WEIGHTED ENUMERATION OF NONBACKTRACKING WALKS ON WEIGHTED GRAPHS *

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Abstract. We extend the notion of nonbacktracking walks from unweighted graphs to graphs whose edges have a nonnegative weight. Here the weight associated with a walk is taken to be the product over the weights along the individual edges. We give two ways to compute the associated generating function, and corresponding node centrality measures. One method works directly on the original graph and one uses a line graph construction followed by a projection. The first method is more efficient, but the second has the advantage of extending naturally to time-evolving graphs. Based on these generating functions, we define and study corresponding centrality measures. Illustrative computational results are also provided.

Key words. Complex network, matrix function, generating function, line graph, combinatorics, evolving graph, temporal network, centrality measure, Katz centrality.

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1. Introduction.

Complex network analysis is an expanding scientific discipline that has recently been producing many research challenges, with applications across several fields of science and engineering [11, 24]. One important question is that of ranking the nodes of a graph by importance, or, in more mathematical terms, defining and studying an appropriate *centrality measure*. Those centrality measures that can be formulated and computed via combinatorial properties of walks on the underlying graph have received special attention [7, 12, 13, 19, 23] because they have convenient formulations in terms of linear algebra that lead to efficient computational methods. In recent years, this paradigm has been refined by studying centrality measures that are based on counting not all walks but only some of them, namely, walks that do not backtrack [1, 2, 5, 14, 28–30] or more generally do not cycle [5]. Nonbacktracking walks are known to be linked to zeta functions of graphs [18, 21, 22, 26]. Their associated centrality measures have been shown to possess attractive computational properties [1, 2, 4, 14] and have been studied both for undirected and directed graphs, and more recently for time evolving graphs [3]. However, in the context of the combinatorics of nonbacktracking walks, so far only unweighted graphs have been studied. We mention that nonbacktracking random walks were previously considered in [20], but the problems studied there are different to the ones analyzed in the present paper. Moreover, [20] focuses only on the special case where the nodes are given themselves a positive weight $\varphi(i)$, and the weight of the edge (i, j) is defined as $\omega((i, j)) = \varphi(i)\varphi(j)$.

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Instead, we do not impose any restriction on the edges' weights. In the theory of graph zeta functions, weighted graphs have been considered by defining the weight of a walk to be the sum (and not the product, as in this paper) of the weights of its edges [18]. We discuss this issue further in section 2.

The main purpose of the present paper is to extend the combinatorial theory of nonbacktracking walks, and corresponding centrality measures, to graphs whose edges carry a positive *weight*. These graphs are associated with generic nonnegative adjacency matrices, in contrast to unweighted graphs that correspond to binary adjacency matrices. While for unweighted graphs one may be interested in the enumeration of walks of a given length, for weighted graphs the combinatorial problem is more sophisticated due to the presence of weights. The edge weights naturally give rise to an overall weight for each walk, a concept that can be used alongside the length (i.e., the number of edges traversed).

The structure of the paper is as follows. In Section 2 we introduce some relevant notation and core concepts. Section 3 sets up and studies the issue of characterizing the classical generating function associated with nonbacktracking weighted walks and using it to compute a centrality measure. In section 4 we introduce an alternative formulation that applies to a wider class of generating functions and centrality measures. Section 5 shows how these ideas can be extended to the case of evolving graphs. Numerical experiments are conducted in Section 6. We finish in Section 7 with a brief discussion.

2. Background and Notation. In this paper, we consider finite graphs. A finite graph is a triple $G = (V, E, \Omega)$ where $V = [n]$ is the set of the nodes (or vertices), $E \subset V \times V$ is the set of (directed) edges, and $\Omega : E \to (0, \infty)$ is a weight function that associates to each edge a positive weight. If $\Omega(e) = 1$ for all $e \in E$, then the graph is said to be *unweighted*; if for any pair $i \neq j$, with $i, j \in V$ we have that $(i, j) \in E \Leftrightarrow (j, i) \in E$ and that $\Omega((i, j)) = \Omega((j, i))$ then the graph is said to be *undirected*; and if, for every $i \in V$, we have that $(i, i) \notin E$ then the graph is said to be without loops. Graphs that are not unweighted are usually called *weighted* and graphs that are not undirected are usually called *directed*. It is, however, convenient (and we will do so within this work) to relax the terminology so that the set of directed (resp., weighted) graphs contains as special cases also undirected (resp., unweighted) graphs, further we will assume that all graphs are without loops.

A *walk of length* ℓ on the graph G is a sequence of nodes $i_1, i_2, \ldots, i_{\ell+1}$ such that $(i_j, i_{j+1}) \in E$ for all $1 \leq j \leq \ell$. Equivalently, it can be seen as a sequence of edges $e_1, \ldots, e_\ell$ such that $e_j \in E$ for all $j = 1, \ldots, \ell - 1$ and the end node of e_j coincides with the starting node of e_{j+1}.

Definition 2.1. *Let $G = (V, E, \Omega)$ be a weighted graph. The* weight *of the walk $e_1, \ldots, e_\ell$ is*

$$\prod_{k=1}^{\ell} \Omega(e_k)$$

where $\Omega(e_k)$ is the weight of the edge $e_k \in E$.

Remark 2.2. *When $\Omega : E \to \{1\}$ is the weight function associated with an unweighted graph, then the weight of all walks in the network is one, regardless of their length.*

In the context of mainstream graph theory, the *weight* (or *length* or *cost*) of a walk is sometimes defined as the *sum*, rather than the product, of the weights of its edges. In that scenario, zeta functions of graphs (which are closely related to the enumeration of nonbacktracking walks) have been studied [18]. However, we argue that within complex network analysis Definition 2.1 has several useful applications. For example, consider a road network where nodes represent towns and a nonnegative integer edge weight A_{ij} records the number of distinct roads connecting town i and town j. Then, the number of distinct routes from i to j that pass through one intermediate town is equal to

$$\sum_{k=1}^{n} A_{ik} A_{kj},$$

that is, the weighted sum of walks of length two, where the weight is the *product* of the weights of its edges. Similarly, in a model where edges represent independent probabilistic events and their weights are their probabilities, as discussed in the original work of Katz [19], it is natural to postulate that the weight of a walk is the product of the weight of its edges, in agreement with the fact that the joint probability of a sequence of independent events is the product of the individual probabilities.

Given a node ordering, the corresponding adjacency matrix of a graph is the matrix $A \in \mathbb{R}^{n \times n}$ entrywise defined as:

$$A_{ij} = \begin{cases} 0 & \text{if } (i,j) \notin E; \\ \Omega((i,j)) & \text{if } (i,j) \in E. \end{cases}$$

Note that a graph is undirected if and only if its adjacency matrix is symmetric; it is without loops if and only if its adjacency matrix has zero diagonal; and it is unweighted if and only if its adjacency matrix has entries all lying in $\{0,1\}$.

The problems of enumerating walks in unweighted graphs and enumerating weighted walks in weighted graphs may both be solved by considering powers of the adjacency matrix: indeed, the (i,j) entry of A^k is equal to, respectively, the number of walks of length k from node i to node j (when the graph is unweighted) or the weighted sum of walks of length k from node i to node j (when the graph is weighted). As a consequence, the generating function for the (possibly weighted) enumeration of walks is given by

$$I + tA + t^2A^2 + \cdots = \sum_{k=0}^{\infty} t^k A^k = (I - tA)^{-1},$$

where we adopt the standard convention that the (weighted) sum of walks of length zero from i to j is 1 if $i = j$ and 0 otherwise. Here, t is a real parameter small enough to ensure convergence of the series which scales by t^k the count for walks of length k.

A walk can also be seen as a sequence of nodes. If the sequence does not contain a subsequence of the form iji for some nodes i and j, then the walk is said to be *nonbacktracking* (NBT). We define $p_k(A)$ to be the matrix whose (i,j) entry contains the sum of the weights of all nonbacktracking walks of length k from node i to node j. By convention, $p_0(A) = I$. Note that, by definition, $p_k(A) \leq A^k$ elementwise. Combinatorially, the problem of computing the (weighted) enumeration of nonbacktracking walks is equivalent to finding an explicit expression for the generating function

$$\Phi(t) = \sum_{k=0}^{\infty} t^k p_k(A) \tag{2.1}$$

for suitable values of the parameter $t > 0$.

This problem was addressed in [14] for unweighted undirected graphs, and later in [1] for unweighted directed graphs. In [2], the solution was extended to the more general generating function

$$\kappa(t) = \sum_{k=0}^{\infty} c_k t^k p_k(A), \tag{2.2}$$

where $(c_k)_k \subset [0, \infty)$ is an arbitrary sequence. In [3], the theory was further extended to consider time evolving graphs. However, so far, the quantities (2.1) and (2.2) have not yet been studied for weighted graphs. Their characterization is the main contribution of this paper.

A corollary of obtaining such computable expressions is a numerical recipe for associated nonbacktracking centralities. Indeed, beyond its algebraic interpretation as a generating function, (2.2) can be interpreted analytically as a function that will converge for sufficiently small values of the variable t. Choosing one such value for t allows us to define a centrality measure based on the weighted sum of edges. For example, if $\mathbf{1}$ is the vector of all ones, then the i-th component of the vector

$$\left(\sum_{k=0}^{\infty} c_k t^k p_k(A) \right) \mathbf{1}$$

computes a nonbacktracking version of Katz centrality [19]. The latter is defined as the doubly weighted sum of all the walks departing from node i, where the weight of each walk within the sum is the product of the weight of the walk itself and t^k, where k is the walk length. Similarly, for the *subgraph centrality* version of nonbacktracking Katz, the doubly weighted sum of all the walks that start and end on node i is given by

$$\left(\sum_{k=0}^{\infty} c_k t^k p_k(A) \right)_{ii}.$$

As a consequence, two additional questions that we address in this paper are to describe the radius of convergence of (2.2) and to derive computable expressions for the associated centrality measures. We refer to [1, 2, 14], and the references therein, for details of the benefits of nonbacktracking in the centrality context.

We consider two approaches to bridge the gap between weighted graphs and current results on the combinatorics of nonbacktracking walks. The first is specialized to the case $c_k \equiv 1$, i.e., to compute (2.1); it leads directly to an expression that has computational advantages as it does not require to go through the edge-level and, thus, it requires the construction of a potentially much smaller matrix than the second approach. The second is based on a technique, described in [3, 5], of forming the line graph, obtaining a generating function there, and finally projecting back to compute (2.2). While, potentially, the second approach may be computationally less efficient, it has the advantages that (i) it is able to solve the more general problem (2.2), (ii) it can be generalized to the setting of time evolving graphs, and (iii) it allows us to easily estimate the convergence radius of (2.2) (including the special case of (2.1)).

3. The generating function of nonbacktracking walks on a weighted graph.

In this section, we assume that G is a finite directed weighted graph with n nodes, without loops, and having adjacency matrix A. The directed edge from node i to node j has weight $A_{ij} > 0$. Following Definition 2.1, to the walk $i_1 i_2 i_3 \dots i_{\ell+1}$

of length ℓ we assign the weight $A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_\ell i_{\ell+1}}$. We note the distinction here between the *length* and the *weight* of a walk.

The goal of this section is to obtain a convenient formula for the generating function $\Phi(t)$ in (2.1). We note that this generalizes the version previously studied for an unweighted graph [1], and the expression $\Phi(t)\mathbf{1}$ is then a natural candidate for a node centrality measure.

3.1. Describing the matrices $p_k(A)$ via a recurrence relation. Let us first set up some further notation: given two square matrices $X, Y \in \mathbb{R}^{n\times n}$ we distinguish between matrix multiplication, XY, and elementwise multiplication, $X \circ Y$, where $(X \circ Y)_{ij} = X_{ij} Y_{ij}$. Similarly, we differentiate between the k-th linear algebraic power X^k and the k-th elementwise power, $X^{\circ k}$, so $(X^{\circ k})_{ij} = (X_{ij})^k$. Moreover, following Matlab notation, $\mathrm{dd}(X) := \mathrm{diag}(\mathrm{diag}(X))$ will denote the diagonal matrix whose diagonal entries are equal to the diagonal entries of X. We first prove the following k-term recurrence, which generalizes previous results that have been derived independently for the unweighted [10, 27] and undirected [26] cases.

THEOREM 3.1. *For all $k \geq 1$,*

$$p_k(A) = \sum_{\substack{\ell=2h+1 \text{ odd} \\ 1\leq \ell \leq k}} (A^{\circ(h+1)} \circ (A^T)^{\circ h}) p_{k-\ell}(A) - \sum_{\substack{\ell=2h \text{ even} \\ 2\leq \ell\leq k}} \mathrm{dd}((A^{\circ h})^2) p_{k-\ell}(A).$$

Proof. For the base case of $k = 1$, the statement reduces to $p_1(A) = (A \circ \mathbf{1}\mathbf{1}^T) p_0(A)$. Since $A \circ \mathbf{1}\mathbf{1}^T = A$ and $p_0(A) = I$, in turn this yields $p_1(A) = A$, which is manifestly true since any walk of length one is nonbacktracking. Let us now give a proof by induction.

We start by considering $Ap_{k-1}(A)$, whose (i, j) entry is equal to the sum of the weights of all walks of length k from i to j that are nonbacktracking if the first step is removed. This value is equal to $p_k(A)_{ij}$ plus the sum of the weights of all backtracking walks of length k from i to j that are nonbacktracking if the first step is removed. Such walks must be of the form $iai \dots j$: the weight of one such walk is $A_{ia}A_{ai}$ times the weight of a certain NBT walk of length $k - 2$ from i to j. Summing over all a adjacent to i yields $\mathrm{dd}(A^2)p_{k-2}(A)$. However, we have subtracted too much, because any such walk of the form $iaia \dots j$, being backtracking after removing the first step, was not present in $(Ap_{k-1}(A))_{ij}$. The weight of one such walk is $A_{ia}A_{ai}A_{ia} = A_{ia}^2 A_{ai}$ times the weight of a certain NBT walk of length $k-3$ from a to j. We can sum again over all a adjacent to i, to obtain $((A^{\circ 2} \circ A^T)p_{k-3}(A))_{ij}$. We should sum this value back, but again we are adding a bit too much, because walks satisfying the previous requirements and being of the form $iaiai \dots j$ should not be there.

It is clear that this sequence of corrections goes on until we exhaust the length of the walk and the statement of the theorem is a consequence of the two following facts, both true for all $h \geq 0$.

1. The total weight of walks of length k from i to j of the form $i(ai)^h a \dots j$, such that the final subwalk (of length $k-(2h+1)$) from a to j is not backtracking, is equal to

$$\sum_{a:(i,a)\in E} (A_{ia})^{h+1} (A_{ai})^h p_{k-2h-1}(A)_{aj} = \Big((A^{\circ(h+1)} \circ (A^T)^{\circ h}) p_{k-2h-1}(A) \Big)_{ij}.$$

2. The total weight of walks of length k from i to j of the form $(ia)^{2h} i \dots j$, such that the final subwalk (of length $k - 2h$) from i to j is not backtracking, is

equal to

$$\sum_{a:(i,a)\in E} (A_{ia})^h (A_{ai})^h p_{k-2h}(A)_{ij} = \left(\mathrm{dd}((A^{\circ h})^2) p_{k-2h-1}(A)\right)_{ij}. \qquad \square$$

3.2. Solving the recurrence relation. Let us continue by giving a combinatorial result in Proposition 3.2. Its statement expresses the generating function of a sequence satisfying a growing recurrence relation in terms of two individual generating functions.

PROPOSITION 3.2. *Let $(\mathcal{P}_k)_k$ and $(\mathcal{C}_\ell)_\ell$ be two sequences in some (possibly non-commutative) ring, and suppose that $(\mathcal{P}_k)_k$ satisfies the growing recurrence*

$$\sum_{\ell=0}^{k} \mathcal{C}_\ell \mathcal{P}_{k-\ell} = 0$$

for all $k \geq 1$. Then, the (formal) generating functions $\Phi(t) = \sum_{k=0}^{\infty} \mathcal{P}_k t^k$ and $\Psi(t) = \sum_{\ell=0}^{\infty} \mathcal{C}_\ell t^\ell$ are related by the formula $\Psi(t)\Phi(t) = \mathcal{C}_0\mathcal{P}_0$.

Proof. Observe that, using the recurrence,

$$\Psi(t)\Phi(t) = \sum_{k=0}^{\infty} t^k \sum_{\ell=0}^{\infty} \mathcal{C}_\ell \mathcal{P}_{k-\ell} = \mathcal{C}_0\mathcal{P}_0. \qquad \square$$

We can now apply the general technique of Proposition 3.2 to the special case of the generating function (2.1), whose coefficients satisfy the recurrence described in Theorem 3.1. In other words, we specialize Proposition 3.2 to sequences in the ring $\mathbb{R}^{n\times n}$ where $\mathcal{P}_k = p_k(A)$ and

$$\mathcal{C}_\ell = \begin{cases} I & \text{if } \ell = 0; \\ -[A^{\circ(h+1)} \circ (A^T)^{\circ h}] & \text{if } \ell = 2h+1; \\ \mathrm{dd}((A^{\circ h})^2) & \text{if } \ell = 2h > 0. \end{cases}$$

In particular, $\mathcal{C}_0 = \mathcal{P}_0 = I$, and hence by Proposition 3.2 $\Phi(t) = \Psi(t)^{-1}$. In turn, we can write $\Psi(t) = \Psi_e(t) - \Psi_o(t)$ by splitting even and odd terms and by extracting the minus sign appearing in the odd terms of $(\mathcal{C}_\ell)_\ell$. It is easy to see that $\Psi_e(t)$ is diagonal while $\Psi_o(t)$ is the off-diagonal part of $\Psi(t)$, since we assume G to be without loops. Moreover,

$$(\Psi_o(t))_{ij} = \sum_{h=0}^{\infty} t^{2h+1} A_{ij}^{h+1} A_{ji}^{h} = \frac{tA_{ij}}{1 - t^2 A_{ij}A_{ji}}.$$

Similarly,

$$(\Psi_e(t))_{ii} = 1 + \sum_{h=1}^{\infty} t^{2h} \sum_{j=1}^{n} A_{ij}^h A_{ji}^h = 1 + \sum_{j=1}^{n} \frac{t^2 A_{ij}A_{ji}}{1 - t^2 A_{ij}A_{ji}}.$$

Let $S = A \circ A^T$, let $Q = S^{\circ 1/2}$, let

$$f_1(x) = \frac{x}{1-x}, \quad f_2(x) = \frac{x}{1+x},$$

and let $f_i(\circ tX)$ denote the *elementwise* application of f_i to the matrix tX, for $i = 1, 2$. Then, if we denote by $\circ/$ the elementwise application of $/$, we can write

$$\Psi_e(t) = I + \mathrm{dd}(f_1(\circ tQ) f_2(\circ tQ)), \qquad \Psi_o(t) = tA \circ /(\mathbf{1}\mathbf{1}^T - t^2 S)$$

and hence $\Psi(t) = I + \mathrm{dd}(f_1(\circ tQ) f_2(\circ tQ)) - tA \circ /(\mathbf{1}\mathbf{1}^T - t^2 S)$.

We can state this more formally as a theorem.

Theorem 3.3. *In the notation above, for all values of t such that (2.1) converges, we have*

$$\Phi(t) = (I + \mathrm{dd}[f_1(\circ tQ) f_2(\circ tQ)] - tA \circ /(\mathbf{1}\mathbf{1}^T - t^2 S))^{-1}. \tag{3.1}$$

As a sanity check, let us see what happens in three distinct interesting limiting cases that have been addressed previously in the literature.

- First, let us verify that in the limit of an unweighted graph we recover [1, equation (3.3)]. In this case, $(A_{ij})^h = A_{ij} \in \{0, 1\}$ for all $h \geq 1$. As a result, if $D = \mathrm{dd}(A^2)$,

$$(\Psi_e(t))_{ii} = 1 + \sum_{h=1}^{\infty} t^{2h} \sum_{j=1}^{n} A_{ij} A_{ji} = 1 + D_{ii} \frac{t^2}{1 - t^2} \Rightarrow \Psi_e(t) = \frac{I - t^2 I + t^2 D}{1 - t^2}$$

  and

$$(\Psi_o(t))_{ij} = tA_{ij} + \sum_{h=1}^{\infty} t^{2h+1} A_{ij} A_{ji} = tA_{ij} + \frac{t^3 S_{ij}}{1 - t^2} \Rightarrow \Psi_o(t) = \frac{tA - t^3(A - S)}{1 - t^2}$$

  which imply the known $\Phi(t) = (1 - t^2)(I - tA + t^2(D - I) + t^3(A - S))^{-1}$ from [1, Equation (3.3)].
- Next, let us observe that if no edge is reciprocated, that is, if there is no $(i, j) \in E$ such that $(j, i) \in E$, then $S = Q = 0$. Hence, we recover the generating function associated with classical Katz centrality, i.e., $\Phi(t) = (I - tA)^{-1}$, which is consistent with the fact that every walk is nonbacktracking under this assumption.
- Finally, if the graph is undirected then $S = A^{\circ 2}$ and $Q = A$. Hence, the formulae simplify to

$$\Psi_e(t) = I + \mathrm{dd}[f_1(\circ tA) f_2(\circ tA)], \qquad \Psi_o(t) = tA \circ /(\mathbf{1}\mathbf{1}^T - t^2 A^{\circ 2})$$

  yielding in particular

$$\Phi(t) = (I + \mathrm{dd}[f_1(\circ tA) f_2(\circ tA)] - tA \circ /(\mathbf{1}\mathbf{1}^T - t^2 A^{\circ 2}))^{-1}.$$

  If we additionally assume that the graph is unweighted, we further reduce to $\Phi(t) = (1 - t^2)(I - At + t^2(D - I))^{-1}$ in agreement with [14, Equation (5.3)].

We now briefly comment on the convergence of $\Phi(t) = \sum_k p_k(A) t^k$ to the right-hand-side of (3.1). Since the series converges to a rational function, its radius of convergence is equal to the smallest of its poles. One way to compute the radius is therefore via the eigenvalues of the rational function $\Psi(t) = \Phi(t)^{-1}$. A more straightforward method (albeit possibly less efficient) to estimate the radius of convergence is available when computing $\Phi(t)$ with a different method. This is described in more detail in Section 3 and, in particular, within Corollary 4.8. In spite of the somewhat awkward notation, (3.1) is in fact quite straightforward to compute given A, by composing elementwise functions and matrix addition and multiplications.

We conclude this section by recalling that we can define a nonbacktracking version of Katz centrality on weighted graphs by summing the value of the generating function over all possible ending nodes, which can be expressed as the linear algebraic matrix-vector multiplication $\Phi(t)\mathbf{1}$.

The following corollary is then an immediate consequence of Theorem 3.1.

Corollary 3.4. *For all values of t such that (2.1) converges, consider the centrality measure where node i is assigned the value x_i according to $\mathbf{x} = \Phi(1)\mathbf{1}$. Then $\mathbf{x}$ may be found by solving the linear system*

$$(I + \mathrm{dd}[f_1(\circ tQ)f_2(\circ tQ)] - tA \circ /(\mathbf{1}\mathbf{1}^T - t^2 S))\mathbf{x} = \mathbf{1}. \tag{3.2}$$

Corollary 3.4 shows in particular that the centrality measure can be found without explicitly computing the inverse in (3.1). We can instead compute the vector of nonbacktracking centralities $\mathbf{x}$ by solving the linear system (3.2). We note that the coefficient matrix in (3.2) is no less sparse than $I - tA$; hence the computational complexity of solving such a linear system is the same as for classical Katz centrality, and the task is feasible with standard tools for sparse linear systems for very large, sparse networks.

4. Generating function by constructing the line graph and projecting back. In this section, we derive an alternative computable expression for the generating function $\Phi(t)$. Although generally this second method is less computationally efficient, it offers three main advantages: (i) it can be extended to nonbacktracking centrality measures other than Katz (for example, based the exponential rather than the resolvent); (ii) it allows for a simple characterization of the radius of convergence of the generating function; and (iii) it can be extended to time evolving graphs.

As before, we consider a finite weighted graph with n nodes. We also assume (directed) edges have been labelled from 1 to m in an arbitrary, but fixed, manner. We may then define the *source matrix* $L \in \mathbb{R}^{m\times n}$ and *target* (or *terminal*) *matrix* $R \in \mathbb{R}^{m\times n}$ as follows [31]:

$$L_{ej} = \begin{cases} 1 & \text{if edge } e \text{ starts from node } j \\ 0 & \text{otherwise} \end{cases} \qquad R_{ej} = \begin{cases} 1 & \text{if edge } e \text{ ends on node } j \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we let Z be an $m \times m$ diagonal matrix such that $Z_{ee} = A_{ij}$, where (in the chosen labelling of the edges) the e-th edge is precisely (i, j).[1] Then, we have the following relationship.

Proposition 4.1. *We have $A = L^T ZR$.*

Proof. Since Z is diagonal, $(L^T ZR)_{ij} = \sum_{e=1}^m L_{ei}Z_{ee}R_{ej}$. But there is at most one value of e such that $L_{ei}R_{ej} \neq 0$, and that is precisely the value identifying the edge $i \to j$, if this is an edge of the graph. If such an edge does not exist then the summation yields 0, as desired. If such an edge exists, then, for that e, $Z_{ee} = A_{ij}$ which concludes the proof. ∎

Now let W be the weighted matrix of the dual graph (or line graph), i.e., the graph whose nodes correspond to the original (directed) edges, and whose edges are

[1] For clarity, we will sometimes use the notation $i \to j$ to denote the edge $(i, j) \in E$.

pairs of edges from the original graph that can form a walk. The pair $(i \to j, j \to k)$ represents a walk that has weight equal to the product of the original edge weights; that is, $A_{ij}A_{jk}$. These values are recorded in the entries of W, with $W_{ef} = A_{ij}A_{jk}$ if e is the label of edge $i \to j$ and f is the label of edge $j \to k$.

THEOREM 4.2. *We have* $W = ZRL^TZ$.

Proof. We proceed entrywise. Suppose for concreteness that edge e is $i \to j$ and edge f is $k \to \ell$, where $i \neq j$, $k \neq \ell$ are (possibly, but not necessarily, all distinct) nodes. Note for a start that $W_{ef} = A_{ij}A_{j\ell}$ if $j = k$ and $W_{ef} = 0$ if $j \neq k$. Now, since Z is diagonal,

$$(ZRL^TZ)_{ef} = Z_{ee}Z_{ff}\sum_{h=1}^{n} R_{eh}L_{fh} = A_{ij}A_{k\ell}\sum_{h=1}^{n} R_{eh}L_{fh}.$$

Suppose $j \neq k$; then there is no h such that $R_{eh}L_{fh} \neq 0$, so the summation above is $0 = W_{ef}$. On the other hand, if $j = k$ then the summation over h yields 1 so that $(ZRL^TZ)_{ef} = A_{ij}A_{j\ell} = W_{ef}$. ◻

In the unweighted case, we have a projection relation $L^TW^kR = A^{k+1}$ [5, Proposition 2.4]. However, for weighted graphs, entries of W^k count walks of length $k+1$, but with incorrect weights. For example, the walk $1 \to 2 \to 3 \to 4$ would be weighted $A_{12}A_{23}^2A_{34}$ rather than $A_{12}A_{23}A_{34}$. We now exhibit a trick that corrects this problem. Coherently with the notation of the previous section, below $M^{\circ 1/2}$ denotes the elementwise nonnegative square root of a nonnegative matrix M; note that generally this does not correspond to the classical matrix square root $\sqrt{M}$ (i.e., the matrix X such that $X^2 = M$), a notable exception being the case of a diagonal square matrix with nonnegative diagonal. We note that Z falls in this latter category, hence the notation in the following result.

THEOREM 4.3. *Let* $0 < k \in \mathbb{N}$. *The* (e, f) *element of* $\sqrt{Z}(W^{\circ 1/2})^k\sqrt{Z}$ *counts, with weights, all walks of length* $k+1$ *from edge* e *to edge* f.

Proof. The crucial observation is that $W^{\circ 1/2} = \sqrt{Z}RL^T\sqrt{Z}$, which is clear by a minor modification of the proof of Theorem 4.2. We now proceed by induction on k. For the base case $k = 1$, it suffices to observe that $\sqrt{Z}W^{\circ 1/2}\sqrt{Z} = ZRL^TZ = W$. Suppose now that the statement holds for $k-1$. Then,

$$\sqrt{Z}(W^{\circ 1/2})^k\sqrt{Z} = \sqrt{Z}(W^{\circ 1/2})^{k-1}\sqrt{Z}(Z^{-1/2})W^{\circ 1/2}\sqrt{Z}.$$

Define for notational simplicity $U := \sqrt{Z}(W^{\circ 1/2})^{k-1}\sqrt{Z}$, $X := Z^{-1/2}$, $Y := W^{\circ 1/2}$, $\Sigma := \sqrt{Z}$. Then, since X and Σ are diagonal,

$$(UXY\Sigma)_{ef} = \sum_{g \in E} U_{eg}X_{gg}Y_{gf}\Sigma_{ff}.$$

Suppose now that edge e is $i \to j$ and edge f is $h \to \ell$; then edges g must be of the form $x \to h$ for some node x. Indeed, $Y_{gf} = 0$ unless the end node of edge g coincides with the start node of edge f, i.e., unless gf is a walk of length two. Hence, in this notation,

$$(UXY\Sigma)_{ef} = \sum_{x: A_{xh} > 0} U_{e,x\to h}\sqrt{A_{h\ell}A_{xh}}\sqrt{\frac{A_{h\ell}}{A_{xh}}} = A_{h\ell}\sum_{x: A_{xh}>0} U_{e,x\to h},$$

where $\sum_{x:A_{xh}>0} U_{e,x\to h}$ is, by the inductive assumption, the count (with weights) of all walks of length $k-1$ from edge e to all edges of the form $x \to h$, i.e., the weighted enumeration of all walks of length $k-1$ from edge e to node h. However, the count with weights of all walks of length k from edge e to edge f is precisely the count with weights of all walks of length k from edge e to node ℓ with node h as the penultimate node, i.e., the right hand side in the latter displayed equation. ◻

We have the following consequence of Theorem 4.3.

Corollary 4.4. *For all* $k \in \mathbb{N}$, $L^T\sqrt{Z}(W^{\circ 1/2})^k\sqrt{Z}R = A^{k+1}$.

Proof. The result follows from Proposition 4.1, if $k = 0$, and from Theorem 4.3, if $k > 0$. ◻

Now let $B \in \mathbb{R}^{m\times m}$ be the nonbacktracking version of W, i.e., $B_{ef} = 0$ if $W_{ef}W_{fe} \neq 0$ and $B_{ef} = W_{ef}$ otherwise. This matrix is often referred to as the *Hashimoto matrix* [16]. Recall, moreover, that $p_k(A) \in \mathbb{R}^{n\times n}$ is the matrix counting all NBT walks of length k (from i to j in its (i, j) element). Now we can observe that all the proofs above hold for B as well, modulo substituting walks with nonbacktracking walks. Hence, the projection relation still holds.

Theorem 4.5. *For all* $k \in \mathbb{N}$, *we have* $L^T\sqrt{Z}(B^{\circ 1/2})^k\sqrt{Z}R = p_{k+1}(A)$.

Proof. We have $p_1(A) = A = L^TZR$, and when $k > 0$ the result follows from a minor modification of the arguments used to prove Corollary 4.4. ◻

Suppose now that $(c_k)_k \subset [0, \infty)$ is a sequence and t is such that

$$\kappa(t, A) = \sum_{k=0}^{\infty} c_k t^k p_k(A)$$

as in (2.2) converges; we are interested in the centrality measure

$$\boldsymbol{v}(t, A) = \kappa(t, A)\mathbf{1}. \tag{4.1}$$

We now derive formulae for $\kappa(t, A)$ and $\boldsymbol{v}(t, A)$. To this end, we introduce the following notation. Given a real-analytic scalar function

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

consider the operator

$$\partial f(x) = \sum_{k=0}^{\infty} c_{k+1} x^k = \frac{f(x) - c_0}{x}.$$

We then have the following.

Theorem 4.6. *It holds that*

$$\sum_{k=0}^{\infty} c_k t^k p_k(A) = c_0 I + tL^T\sqrt{Z}\partial f(tV)\sqrt{Z}R,$$

for $V = B^{\circ 1/2}$ *and* $|t| < r/\rho(V)$, *where* $\rho(V)$ *is the spectral radius of* V *and* r *is the radius of convergence of the scalar function* $f(x) = \sum_{k=0}^{\infty} c_k t^k$.

Hence, for the centrality associated with $f(x)$ *and* t *small enough to give convergence in the matrix power series, in* (4.1) *we have*

$$\boldsymbol{v}(t, A) = c_0\mathbf{1} + tL^T\sqrt{Z}\partial f(tV)\sqrt{Z}\mathbf{1}.$$

Proof. By Theorem 4.5 we easily see that

$$\kappa(t, A) = c_0 I + tL^T \sqrt{Z} \left(\sum_{k=0}^{\infty} c_{k+1} t^k (B^{\circ 1/2})^k \right) \sqrt{Z} R.$$

As a consequence,

$$\boldsymbol{v}(t, A) = c_0 \mathbf{1} + tL^T \sqrt{Z} \left(\sum_{k=0}^{\infty} c_{k+1} t^k (B^{\circ 1/2})^k \right) \sqrt{Z} \mathbf{1}.$$

Observing that the resolvent is an eigenfunction (with eigenvalue 1) of ∂, we note in particular that for Katz centrality, i.e., $c_k = 1$ for all k, $\partial f(x) = f(x) = (1-x)^{-1}$. Hence, we have the following special case.

Corollary 4.7. *In the notation of Theorem 4.6, we have that the generating function $\Phi(t)$ defined in (2.1) can be expressed as*

$$\Phi(t) = I + tL^T \sqrt{Z} (I - tV)^{-1} \sqrt{Z} R. \tag{4.2}$$

This analysis in particular yields a lower bound for the radius of convergence for (2.1).

Corollary 4.8. *If $|t| < \rho(V)^{-1}$, where $V = B^{\circ 1/2}$, then the sequence $\Phi(t) = \sum_{k=0}^{\infty} p_k(A) t^k$ converges.*

Remark 4.9. *Letting r denote the radius of convergence of (2.1), Corollary 4.8 shows that $r \geq \rho(V)^{-1}$. It is possible to strengthen this result and prove that $r = \rho(V)^{-1}$. A proof of this fact, which is beyond the scope of the present article, appears in [25, Theorem 5.2].*

5. Nonbacktracking centralities for evolving weighted graphs. In Sections 3 and 4, we obtained formulae for the generating function $\Phi(t)$ in (2.1) by working, respectively, at node and edge level. For a static network, i.e., one which does not evolve in time, working at the node level is clearly preferable as, for large n, we may have that $n \ll m$. However, a significant advantage of the latter, edge-level, formula is that it easily extends to the case of temporal networks in all backtracking regimes, whereas a direct node-level formula which forbids backtracking in time is generally unavailable [3]. Let us first generalize the definition of graph, walk, and NBT walk to the dynamic case.

Definition 5.1. *A finite time-evolving graph $\mathcal{G}$ is a finite collection of graphs $(G^{[1]}, \ldots, G^{[N]})$, associated with the non-decreasing time stamps $(t_1, \ldots, t_N) \in \mathbb{R}^N$, such that the set of nodes of $G^{[i]}$ does not depend on i and when observed at time t_i the structure of $\mathcal{G}$ is identical to that of $G^{[i]}$.*

We remark that the concept of a graph can be extended to the dynamic setting in a number of ways [17]. The discrete-time framework of Definition 5.1 covers a range of realistic scenarios where interactions take place, or are recorded, at specific points in time. For example, in an on-line social media platform, an edge may represent a form of communication between users, and $G^{[i]}$ may count the number of interactions between each pair of individuals over time $(t_{i-1}, t_i]$.

The definition of walk across a network can be extended to the setting of temporal graphs as follows.

Definition 5.2. A walk of length ℓ *across a temporal network is defined as an ordered sequence of* ℓ *edges* $e_1 e_2 \dots e_\ell$ *such that for all* $k = 1, \dots, \ell-1$ *the end node of* e_k *coincides with the start node of* e_{k+1} *and, moreover, that* $e_k \in E^{[\tau_1]}, e_{k+1} \in E^{[\tau_2]}$ *for some* $1 \le \tau_1 \le \tau_2 \le N$*, where* $E^{[\tau_i]}$ *denotes the set of edges of the graph* $G^{[\tau_i]}$*.*

It is useful to make an equivalent definition.

Definition 5.3. A walk of length ℓ *across a temporal network is defined as an ordered sequence of* $\ell+1$ *nodes* $i_1 i_2 \dots i_{\ell+1}$ *such that for all* $k = 2, \dots, \ell$ *it holds that* $i_{k-1} \to i_k \in E^{[\tau_1]}$ *and* $i_k \to i_{k+1} \in E^{[\tau_2]}$ *for some* $1 \le \tau_1 \le \tau_2 \le N$*.*

We want to stress that multiple edges can be crossed at one given time stamp and, moreover, that a walk is allowed to remain inactive for some of the time stamps. We also recall here that there is not just one definition of backtracking for temporal networks; indeed, three arise naturally [3]:

- backtracking happens within a certain time-stamp; we will refer to this as *backtracking in space*,
- backtracking happens across time-stamps; we will refer to this as *backtracking in time*,
- backtracking happens both within a time-stamp and across time-stamps (not necessarily in this order); we will refer to this as *backtracking in time and space*.

Given any finite time-evolving graph $\mathcal{G}$, we can associate with it a matrix M called the *global temporal transition matrix* which was defined in [3] for unweighted graphs. Definition 5.4 below generalizes the definition of the global temporal transition matrix to the weighted case.

Definition 5.4. *Let* $\mathcal{G} = (G^{[1]}, G^{[2]}, \dots, G^{[N]})$ *be a time-evolving graph with* N *time stamps. The* weighted global temporal transition matrix *associated with* $\mathcal{G}$ *is the* $m \times m$ *block matrix*

$$M = M^{[1,\dots,N]} = \begin{bmatrix} C^{[1]} & C^{[1,2]} & C^{[1,3]} & \dots & C^{[1,N]} \\ 0 & C^{[2]} & C^{[2,3]} & \dots & C^{[2,N]} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & C^{[N]} \end{bmatrix}^{\circ 1/2}, \tag{5.1}$$

where the definition of the blocks depends on the chosen backtracking regime in the following way:

(i) $C^{[\tau_1]} = W^{[\tau_1]}$ *and* $C^{[\tau_1,\tau_2]} = W^{[\tau_1,\tau_2]} := S^{[\tau_1]} R^{[\tau_1]} (L^{[\tau_2]})^T S^{[\tau_2]}$ *for all* $\tau_1, \tau_2 = 1, 2, \dots, N$ *(*$\tau_1 < \tau_2$*) if backtracking in both space and time is permitted;*

(ii) $C^{[\tau_1]} = B^{[\tau_1]}$ *and* $C^{[\tau_1,\tau_2]} = W^{[\tau_1,\tau_2]}$ *for all* $\tau_1, \tau_2 = 1, 2, \dots, N$ *(*$\tau_1 < \tau_2$*) if backtracking in space is forbidden but backtracking in time is permitted;*

(iii) $C^{[\tau_1]} = W^{[\tau_1]}$ *and* $C^{[\tau_1,\tau_2]} = B^{[\tau_1,\tau_2]} := W^{[\tau_1,\tau_2]} - (W^{[\tau_1,\tau_2]} \circ W^{[\tau_2,\tau_1]^T})^{\circ 1/2}$ *for all* $\tau_1, \tau_2 = 1, 2, \dots, N$ *(*$\tau_1 < \tau_2$*) if backtracking in time is forbidden but backtracking in space is permitted; and*

(iv) $C^{[\tau_1]} = B^{[\tau_1]}$ *and* $C^{[\tau_1,\tau_2]} = B^{[\tau_1,\tau_2]}$ *for all* $\tau_1, \tau_2 = 1, 2, \dots, N$ *(*$\tau_1 < \tau_2$*) if backtracking in both time and space is forbidden.*

It was further shown in [3] that the global temporal transition matrix provides an accurate way of counting walks in all backtracking regimes across a finite unweighted time-evolving graph and thereby allows for the computation of the (nonbacktracking)

Katz centrality $\boldsymbol{v}$ via the formula:

$$\boldsymbol{v}(t) = (I + t\mathcal{L}^T(I - tM)^{-1}\mathcal{R})\mathbf{1}, \tag{5.2}$$

where $\mathcal{L}, \mathcal{R}$ are the global source and target matrices respectively as defined in [3, Definition 4.4].

To handle the weighted case, we may extend Theorem 4.5 naturally to the global temporal transition matrix M in the following way.

Theorem 5.5. *For a finite time-evolving graph with N-many time frames, let the* global weight matrix Z *be defined block-wise as*

$$Z := Z^{[1,2,\dots,N]} = \operatorname{diag}(Z^{[1]}, Z^{[2]}, \dots, Z^{[N]}),$$

where $Z^{[\tau_i]}$ is the diagonal matrix associated with time stamp $1 \le \tau_i \le N$. For each of these matrices, their diagonal entries are given by $Z^{[\tau_i]}_{ee} = {w_e}^{[\tau_i]}$, with ${w_e}^{[\tau_i]}$ being the weight of edge e at time stamp τ_i. Further let the backtracking regime be fixed such that the weighted global temporal transition matrix M is fixed. Then, for $0 < k \in \mathbb{N}$, the (e, f)-th entry of $\sqrt{Z}M^k\sqrt{Z}$ counts, with weights, all permitted walks of length $k + 1$ from edge e to edge f across the time evolving graph given the backtracking regime.

Proof. Suppose the backtracking regime is given such that the structure of M is fixed as specified in Definition 5.4. We prove the theorem by induction on the length of permitted walks $k \in \mathbb{N}$. Consider the basis case $k = 1$:

$$\sqrt{Z}M\sqrt{Z} = \sqrt{Z}\begin{bmatrix} C^{[1]} & C^{[1,2]} & C^{[1,3]} & \dots & C^{[1,N]} \\ 0 & C^{[2]} & C^{[2,3]} & \dots & C^{[2,N]} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & C^{[N]} \end{bmatrix}^{\circ 1/2} \sqrt{Z}.$$

The (e, f)-th element of this matrix correctly counts the unique walk of length two with weights from edge e to edge f. In the following we will omit the temporal superscript, since the indices e and f also uniquely determine the time frame. With this convention:

$$\begin{aligned}\left(\sqrt{Z}M\sqrt{Z}\right)_{ef} &= \sum_{r,s} \sqrt{Z}_{er} M_{rs} \sqrt{Z}_{sf} \\ &= (\sqrt{Z})_{ee}(\sqrt{Z})_{ff} M_{ef} \\ &= \begin{cases} \sqrt{w_e}\sqrt{w_f}\sqrt{w_e w_f} = w_e w_f & ef \text{ is a permitted walk of length two} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

where we have used the fact that, if ef is an admissible walk of length two, then $(M)_{ef} = \sqrt{w_e w_f}$.

Suppose now that the result holds for $k-1$ and for brevity denote by P the matrix $\sqrt{Z}M^{k-1}\sqrt{Z}$, which, by the inductive assumption, correctly counts in its entries the number of weighted temporal walks of length k. Then, using the fact that Z is diagonal,

$$(\sqrt{Z}M^k\sqrt{Z})_{ef} = \sum_{r,s,t}\left(\sqrt{Z}M^{k-1}\sqrt{Z}\right)_{er}(Z^{-1/2})_{rs}M_{st}(\sqrt{Z})_{tf}$$
$$= \sum_r P_{er}(Z^{-1/2})_{rr}M_{rf}(\sqrt{Z})_{ff}$$
$$= \begin{cases} w_f \sum_r P_{er} & e\dots rf \text{ is a permitted walk of length } k+1 \\ 0 & \text{otherwise.}\end{cases}$$

By the inductive assumption P_{er} counts the permitted weighted walks of length k from edge e to edge r. Therefore the above formula does indeed count the weighted permitted walks beginning with edge e and ending on edge f with length $k+1$ correctly. ∎

THEOREM 5.6. *Given global source, target and weight matrices, $\mathcal{L}$, $\mathcal{R}$ and Z respectively, we can compute M when backtracking is entirely forbidden in two steps:*
1. $\widehat{M} = \left(\sqrt{Z}(\mathcal{R}\mathcal{L}^T - \mathcal{R}\mathcal{L}^T \circ \mathcal{L}\mathcal{R}^T)\sqrt{Z}\right)$*;*
2. *Obtain M from $\widehat{M}$ by setting all entries below the block diagonal to* 0.

Proof. The above theorem is easy to prove by observing that the (i,j)-th block of the matrix $\mathcal{R}\mathcal{L}^T$ is equal to $R^{[i]}{L^{[j]}}^T$, whereas the (i,j)-th block of the matrix $\mathcal{L}\mathcal{R}^T$ is equal to $L^{[i]}{R^{[j]}}^T = (R^{[j]}{L^{[i]}}^T)^T$; whereupon the (i,j)-th block of the matrix $\widehat{M}$ becomes

$$\widehat{M}_{ij} = \sqrt{Z^{[i]}}\left(R^{[i]}{L^{[j]}}^T - R^{[i]}{L^{[j]}}^T \circ (R^{[j]}{L^{[i]}}^T)^T\right)\sqrt{Z^{[j]}}.$$

The central term here in brackets can be seen as the binarized $B^{[i,j]}$, i.e., $B^{[i,j]}$ where all non-zero weights are uniformly equal to 1, thus the presence of a non-zero entry $(B^{[i,j]})_{ef}$ simply reflects whether or not the concatenation of edges e and f forms a non-backtracking walk of length two. Matrix multiplication from the left by $\sqrt{Z^{[i]}}$ and on the right by $\sqrt{Z^{[j]}}$ then provides the appropriate weighting for the (e,f)-th entry, namely $\sqrt{w_e}\sqrt{w_f}$, as required. Finally, the second step of setting all blocks below the block diagonal, i.e., $\widehat{M}_{ij}$ with $i > j$, to zero reflects the requirement that walks may not move back in time. ∎

We can also compute the f-total communicability of the time-evolving graph with weights by using the global temporal transition matrix M.

THEOREM 5.7. *Given a function f with series expansion $f(t) = \sum_{k=0}^{\infty} c_k t^k$ having radius of convergence r, and some fixed backtracking regime, the f-total communicability $\boldsymbol{v}_f(t)$ of the time-evolving graph $\mathcal{G} = (G^{[1]}, \dots, G^{[N]})$ with N time stamps is given by the formula:*

$$\boldsymbol{v}_f(t) = (c_0 I + t\partial\mathcal{L}^T\sqrt{Z}f(tM)\sqrt{Z}\mathcal{R})\mathbf{1} \tag{5.3}$$

for $0 < |t| < r/\max_{i=1,\dots,N}\{\rho(C^{[\tau_i]})\}$.

Proof. By Proposition 5.5, we have that $\sqrt{Z}M^k\sqrt{Z}$ counts with weights all walks of length $k+1$, thus

$$\boldsymbol{v}_f(t) = (c_0 I + t\partial\mathcal{L}^T\sqrt{Z}f(tM)\sqrt{Z}\mathcal{R})\mathbf{1} = c_0 I + t\sum_{k=0}^{\infty} c_{k+1}t^k\mathcal{L}^T\sqrt{Z}M^k\sqrt{Z}\mathcal{R}\mathbf{1}.$$

Table 6.1: Static network convergence information.

Non-binarized graph $\rho(B^{\circ 1/2})$	926.9
Binarized graph $\rho(B)$	48.61
Non-binarized nonbacktracking permitted range of t	$t \in [0, 1.079 \cdot 10^{-3})$
Binarized nonbacktracking permitted range of t	$t \in [0, 2.057 \cdot 10^{-2})$
Non-binarized $\rho(A)$	1038
Binarized $\rho(A)$	51.26
Non-binarized backtracking permitted range of t	$t \in [0, 9.635 \cdot 10^{-4})$
Binarized backtracking permitted range of t	$t \in [0, 1.951 \cdot 10^{-2})$

In the above formula we see the number of walks of length $k+1$ correctly counted with weights that are further weighted by the coefficient $c_{k+1}t^{k+1}$, which is provided by the series expansion of $f(x)$. ∎

6. Numerical Experiments.

In this section we show how the formulae for nonbacktracking Katz centrality from sections 3, 4 and 5 may produce significantly different node-rankings for real-world social networks when compared with Katz centrality which permits backtracking walks. We further examine the effect of weighting edges on the rankings produced by both centrality measures. To this end, we consider the Katz centrality formula (5.2) as applied to one static network and one temporal network, both derived from the same data set (Fauci's email release [6]) [2]. The original dataset is a collection of over 3000 pages of emails involving Anthony Fauci and his staff during the COVID-19 pandemic. Data includes sender and receivers (including CC'd) of emails, as well as time stamps of when the emails were sent. Both networks used in the following were presented in [6].

6.1. Analysis on Static Networks.

In this section, we analyze a static network produced by [6] which is both undirected and weighted. We have an edge (i, j) if there exists an email which involves both nodes i and j as any combination of sender and recipient (including CC'd recipients). The weight assigned to such an edge, $\Omega((i, j))$, is a positive integer equal to the number of such emails that were sent.

In our analysis, we apply Corollary 3.4 to obtain the NBT Katz centrality vector for our network, which is then contrasted with the classical Katz centrality vector for attenuation factor values $t = 0.5/\rho$ and $t = 0.95/\rho$, where $1/\rho$ is the radius of convergence for the respective centrality measure. In particular $1/\rho$ is equal to $1/\rho(A)$, where A is the adjacency matrix of the graph in the case of classical Katz centrality; whereas $1/\rho = 1/\rho(B^{\circ 1/2})$ in the case of weighted nonbacktracking Katz centrality [25, Theorem 5.2], where B is the Hashimoto matrix associated to the graph. These values are given in Table 6.1. We also analyze the binarized graph which is produced from the static graph by setting all edge weights to 1. In the context of the binarized network $1/\rho$ equals $1/\rho(A)$ in the case of classical Katz centrality, and $1/\rho(B)$ for nonbacktracking Katz centrality, where A and B are the adjacency and Hashimoto matrices associated with the binarized network, respectively.

The results are visualised in Figures 6.1, 6.2 and 6.3. Figure 6.1 shows that NBT Katz centrality emphasizes a clique not containing the node corresponding to Antony Fauci, and that for large values of the attenuation factor this clique begins to

[2] The code used in the following analysis can be found at https://github.com/rwood12347/Weighted-enumeration-of-nonbacktracking-walks-on-weighted-graphs

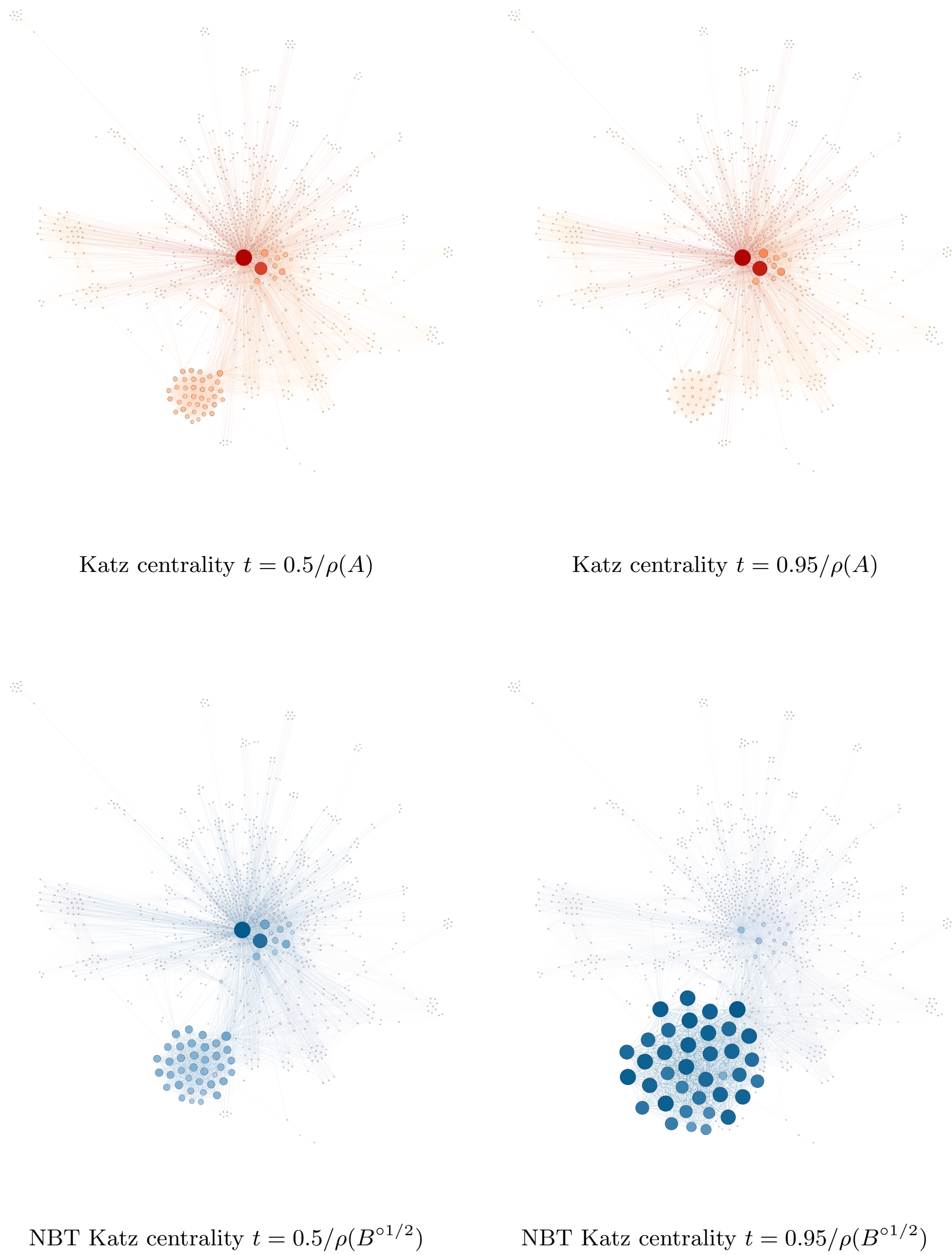

Fig. 6.1: Visualizations of classical (top/red) and NBT Katz (bottom/blue) across the static email network with large node size and dark colour indicating large centrality values; darker edges indicate a larger weight.

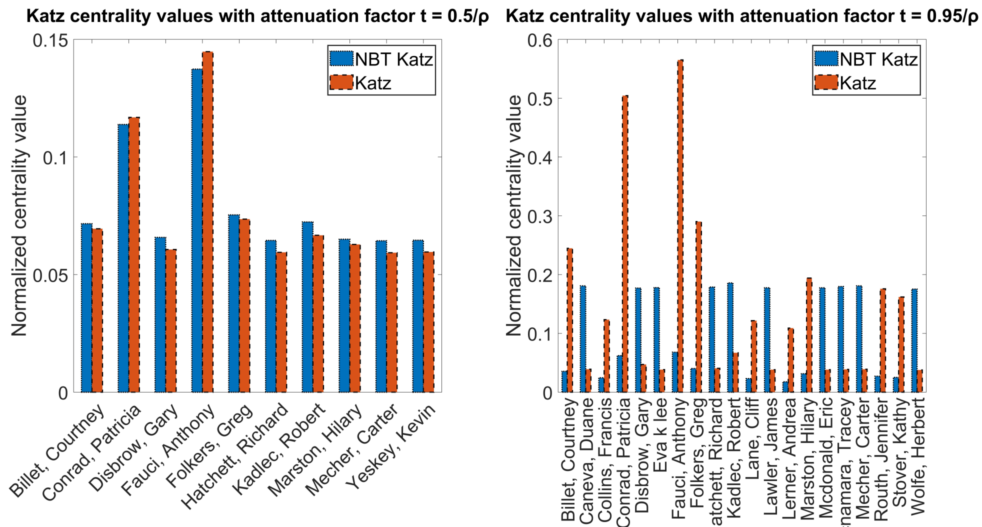

Fig. 6.2: Classical and nonbacktracking Katz centrality vector values for backtracking fully forbidden with attenuation factor $t = 0.5/\rho$ and $t = 0.95/\rho$, respectively. In each plot we display the union of the 10 most central nodes according to each centrality measure.

dominate the ranking to such an extent that the node corresponding to Anthony Fauci, which occupies the central position in the network visualisation, is no longer counted among the 10 most central nodes. This can be seen in Figure 6.2 which depicts the nonbacktracking and classical Katz normalized centrality values for the union of the 10 most central nodes in the static network. The left bar chart in Figure 6.2 indicates that both classical and nonbacktracking Katz agree on the 10 most central nodes of which 'Anthony Fauci' is most central when $t = 0.5/\rho$. However the rightmost figure depicts a complete divergence in the ten most highly-ranked nodes produced by classic and nonbacktracking Katz centralities respectively. In particular we see that while the 'Anthony Fauci' node remains fairly central according to both measures, nodes belonging to the clique shown in Figure 6.1 have overtaken it in the ranking induced by nonbacktracking Katz centrality. The clique identified in this case consists exclusively of participants (i.e., either directly sent or received an email within the thread, or were CC'd in an email within the thread) in the so-called 'Red Dawn' email thread that was used throughout the pandemic "to provide thoughts, concerns, raise issues, share information across various colleagues responding to Covid-19" [8].

The effect of weighted edges on the rankings produced by nonbacktracking and classical Katz centralities for the static network is demonstrated in Figure 6.3. The figure contains two scatter graphs of the normalized nonbacktracking Katz centrality vector ($t = 0.95/\rho(B^{\circ 1/2})$) plotted against the Katz centrality vector ($t = 0.95/\rho(A)$) for both the original network (right) and a binarized modified network (left), which is formed from the original network by setting all edge weights to 1.

In particular we see that the presence of non-uniformly weighted edges in the network produces greater variation in the nonbacktracking and classical Katz centrality vectors.

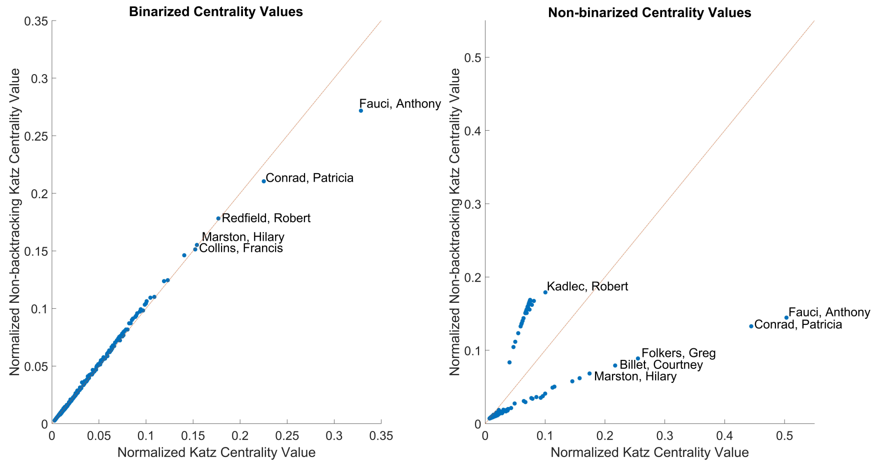

Fig. 6.3: Scatter plots of normalized NBT Katz centrality against normalized classical Katz centrality corresponding to binarized and non-binarized static networks with attenuation factor $t = 0.95/\rho$.

6.2. Analysis on Temporal Networks. We now move on to the case of a time-dependent network, and we note that the special case of an unweighted network with backtracking permitted corresponds to the work in [15] wherein the *dynamic communicability matrix* $\mathcal{Q}(t)$ associated to such a network is defined as the product of the successive resolvents

$$\mathcal{Q}(t) = (I - tA^{[1]})^{-1}(I - tA^{[2]})^{-1} \cdots (I - tA^{[N]})^{-1}. \tag{6.1}$$

Here $A^{[i]}$ is the adjacency matrix associated to the i-th time-stamp of the temporal network $\mathcal{G}$. Katz centrality can then be computed via the formula

$$\boldsymbol{x}(t) = \mathcal{Q}(t)\mathbf{1}. \tag{6.2}$$

This formula accounts for all walks across the temporal network $\mathcal{G}$ including those that backtrack in space and between time-stamps.

The temporal network $\mathcal{G}$ analyzed in this section is the largest temporal strong component [9] of the provided email data, i.e., the largest component that is connected in the sense that there exists a time-respecting path between any two nodes contained within. This network consists of a collection of 100 directed networks associated with the date 2018-09-04 and the 99 consecutive days between 2020-01-26 and 2020-05-05. In this network we have a directed weighted edge $(i, j) \in E(G^{[\tau_t]})$, if node j is a recipient of, or is CC'd in, an email sent by node i. The weight of such an edge is equal to the number of such emails sent during the t-th timestamp.

We reiterate here that when treating temporal networks there is a range of possible nonbacktracking regimes, as outlined in Definition 5.4. The choice of appropriate backtracking regime is highly context-dependent. For the data set analyzed here, it is reasonable to forbid backtracking entirely, since the time-stamps associated with the temporal network have an almost uniform spacing of one day, and the time taken to reply to an email is on a similar scale to the spacing between time-stamps. It is

Table 6.2: Temporal network convergence information.

$\rho(M)$	5.025
nonbacktracking permitted range of t	$t \in [0, 0.1990)$
$\max_i(\rho(A^{[i]}))$	8.832
Backtracking permitted range of t	$t \in [0, 0.1132)$

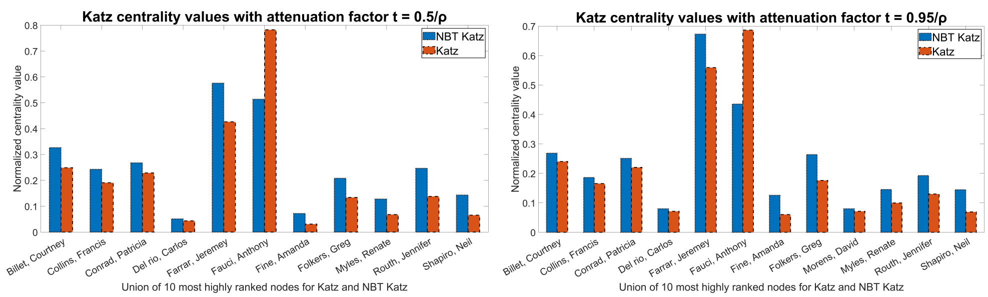

Fig. 6.4: The time-evolving network centrality vector values for both NBT and classical Katz with attenuation factor $t = 0.5/\rho$ and $t = 0.95/\rho$ respectively. In each plot we display the union of the ten most central nodes according to each centrality measure.

worth mentioning that this choice to fully forbid backtracking is subjective and other regimes may also be reasonable.

Our analysis of the spectrum of the global temporal transition matrix M associated to the graph $\mathcal{G}$ with backtracking fully-forbidden yields the permitted ranges of attenuation factor t shown in Table 6.2. We contrast this with the permitted range of t in the case of classical Katz centrality via the dynamic communicability matrix $\mathcal{Q}$ as defined in (6.1).

Figure 6.4 depicts two bar charts which display the normalized centrality values for both classical and nonbacktracking Katz centralities for $t = 0.5/\rho$ and $t = 0.95/\rho$ respectively, where $1/\rho$ is the upper-limit of the respective regime as given in Table 6.2. In particular $1/\rho$ is equal to $1/\rho(M)$ (see the proof of [25, Theorem 5.2]) in the case of nonbacktracking Katz centrality, where M is the matrix described in Definition 5.4 *(iv)* that is, the form of M in which all forms of backtracking are forbidden. In the case of classical Katz centrality $1/\rho$ is given by $1/\max_i(\rho(A^{[i]})$, the reciprocal of the largest principal eigenvalue of the adjacency matrices. In Figure 6.4 we report results for 12 nodes, which are selected by taking the union of the 10 most highly ranked nodes for classical Katz and the 10 most highly ranked nodes for NBT Katz, when $t = 0.95/\rho$.

In Figure 6.5 we plot for the weighted temporal network both the classical and nonbacktracking Katz centrality values of 10 selected nodes against the attenuation factor t which ranges from 0% to 99% of its permitted range (as given in Table 6.2). The 10 nodes were selected such that they are the most central for large values of t.

Figure 6.6 presents results for the same experiment, this time carried out with the binarized version of the temporal network, i.e., the temporal network with all non-zero weights set to 1.

It is interesting to note that nonbacktracking Katz identifies a node distinct from "Anthony Fauci" as the most central node for large values of t, favouring instead the node "Jeremy Farrar" which is considerably lower ranked in the static networks produced from the same data set. Furthermore by comparing Figures 6.5 and 6.6, we observe the large effect that weighting has on the two centrality measures.

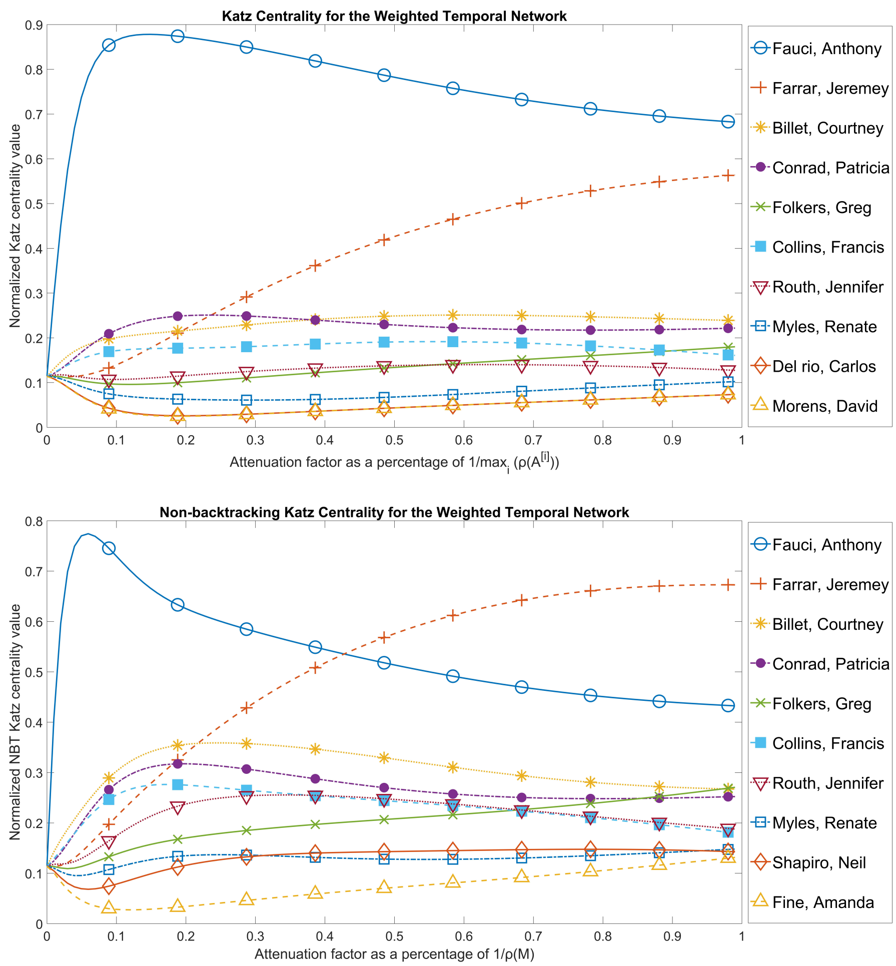

Fig. 6.5: Plots of the normalized Katz (upper) and nonbacktracking Katz (lower) centralities vector values for 10 most prominent nodes (i.e., those with the largest centrality value as of the upper limit of the attenuation factor t) within the weighted temporal network

7. Discussion. Our aim in this work was to develop a useful theory for the enumeration of nonbacktracking walks as well as for associated centrality measures, in the case of edge weights that are combined multiplicatively. We showed in Theorem 3.1 that in contrast to the unweighted case where a four-term recurrence is sufficient to count nonbacktracking walks of different lengths, the weighted case gives rise to a recurrence where the walk count at length k depends on walk counts for all

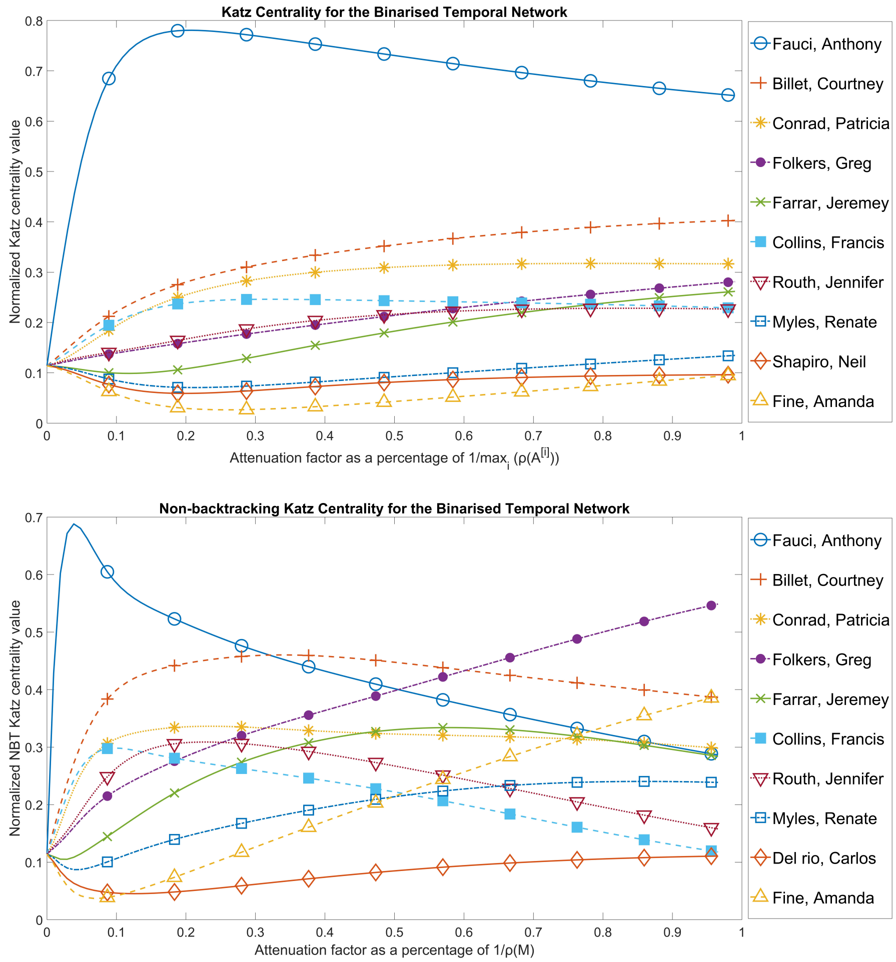

Fig. 6.6: Plots of the normalized Katz (upper) and nonbacktracking Katz (lower) centralities vector values for 10 most prominent nodes (i.e., those with the largest centrality value as of the upper limit of the attenuation factor t) within the binarized temporal network.

shorter lengths. Despite this added complexity, the resulting formulas for the standard generating function in Theorem 3.3 and corresponding node centrality measure in Corollary 3.4 are straightforward to evaluate.

We also showed in Theorem 4.5 that when working at the line graph level, the introduction of appropriate componentwise square roots allows us to develop a theory that extends to the unweighted case, with Theorem 4.6 summarizing the results, and Theorem 5.7 dealing with more general time-evolving graph sequences.

A practical take-home message is that a theory of nonbacktracking walk counts for static or dynamic weighted graphs is available, with corresponding computational algorithms that have the same complexity as in the unweighted case.

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