Low-Rank Para-Hermitian Matrix EVD via Polynomial Power Method with Deflation

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***Abstract*—The power method in conjunction with deflation provides an economical approach to compute an eigenvalue decomposition (EVD) of a low-rank Hermitian matrix, which typically appears as a covariance matrix in narrowband sensor array processing. In this paper, we extend this idea to the broadband case, where a polynomial para-Hermitian matrix needs to be diagonalised. For the low-rank case, we combine a polynomial equivalent of the power method with a deflation approach to subsequently extract eigenpairs. We present perturbation analysis and simulation results based on an ensemble of low-rank randomized para-Hermitian matrices. The proposed approach demonstrates higher accuracy, faster execution time, and lower implementation cost than state-of-the-art algorithms.**

I. Introduction

In broadband sensor arrays processing, problems can be formulated using polynomial matrices, and polynomial matrix decomposition techniques have proven useful in obtaining optimal solutions [1–5]. One of the most popular decompositions is the eigenvalue decomposition (EVD). Due to its complexity even with efficient implementations [6–8], a partial or reduced EVD can be considered useful for low rank applications, such as in speech enhancement where a large number of microphones may record only a very limited number of speakers [9–11]. In the narrowband case, the power method in conjunction with Hotelling's deflation approach [12] is well suited for factorising rank-deficient matrices, where the number of eigenvalues and eigenvectors to be determined is smaller than the dimension of the matrix; hence this paper aims to extend this utility to the broadband case.

Due to the existence of the analytic EVD for para-Hermitian polynomial matrices [13–15], and the fact that analytic functions can be arbitarily closely approximated by polynomials of sufficiently high order, the deflation concept appears viable for polynomial matrices. Therefore, the recently proposed polynomial equivalent of the power method for para-Hermitian polynomial matrices [16], and its extension to general polynomial matrices [17], motivate the extension of Hotelling's deflation to polynomial matrices. With this extension, the polynomial power method can be utilized for a partial or full PEVD of a para-Hermitian polynomial matrix.

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The paper is outlined as follows: Sec. II briefly describes deflation and the power method. Sec. III reviews the analytic PEVD , whereas Sec. IV summarizes the polynomial extension of the power method. Sec. V combines the polynomial power method with deflation, with numerical examples in Sec. VI and the application to a low-rank PEVD in Sec. VII. Conclusions are drawn in Sec. VIII.

II. EVD via Power Method and Deflation

A Hermitian matrix $\mathbf{R} \in \mathbb{C}^{M\times M}$ with $p \leq M$ non-zero eigenvalues $\lambda_m \in \mathbb{R}$, $m = 1,\dots p$ can be represented as a sum of rank one terms

$$\mathbf{R} = \sum_{m=1}^{p} \mathbf{q}_m \mathbf{q}_m^{\mathrm{H}} \lambda_m = \sum_{m=1}^{p} \mathbf{R}_m \tag{1}$$

where $\mathbf{q}_m$ is mth eigenvector and $\mathbf{R}_m$ is a rank one Hermitian matrix, whose columns are spanned by $\mathbf{q}_m$. We assume that $\mathbf{R}$ is positive semi-definite, and its p non-zero eigenvalues are distinct and majorised as $\lambda_m > \lambda_{m+1}, m = 1,\dots,(p-1)$. The power method [18] can be used to determine the dominant eigenpair, i.e. $\{\mathbf{q}_1, \lambda_1\}$. In the power method, an initial random unit-norm vector $\mathbf{x}_1^{(0)} \in \mathbb{C}^M$ is assumed to be some linear combination of the eigenvectors, $\mathbf{x}^{(0)} = \mathrm{diag}\{c_1,\dots,c_M\}\,[\mathbf{q}_1,\dots,\mathbf{q}_M]$, whereby we assume $c_1 \neq 0$. Then the iteration

$$\mathbf{x}^{(k)} = \mathbf{R}\mathbf{x}^{(k-1)} = \mathbf{R}^k \mathbf{x}^{(0)} \tag{2}$$

can be shown to converge to $\lim_{k\to\infty} \mathbf{x}^{(k)} = A\mathbf{q}_1$, with some constant A that can be determined through normalisation since $\mathbf{q}_1$ must have unit norm. The corresponding principal eigenvalue can then be obtained as $\lambda_1 = \mathbf{q}_1^{\mathrm{H}}\mathbf{R}\mathbf{q}_1$.

The matrix $\mathbf{R}$ can be deflated by removing the contribution of the dominant eigenpair as

$$\mathbf{R}^{(2)} = \mathbf{R} - \mathbf{R}_1 = \mathbf{R} - \mathbf{q}_1\lambda_1\mathbf{q}_1^{\mathrm{H}}\ . \tag{3}$$

If the estimated eigenpair is sufficiently accurate, then the deflated matrix $\mathbf{R}^{(2)}$ has the decremented rank $(p-1)$, and its dominant eigenpair is now $\{\mathbf{q}_2, \lambda_2\}$. This second eigenpair can be extracted by a repeat of the power method on the matrix $\mathbf{R}^{(2)}$. In turn, $\mathbf{R}^{(2)}$ can now be deflated, and through a total of p iteration, all eigenpairs of $\mathbf{R}$ can be determined. As a recursive formulation with the initialisation $\mathbf{R}^{(1)} = \mathbf{R}$, the scheme operates via

$$\mathbf{R}^{(m+1)} = \mathbf{R}^{(m)} - \lambda_m\mathbf{q}_m\mathbf{q}_m^{\mathrm{H}}\ , \tag{4}$$

with $m = 1, \ldots, p$. Ideally, by exactly extracting the dominant eigenpair $\{\mathbf{q}_m, \lambda_m\}$ at the mth iteration, we finally end up with $\mathbf{R}^{(p+1)} = \mathbf{0}$.

If an eigenpair, such as the first one in (4), is inaccurate, i.e. we obtain an estimate $\{\hat{\mathbf{q}}_m, \hat{\lambda}_m\}$ via a limited number of iterations k instead of the exact $\{\mathbf{q}_m, \lambda_m\}$, then $\mathbf{R}^{(m+1)}$ will be perturbed by the error $\mathbf{R}_m - \hat{\mathbf{R}}_m$, with $\hat{\mathbf{R}}_m = \hat{\mathbf{R}}^{(m)} - \hat{\mathbf{q}}_m \hat{\mathbf{q}}_m^{\mathrm{H}} \hat{\lambda}_m$. This perturbation term will (i) lead to an insufficient rank reduction, and (ii) cause error propagation as subsequent eigenpairs are estimated with increasing inaccuracy. To investigate such perturbation effects, and hence potential bounds on the estimation error of subsequently extracted eigenpairs, a perturbation analysis [19], in part for reduced-rank perturbations [20, 21], can be employed.

III. Analytic EVD of Para-Hermitian Matrices

A matrix $\boldsymbol{R}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$, that satisfies the para-Hermitian property $\boldsymbol{R}(z) = \boldsymbol{R}^{\mathrm{P}}(z) = \{\boldsymbol{R}(1/z^*)\}^{\mathrm{H}}$, is analytic in $z \in \mathbb{C}$, and is not connected to any multiplexing operation, admits an analytic EVD [13, 22]

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \; . \tag{5}$$

In (5), it is possible to select right hand side factors that are analytic in z. The columns of the paraunitary matrix $\boldsymbol{Q}(z)$, such that $\boldsymbol{Q}(z)\boldsymbol{Q}^{\mathrm{P}}(z) = \mathbf{I}$, represent analytic eigenvectors and the diagonal, para-Hermitian matrix $\boldsymbol{\Lambda}(z) = \mathrm{diag}\{\lambda_1(z), \ldots, \lambda_M(z)\}$ contains the analytic eigenvalues. While the eigenvalues are unique up to a permutation, the eigenvectors are subject to an allpass ambiguity: if $\boldsymbol{q}_m(z)$ is the mth column of $\boldsymbol{Q}(z)$ and therefore the mth eigenvector, then $\phi_m(z)\boldsymbol{q}_m(z)$ is also a valid mth eigenvector, where $\phi_m(z)$ is an arbitrary allpass filter.

If $\boldsymbol{R}(z)$ is estimated from finite data, the resulting eigenvalues will be strictly spectrally majorised [23], such that on the unit circle for $z = \mathrm{e}^{\mathrm{j}\Omega}$, we have

$$\lambda_1(\mathrm{e}^{\mathrm{j}\Omega}) > \lambda_2(\mathrm{e}^{\mathrm{j}\Omega}) > \cdots > \lambda_p(\mathrm{e}^{\mathrm{j}\Omega}) , \quad \forall\, \Omega \; . \tag{6}$$

This assumes that $\boldsymbol{R}(z)$ possesses rank $p \leq M$, in which case $\lambda_{p+1}(\mathrm{e}^{\mathrm{j}\Omega}) = \ldots = \lambda_M(\mathrm{e}^{\mathrm{j}\Omega}) = 0 \;\forall\Omega$, i.e. that there are $(M-p)$ eigenvalues that are identical to zero. In the case of $p < M$ for $\boldsymbol{R}(z)$, we refer to $\boldsymbol{R}(z)$ as a low-rank polynomial matrix.

In the remainder of this paper, we want to address the analytic EVD in (5) via iterative applications of a polynomial power method [24] and deflation of a rank-deficient ($p < M$) or even low rank ($p \ll M$) matrix $\boldsymbol{R}(z)$ in order to avoid the computational cost incurred by methods that evaluate a full analytic EVD [25–29].

IV. Polynomial Equivalent of the Power Method

In order to extract the dominant eigenpair $\{\boldsymbol{q}_1(z), \lambda_1(z)\}$ from $\boldsymbol{R}(z)$, an extension of the power method [18] to polynomial matrices can be utilised [24], which we briefly summarise below. In the method of [24], an arbitrary polynomial vector $\boldsymbol{x}^{(0)}(z)$ is repeatedly multiplied with a para-Hermitian matrix $\boldsymbol{R}(z)$ to generate a sequence of polynomial vectors such that after k iterations, it produces

$$\boldsymbol{x}^{(k)}(z) = \boldsymbol{R}(z)\boldsymbol{x}^{(k-1)}(z) = \boldsymbol{R}^k(z)\boldsymbol{x}^{(0)}(z) \; . \tag{7}$$

Analogous to the power method, a normalisation of the vector $\boldsymbol{x}^{(k)}(z)$ to $\boldsymbol{x}^{(k)}_{\mathrm{norm}}(z)$ is required such that $\boldsymbol{x}^{(k),\mathrm{P}}_{\mathrm{norm}}(z)\boldsymbol{x}^{(k)}_{\mathrm{norm}}(z) = 1$. Due to analyticity of $\boldsymbol{Q}(z)$ in (5), the normalization can be carried out in the discrete Fourier transform (DFT) domain since with a sufficient DFT size K, the approximation can be arbitrarily accurate. With each iteration k, the order of the vector $\boldsymbol{x}^{(k)}(z)$ grows, which may need to be limited through truncation. This truncation can either be in the form of limiting the order to the estimated support of the eigenvector, which can be obtained from [30] by shifted-truncation [31], or by removing any trailing coefficients that fall below some small threshold [32, 33]. The iterative procedure is terminated either the Hermitian angle between the consecutive iterations normalized vector falls below some threshold ϵ or a maximum number of iterations $k_{\max}$ is expended. For further details, please refer to [24].

V. Analytic EVD via Polynomial Power Method

A. Rank One Representation and Deflation

Similar to a Hermitian matrix $\mathbf{R}$, a para-Hermitian matrix can be represented as the sum of rank one para-Hermitian matrices $\boldsymbol{R}_m(z)$,

$$\boldsymbol{R}(z) = \sum_{m=1}^{p} \boldsymbol{q}_m(z)\boldsymbol{q}^{\mathrm{P}}{}_m(z)\lambda_m(z) = \sum_{m=1}^{p} \boldsymbol{R}_m(z) \; , \tag{8}$$

where $p \leq M$ is the number of non-zero analytic eigenvalues. This shows that if an eigenpair is available, deflation can be performed to reduce the rank of $\boldsymbol{R}(z)$. For instance, if $\{\boldsymbol{q}_1(z), \lambda_1(z)\}$ is extracted via the polynomial power method of Sec. IV, its contribution can be removed from the original para-Hermitian matrix as

$$\boldsymbol{R}^{(2)}(z) = \boldsymbol{R}(z) - \boldsymbol{q}_1(z)\boldsymbol{q}_1^{\mathrm{P}}(z)\lambda_1(z) = \boldsymbol{R}(z) - \boldsymbol{R}_1(z) \; , \tag{9}$$

The allpass ambiguity of the extracted eigenvector mentioned in Sec. III does not cause any issue since with $\phi_1(z)\phi_1^{\mathrm{P}}(z) = 1$ this ambiguity drops out.

The polynomial power method can be repeated on $\boldsymbol{R}^{(2)}(z)$, if the dominant eigenpair is accurate. Thus over $p-1$ deflations and p application of the polynomial power method, an analytic EVD can be computed using a recursive procedure akin to (4), such that with $\boldsymbol{R}^{(1)}(z) = \boldsymbol{R}(z)$

$$\boldsymbol{R}^{(m+1)}(z) = \boldsymbol{R}^{(m)} - \underbrace{\boldsymbol{q}_m(z)\boldsymbol{q}^{\mathrm{P}}(z)\lambda_m(z)}_{\boldsymbol{R}_m(z)} \; . \tag{10}$$

The approach in (10) requires the accurate determination of eigenpairs $\{\boldsymbol{q}_m(z), \lambda_m(z)\}$ via the polynomial power method of Sec. IV at every stage. Estimation errors due to a limited number of iterations k will result not only in estimated eigenpairs $\{\hat{\boldsymbol{q}}_m(z), \hat{\lambda}_m(z)\}$ that may differ from the desired quantities, but will also lead to potentially inaccurate estimates $\hat{\boldsymbol{R}}_m(z)$ of the rank one matrices and $\hat{\boldsymbol{R}}^{(m+1)}(z)$ of the deflated matrices w.r.t. the quantities defined in (10). We therefore want to next investigate how incorrect eigenpairs and rank-one estimates perturb the subsequent extraction of any remaining eigenpairs.

B. Perturbation Analysis and Error Propagation

We define a perturbation and error propagation analysis per frequency bin, i.e. for any specific frequency Ω on the unit circle. We assess the difference $\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})$ between the correctedly deflated matrix $\boldsymbol{R}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})$ after the mth rank deflation, and its estimate, $\hat{\boldsymbol{R}}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})$:

$$\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{R}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega}) - \hat{\boldsymbol{R}}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega}) \;, \tag{11}$$

for $m = 1, \ldots, (p-1)$. Since due to the rank one deflations

$$\boldsymbol{R}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega}) - \boldsymbol{R}_1(\mathrm{e}^{\mathrm{j}\Omega}) - \ldots - \boldsymbol{R}_m(\mathrm{e}^{\mathrm{j}\Omega}) \;, \tag{12}$$

$$\hat{\boldsymbol{R}}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega}) - \hat{\boldsymbol{R}}_1(\mathrm{e}^{\mathrm{j}\Omega}) - \ldots - \hat{\boldsymbol{R}}_m(\mathrm{e}^{\mathrm{j}\Omega}) \;, \tag{13}$$

we have

$$\begin{aligned}\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega}) = \sum_{\mu=1}^{m} & \hat{\mathbf{q}}_\mu(\mathrm{e}^{\mathrm{j}\Omega})\hat{\mathbf{q}}_\mu^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\hat{\lambda}_\mu(\mathrm{e}^{\mathrm{j}\Omega}) \\ & - \mathbf{q}_\mu(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{q}_\mu^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\lambda_\mu(\mathrm{e}^{\mathrm{j}\Omega}) \;. \end{aligned} \tag{14}$$

This can also be written recursively as

$$\begin{aligned}\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{E}^{(m)}(\mathrm{e}^{\mathrm{j}\Omega}) + \Big(& \hat{\mathbf{q}}_m(\mathrm{e}^{\mathrm{j}\Omega})\hat{\mathbf{q}}_m^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\hat{\lambda}_m(\mathrm{e}^{\mathrm{j}\Omega}) \\ & - \mathbf{q}_m(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{q}_m^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\lambda_m(\mathrm{e}^{\mathrm{j}\Omega})\Big) \;. \end{aligned} \tag{15}$$

Since the eigenvectors of a para-Hermitian matrix can be selected to be orthonormal, the different terms in the sum of (14) are approximately orthogonal, and for sufficiently small perturbations we can show that

$$\begin{aligned}\|\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})\|_{\mathrm{F}}^2 \approx\ & \|\boldsymbol{E}^{(m)}(\mathrm{e}^{\mathrm{j}\Omega})\|_{\mathrm{F}}^2 + \\ & + \Big\| \hat{\mathbf{q}}_m(\mathrm{e}^{\mathrm{j}\Omega})\hat{\mathbf{q}}_m^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\hat{\lambda}_m(\mathrm{e}^{\mathrm{j}\Omega}) \\ & \quad - \mathbf{q}_m(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{q}_m^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\lambda_m(\mathrm{e}^{\mathrm{j}\Omega}) \Big\|_{\mathrm{F}}^2 \\ \geq\ & \|\boldsymbol{E}^{(m)}(\mathrm{e}^{\mathrm{j}\Omega})\|_{\mathrm{F}}^2 \;. \end{aligned} \tag{16}$$

Hence, the error norm does not improve over subsequent deflation operations, and generally tends to grow.

We can now assess the effect of the above error on the eigenpair $\{\boldsymbol{q}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega}), \lambda_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})\}$ that is to be extracted from $\boldsymbol{R}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})$. Likewise, the estimate $\{\hat{\boldsymbol{q}}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega}), \hat{\lambda}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})\}$ is extracted from $\hat{\boldsymbol{R}}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})$. Using (11) and the Bauer-Fike theorem [34], we find that the accuracy of the extracted $(m+1)$st eigenvalue is upper-bounded as

$$|\lambda_{m+1}(\mathrm{e}^{\mathrm{j}\Omega}) - \hat{\lambda}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})| \leq \|\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})\|_{\mathrm{F}}^2 \;., \tag{17}$$

Therefore, the worst case accuracy of the $(m+1)$st eigenvalue is determined by the cummulative error $\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})$. For the eigenvectors, we can define the subspace distance [18] via the spectral norm $\|\cdot\|_2$ of the difference of projections

$$\begin{aligned}\mathcal{U}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega}) = \Big\| & \boldsymbol{q}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{q}_{m+1}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega}) \\ & - \hat{\boldsymbol{q}}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})\hat{\boldsymbol{q}}_{m+1}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega}) \Big\|_2 \;. \end{aligned} \tag{18}$$

Then perturbation theory [18] provides an upper bound

$$\mathcal{U}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega}) \leq \frac{4}{d}\|\boldsymbol{e}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})\|_2 \;, \tag{19}$$

where $d = \lambda_{m+1}(\mathrm{e}^{\mathrm{j}\Omega}) - \lambda_{m+2}(\mathrm{e}^{\mathrm{j}\Omega})$ is the distance to the next-nearest eigenvalue, and $\boldsymbol{e}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{C}^{M-1}$ comes from a partition of $\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})$,

$$\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega}) = \begin{bmatrix} e_{m+1}(\mathrm{e}^{\mathrm{j}\Omega}) & \boldsymbol{e}_{m+1}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega}) \\ \boldsymbol{e}_{m+1}(\mathrm{e}^{\mathrm{j}\Omega}) & \boldsymbol{E}_{2,m+1}(\mathrm{e}^{\mathrm{j}\Omega}) \end{bmatrix} \;. \tag{20}$$

Thus, the upper bound on the accuracy of the $m+1$st eigenvector extracted by deflation also depends on the accummulated errors in $\boldsymbol{E}^{(m+1)}(\mathrm{e}^{\mathrm{j}\Omega})$. Hence, any inaccuracies in a rank one estimate will impact on and further degrade the precision bounds with which any remaining eigenpairs can be determined.

VI. Numerical Example

To demonstrate the deflation concept combined with the polynomial power method, we consider a spectrally majorised para-Hermitian matrix $\boldsymbol{R}(z)$ where the analytic eigenvalues in $\boldsymbol{\Lambda}(z) = \mathrm{diag}\{\lambda_1(z), \lambda_2(z), \lambda_3(z)\}$ are

$$\begin{aligned}\lambda_1(z) &= z(6+\mathrm{j})/100 + 1.01 \;+\; z^{-1}(6-\mathrm{j})/100 \\ \lambda_2(z) &= -z(1-2\mathrm{j})/100 + 0.86 - z^{-1}(1+2\mathrm{j})/100 \\ \lambda_3(z) &= z(5-2\mathrm{j})/100 + 0.71 + z^{-1}(5+2\mathrm{j})/100 \;,\end{aligned}$$

and the matrix of eigenvectors $\boldsymbol{Q}(z)$ is defined by a sequence of elementary paraunitary operations [35],

$$\boldsymbol{Q}(z) = \prod_{i=1}^{4}(\mathbf{I} + \frac{1}{2}(z^{-1}-1)\mathbf{e}_i\mathbf{e}_i^{\mathrm{H}}) \;, \tag{21}$$

with $\mathbf{e}_{i=\{1,3\}} = [1,\ 0,\ \mp 1]^{\mathrm{T}},\ \mathbf{e}_{i=\{2,4\}} = [\pm 1,\ 1,\ 0]^{\mathrm{T}}$.

The exact eigenvalues of $\boldsymbol{R}(z)$ are shown in Fig. 1. These are compared to eigenvalues extracted by the deflation approach based on the polynomial power method executed with $k_{\max} = 5\mathrm{e}3$, $\epsilon = 10^{-4}$ and $\boldsymbol{x}^{(0)}(z) = \sum_{i=0}^{4} z^{-i}$. The difference between the estimated and the ground-truth eigenvalues can be seen in Fig. 1, and be measured for the mth eigenvalue $\lambda_m[\tau] \circ\!\!-\!\!\bullet\, \lambda_m(z)$ via

$$\xi_{\lambda_m} = \sum_{\tau} |\lambda_m[\tau] - \hat{\lambda}_m[\tau]|^2 \;. \tag{22}$$

For $\hat{\lambda}_1(z)$, which is extracted by the polynomial power method from $\boldsymbol{R}^{(1)}(z) = \boldsymbol{R}(z)$, we obtain $\xi_{\lambda_1} = 6.8 \times 10^{-5}$. By subsequent deflation, from $\hat{\boldsymbol{R}}^{(2)}$ we obtain the second eigenpair with $\xi_{\lambda_2} = 1.55 \times 10^{-4}$. This shows that the extracted second eigenvalue is not as accurate as the first one. The third eigenvalues is then obtained from $\hat{\boldsymbol{R}}^{(3)}(z)$ with $\xi_{\lambda_3} = 3.6 \times 10^{-4}$. It can be seen that due to error propagation,

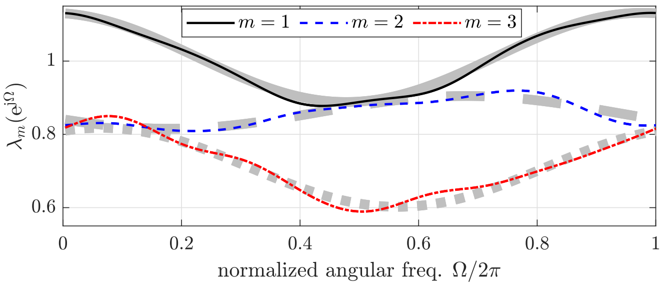

Fig. 1. Example for the ground truth (shaded grey) and estimated eigenvalues using the proposed deflation approach (in colour) $\hat{\lambda}_m(e^{j\Omega}), m = 1, 2, 3$ for the example matrix $\boldsymbol{R}(z)$ when injecting perturbation through insufficient convergence of the polynomial power method.

indeed $\xi_{\lambda_3} > \xi_{\lambda_2} > \xi_{\lambda_3}$, and the error is increasing with each extraction. The impact of perturbations and error propagation can also be seen in the estimated eigenvalues shown in Fig. 1, where the third eigenvalue appears to have the largest estimation error.

VII. Application and Ensemble Simulation

This section provides an ensemble test to demonstrate the enhanced performance of the proposed approach for the PEVD of low-rank para-Hermitian matrices. The performance metrics selected for comparison are the resulting order of the estimated paraunitary matrix $\hat{\boldsymbol{Q}}(z)$, denoted as $\mathcal{O}(\hat{\boldsymbol{Q}}(z))$, the execution time t of the approach, and the reconstruction metric ξ_R. The latter is defined as $\xi_R = \sum_\tau ||\mathbf{R}[\tau] - \hat{\mathbf{R}}[\tau]||_F^2 / \sum_\tau ||\mathbf{R}[\tau]||_F^2$ which measures the accuracy of the decomposition, where with the convolution operator $*$, $\hat{\mathbf{R}}[\tau] = \hat{\mathbf{Q}}[\tau] * \hat{\mathbf{\Lambda}}[\tau] * \hat{\mathbf{Q}}^H[-\tau]$.

For an exhaustive test, we have constructed an ensemble comprising of 100 instantiations of 6×6 para-Hermitian matrices of rank two, where each instance represents a system of two spectrally majorised broadband sources illuminating an array of $M = 6$ sensors through a convolutive mixing system. The instantiations are generated using the source model in [3], with the source power spectral densities and the convolutive paraunitary mixing defining the ground truth analytic EVD. The concatenation of spectral shaping and mixing forms a system $\boldsymbol{H}(z) : \mathbb{C} \rightarrow \mathbb{C}^{6\times 2}$ of order 100. The resulting cross-spectral density matrix $\boldsymbol{R}(z) = \boldsymbol{H}(z)\boldsymbol{H}^P(z)$ is therefore of order 200.

The polynomial power algorithm is executed with $k_{\max} = 10^3$ and $\epsilon = 10^{-7}$. The trailing coefficients of the normalized vector are truncated once they fall below a threshold of 10^{-3}. The support of the initial vector $\boldsymbol{x}^{(0)}(z)$ is set to the estimated support of the eigenvectors, which can be evaluated via [30]; its coefficients are drawn from a complex-valued normal distribution. The state-of-the-art algorithms SBR2 [2] and SMD [3] are run for comparison, and are permitted to reach a maximum of 500 iterations or run until the maximum off-diagonal element magnitude falls below 10^{-6}. The intermediate para-Hermitian and paraunitary matrices are truncated by removing the outer lags via a threshold $\mu_{PH} = \mu_{PU} = 10^{-6}$ [2, 32, 33].

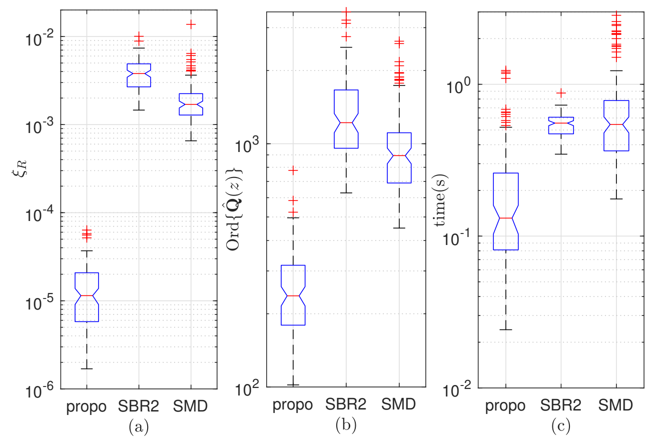

Fig. 2. Ensemble results illustrated as box-plots for (a) reconstruction error, (b) $\mathrm{Ord}\{\hat{\boldsymbol{Q}}(z)\}$, and (c) execution time. (red marks show outliers)

The ensemble results in the form of the three metrics are illustrated as box-plots in Fig.. 2. In Fig. 2(a), we can see that the reconstruction metric ξ_R is orders of magnitude lower for the proposed approach than for SBR2 and SMD. This suggests that the proposed combination of polynomial power method and deflation can compute the PEVD of a para-Hermitian matrix significantly more accurately than both benchmark algorithms. Moreover, in Fig. 2(b) the order of the estimated paraunitary $\hat{\boldsymbol{Q}}(z)$ is lower for the proposed method. This indicates that the perturbation potentially introduced by the deflation process is negligible, as otherwise the order might grow as successive eigenpairs are extracted. The lower order is also significant, since this determines the complexity of implementing the paraunitary $\boldsymbol{Q}(z)$ for subspace projection-type applications. Lastly, the proposed approach executes faster compared to SBR2 and SMD, as evident from Fig. 2(c).

VIII. Conclusion

An approach of combining the polynomial power method with deflation for the PEVD of lowrank para-Hermitian polynomial matrices has been presented. We have shown that it is possible for almost all para-Hermitian matrices to apply deflation similarly the approach for ordinary matrices. The perturbation of the eigenpairs of the deflated matrix has been studied and has been shown to relate directly to the accuracy of the successively extracted eigenpairs. Over an ensemble of low-rank para-Hermitian matrices, the proposed method has outperformed state-of-the-art algorithms in terms of accuracy, speed, and implementation complexity. The algorithm can similarly be extended to compute the PSVD of low-rank general polynomial matrices based on a generalized polynomial power method [17] as an alternative to a full PSVD in [36]. The proposed technique can also be directly applied to a number of low-rank applications where the number of channels can substantially exceed the number of sources [37–39] or in problems that are rank one [40].

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