \rangle$ remains zero for all time, which can be seen from taking the expectation value of:

$$\frac{d(\delta n + \delta N)}{dt} = \frac{i}{\hbar}[H_{cav}, \delta n + \delta N] = 0. \quad (\text{S50})$$

Using the identity $[A, BC] = [A, B]C + B[A, C]$ cyclically, the unitary temporal evolution gives also zero values for the powers of the $\delta n + \delta N$ operator at all t , which means that all moments of this operator are also zero at all t . If one includes the cavity photon dissipation into the picture, the Lindblad-type evolution gives:

$$\frac{d(\delta n + \delta N)}{dt} = -2\kappa \delta n. \quad (\text{S51})$$

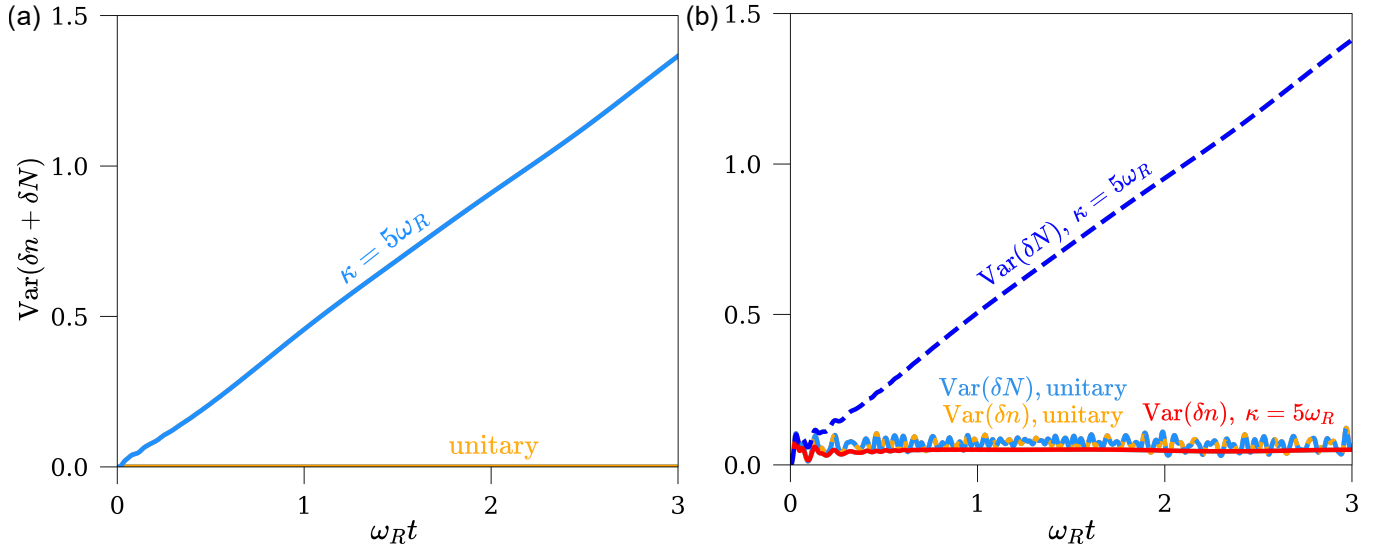


FIG. S2. Temporal evolution of the variances for the unitary and dissipative case with $\kappa = 5\omega_R$ for the ring cavity Hamiltonian H_{cav} . (a) Variance of $\delta n + \delta N$ for the unitary (orange) and $\kappa = 5\omega_R$ case (blue). (b) Variance of δn for the unitary (orange) and $\kappa = 5\omega_R$ (red) case, along with the variance of δN for the unitary (light blue) and $\kappa = 5\omega_R$ (dark blue) case. Simulation parameters: $N = 8$, $(\eta, \bar{\Delta}_c, \bar{\Delta}'_c, U_0) = (40, 110, -45, 10)\omega_R$, with $\hbar = 1$.

As $\langle \delta n \rangle = 0$ for all t due to the parity symmetry of the Hamiltonian, the $\langle \delta n + \delta N \rangle$ will also vanish in the dissipative case. For the variance, we look at the evolution of $(\delta n + \delta N)^2$, described by:

$$\frac{d(\delta n + \delta N)^2}{dt} = 2\kappa(n_+ + n_-) - 4\kappa(\delta n + \delta N)\delta n. \quad (\text{S52})$$

The expectation values of the operators on the right hand side no longer vanish, which means $\langle (\delta n + \delta N)^2 \rangle$ no longer vanishes, leading to a nonvanishing variance of $\delta n + \delta N$ in the dissipative case.

Fig. S2a) shows the behavior of the $\delta n + \delta N$ variances in the case of unitary and dissipative evolution. In the unitary case, the variance vanishes (see Eq. (S50)), while in the dissipative case the variance increases almost linearly, indicating that the random dissipation of photons from the cavity increases the overall noise for both the photonic and atomic variables.

The temporal evolution of the δn and δN operators is correlated due to $[H_{cav}, \delta n] = -[H_{cav}, \delta N]$. In the unitary case the evolution of the δn and δN operators is related by:

$$\frac{d\delta n}{dt} = \frac{i}{\hbar}[H_{cav}, \delta n] = -\frac{i}{\hbar}[H_{cav}, \delta N] = -\frac{d\delta N}{dt}, \quad (\text{S53})$$

which, upon integration over t , leads to $\delta n = -\delta N$ for all t . This in turn means that $\langle \delta n \rangle^2 = \langle \delta N \rangle^2$ for all t , leading to equality of the variances of δn and δN for all t in the unitary case (see Fig. S2b)). Smaller η 's lead to smaller $\langle n_0 \rangle$, $\langle n_{\pm} \rangle$ and δn , meaning that the J_z variance in the unitary case will be reduced for a smaller number of photons in the cavity.

For the dissipative case, the δn evolves as:

$$\frac{d\delta n}{dt} = \frac{i}{\hbar}[H_{cav}, \delta n] - 2\kappa\delta n, \quad (\text{S54})$$

while

$$\frac{d\delta N}{dt} = \frac{i}{\hbar}[H_{cav}, \delta N] = -\frac{d\delta n}{dt} - 2\kappa\delta n. \quad (\text{S55})$$

For the variances in the dissipative case we look at the evolution of $(\delta n)^2$, given by:

$$\frac{d(\delta n)^2}{dt} = \frac{i}{\hbar}[H_{cav}, (\delta n)^2] + 2\kappa(n_+ + n_-) - 4\kappa(\delta n)^2, \quad (\text{S56})$$

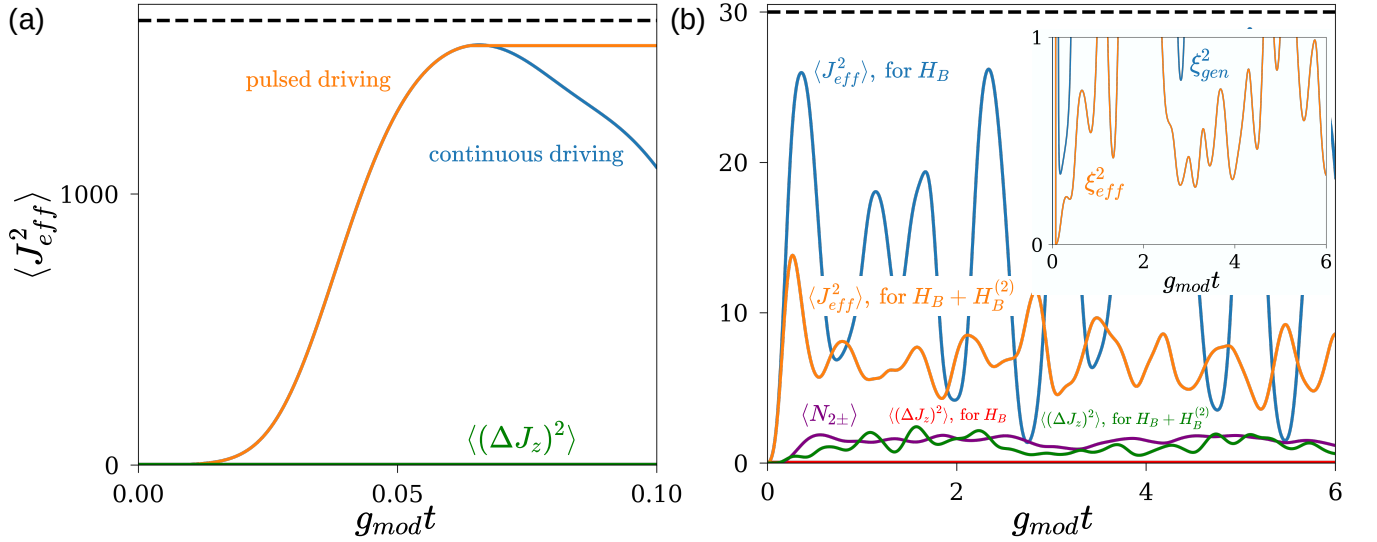


FIG. S3. Simulation results for unitary evolution with H_B and $H_B + H_B^{(2)}$. (a) Temporal evolution of $\langle J_{eff}^2 \rangle$ for pulsed driving switched off at $t_{OFF} = 0.064/g_{mod}$ (orange) and continuous driving (blue) for $N = 80$ atoms, see text. The green line represents $\langle (\Delta J_z)^2 \rangle$ for both cases. (b) Influence of adding the second order terms $H_B^{(2)}$ on the generation of entangled states for $N = 10$ atoms. Temporal evolution of the variables $\langle J_{eff}^2 \rangle$ (blue for H_B , orange for $H_B + H_B^{(2)}$), $\langle (\Delta J_z)^2 \rangle$ (red for H_B , green for $H_B + H_B^{(2)}$) and $\langle N_{2\pm} \rangle$ (purple), where $N_{2\pm} = a_{2\pm}^\dagger a_{2\pm}$. Inset: Temporal evolution of ξ_{gen}^2 (blue) and ξ_{eff}^2 (orange) for the Hamiltonian $H_B + H_B^{(2)}$. Horizontal dashed black line in both plots represents the largest achievable value of $\langle J_{eff}^2 \rangle$.

and the evolution of $(\delta N)^2$, given by

$$\frac{d(\delta N)^2}{dt} = \frac{i}{\hbar} [H_{cav}, (\delta N)^2] = -2 \left[\frac{d\delta n}{dt} + 2\kappa\delta n \right] \delta N. \quad (\text{S57})$$

where we have again used $[A, BC] = [A, B]C + B[A, C]$. The variances of δn and δN will no longer be equal at all t in the dissipative case, although the temporal evolution of these operators is still coupled.

In Fig. S2b) we plot the evolution of δn and δN variances for the unitary and dissipative cases. As expected from Eq. (S53), the variances of δn and δN are equal for the unitary case, and not equal for the dissipative case, with the variance of δN increasing almost linearly with time, and the δn variance fluctuating around a small constant value. The reason for a relatively low δn variance at the used parameters is the low intracavity photon number (see e.g. Fig. S5), occurring due to the laser-cavity detunings $\bar{\Delta}_c$, $\bar{\Delta}_c'$ being much larger in absolute value than the cavity linewidth.

From Eq. (S57) it is also clear that when δn is a zero matrix, which happens e.g. when there is no light in the cavity, the variance of δN stays constant. This fact can be used to create steady state motional Dicke squeezing by turning off the pump drive at a suitable time (see Fig. S4).

STEADY STATE DICKE ENTANGLEMENT GENERATION

In order to generate Dicke entanglement in the steady state, we move away from continuous wave (cw) driving and instead apply a temporally tailored driving field, given by a single square pulse amplitude starting instantaneously at $t = 0$ and switching off instantaneously at $t = t_{OFF}$, where t_{OFF} is the time for which strongest Dicke state entanglement is observed.

For unitary evolution with $N = 80$ atoms under H_B , this switch off time is $t_{OFF} = 0.064/g_{mod}$, as shown in Fig. S3a). After switching off the driving g_{mod} , the $\langle J_{eff}^2 \rangle$ stays near its maximum value (black dashed line) with $\langle (\Delta J_z)^2 \rangle = 0$, meaning the system is very near the ideal Dicke state. Note that we have here neglected the fact that a square-shaped driving will have in the spectrum the Fourier components with frequencies not equal to ω_{mod} . The undesired part of the driving spectrum can however be readily suppressed by using a suitable window function for the pulse shape. Indeed, apparently steady state sideband momentum distributions with peaks at discrete momentum states at a ring of radius $\hbar k_f$ were measured after rapidly switching off the magnetic field driving in the 2D experiment of Ref. [S1].

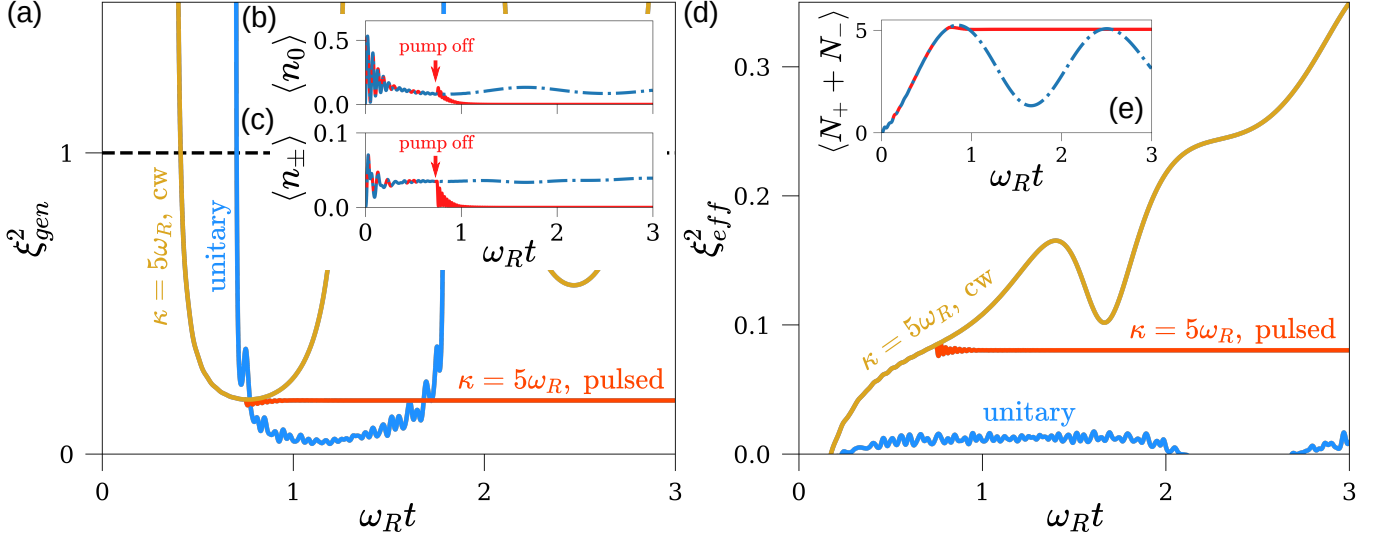


FIG. S4. (a) Temporal evolution of the ξ_{gen}^2 for the unitary (blue), dissipative cw (yellow) and pulsed (orange) cases with the largest entanglement seen in our simulations. The ξ_{gen}^2 reaches negative values, for which the $\xi_{gen}^2 < 1$ criterion is not valid, for $\langle J_{eff}^2 \rangle < N/2$, so these points were excluded from the plot. The temporal evolution of (b) $\langle n_0 \rangle$ and (c) $\langle n_{\pm} \rangle$ for the cw pump (blue, dot-dashed) and the square-pulse pump (red, solid). (d) Temporal evolution of ξ_{eff}^2 for the 3 cases shown in (a). (e) Temporal evolution of the number of sideband atoms for cw driving in the cw pumped case (blue, dot-dashed) and square-pulse pump (red, solid). Simulation parameters: $N = 8$, $(\bar{\Delta}_c, \bar{\Delta}'_c, U_0) = (110, -45, 10)\omega_R$, with $\hbar = 1$. Unitary case: $\eta = 40\omega_R$, dissipative case: $\eta = 50\omega_R$, $\kappa = 5\omega_R$. The pulse starts at $t = 0$ and is turned off instantaneously at $t_{OFF} = 0.75/\omega_R$.

For the dissipative evolution with $\kappa = 5\omega_R$, $N = 8$ atoms under H_{cav} , this switch off time is $t_{OFF} = 0.75/\omega_R$, see Fig. S4a). After switching off the laser pump, the light field inside the cavity drops to zero in a time $\sim 1/\kappa$ (see Fig. S4b,c). In contrast, the atoms are left with a given momentum state distribution, and their kinetic energy at a constant value, as the light-matter interaction vanishes and there are no other channels for energy exchange in the system, since interatomic collisions and atom losses are neglected in our model. Indeed, ξ_{gen}^2 stays at a steady-state value of 0.18, see Fig. S4a), which is also the lowest value attained for continuous driving. The lowest value attained in our simulations for the unitary case is $\xi_{gen}^2 = 0.03$.

To compare to the experimental results of Ref. [S18], we introduce ξ_{eff}^2 , given by:

$$\xi_{eff}^2 = \frac{(\langle N_+ + N_- \rangle - 1)\langle (\Delta J_z)^2 \rangle}{\langle J_{eff}^2 \rangle - \langle N_+ + N_- \rangle / 2}, \quad (\text{S58})$$

where the number of atoms in the sideband Dicke state has been postselected [S21], by replacing N with $\langle N_+ + N_- \rangle$ in ξ_{gen}^2 . As shown in Fig. Fig. S4e), the population of sideband atoms $\langle N_+ + N_- \rangle$ initially increases and then oscillates in time for cw pumping, while for the square pulse the number stays at a constant value of $\langle N_+ + N_- \rangle \approx 5$. For the square pulse case, the effective Dicke entanglement stays in our model at a constant value of $\xi_{eff}^2 = 0.08$ (-11 dB), which is comparable to -11.4 dB of Ref. [S18], see Fig. S4d).

For both models the random collisions of the atoms in a BEC were neglected. The influence of such random collisions on the lifetime of motional state multiparticle entanglement in a BEC is currently under investigation.

SUPPRESSION OF HIGHER ORDER SIDEBAND EXCITATION

Note that in writing the ansatz (S2), we have neglected the Fourier components with spatial periodicity of $\Lambda_c/2$, $\Lambda_c/3$, ... in the atomic field, as was also done e.g. in Refs. [S12, S22]. In a transversely pumped Fabry-Perot cavity, the occurrence of these higher order components in $\psi(x)$ is suppressed by the excitation of effectively only a single longitudinal optical mode. This leads to a preference of the atoms to bunch into the optical lattice with the spatial periodicity set by the longitudinal cavity mode. Similarly, in the setup for H_{cav} , the suppression of higher order Fourier components of $\psi(x)$ is done by Fourier filtering of the intracavity light, which again leads to the preference for formation of an atomic field with periodicity Λ_c . Indeed, taking the self-organized potential depth in our calculations to be equal to $\hbar U_0 \langle n_+ + n_- \rangle \approx 1\hbar\omega_R$ (see Fig. S5i)), and the atomic kinetic energy to be

$\hbar\omega_R(N_+ + N_-) \approx 5\hbar\omega_R$ (see Fig. S4e)), the ratio of the potential depth to kinetic energy is smaller than unity, which indicates weak localization of the atoms [S23]. This relatively shallow potential means that the atomic field should not be deformed with respect to the shape of the self-organized optical lattice, and the Eq. (S2) should be a good ansatz for the atomic field operator.

In contrast, the excitation of higher order sidebands in the experiment of Ref. [S1] is precluded only for $\langle N_0 \rangle \gg \langle N_{\pm} \rangle$. To clarify this, we first write the second order atomic field operator as:

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{V}} \left(b_0 + b_+ e^{ik_f x} + b_- e^{-ik_f x} + b_{2+} e^{2ik_f x} + b_{2-} e^{-2ik_f x} \right). \quad (\text{S59})$$

Integration of the corresponding interatomic scattering term of the Hamiltonian, $\int d^3r \psi(\mathbf{r})^\dagger \psi(\mathbf{r})^\dagger \psi(\mathbf{r}) \psi(\mathbf{r})$, over the volume of the cloud, yields here the terms containing the second order sideband operators:

$$b_{2+}^\dagger b_{2+}^\dagger b_{2-} b_{2-} + b_{2-}^\dagger b_{2-}^\dagger b_{2+} b_{2+} + 2b_0^\dagger b_0^\dagger b_{2+} b_{2-} + 4b_{2+}^\dagger b_{2-}^\dagger b_{2+} b_{2-} + 4b_{2+}^\dagger b_{2+}^\dagger b_{2+} b_{-} + 4b_0^\dagger b_{2+}^\dagger b_{2+} b_{-} \quad (\text{S60})$$

$$+ 4b_{2+}^\dagger b_{2+}^\dagger b_{2+} b_{+} + 4b_{2+}^\dagger b_{2+}^\dagger b_{2+} b_{2-} + 4b_0^\dagger b_{2+}^\dagger b_0 b_{2+} + 2b_{2+}^\dagger b_{2+}^\dagger b_0 b_{2+} + 2b_0^\dagger b_{2+}^\dagger b_{2+} b_{+} + 2b_{2+}^\dagger b_{2-}^\dagger b_0 b_0 \quad (\text{S61})$$

$$+ 4b_{2+}^\dagger b_{2-}^\dagger b_{+} b_{-} + 4b_{2+}^\dagger b_{2+}^\dagger b_{+} b_0 + 4b_0^\dagger b_{2+}^\dagger b_{+} b_{2-} + 4b_{2+}^\dagger b_{2-}^\dagger b_{-} b_{2-} + 4b_0^\dagger b_{2-}^\dagger b_0 b_{2-} + 2b_{2+}^\dagger b_{2+}^\dagger b_0 b_{2-} \quad (\text{S62})$$

$$+ 4b_{2+}^\dagger b_{2-}^\dagger b_{+} b_{2-} + 2b_0^\dagger b_{2-}^\dagger b_{-} b_{-} + 4b_{2+}^\dagger b_{2-}^\dagger b_0 b_{-}. \quad (\text{S63})$$

The dispersion for momentum states is $\varepsilon(k) = \hbar^2 k^2 / 2m$, which gives for the single excitation states $\varepsilon_{\pm} = \hbar^2 k_f^2 / 2m = \varepsilon$, while for the double excitation states one has $\varepsilon_{2\pm} = 4\hbar^2 k_f^2 / 2m = 4\varepsilon$. Transforming to the rotating frame with $b_{\pm} \rightarrow b_{\pm} e^{-i\varepsilon t / \hbar}$, $b_{2\pm} \rightarrow b_{2\pm} e^{-4i\varepsilon t / \hbar}$, and keeping only the terms resonant with the driving $\pm \hbar\omega_{mod} = 2\varepsilon$, leads now to the second order Hamiltonian:

$$H_B^{(2)} = i\hbar g_{mod} (b_0^\dagger b_{2+}^\dagger b_{+} b_{+} + b_0^\dagger b_{2-}^\dagger b_{-} b_{-}) + \text{H.c.} \quad (\text{S64})$$

It is clear from the Eq. (S64) that the higher order sideband terms are small with respect to the first order terms only if the number of single sideband excitations is much smaller than the number of ground state atoms, i.e. $\langle N_0 \rangle \gg \langle N_{\pm} \rangle$, which is satisfied for the weakly excited (collisionally thin) medium of Ref. [S1].

In the results plotted in Fig. S3b), we numerically demonstrate that the second order terms increase the ξ_{gen}^2 parameter for the two first order sidebands, by increasing $\langle (\Delta J_z)^2 \rangle$ and reducing $\langle J_{eff}^2 \rangle$. However, the ξ_{eff}^2 parameter still reaches quite small values (at short times after switching on the driving). We note here that it may turn out to be possible to filter out the higher order excitations in experiment, e.g. by judiciously tailoring the $a(t)$ or by removing the atoms in the second order modes from the self-organized lattice, which is a highly intriguing topic for future research, given the very large values of $\langle J_{eff}^2 \rangle$ reached for the results of Fig. 2 of the main text.

EXPERIMENTAL DESIGN OF THE RING CAVITY SETUP

To estimate the attainability of Dicke squeezing for realistic parameters, we first note the experimentally available κ values of $2\pi \times 0.13$ MHz [S24] and 2π MHz [S25]. Also, in state of the art experiments, cloud sizes for cigar-shaped BECs can be on the order of $\sim 250 \mu\text{m}$ [S26], which limits Λ_c to values between approximately $1\text{-}50 \mu\text{m}$. On the short side, Λ_c is limited by numerical aperture of the light collection system, while on the long side it is limited by the requirement of having at least a few periods of the transverse pattern in a cloud.

Tuning the sideband frequency ω'_0 by varying effective cavity length L

The cavity configuration envisaged in Fig. S1a) consists of an afocal telescope in a ring cavity, such that the front focal plane of the right lens is a distance L apart from the back focal plane of the left lens. This enables the tuning of the diffractive length of the cavity independent of the physical cavity length and thus enables flexibility with regard to distances of the optical elements to the BEC. In the focal plane between the two lenses, the spatial Fourier spectrum of the intra-cavity field at the position of the BEC is available for spatial filtering. The filter is transmitting the on-axis mode and the off-axis spatial sidebands at a particular transverse wavenumber q_c . The schematic with two intracavity lenses is conceptually the simplest, however, to minimize Fresnel losses the system may be implemented using curved mirrors (see below).

The sideband cavity mode wavevector \mathbf{k}'_0 (with length $k'_0 = 2\pi/\lambda'_0$) is related to the on-axis cavity mode wavevector $\mathbf{k}_0 = k_0 \hat{\mathbf{z}}$ (with length $k_0 = 2\pi/\lambda_0$) via the relation $\mathbf{k}'_0 = \mathbf{k}_0 + \mathbf{q}_c$, where $\mathbf{q}_c = q_c \hat{\mathbf{x}}$ (with length $q_c = 2\pi/\Lambda_c$) is the transverse component of the sideband mode wavevector, which is in our case selected by Fourier filtering (see Fig. S1a)). For a ring cavity without

the intracavity lenses, the dispersion relation of the on-axis modes is $\omega_0 = ck_0$, while for the sideband with q_c it is $\omega'_0 = ck'_0 = c(k_0^2 + q_c^2)^{1/2} = \omega_0(1 + q_c^2/k_0^2)^{1/2}$.

For the cavity with two intracavity lenses of same focal length f , the diffractive length is given by L (see Fig. S1a), where L is also the combined distance of the two lenses from the $4f$ configuration. In both the cavity with and without the intracavity lenses, the phase difference between the selected sideband mode and the on-axis mode after one round trip of duration L_{cav}/c through the cavity is given by $\delta\phi = (\omega'_0 - \omega_0)L_{cav}/c$, where L_{cav} is the length of the cavity. On the other hand, the diffraction of the sideband mode through the cavity with a diffractive length L leads, in the paraxial limit (valid for small q_c/k_0), to a diffractive phase shift with respect to the on-axis mode of $\delta\phi \approx q_c^2 L/2k_0 = q_c^2 cL/2\omega_0$, see e.g. Eq. (2) of Ref. [S9]. Equating the two expressions for the phase shift after one round trip through the cavity, the frequencies ω'_0 and ω_0 are in this case related via the relations (valid for small q_c/k_0):

$$\omega_0 \approx \omega'_0 \left(1 - \frac{q_c^2 c^2}{2\omega_0^2} \frac{L}{L_{cav}}\right), \quad \omega'_0 \approx \frac{\omega_0}{2} \left(1 + \sqrt{1 + \frac{2q_c^2}{k_0^2} \frac{L}{L_{cav}}}\right). \quad (\text{S65})$$

The frequency ω'_0 of the cavity mode with transverse wavevector q_c , selected by Fourier filtering, can thus be tuned relative to the on-axis mode frequency ω_0 , by translating the position of the intracavity lenses with respect to the $4f$ configuration, which changes the effective cavity length L . In the case when $L = 0$ ($4f$ condition perfectly satisfied), all sidebands have equal frequencies, see e.g. Ref. [S27]. On the other hand, for $L = L_{cav}$ (no intracavity lenses), we regain the above stated dispersion relation by a Taylor expansion for small q_c/k_0 , which gives $\omega'_0 \approx \omega_0(1 + q_c^2/2k_0^2)$.

We note here that the cavity mirrors might need to be slightly concave to allow intracavity propagation of sidebands with a slight tilt from the cavity axis in a system of finite extent. The stability of such cavities has been demonstrated e.g. in Refs. [S28, S29], where Fabry-Perot cavities were used for experimental measurements of transverse self-organization.

Estimating the threshold for realistic parameter values

To start, we take the relevant on-axis mode of the ring cavity to be detuned by $\Delta_a = 2\pi \times 50$ GHz from the ^{87}Rb D2 line ($\omega_{Rb} = 2\pi \times 384.23$ THz, $\lambda_{Rb} = 780$ nm, transition linewidth $\Gamma = 2\pi \times 6.066$ MHz, recoil frequency $\omega_r = 2\pi \times 3.77$ kHz). Taking the cavity length of $L_{cav} = 20$ cm and the effective cavity length of $L = 0.18$ mm (i.e. very near the perfect $4f$ condition), the frequency difference of the on-axis mode and the sideband with $\Lambda_c = 30 \mu\text{m}$ is equal to $\omega'_0 - \omega_0 = 2\pi \times 116.89$ MHz, which is an order of magnitude smaller than the free spectral range of this ring cavity, given by $2\pi \times c/L_{cav} = 2\pi \times 1.5$ GHz.

In deriving H_{cav} we approximate the atom-cavity coupling g_0 for all three cavity modes to be equal, which is a good approximation in this case, as the mode frequencies and their volumes at the location of the atoms are approximately equal. Allowing for a reduction of the cavity finesse by the intracavity optics in the $4f$ configuration ("bad cavity" regime), we take $\kappa = 2\pi \times 100$ MHz and $g_0 = 2\pi \times 80$ kHz, and get $U_0 = g_0^2/\Delta_a = 2\pi \times 0.128$ Hz.

The transverse pattern recoil frequency ω_R can be related to the optical transition recoil frequency ω_r via the relation $\omega_R = \omega_r \lambda_{Rb}^2/\Lambda_c^2 = 2\pi \times 2.55$ Hz, and we take the number of atoms to be $N = 5 \times 10^5$. Finally, tuning the laser frequency at $\bar{\Delta}'_c = -\kappa$ (with $\Delta'_c = -2\pi \times 99.94$ MHz), we get $\bar{\Delta}_c = 2\pi \times 16.89$ MHz, and the critical pump rate of $\eta_c = 2\pi \times 12.65$ GHz.

Following Ref. [S30], the detuned saturation parameter of the intracavity beam is given by $s_\Delta = I/I_s/[1 + (\Delta_a/\Gamma)^2] = g_0^2 \langle n_0 \rangle / (\Delta_a^2 + \Gamma^2)$, which gives for the threshold intracavity zero-order light intensity $I_c = g_0^2 N |\alpha_0^c|^2 I_s / \Gamma^2 = 6.76$ mW/cm². The number of zero-order photons per atom at threshold can be estimated from Eq. (S30) to be equal to $|\alpha_0^c|^2 = \langle n_0 \rangle / N = \eta_c^2 / N / (\bar{\Delta}_c^2 + \kappa^2) = 0.031$, which is comparable to the results in Fig. S5.

We note at the end of this section that it may be possible to avoid the "bad cavity" parameters in a state-of-the-art experiment, by replacing the intracavity lens-mirror pairs by suitably curved mirrors. This would reduce the scattering losses caused by finite reflectivity of the lens elements, and thus lead to an increase of the cavity finesse as compared to the system with intracavity lenses, which should benefit both the reduction of κ and the increase of g_0 . There will be other experimental challenges which will need consideration and engineering efforts, such as the losses at the Fourier filtering stage and the finite extension of the medium and beam.

LIMITATIONS FOR THE NUMERICAL SIMULATIONS WITH H_{cav}

When solving the Schrödinger and master equations for H_{cav} numerically, we truncate the infinite-dimensional Fock space of the photonic degrees of freedom into a finite-sized Fock space. This limits the maximal pump strength η that can be used in our simulations, as higher pump rates will naturally lead to larger $\langle n_0 \rangle$ and $\langle n_\pm \rangle$ values, such that higher dimensional photonic Fock

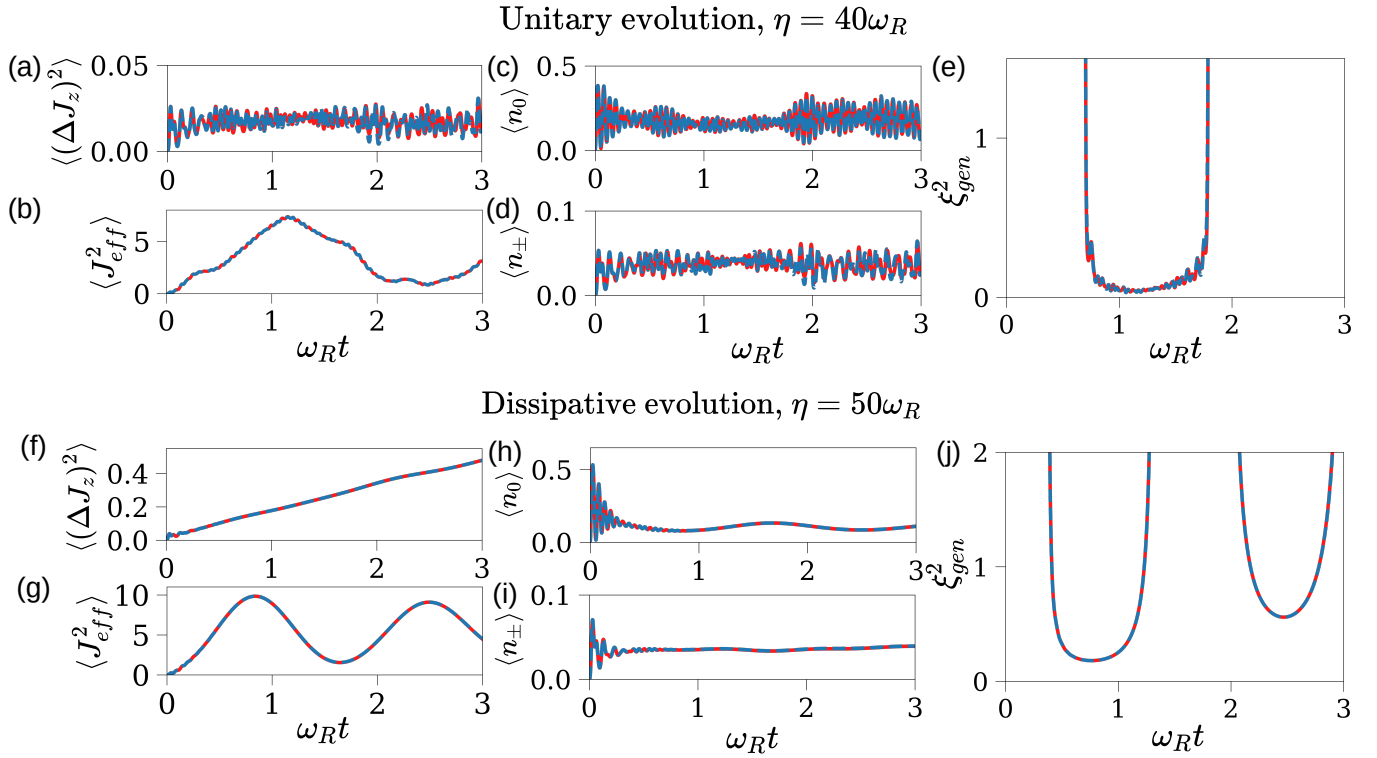


FIG. S5. Comparison of the temporal evolution under H_{cav} of relevant system observables for the case when the state with maximal photon number in all modes is $|4\rangle_0|3\rangle_+|3\rangle_-$ (red, solid) and the case when the state with maximal photon number in all modes is $|5\rangle_0|3\rangle_+|2\rangle_-$ (blue, dashed), see text for details. The evolution of observables for the unitary case with $\eta = 40\omega_R$: (a) $\langle(\Delta J_z)^2\rangle$, (b) $\langle J_{eff}^2\rangle$, (c) $\langle n_0\rangle$, (d) $\langle n_{\pm}\rangle$ and (e) ξ_{gen}^2 . The evolution of observables for the dissipative case with $\eta = 50\omega_R$: (f) $\langle(\Delta J_z)^2\rangle$, (g) $\langle J_{eff}^2\rangle$, (h) $\langle n_0\rangle$, (i) $\langle n_{\pm}\rangle$ and (j) ξ_{gen}^2 . Simulation parameters: $N = 8$, $(\bar{\Delta}_c, \bar{\Delta}'_c, U_0, \kappa) = (110, -45, 10, 5)\omega_R$, with $\hbar = 1$.

spaces are needed to correctly capture the system dynamics. The limits of the calculations are in our case set by the size of the working memory of the computational nodes.

To demonstrate the numerical accuracy of the plots in the main text, we have compared the results for the dynamical evolution of the relevant observables for the case where the maximal photon number state in all modes is $|4\rangle_0|3\rangle_+|3\rangle_-$, to the case with the same total maximal photon number but where the state with the maximal photon number in all modes is $|5\rangle_0|3\rangle_+|2\rangle_-$, both at $N = 8$ atoms, and at largest η used in the unitary and dissipative cases.

In Fig. S5 we compare the two Fock space truncations by plotting the temporal evolution of relevant variables for the maximum η values used in the unitary ($\eta = 40\omega_R$) and dissipative ($\eta = 50\omega_R$) cases of the main text. The plots confirm that the truncation of Fock space where the state with the maximal number of photons in each mode is $|4\rangle_0|3\rangle_+|3\rangle_-$, accurately describes the dynamics in the simulated time interval.

In the unitary case the evolution of both the atomic and photonic variables exhibit fast oscillatory motion, a signature of Vacuum Rabi oscillations occurring in the quantum electrodynamic treatment of light-matter interaction of atoms in a cavity [S31]. In the dissipative case with $\kappa = 5$, these fast oscillations are averaged out. However, the slow oscillations, a signature of the sloshing dynamics where atoms slosh around the minima of the dynamical potential and periodically amplify the transverse patterns [S13], still persist.

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