SIMPLIFIED MODEL FOR THE STIFFNESS OF SUCTION CAISSON FOUNDATIONS UNDER 6 DOF LOADING

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Abstract
Suction caisson foundations are increasingly used as foundations for offshore wind turbines. This paper presents a new, computationally efficient, model to determine the stiffness of caisson foundations embedded in linearly elastic soil, when subjected to six degree-of-freedom loading; vertical (V), horizontal (H_x, H_y), overturning moment (M_x, M_y) and torsion (T). This approach is particularly useful for fatigue limit analyses, where the constitutive behaviour of the soil can be modelled as linearly elastic. The paper describes the framework on which the new model is based and the 3D finite element modelling required for calibration. Analyses conducted using the proposed approach compare well with results obtained using 3D finite element analysis. The possibility of low-cost analysis, coupled with a simple calibration process, makes the proposed design method an attractive candidate for intensive applications such as foundation design optimisation.

1. Introduction

Despite the current dominance of monopile foundations for offshore wind turbines, there is increasing interest in deploying suction caisson (or suction bucket) foundations (e.g. for jacket structures) for offshore wind farms located in deeper waters due to economic advantages. Once installed, the caisson foundations will experience vertical (V), horizontal (H_x, H_y), overturning moment (M_x, M_y) and torsional (T) loading during normal operations.

Although the ultimate capacity of the foundation is important, the general operation of a wind turbine means assessment of the dynamic and fatigue performance of the foundation and structure is particularly important. For such assessments, the soil response can be approximated as linear elastic, as the applied loads are within the lower ends of the expected range during the lifetime. Care is needed, however, in the selection of appropriate soil stiffness parameters for use in these assessments.

For a linear elastic soil, it is known from previous research (Doherty et al., 2005) that the resultant forces (H_x, H_y, V, M_x, M_y, T) acting on a caisson foundation are related to the displacements (U_x, U_y, U_z, Θ_x, Θ_y, Θ_z) through global stiffness coefficients, as shown in Eq. 1. K_V, K_H, K_M, K_T and K_C are the vertical, lateral, rotational, torsional and lateral-rotational coupling stiffness respectively.

$$\begin{Bmatrix} H_x \\ H_y \\ V \\ M_x \\ M_y \\ T \end{Bmatrix} = \begin{bmatrix} K_H & 0 & 0 & 0 & -K_C & 0 \\ 0 & K_H & 0 & K_C & 0 & 0 \\ 0 & 0 & K_V & 0 & 0 & 0 \\ 0 & K_C & 0 & K_M & 0 & 0 \\ -K_C & 0 & 0 & 0 & K_M & 0 \\ 0 & 0 & 0 & 0 & 0 & K_T \end{bmatrix} \begin{Bmatrix} U_x \\ U_y \\ U_z \\ \Theta_x \\ \Theta_y \\ \Theta_z \end{Bmatrix} \tag{1}$$

The main design challenge is that a large number of analyses are required for fatigue assessments. Unlike the small number of caisson foundations used for bespoke offshore structures in oil and gas projects, a typical new offshore wind farm may have hundreds of such foundations (Byrne et al., 2015). Optimisation of the caisson foundations for this new application therefore requires design methods that are both fast and reliable.

Unfortunately, existing design methods for assessing stiffness of suction caisson foundations under low operational loads are limited either by their efficiency or the level of detail of soil profiles that can be modelled. For example, some methods are not efficient enough to handle the large number of analyses required for fatigue design while others are applicable only for relatively simple ground profiles. There is clearly a need for new design methods that are robust, fast and general enough to handle the widely varying heterogeneities in most real-world ground profiles. This paper sets out a new design method that addresses this need, and which can be applied to large scale projects that require optimisation, such as offshore wind farms.

2. Existing Design Methods

2.1 Macro Element model

The macro element model (e.g. Doherty et al., 2005) represents the caisson foundation as a single element, where the behaviour is described purely in terms of the resultant forces acting on it and the corresponding displacements. In other words, this model directly provides the stiffness coefficients in Eq. 1. This model has several key advantages such as computational efficiency and easy integration with most structural analysis programs.

However, there are some notable limitations. First, the calibration process for this model is cumbersome as a different set of stiffness coefficients is required for every unique combination of soil and caisson stiffness. Second, this model is accurate only for soils where the stiffness increases continuously with depth. As most ground conditions encountered in practice involve layered soils, using simplified soil profiles may introduce significant errors. This is especially true when there is a stiff layer overlying softer layers (Suryasentana et al., 2017).

2.2 3D Finite Element (3D FE) method

The 3D FE method is a rigorous design method and is often the standard against which other design methods are benchmarked. It can provide accurate stiffness predictions for complex ground profiles, soil constitutive behaviour and structural geometries. However, it is limited by the high computational cost and modelling complexities, relative to other design methods. It is generally unsuitable for the design and optimisation of foundations in large scale projects such as for an offshore wind farm.

2.3 Winkler model

Winkler based models have been used successfully for the design of deep foundations such as monopiles (API, 2010; DNV, 2014). More recently, this approach has been applied to shallow foundations (e.g. Houlsby et al., 2005; Gerolymos & Gazetas, 2006). In this modelling approach, the soil continuum is represented by a series of independent springs, each of which captures the local soil reaction. This approach has several key advantages. Similar to the macro element model, it is computationally efficient and easily coupled with structural analysis programs. However, a major advantage over the macro element model is the localised nature of the soil reactions, which allows models that are based on the Winkler approach to be used for any type of non-homogeneous elastic soil, including layered soil.

Nevertheless, the Winkler approach is not without limitations. The assumption that the springs are independent ignores the continuum nature of soil. This assumption may introduce significant errors in stiffness predictions for highly heterogeneous ground profiles. Furthermore, an issue with the Winkler model, specific to caisson foundations, is that it is incomplete. The available Winkler models for caisson foundations are limited to lateral loading only (Davidson et al., 1982; Gerolymos & Gazetas, 2006). The current Winkler approaches, unlike other design methods, cannot be readily used to assess the stiffness of caisson foundations under 6 degree-of-freedom (dof) loading.

3. Proposed Design Method

This paper proposes a new Winkler-based model, termed '1D caisson model', to predict the stiffness of caisson foundations in linear elastic soil. Unlike existing Winkler models for caisson foundations, this model is complete and can provide stiffness predictions for 6 dof loading. Moreover, the formulations of the Winkler spring forces, hereafter referred to as 1D soil reactions, are calibrated against rigorous 3D FE solutions. The model offers the speed of the Winkler modelling approach and the accuracy of the 3D FE method.

3.1 Theory

In this model, the global coordinate system is defined at the centre of the suction caisson lid base (i.e. the interface between the lid and the soil medium). Furthermore, this origin is adopted as the point of applied loading (LRP), as shown in Fig. 1.

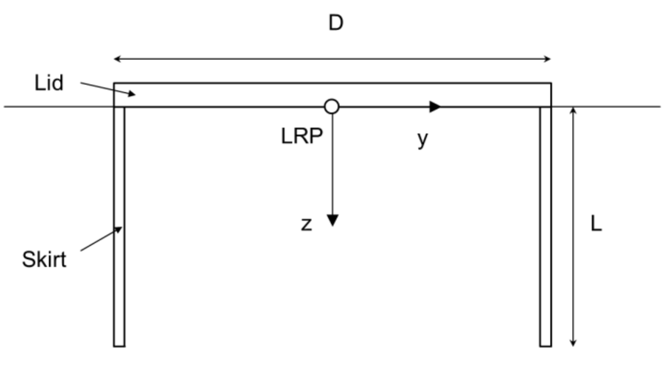


Figure 1: Schematic representation of a caisson foundation and the point of applied loading (D is the caisson diameter, L is the skirt length and z is the depth below ground surface)

The caisson is assumed to be fully rigid and no slip or gap is allowed between the foundation and soil.

For each cross section along the caisson skirt, the 3D soil stresses acting on it can be resolved into 1D soil reactions, which are essentially the resultant soil forces acting on each cross section.

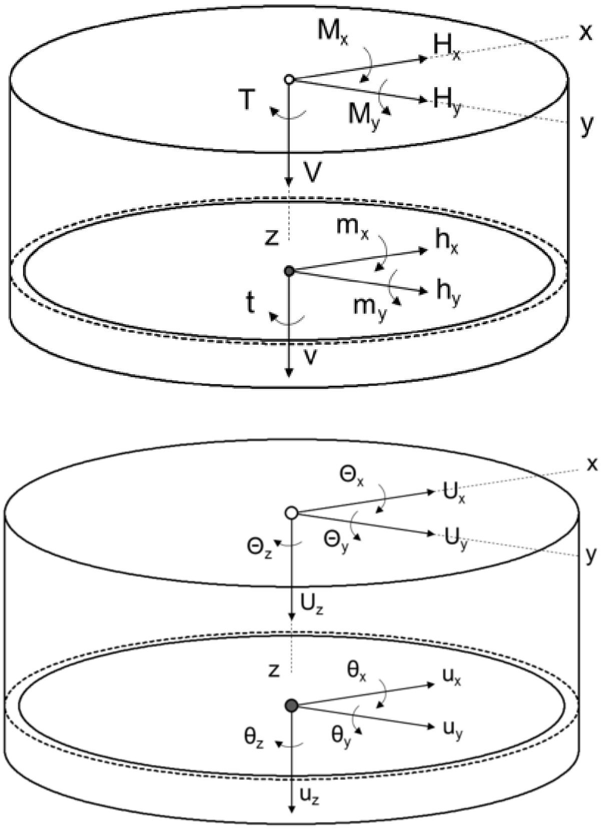


Figure 2: Sign conventions for the global applied loads and dof of the foundation, with respect to the 1D soil reactions and local dof of each cross section

There are six 1D soil reactions (h_x, h_y, v, m_x, m_y, t) and six local dof (u_x, u_y, u_z, θ_x, θ_y, θ_z) associated with each cross section. Figure 2 shows the sign convention for the global and local dof with the relation between the two defined by Eq. 2.

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & z & 0 \\ 0 & 1 & 0 & -z & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_x \\ U_y \\ U_z \\ \Theta_x \\ \Theta_y \\ \Theta_z \end{Bmatrix} \tag{2}$$

Since the caisson skirt can be divided into an infinite number of infinitesimally thin cross sections, the 1D soil reactions acting on it, henceforth known as the skirt 1D soil reactions, would be distributed in nature. Furthermore, there is an additional non-distributed 1D soil reaction acting at the base of the foundation, termed the base 1D soil reaction. This is the resultant force acting across the base cross section, which includes both the skirt tip annulus and the soil plug base.

Fig. 3 shows a schematic diagram of the transfer of the applied vertical load (V) into the respective 1D soil reactions. As shown in Fig. 3a, V is balanced by the soil reactions as follows:

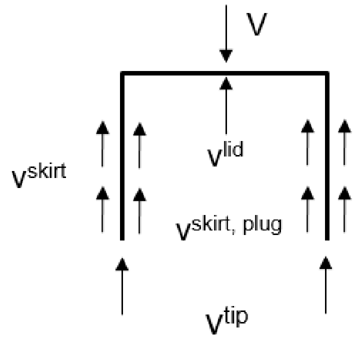
$$V = v^{tip} + \int_0^L v^{skirt} dz + \int_0^L v^{skirt,plug} dz + v^{lid} \tag{3}$$

where v^{tip}, v^{skirt}, $v^{skirt,\, plug}$, v^{lid} are the soil reactions on the skirt tip annulus, skirt exterior, skirt interior and the lid base respectively. The skirt 1D soil reaction is v^{skirt} while the base 1D skirt reaction (v^{base}) is:

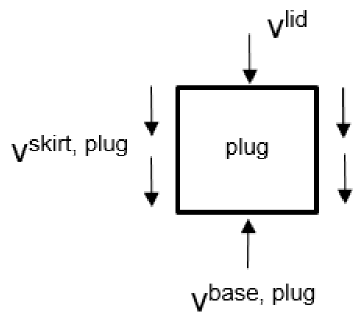
$$v^{base} = v^{tip} + v^{base,plug} \tag{4}$$

where $v^{base,\, plug}$ is the soil reaction on the base of the soil plug. From Fig. 3b, it is shown that:

$$v^{base,plug} = \int_0^L v^{skirt,plug} dz + v^{lid} \tag{5}$$



(a) Force equilibrium between caisson foundation, applied load and local soil reactions



(b) Force equilibrium between internal soil plug and local soil reactions

Figure 3: Transfer of vertical load into the 1D soil reactions

Substituting Eqs. 4 and 5 into Eq. 3 gives:

$$V = \int_0^L v^{skirt} dz + v^{base} \tag{6}$$

Thus, the skirt and base 1D soil reactions complete the set of soil reactions acting on the caisson.

3.2 *Calibration of 1D soil reactions*

To calibrate the 1D soil reactions, the caisson-soil interaction problem is analysed using the 3D FE method. Then, the 3D soil stresses from the adjacent soil elements are resolved into 1D soil reactions.

It is assumed that the 1D soil reactions at each depth, z, depend only on the soil properties at that depth. This assumption implies that the model only needs to be calibrated against the 3D FE results for a homogeneous elastic soil, with the calibrated reactions being applicable to non-homogeneous elastic soil too. This model also assumes that the 1D soil reactions are independent of the caisson stiffness properties. Therefore, the model only needs to be calibrated against a rigid caisson, with the calibrated reactions also applying to caissons with flexible skirts. An examination of these assumptions is not provided here but will be addressed in future work.

To determine the 1D soil reactions, the nodal force results from the 3D FE analyses are used. Specifically, the 1D soil reactions are computed from the contact nodal forces of the soil elements adjacent to the foundation (including the soil plug).

For the skirt 1D soil reactions, contact nodal forces refer to nodal forces from nodes shared by the skirt exterior and surrounding soil elements. For each 'ring' of soil elements in contact with the skirt exterior, the skirt 1D vertical and lateral reactions are computed as the sum of the contact nodal forces in the respective axes, divided by the soil element thickness. The computed value corresponds to the local soil reaction at the depth of the 'ring' of soil elements. For the skirt 1D moment and torsional reactions, the computation involves the sum of the moment induced by each contact nodal force about the centre of the cross section, divided by the soil element thickness.

For the base 1D soil reactions, contact nodal forces refer to nodal forces from nodes shared by the interface between the bases of the internal soil plug and skirt tip annulus and the soil elements directly below them. The base 1D vertical and lateral soil reactions are the sum of the contact nodal forces in the respective axes while the base 1D moment and torsional soil reactions are the sum of the moment induced by each contact nodal force about the centre of the cross section.

Finally, mathematical formulations are derived to approximate these 1D soil reactions; these formulations form the predictive basis of the 1D caisson model.

3.3 *Global stiffness equations*

One advantage of the Winkler assumption is the availability of analytical solutions to derive the global stiffness of a rigid caisson directly from the 1D soil reactions, which are shown in Table 1.

Table 1: Analytical solutions to compute the global stiffness of the foundation directly from the 1D soil reactions. L refers to the caisson skirt length. K_H, K_M and K_C can be similarly defined in terms of h_x and m_y, but with some minor modifications

	Equation
K_V	$\int_0^L \frac{\partial v^{skirt}}{\partial u_z} dz + \frac{\partial v^{base}}{\partial u_z}$
K_H	$\int_0^L \frac{\partial h_y^{skirt}}{\partial u_y} dz + \frac{\partial h_y^{base}}{\partial u_y}$
K_M	$\int_0^L \frac{\partial m_x^{skirt}}{\partial \theta_x} + \frac{\partial m_x^{skirt}}{\partial u_y}(-z) dz + \frac{\partial m_x^{base}}{\partial \theta_x} + \frac{\partial m_x^{skirt}}{\partial u_y}(-L)$ $+ \int_0^L \left(\frac{\partial h_y^{skirt}}{\partial \theta_x} + \frac{\partial h_y^{skirt}}{\partial u_y}(-z) \right)(-z) dz$ $+ \left(\frac{\partial h_y^{base}}{\partial \theta_x} + \frac{\partial h_y^{base}}{\partial u_y}(-L) \right)(-L)$
K_T	$\int_0^L \frac{\partial t^{skirt}}{\partial \theta_z} dz + \frac{\partial t^{base}}{\partial \theta_z}$
K_C	$\int_0^L \frac{\partial m_x^{skirt}}{\partial u_y} + \frac{\partial h_y^{skirt}}{\partial u_y}(-z) dz + \frac{\partial m_x^{base}}{\partial u_y} + \frac{\partial h_y^{base}}{\partial u_y}(-L)$ or $\int_0^L \frac{\partial h_y^{skirt}}{\partial \theta_x} + \frac{\partial h_y^{skirt}}{\partial u_y}(-z) dz + \frac{\partial h_y^{base}}{\partial \theta_x} + \frac{\partial h_y^{base}}{\partial u_y}(-L)$

3.4 *Relation to the approach of Byrne et al. (2015)*

Whilst the 1D caisson model is similar to the PISA design approach for short monopile foundations (Byrne et al., 2015), there are also important differences. First, it provides the 1D soil reactions corresponding to the vertical and torsional dof. Thus, it can handle fully three-dimensional loading.

Second, unlike the PISA approach, this model has coupling between the lateral and rotational dof. A local cross section rotation induces a local lateral soil reaction and a local lateral displacement would induce a local moment soil reaction. This coupling has thus far been ignored by existing Winkler models, such as the *p-y* method for pile foundations (API, 2010; DNV, 2014).

4. Numerical Example

This section illustrates the process of calibrating the 1D soil reactions using the solutions of 3D FE analyses. In this numerical example, a 3D FE model of a caisson foundation embedded in incompressible linear elastic soil was implemented in the finite element program ABAQUS (version 6.13). The global coordinate system adopted in the FE model is the same as defined in Fig. 1.

The foundation has a unit diameter (D), a unit skirt length ($L = D$) and a skirt thickness of $0.0025D$. Mesh convergence analyses were carried out to determine the required mesh fineness. Moreover, a mesh domain of $80D$ for both diameter and depth was found to be sufficient to avoid boundary effects. A typical mesh of the FE model is shown in Fig. 4.

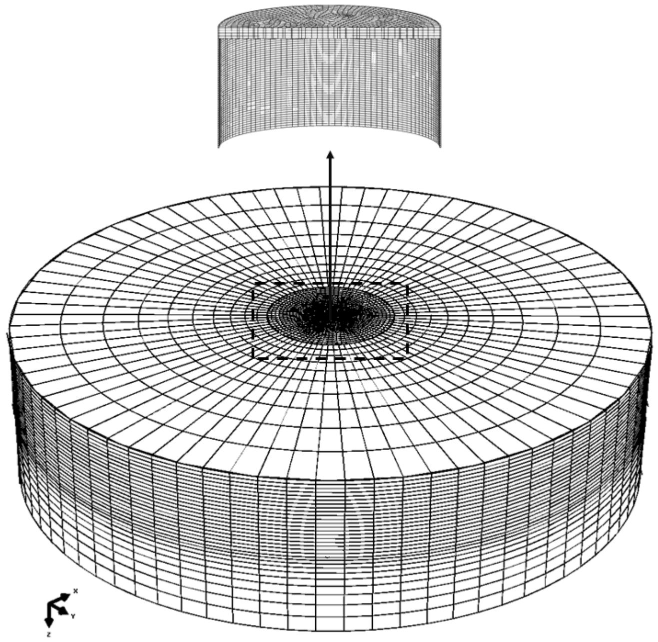


Figure 4: Mesh of the complete 3D FE model, with an enlarged partial view of caisson foundation

Displacements were fixed in all directions at the bottom of the mesh domain and in the radial directions on the periphery. Contact breaking between the foundation and soil was not allowed and this was implemented using tie constraints at the foundation-soil interface.

The soil was weightless and homogeneous isotropic linear elastic. A Young's modulus of 100MPa and a Poisson's ratio of 0.49 was assigned to the soil elements, for which eight-noded linear brick elements C3D8RH (Dassault Systèmes, 2010) were used. The foundation was assumed to be entirely rigid and the rigid behaviour was simulated using rigid body constraints. The reference point was set to be the point of applied loading as defined in Fig. 1.

To fully calibrate the 1D soil reactions, four sets of 3D FE results are required. These four sets of results are obtained from the 3D FE analyses of the caisson foundation under four different prescribed displacements. These prescribed displacements were implemented by applying different boundary conditions to the reference point of the caisson foundation, as detailed in Table 2.

Table 2: Boundary conditions for the four types of prescribed displacements applied in the 3D FE analyses

	U_x/D	U_y/D	U_z/D	Θ_x	Θ_y	Θ_z
Vertical	0	0	0.1	0	0	0
Lateral	0	0.1	0	0	0	0
Rotational	0	0	0	0.1	0	0
Torsional	0	0	0	0	0	0.1

To verify that the 3D FE model has been set up correctly, the normalized global stiffness coefficients resulting from the prescribed displacements are compared against known results from previous work (in this case Doherty et al., 2005), as shown in Table 3.

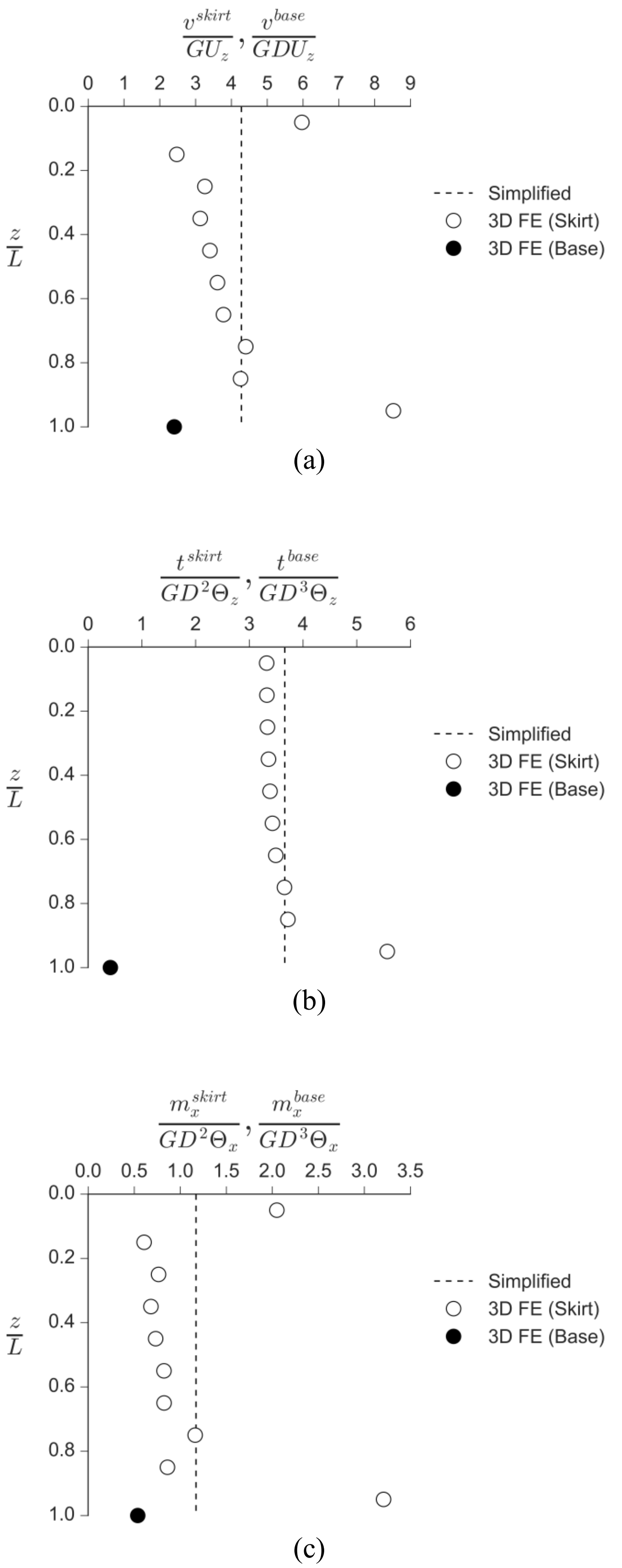
Table 3: Comparison of normalised stiffness coefficients from the 3D FE results and values reported in previous work

Stiffness	Doherty et al. (2005)	3D FE	Difference
K_V/GD	6.64	6.68	0.60 %
K_H/GD	7.54	7.68	1.86 %
K_M/GD^3	7.40	7.11	-3.92 %
K_T/GD^3	4.04	4.07	0.74 %
K_C/GD^2	-4.69	-4.66	-0.64 %

The normalised stiffness coefficients computed by the 3D FE model matched the values reported by Doherty et al. (2005) well, with the maximum deviation being only 3.92%. This provides confidence that the FE model had been set up correctly.

5. Results

Fig. 5 shows the 1D soil reactions profile that resulted from the 3D FE analyses of the four sets of prescribed displacements given in Table 2. Note that the values depicted in Fig. 5 are with respect to the global dof, and not the local dof.



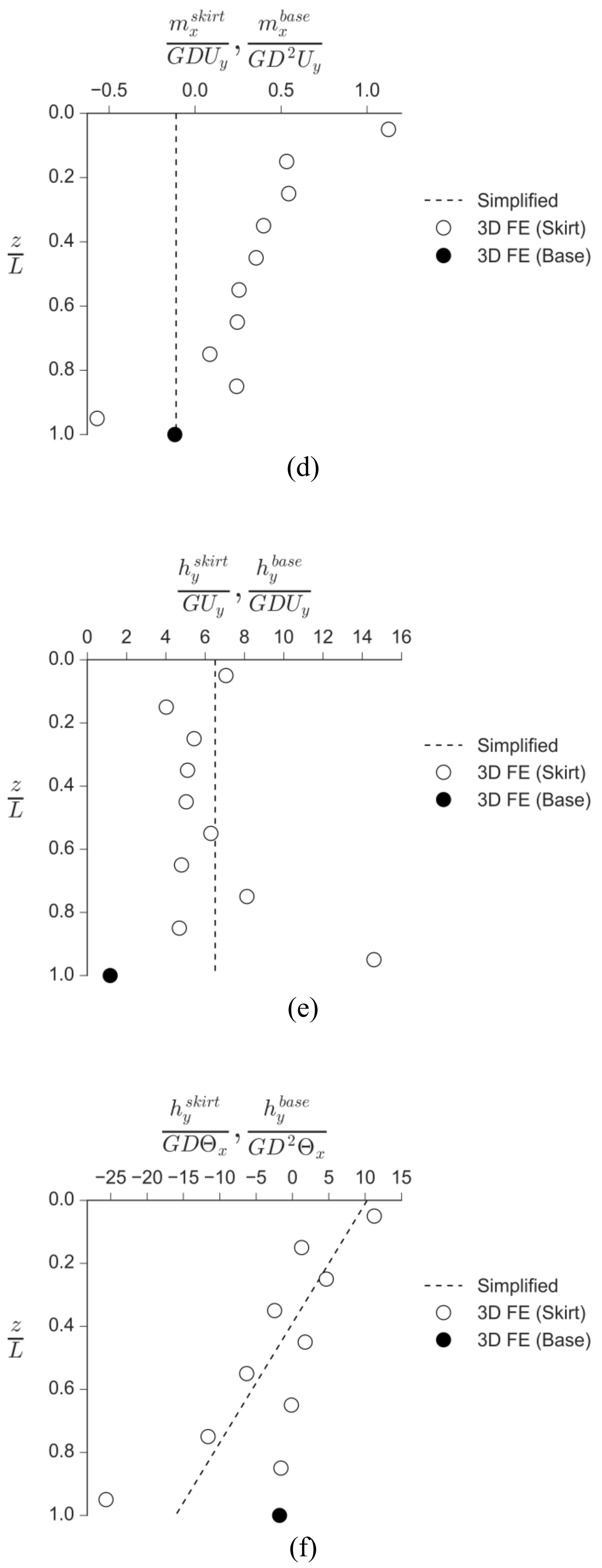


Figure 5: Skirt and base 1D soil reactions from 3D FE results

As shown in the figure, most skirt 1D soil reactions appear to be constant along the skirt length, apart from the reactions nearest to the ground surface and skirt tip. The only exception is the h_y coupling reaction, which changes with depth (see Fig. 5f). This is expected as it is evident from Eq. 2 that a pure global rotation Θ_x would result in local lateral displacements that increase with depth.

Furthermore, Fig. 5d shows that a rigid lateral displacement induces local moment reactions along the skirt and at the base. Most existing Winkler formulations for monopiles (API, 2010) or suction caissons (Gerolymos & Gazetas, 2006) ignore these coupling terms and doing so might introduce errors in reproduction of the 3DFE results.

Next, simplifying approximations were made when deriving the mathematical formulations for these 1D soil reactions. Specifically, all the skirt 1D soil reactions were assumed to be constant along the skirt, apart from the h_y coupling reaction, which varies linearly with depth. This extra complexity is necessary for accurate K_M computations. The constant or linearly varying profiles were found using ordinary least square regression against the true skirt 1D soil reactions and the best fit, simplified skirt 1D soil reaction profiles are shown in Fig. 5.

Table 4 shows the formulations that were derived based on the simplified 1D soil reactions. A two-step process was used to derive the formulations of these reactions. First, these reactions were formulated with respect to the global dof. Thereafter, these formulations were transformed into the local dof space using Eq. 2. The finalised formulations of the 1D soil reactions are as follows.

Table 4: Approximate formulations of the 1D soil reactions following calibration against the 3D FE results. Formulations for h_x and m_y are similar to that of h_y and m_x

	Formulations
Vertical	$v^{skirt} = 4.28\ G\ u_z$
	$v^{base} = 2.4\ GD\ u_z$
Torsional	$t^{skirt} = 3.66\ GD^2\ \theta_z$
	$t^{base} = 0.41\ GD^3\ \theta_z$
Rotational	$m_x^{skirt} = GD^2\ (-0.12\ u_y/D + (1.17 - 0.12\ z/D)\ \theta_x)$
	$m_x^{base} = GD^3\ (-0.12\ u_y/D + 0.42\ \theta_x)$
Lateral	$h_y^{skirt} = GD\ (6.51\ u_y/D + (-19.83 * z/D + 10.28)\ \theta_x)$
	$h_y^{base} = GD^2\ (1.17\ u_y/D - 0.6\ \theta_x)$

6. Discussion

To verify that the calibration was robust and that the simplification in the formulations did not introduce significant errors, the normalised global stiffness coefficients computed using the formulated 1D soil reactions were compared against the actual 3D FE results. For this exercise, the global stiffness coefficients were computed using the analytical solutions in Table 1 and the 1D soil reaction formulations in Table 4. Table 5 shows the normalised global stiffness coefficients predicted by the 1D caisson model and the original 3D FE results. As can be observed, the formulated 1D soil reactions, albeit simplified, can reproduce the 3D FE results well.

Table 5: Comparison of normalised stiffness coefficients computed by the formulated 1D soil reactions and the 3D FE model. The first and second K_C predictions by the 1D soil reactions were computed using the m_x and h_y based equations in Table 1 respectively

Stiffness	3D FE	1D Soil Reactions	Difference
K_V/GD	6.68	6.68	0 %
K_H/GD	7.68	7.68	0 %
K_M/GD^3	7.11	7.12	0.06 %
K_T/GD^3	4.07	4.07	0 %
K_C/GD^2 (1)	-4.66	-4.66	0 %
K_C/GD^2 (2)	-4.66	-4.66	0 %

An important result to note is the computational time required for each set of predictions. While the 3D FE analyses took an hour in total to compute the stiffness coefficients shown in Table 3, the proposed 1D model takes only milliseconds. This shows the potential of an efficient design process, which can be broken down into an offline and online stage.

In the offline stage, the time intensive 3D FE analyses are carried out to calibrate the proposed model (which only needs to be done once). In the online stage, the calibrated 1D caisson model is used with minimal computational effort. This allows a rapid turnover of design evaluations, which is a crucial part of many time critical design activities such as foundation design optimisation. This is a very significant improvement over the current state of practice.

Nevertheless, the proposed model does have some limitations. The paper has shown that the modelling approach is satisfactory and has been implemented correctly; given the excellent agreement between the model predictions and the 3D FE results as shown in Table 5. However, these results do not provide any evidence that the proposed model, in its current form, has any predictive capabilities beyond the single case of a caisson foundation of L/D = 1 in incompressible elastic soil. Nevertheless, it is not difficult to run a more extensive offline stage with more 3D FE analyses to derive generalised 1D soil reaction formulations for different caisson dimensions and elastic soil properties. Although this work has been completed it is not reported here as the focus of this paper is on the underlying modelling approach. The work on generalised 1D soil reactions will be reported at a later stage.

7. Conclusion

Fatigue design of caisson foundations usually requires a large number of analyses. Thus, a suitable design method for fatigue design must be efficient, in addition to being accurate. However, existing design methods are limited by efficiency or the level of detail of soil profiles that can be modelled.

This paper addresses this issue by proposing a computationally efficient design method that can provide accurate predictions of the stiffness of caisson foundations in elastic soil. Compared to the 3D FE model, the proposed model can provide stiffness predictions at similar levels of accuracy but at a small fraction of the computational cost. Unlike existing macro element models, the proposed model is applicable for any non-homogeneous soil, including layered soil.

It is evident that the proposed model offers significant advantages over existing methods, especially ease of calibration and computational efficiency. Most of the limitations of the model are related to the incomplete formulations of the 1D soil reactions, which can be rectified with further calibration against more 3D FE results, following the methodology illustrated in this paper.

8. Acknowledgments

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