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Brief paper

Structured stabilisation of superlinear delay systems by bounded discrete-time state feedback control ☆

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ABSTRACT

Taking different system structures in different Markovian modes into consideration, this paper studies the structured stabilisation of a class of superlinear hybrid stochastic delay systems by feedback control based on discrete-time state observations. The controller is designed in a bounded state area, rather than every observable state, in order to reduce control cost. The time delay is more general in terms of the classical differentiability assumption being relaxed. Compared with the existing papers on discrete-state-feedback stabilisation problem, a new method to estimate the difference between current-time state and discrete-time state is presented, as a result of which the conditions imposed on the underlying system and the control function are less restrictive. Meanwhile, the Lyapunov functional used in this paper is modified to adapt to this change. Finally, an application to stochastic structured neural networks is given to demonstrate the practicability of the developed theory.

1. Introduction

Given an unstable hybrid stochastic delay differential equation (SDDE)

$$\begin{aligned} dx(t) =& f(x(t), x(t-\delta), t, r(t))dt \\ &+ g(x(t), x(t-\delta), t, r(t))dW(t), \end{aligned} \tag{1.1}$$

compared with the continuous-time state feedback control $u(x(t), t, r(t))$, it is more practical and less costly to use a state feedback control based on discrete-time observations, say at times $0, \tau, 2\tau, \ldots$, to achieve the stabilisation of the controlled system

$$\begin{aligned} dx(t) =& (f(x(t), x(t-\delta), t, r(t)) + u(x(t_\tau), t, r(t)))dt \\ &+ g(x(t), x(t-\delta), t, r(t))dW(t). \end{aligned} \tag{1.2}$$

Here $x(t) \in \mathbb{R}^d$, $r(t)$ is a Markov chain taking values in $\mathbb{S}$, $W(t)$ is a Brownian motion, the non-negative constant δ stands for system time lag, $t_\tau = [t/\tau]\tau$, where $[t/\tau]$ is the integer part of t/τ. This stabilisation problem for stochastic systems was first proposed by Mao (2013). Traditionally, the system coefficients f and g should satisfy the linear growth condition (see, e.g. Li and Kou (2017) and You, Liu, Lu, Mao, and Qiu (2015)). But recently, Mei, Fei, Fei, and Mao (2020) eased this restriction and brought this stabilisation problem into superlinear area. Although the theory developed therein has made great progress and more real models could be included, such as competitive model (Liu & Bai, 2017; Zhang & Teng, 2011) and ocean temperature oscillator (Suarez & Schopf, 1988), there are still four questions deserved our further discussion.

Q1. Structured stabilisation

Firstly, we emphasise the key ingredient in Mei et al. (2020) for stabilisation purpose, namely, the condition:

$$\begin{cases} x^T f(x, y, t, i) + \dfrac{1}{2}|g(x, y, t, i)|^2 \\ \le \chi_{i1}|x|^2 + \chi_{i2}|y|^2 - \chi_{i3}|x|^{p+1} + \chi_{i4}|y|^{p+1} \\ x^T f(x, y, t, i) + \dfrac{p}{2}|g(x, y, t, i)|^2 \\ \le \bar{\chi}_{i1}|x|^2 + \bar{\chi}_{i2}|y|^2 - \bar{\chi}_{i3}|x|^{p+1} + \bar{\chi}_{i4}|y|^{p+1} \end{cases} \tag{1.3}$$

for every $(x, y, t, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$. Condition (1.3) is indeed more advanced than the conventional linear growth condition. However, it is required for all modes, in particular, χ_{i3} and $\bar{\chi}_{i3}$ should be strictly positive for any $i \in \mathbb{S}$. This seems a little restrictive in reality as this structure might be lost in some modes. For example, Fei, Hu, Mao, and Shen (2018) studied a population system described by $dx(t) = -2x(t)dt + 0.8x(t-\delta)dW(t)$ in mode 1 (dry); $dx(t) = \left(x(t) - 2x^3(t)\right)dt + 1.2x^2(t-\delta)dW(t)$ in mode 2 (rain). It is clear that condition (1.3) cannot be satisfied in mode

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1 since $\chi_{13} = \bar{\chi}_{13} = 0$. Thus to deal with this situation, we need to consider structured stabilisation.

To the best of the authors' knowledge, although the structured stability has drawn many researchers' interest (e.g. Fei et al. (2018), Lu, Song, and Zhu (2022) and Shen, Mei, and Deng (2019)), there are few results on structured stabilisation. While recently, Shi, Mao, and Wu (2022) made some efforts to this problem. They successfully designed a discrete-time state feedback control for hybrid stochastic differential equations with different structures in different modes. But they have not considered time delay in their systems, which could actually influence the mode-structure classification (see Assumption 4, Example 1 later on). As a result, the structured stabilisation of hybrid SDDEs deserves our investigation.

Q2. Estimation of discrete-time state observations

Secondly, let us say more about condition (1.3). The reader might wonder why we need to give two similar inequalities at the same time, particularly, the first one can be deduced from the other. It is actually arisen from the effect of discrete-time state observations. To deal with this effect, we usually decompose the drift coefficient of the controlled system (1.2) as $(f(x(t), x(t-\delta), t, r(t)) + u(x(t), t, r(t))) + (u(x(t_\tau), t, r(t)) - u(x(t), t, r(t)))$ and hope the second term (or $|x(t) - x(t_\tau)|$) could be small enough if the observation duration τ is sufficiently small. Currently, one popular method to estimate the second term is to compute $E|x(t) - x(t_\tau)|^2$. Then the estimation result (see, e.g. equation (56) in Mei et al. (2020) and equation (4.27) in Shi et al. (2022)) forces us to give two inequalities in condition (1.3) unavoidably. But is it possible for us to modify the estimation so that condition (1.3) could be relaxed? In this paper, we will give a positive answer to this question (see Lemma 5). Owing to this modification, only the first inequality of condition (1.3) is required in the stability analysis.

Q3. Bounded-state-area control

Thirdly, we highlight that in many papers studying discrete-state-feedback stabilisation such as Mao (2013), Mei et al. (2020), Ren, Yin, and Sakthivel (2018), Shi et al. (2022) and You et al. (2015), the control function $u(x, t, i)$ is usually designed on every observable discrete-time state, such as the linear form $\nu_i x(t_\tau)$ in the example of Mei et al. (2020). But this sometimes seems a little rough and would lead to some unnecessary cost. In general, the control cost is proportional to $|u(x(t_\tau), t, r(t))|$. Thus the control cost goes up as system state value $|x(t_\tau)|$ increases. Particularly, if the initial data is given large, the cost on the beginning stage will be relatively high. This then begs a question: is it really necessary to impose control on every discrete-time state? The answer at least in this paper is negative. We will design the feedback control in a bounded state area (see Rule 1 and Remark 1).

Q4. Variable time delays

Finally, let us comment on the delay function in Mei et al. (2020), which is assumed to be a constant δ. In a slew of real-world SDDE models, the time delay is a variable function of time such as Dong and Mao (2022), Gugat and Dick (2011), Gugat, Dick, and Leugering (2013), Gugat and Tucsnak (2011), Min, Xu, Zhang, and Ma (2018), Sun, Sun, and Chen (2020) and Wang, Liu, and Liu (2008). Therefore, it seems a little unreasonable to continue considering the constant delay. Moreover, rather than the widely imposed condition on delay systems that the time delay is differentiable with derivative less than one (see, e.g. Min et al. (2018) and Wang et al. (2008)), in this paper, we will consider the time delays recently studied in Dong and Mao (2022), which meet a weaker assumption (namely Assumption 1). This allows us to include more practical time delays, such as periodic switching delay (Gugat & Tucsnak, 2011) and sawtooth delay (Sun et al., 2020). But differently, the delay function here is no longer needed to be bounded below by a positive number.

This paper is devoted to addressing these four issues. In theory, conditions on the original system and the control function are less restrictive. In reality, not only could a much wider class of hybrid stochastic systems be covered, but could also the control costs be reduced significantly.

2. Model description

2.1. Notation

Throughout this paper, we will work on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (that is, it is increasing, right-continuous and $\mathcal{F}_0$ contains all P-null sets). We let $W(t) = (W_1(t), \dots, W_m(t))^{\mathrm{T}}$ be an m-dimensional Brownian motion, and $r(t)$ a right-continuous Markov chain taking values in a finite state space $\mathbb{S} = \{1, \dots, N\}$ with transition rate matrix $Q = (q_{ij})_{N\times N}$ given by

$$P\left(r(t+\epsilon) = j | r(t) = i\right) = \begin{cases} 1 + q_{ij}\epsilon + o(\epsilon), & \text{if } i = j, \\ q_{ij}\epsilon + o(\epsilon), & \text{if } i \ne j, \end{cases}$$

as $\epsilon \downarrow 0$. Here $q_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$, while $q_{ii} = -\sum_{j\ne i} q_{ij}$. We assume that the Markov chain $r(t)$ and the Brownian motion $W(t)$ are independent.

We use $\mathbb{R}^d$ to signify the d-dimensional Euclidean space with Euclidean norm $|\cdot|$. For any $a, b \in \mathbb{R}$, we denote by $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Everyone agrees that $\mathbb{R}_+ = [0, \infty)$. If A is a vector or matrix, A^{T} is its transpose. If A is a matrix, $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$ is its trace norm. For a subset F_1 included in some universal set F, 1_{F_1} denotes its indicator function, that is, $1_{F_1}(a) = 1$ if $a \in F_1$, otherwise, 0. For some positive constant h, we let $\mathbb{C}\left([-h, 0]; \mathbb{R}^d\right)$ represent the family of all continuous functions ϕ from $[-h, 0]$ to $\mathbb{R}^d$ with norm $\|\phi\| = \sup_{-h\le\theta\le 0} |\phi(\theta)|$.

2.2. Structures on original system

A general hybrid SDDE is described by

$$\begin{aligned} \mathrm{d}x(t) =& f(x(t), x(t-\delta(t)), t, r(t))\mathrm{d}t \\ &+ g(x(t), x(t-\delta(t)), t, r(t))\mathrm{d}W(t). \end{aligned} \tag{2.1}$$

Here, $\delta : \mathbb{R}_+ \to [0, \Delta]$ denotes the system delay, $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^d$ is the drift coefficient, and $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{d\times m}$ is the diffusion coefficient. As a standing hypothesis, we assume that the coefficients $f(x, y, t, i)$ and $g(x, y, t, i)$ are locally Lipschitz continuous in x and y (see Theorem 3.15 in Mao and Yuan (2006)). In order to drive this equation, we need to know the initial data, which is given by $\{x(t)| -\Delta \le t \le 0\} = \xi \in \mathbb{C}([-\Delta, 0]; \mathbb{R}^d)$ and $r(0) = r_0 \in \mathbb{S}$.

The delay function considered in this paper should satisfy the following assumption, which is clearly less restrictive than the widely imposed differentiability condition.

Assumption 1. Assume the delay function $\delta(t)$ is a Borel measurable function satisfying that

$$\Delta^* = \limsup_{\epsilon\to 0^+} \left(\sup_{s\ge -\Delta} \mathrm{Leb}(I_{s,\epsilon})/\epsilon \right) < \infty, \tag{2.2}$$

where $\mathrm{Leb}(\cdot)$ denotes the Lebesgue measure on the real line and $I_{s,\epsilon} = \{t \in \mathbb{R}_+ | t - \delta(t) \in [s, s+\epsilon)\}$.

Assumption 1 is not so strong and can be met by many time-variable delay functions in practice. For example, the piecewise constant function $\delta(t) = \sum_{k=0}^{\infty} 1_{[(2k+1),(2k+2))}(t)$ satisfies with $\Delta^* = 2$. Moreover, if $\delta(t)$ is a Lipschitz continuous function with Lipschitz coefficient $\hat{h} \in (0, 1)$, then Assumption 1 is satisfied with $\Delta^* = 1/(1-\hat{h})$. For more details about Assumption 1, we refer the reader to Dong and Mao (2022). But differently, the delay function $\delta(t)$ considered in this paper is not needed to be bounded below by a positive constant. Next, we need present a useful lemma to tackle time delay effect under our new Assumption 1. One can use the similar analysis of Lemma 2.2 in Dong and Mao (2022) to show it, so we omit the proof given the page limit.

Lemma 1. *Under Assumption 1, for any $T > 0$ and continuous function $\phi : [-\Delta, T] \to \mathbb{R}_+$, we have*

$$\int_0^T \phi(v-\delta(v))\mathrm{d}v \le \Delta^* \int_{-\Delta}^T \phi(v)\mathrm{d}v. \tag{2.3}$$

Although the linear growth condition is not of our interest, we still do not want system coefficients to grow very sharply. Hence the following polynomial growth condition is required.

Assumption 2. Assume that there exist non-negative constants K_j $(j = 1,\dots,8)$ and $p > 1$ such that for every $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,

$$|f(x,y,t,i)| \le K_1|x| + K_2|y| + K_3|x|^p + K_4|y|^p, \tag{2.4}$$

$$|g(x,y,t,i)|^2 \le K_5|x|^2 + K_6|y|^2 + K_7|x|^{p+1} + K_8|y|^{p+1}. \tag{2.5}$$

But note that Assumption 2 cannot guarantee hybrid SDDE (2.1) has a unique global solution. For this purpose, the Khasminskii-type condition is always needed, which arises widely now in the study of superlinear stochastic systems (see, e.g. Fei et al. (2018), Mao and Yuan (2006) and Mei et al. (2020)). But differently from these references, in this paper, we are more interested in that the structure of hybrid SDDE (2.1) does not always remain the same type in all modes. For simplicity, we divide the mode space $\mathbb{S}$ into two parts, $\mathbb{S}_1 = \{1,\dots,N_1\}$ and $\mathbb{S}_2 = \{N_1+1,\dots,N\}$, where $1 \le N_1 < N$. The subsystems of hybrid SDDE (2.1) in $\mathbb{S}_1$-modes and $\mathbb{S}_2$-modes satisfy the classical Khasminskii-type condition and the generalised Khasminskii-type condition, respectively.

Assumption 3. Let $q \ge p+1$. For $i \in \mathbb{S}_1$, suppose that there exist constants $\tilde{a}_i \in \mathbb{R}$ and $\tilde{b}_i \ge 0$ such that for all $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$

$$x^{\mathrm{T}}f(x,y,t,i) + \frac{q+p-2}{2}|g(x,y,t,i)|^2 \le \tilde{a}_i|x|^2 + \tilde{b}_i|y|^2 \tag{2.6}$$

and for $\tilde{A} = -(q+p-1)\mathrm{diag}(\tilde{a}_1,\dots,\tilde{a}_{N_1}) - (q_{ij})_{i,j\in\mathbb{S}_1}$ to be a non-singular M-matrix. For $i \in \mathbb{S}_2$, assume that there exist constants $\tilde{\gamma}_i, \tilde{b}_i, \tilde{d}_i \ge 0$ and $\tilde{c}_i > 0$ such that for any $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$

$$\begin{aligned} &x^{\mathrm{T}}f(x,y,t,i) + \frac{q-1}{2}|g(x,y,t,i)|^2 \\ \le&\tilde{\gamma}_i|x|^2 + \tilde{b}_i|y|^2 - \tilde{c}_i|x|^{p+1} + \tilde{d}_i|y|^{p+1}. \end{aligned} \tag{2.7}$$

For the theory of M-matrix, the reader can refer to Section 2.6 in Mao and Yuan (2006). We have mentioned before that time delay could influence our mode-structure classification. Let us now give an example to explain this.

Example 1. Consider a scalar hybrid SDDE with $f(x,y,t,1) = -x - 3xy^2$, $g(x,y,t,1) = 0.5xy$; $f(x,y,t,2) = x - 4.5x^3$, $g(x,y,t,2) = x^2$; $f(x,y,t,3) = -x^3$, $g(x,y,t,3) = y$, with $Q = [-2, 1, 1; 9, -18, 9; 5, 5, -10]$. In this situation, time delay vanishes in mode 2. There are actually two classification schemes. Case 1: $\mathbb{S}_1 = \{1,2\}$, $\mathbb{S}_2 = \{3\}$. Case 2: $\mathbb{S}_1 = \{1\}$, $\mathbb{S}_2 = \{2,3\}$. However, if we consider time delay into subsystem in mode 2 and let $g(x,y,t,2) = y^2$, then we only have one scheme, $\mathbb{S}_1 = \{1\}$, $\mathbb{S}_2 = \{2,3\}$.

2.3. Existence of global solution

Let $\left(\tilde{\eta}_1,\dots,\tilde{\eta}_{N_1}\right)^{\mathrm{T}} = \tilde{A}^{-1}(1,\dots,1)^{\mathrm{T}}$. Since $\tilde{A}$ is a non-singular M-matrix, all $\tilde{\eta}_i$ are positive $(i \in \mathbb{S}_1)$. Along with the properties of transition rate matrix and $\tilde{c}_i > 0$ for all $i \in \mathbb{S}_2$, $\tilde{\mu} := q\tilde{c}_m/(1+\max_{i\in\mathbb{S}_2}(\sum_{j=1}^{N_1} q_{ij}\tilde{\eta}_j))$ is also positive, where $\tilde{c}_m = \min_{i\in\mathbb{S}_2}\tilde{c}_i$. Now, we show that hybrid SDDE (2.1) has a unique global solution.

Theorem 1. *Let Assumptions 1, 2, 3 hold. Further assume that $\tilde{D} = 1-(q+p-3+2\Delta^*)\max_{i\in\mathbb{S}_1}(\tilde{b}_i\tilde{\eta}_i) > 0$ and $q(q-2+(p+1)\Delta^*)\tilde{d}_M/(q+p-1) \le \tilde{\mu}\tilde{D}$. Then hybrid SDDE (2.1) has a unique global solution $x(t)$ such that for all $t \ge 0$*

$$\sup_{-\Delta\le s\le t} E\left(|x(s)|^q + |x(s)|^{q+p-1}1_{\{r(s)\in\mathbb{S}_1\}}\right) < \infty. \tag{2.8}$$

Proof. Set a function $\tilde{U} : \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}_+$ by $\tilde{U}(x,i) = |x|^q + \tilde{\mu}\tilde{\eta}_i|x|^{q+p-1}1_{\{i\in\mathbb{S}_1\}}$ and define a function $\mathbb{L}\tilde{U} : \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{S} \to \mathbb{R}$ by $\mathbb{L}\tilde{U}(x,y,t,i) = \tilde{U}_x(x,i)f(x,y,t,i) + \frac{1}{2}\mathrm{trace}(g^{\mathrm{T}}(x,y,t,i)\tilde{U}_{xx}(x,i)g(x,y,t,i)) + \sum_{j=1}^N q_{ij}\tilde{U}(x,j)$. For $i \in \mathbb{S}_1$, we could derive from condition (2.6) that $\mathbb{L}\tilde{U}(x,y,t,i) \le q|x|^{q-2}(\tilde{a}_i|x|^2+\tilde{b}_i|y|^2) + \tilde{\mu}((q+p-1)\tilde{a}_i\tilde{\eta}_i + \sum_{j=1}^{N_1} q_{ij}\tilde{\eta}_j)|x|^{q+p-1} + (q+p-1)\tilde{\mu}\tilde{b}_i\tilde{\eta}_i|x|^{q+p-3}|y|^2$. Since $(q+p-1)\tilde{a}_i\tilde{\eta}_i + \sum_{j=1}^{N_1} q_{ij}\tilde{\eta}_j = -1$, we further have

$$\begin{aligned} &\mathbb{L}\tilde{U}(x,y,t,i) \\ =&(q\tilde{a}_i + (q-2)\tilde{b}_i)|x|^q + 2\tilde{b}_i|y|^q - \tilde{\mu}|x|^{q+p-1} \\ &+ (q+p-3)\tilde{\mu}\tilde{b}_i\tilde{\eta}_i|x|^{q+p-1} + 2\tilde{\mu}\tilde{b}_i\tilde{\eta}_i|y|^{q+p-1}. \end{aligned} \tag{2.9}$$

For $i \in \mathbb{S}_2$, we have $\mathbb{L}\tilde{U}(x,y,t,i) \le q|x|^{q-2}(\tilde{\gamma}_i|x|^2 + \tilde{b}_i|y|^2 - \tilde{c}_i|x|^{p+1} + \tilde{d}_i|y|^{p+1}) + \tilde{\mu}\sum_{j=1}^{N_1} q_{ij}\tilde{\eta}_j|x|^{q+p-1}$ by (2.7). From the definition of $\tilde{\mu}$, we deduce that for $i \in \mathbb{S}_2$, $\tilde{\mu} + \tilde{\mu}\sum_{j=1}^{N_1} q_{ij}\tilde{\eta}_j \le \tilde{\mu} + \tilde{\mu}\max_{i\in\mathbb{S}_2}\left(\sum_{j=1}^{N_1} q_{ij}\tilde{\eta}_j\right) = q\tilde{c}_m \le q\tilde{c}_i$. This implies that

$$\begin{aligned} &\mathbb{L}\tilde{U}(x,y,t,i) \\ \le&\left(q\tilde{\gamma}_i + (q-2)\tilde{b}_i\right)|x|^q + 2\tilde{b}_i|y|^q - \tilde{\mu}|x|^{q+p-1} \\ &+ \frac{q\tilde{d}_i}{q+p-1}((q-2)|x|^{q+p-1} + (p+1)|y|^{q+p-1}). \end{aligned} \tag{2.10}$$

Combining (2.9) with (2.10), we could derive that

$$\mathbb{L}\tilde{U}(x,y,t,i) \le \zeta_1|x|^q + \zeta_2|y|^q - \zeta_3|x|^{q+p-1} + \zeta_4|y|^{q+p-1}, \tag{2.11}$$

where $\zeta_1 = q\tilde{a}_{\max} + (q-2)\tilde{b}_{\max}$, $\zeta_2 = 2\tilde{b}_{\max}$, $\zeta_3 = \tilde{\mu} - (q+p-3)\tilde{\mu}\max_{i\in\mathbb{S}_1}(\tilde{b}_i\tilde{\eta}_i) - q(q-2)\tilde{d}_M/(q+p-1)$, $\zeta_4 = 2\tilde{\mu}\max_{i\in\mathbb{S}_1}(\tilde{b}_i\tilde{\eta}_i) + (p+1)\tilde{d}_M/(q+p-1)$. Here $\tilde{a}_{\max} = \left(\max_{i\in\mathbb{S}_1}\tilde{a}_i\right) \vee \left(\max_{i\in\mathbb{S}_2}\tilde{\gamma}_i\right)$, $\tilde{b}_{\max} = \max_{i\in\mathbb{S}}\tilde{b}_i$, $\tilde{d}_M = \max_{i\in\mathbb{S}_2}\tilde{d}_i$.

Since the system coefficients are locally Lipschitz continuous, there is a unique maximal local solution $x(t)$ on $t \in [0,\sigma_e)$ by Theorem 7.12 in Mao and Yuan (2006), where σ_e is the explosion time. Let $k_0 > 0$ be sufficiently large for $k_0 \ge \|\xi\|$. For each integer $k \ge k_0$, define stopping time $\sigma_k = \inf\left\{t \in [0,\sigma_e)\big||x(t)| \ge k\right\}$. Clearly, σ_k is increasing as $k \to \infty$. Set $\sigma_\infty = \lim_{k\to\infty}\sigma_k$, whence $\sigma_\infty \le \sigma_e$ a.s. If we can show that $\sigma_\infty = \infty$ a.s., then $\sigma_e = \infty$ a.s., and the solution $x(t)$ is global. Then, for any $k \ge k_0$ and $t \ge 0$, we derive from the Itô formula and (2.11) that

$$\begin{aligned} &E\tilde{U}(x(t\wedge\sigma_k), r(t\wedge\sigma_k)) - \tilde{U}(\xi(0), r_0) \\ \le&E\int_0^{t\wedge\sigma_k}(\zeta_1|x(s)|^q + \zeta_2|x(s-\delta(s))|^q - \zeta_3|x(s)|^{q+p-1} \\ &+ \zeta_4|x(s-\delta(s))|^{q+p-1})\mathrm{d}s. \end{aligned} \tag{2.12}$$

Making use of Lemma 1, we have that $E\tilde{U}(x(t\wedge\sigma_k), r(t\wedge\sigma_k)) \le C_1 + E\int_0^{t\wedge\sigma_k}(\zeta_1|x(s)|^q + \zeta_2|x(s-\delta(s))|^q - (\zeta_3 - \zeta_4\Delta^*)|x(s)|^{q+p-1})\mathrm{d}s$, where $C_1 = \tilde{U}(\xi(0), r_0) + \zeta_4 \Delta^* \Delta \|\xi\|^{q+p-1}$. It is easy to compute $\zeta_3 - \zeta_4\Delta^* \geq 0$. Then

$$\begin{aligned}&\sup_{-\Delta\le s\le t} E\tilde{U}(x(s\wedge\sigma_k), r(s\wedge\sigma_k))\\ \le& C_1 + (\zeta_1+\zeta_2)\int_0^t \sup_{-\Delta\le u\le s} E|x(u\wedge\sigma_k)|^q \mathrm{d}s. \end{aligned} \tag{2.13}$$

This implies that $\sup_{-\Delta\le s\le t} E|x(s\wedge\sigma_k)|^q \le (C_1 + \|\xi\|^q) + (\zeta_1 + \zeta_2)\int_0^t \sup_{-\Delta\le u\le s} E|x(u\wedge\sigma_k)|^q \mathrm{d}s$. The Gronwall inequality gives $E|x(t\wedge\sigma_k)|^q \le \sup_{-\Delta\le s\le t} E|x(s\wedge\sigma_k)|^q \le (C_1+\|\xi\|^q)e^{(\zeta_1+\zeta_2)t} < \infty$. This implies that $k^q P(\sigma_k \le t) = E\left(|x(t\wedge\sigma_k)|^q 1_{\{\sigma_k\le t\}}\right) \le E|x(t\wedge\sigma_k)|^q < \infty$. We can let $k\to\infty$ to obtain that $P(\sigma_\infty \le t) = 0$, namely, $P(\sigma_\infty > t) = 1$. Since t is arbitrary, we have $P(\sigma_\infty = \infty) = 1$ as required. Letting $k\to\infty$ in (2.13) gives $\sup_{-\Delta\le s\le t} E\tilde{U}(x(s), r(s)) < \infty$. Then the required assertion (2.8) follows since $\left(1\wedge\tilde{\mu}\min_{i\in\mathbb{S}_1}\tilde{\eta}_i\right)\left(|x|^q + |x|^{q+p-1}1_{\{i\in\mathbb{S}_1\}}\right) \le \tilde{U}(x,i)$. The proof is therefore complete. □

From now on, since the subsequent stability analysis will be our main focus, we will not mention the conditions of Theorem 1 explicitly and assume they are true.

3. Design of discrete-time feedback control

Suppose hybrid SDDE (2.1) is unstable, we want to design a feedback control to stabilise it. In theory, the design of the feedback control is based on continuous-time state observations. But in practice, the state can only be observed at discrete times, say $0, \tau, 2\tau, \ldots$. Letting $t_\tau = [t/\tau]\tau$, our controlled system actually becomes

$$\begin{aligned}\mathrm{d}x(t) =& (f(x(t), x(t-\delta(t)), t, r(t)) + u(x(t_\tau), t, r(t)))\mathrm{d}t\\ &+ g(x(t), x(t-\delta(t)), t, r(t))\mathrm{d}W(t)\end{aligned} \tag{3.1}$$

on $t\ge 0$ with initial data ξ and r_0.

3.1. Bounded-state-area feedback control

Before giving our control design, in addition to all conditions in Theorem 1, we need to give another assumption on our original unstable system (2.1) for the stabilisation aim.

Assumption 4. For $i\in\mathbb{S}_1$, there are constants $a_i, \hat{a}_i \in \mathbb{R}$ and $b_i, \hat{b}_i \ge 0$ such that for any $(x,y,t)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+$,

$$\begin{cases} x^{\mathrm{T}} f(x,y,t,i) + \dfrac{1}{2}|g(x,y,t,i)|^2 \le a_i|x|^2 + b_i|y|^2 \\ x^{\mathrm{T}} f(x,y,t,i) + \dfrac{p}{2}|g(x,y,t,i)|^2 \le \hat{a}_i|x|^2 + \hat{b}_i|y|^2 \end{cases} \tag{3.2}$$

and for $A_1 = -2\mathrm{diag}(a_1,\ldots,a_{N_1}) - (q_{ij})_{i,j\in\mathbb{S}_1}$, $\hat{A} = -(p+1)\mathrm{diag}(\hat{a}_1,\ldots,\hat{a}_{N_1}) - (q_{ij})_{i,j\in\mathbb{S}_1}$ to be non-singular M-matrices. Assume that $\hat{D} := 1-(p-1+2\Delta^*)\max_{i\in\mathbb{S}_1}(\hat{b}_i\hat{\eta}_i)$ and $D_b := 1 - 2\Delta^*\max_{i\in\mathbb{S}_1}(b_i\sum_{j=1}^{N_1}(A_1^{-1})_{ij})$ are positive, where $(\hat{\eta}_1,\ldots,\hat{\eta}_{N_1})^{\mathrm{T}} = \hat{A}^{-1}(1,\ldots,1)^{\mathrm{T}}$.

For $i\in\mathbb{S}_2$, there are non-negative constants γ_i, b_i, d_i, positive constant c_i so that for all $(x,y,t)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+$

$$\begin{aligned}&x^{\mathrm{T}} f(x,y,t,i) + \frac{1}{2}|g(x,y,t,i)|^2\\ \le& \gamma_i|x|^2 + b_i|y|^2 - c_i|x|^{p+1} + d_i|y|^{p+1}.\end{aligned} \tag{3.3}$$

Further, letting $D_q = 1 + \max_{i\in\mathbb{S}_2}(\sum_{j=1}^{N_1} q_{ij}\hat{\eta}_j)$, assume that $D_d := \min_{i\in\mathbb{S}_2} c_i - D_q\Delta^*\max_{i\in\mathbb{S}_2} d_i/\hat{D} > 0$.

But the reader might find that condition (3.2) could be deduced by condition (2.6) with $a_i = \hat{a}_i = \tilde{a}_i$ and $b_i = \hat{b}_i = \tilde{b}_i$. Is it necessary to give them at the same time? Actually, these parameters also influence the value of τ. Thus we need to give them separately. The same reason is applicable for condition (3.3). On the other hand, the positivity of D_b, $\hat{D}$ and D_d cannot be derived from Assumption 3, whose roles will be explained in Remark 3. In other words, Assumptions 3 and 4 are different, and any one cannot be deduced from the other. As a result, Assumption 4 is indeed required.

Next, we will introduce how to design the feedback control $u(x,t,i)$ according to mode-structure classification in Assumption 4. For convenience, for any $0<a<b$, we denote by $B_a = \{x\in\mathbb{R}^d \big| |x|\le a\}$, $B_a^c = \{x\in\mathbb{R}^d \big| |x| > a\}$, $B_b - B_a = \{x\in\mathbb{R}^d \big| a<|x|\le b\}$.

Rule 1. *For $i\in\mathbb{S}_1$, let $u(x,t,i)\equiv 0$. For $i\in\mathbb{S}_2$, choose a constant $\kappa_i\ge 0$ and set $R_i = (2(\gamma_i+\kappa_i)/c_i)^{1/(p-1)}$. The control in this mode can be designed as: (i) when $x\in B_{R_i}$, design $u(x,t,i)$ such that we can find a constant $K>0$ to let*

$$|u(x,t,i) - u(y,t,i)| \le K|x-y|, \tag{3.4}$$

$$x^{\mathrm{T}} u(x,t,i) \le -\kappa_i|x|^2 \tag{3.5}$$

hold for any $(x,y,t)\in B_{R_i}\times B_{R_i}\times\mathbb{R}_+$, and moreover $u(0,t,i)\equiv 0$; (ii) when $x\in B_{2R_i} - B_{R_i}$, let $u(x,t,i) = u((2R_i/|x|-1)x,t,i)$, where $(2R_i/|x|-1)x\in B_{R_i}$; (iii) when $x\in B_{2R_i}^c$, let $u(x,t,i) = 0$ for all $t\in\mathbb{R}_+$.

Here, when $R_i = 0$, we think $u(x,t,i) = 0$ for all $x\in\mathbb{R}^d$. Next, let us make some comments on this control rule.

Remark 1. If we pay attention to hybrid SDDE (2.1) on $\mathbb{S}_1$, we find these subsystems might become stable. There is no need to impose any control when $i\in\mathbb{S}_1$. But this does not mean the whole system is stable. Thus we need to design control for $\mathbb{S}_2$-modes, which is imposed in the bounded state area. In fact, we could rewrite the right-hand side of (3.3) as $-\kappa_i|x|^2 + b_i|y|^2 - \frac{c_i}{2}|x|^{p+1} - d_i|y|^{p+1} + ((\gamma_i+\kappa_i)|x|^2 - \frac{c_i}{2}|x|^{p+1})$. It is then easy to see that $(\gamma_i+\kappa_i)|x|^2 - \frac{c_i}{2}|x|^{p+1} < 0$ when $|x|>R_i$. This implies that $x^{\mathrm{T}} f(x,y,t,i) + \frac{1}{2}|g(x,y,t,i)|^2 \le -\kappa_i|x|^2 + b_i|y|^2 - \frac{c_i}{2}|x|^{p+1} - d_i|y|^{p+1}$. We hence do not need to impose any control when $|x|$ exceeds R_i.

Remark 2. It should also be pointed out that we could in fact let $u(x,t,i) = 0$ for $x\notin B_{R_i}$ and $(t,i)\in\mathbb{R}_+\times\mathbb{S}_2$. But in our control scheme, we set an additional connect area $B_{2R_i} - B_{R_i}$ and require $u(x,t,i)$ to vanish when $|x|\ge 2R_i$. This is needed for the purpose of continuity of $u(x,t,i)$ in x to guarantee the existence of unique global solution of the controlled system (3.1), and can also guarantee the global Lipschitz continuity of $u(x,t,i)$ in x with the same Lipschitz coefficient K assumed in $x\in B_{R_i}$, which is stated as Lemma 2.

From Rule 1 and the above discussions, after giving an appropriate κ_i, we see that the design of $u(x,t,i)$ for $\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{S}_1$ and $B_{R_i}^c\times\mathbb{R}_+\times\mathbb{S}_2$ is very clear. The remaining question is whether we could design a $u(x,t,i)$ for $B_{R_i}\times\mathbb{R}_+\times\mathbb{S}_2$. Actually, there are lots of control functions available. For example, design the control function in the linear form $u(x,t,i) = -\mathcal{A}_i x$ with constant $\mathcal{A}_i \ge \kappa_i$. Then (3.4) and (3.5) are satisfied. Certainly, for the stability purpose, the constant κ_i we pick up should satisfy some additional rules later on.

Lemma 2. *Let Rule 1 hold. Then for all $(x,y,t,i)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{S}$,*

$$|u(x,t,i) - u(y,t,i)| \le K|x-y|. \tag{3.6}$$

By discussing the positions of x and y, it is easy to show this lemma so we omit it. We also observe from Lemma 2 that $u(x,t,i)$ meets the linear growth condition, namely,

$$|u(x,t,i)| \le K|x|, \quad \forall (x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}. \tag{3.7}$$

Conditions (2.4) and (2.5) tell us that $f(0,0,t,i) \equiv 0$ and $g(0,0,t,i) \equiv 0$. Therefore, the controlled system (3.1) admits a trivial solution. Moreover, in analogy to the proof of Theorem 1, we can show that there exists a global solution of the controlled system (3.1), satisfying $\sup_{-\Delta\le s\le t} E\left(|x(s)|^q + |x(s)|^{q+p-1}1_{\{r(s)\in\mathbb{S}_1\}}\right) < \infty$ for any $t \ge 0$, under Assumption 4 and Rule 1.

3.2. Additional rules on control function

From Assumption 4 and Rule 1, we observe that the controlled system (3.1) also has different structures in different modes. For $i \in \mathbb{S}_1$, since $u(x,t,i) \equiv 0$, we then derive that for every $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$, $x^{\mathrm{T}}(f(x,y,t,i)+u(x,t,i)) + \frac{1}{2}|g(x,y,t,i)|^2 \le a_i|x|^2 + b_i|y|^2$, $x^{\mathrm{T}}(f(x,y,t,i)+u(x,t,i)) + \frac{p}{2}|g(x,y,t,i)|^2 \le \hat{a}_i|x|^2 + \hat{b}_i|y|^2$. For $\mathbb{S}_2$-modes, we have the following lemma.

Lemma 3. *Let Assumption 4 and Rule 1 hold. For $i \in \mathbb{S}_2$, let $a_i = \gamma_i - \kappa_i$. Then for all $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$*

$$\begin{aligned} &x^{\mathrm{T}}(f(x,y,t,i)+u(x,t,i)) + \frac{1}{2}|g(x,y,t,i)|^2 \\ \le& a_i|x|^2 + b_i|y|^2 - c_i|x|^{p+1} + \frac{d_i}{2}|y|^{p+1}. \end{aligned}$$

It is easy to show this lemma by discussing the positions of x, so we leave it to the reader. The control function $u(x,t,i)$ designed by Rule 1 might still not stabilise the original system (2.1). Hence we need to impose some additional conditions.

Rule 2. *Ensure that κ_i we choose in Rule 1 make $A = -2diag(a_1,\ldots,a_N) - Q$ be a non-singular M-matrix, where a_i are the same in Assumption 4 or Lemma 3.*

Let $(\eta_1,\ldots,\eta_N)^{\mathrm{T}} = A^{-1}(1,\ldots,1)^{\mathrm{T}}$. Define a function $U : \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}_+$ by $U(x,i) = \eta_i|x|^2 + \hat{\mu}\hat{\eta}_i|x|^{p+1}1_{\{i\in\mathbb{S}_1\}}$ with $\hat{\mu} = \min_{i\in\mathbb{S}_2}(c_i\eta_i)/D_q$, where $\hat{\eta}_i$ and D_q have been given in Assumption 4. While define a function $LU : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ by $LU(x,y,t,i) = U_x(x,i)(f(x,y,t,i)+u(x,t,i)) + \frac{1}{2}\mathrm{trace}(g^{\mathrm{T}}(x,y,t,i)U_{xx}(x,i)g(x,y,t,i)) + \sum_{j=1}^N q_{ij}U(x,j)$. The estimation of $LU(x,y,t,i)$ is the key ingredient for subsequent stability analysis. Here, for the convenience of the reader, we state it as the following lemma.

Lemma 4. *Let Assumption 4 and Rules 1 and 2 hold. Let $\beta_1 = \max_{i\in\mathbb{S}}(b_i\eta_i)$, $\beta_2 = \max_{i\in\mathbb{S}_1}(\hat{b}_i\hat{\eta}_i)$, $\beta_3 = \max_{i\in\mathbb{S}_2}(d_i\eta_i)$. Then for any $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,*

$$\begin{aligned} LU(x,y,t,i) \le& -|x|^2 + 2\beta_1|y|^2 - (1-(p-1)\beta_2)\hat{\mu}|x|^{p+1} \\ &+ (2\beta_2\hat{\mu} + 2\beta_3)|y|^{p+1}. \end{aligned} \tag{3.8}$$

The proof of Lemma 4 is quite similar to the estimation of $\mathbb{L}\tilde{U}(x,y,t,i)$ in (2.11), so we omit it. For the stability aim, we always want $LU(x,y,t,i)$ to be negative, which forces us to give the following rule.

Rule 3. *Also ensure that κ_i in Rule 1 can make the numbers $D_1 = 1 - 2\Delta^*\beta_1$ and $D_2 = \hat{D}\hat{\mu} - 2\Delta^*\beta_3$ positive.*

But the reader may wonder if we can find the appropriate κ_i to make Rules 2 and 3 fulfilled. The following remark will deny this worry.

Remark 3. Since A_1 is a non-singular M-matrix required in Assumption 4, there is a constant κ large enough such that $-2\mathrm{diag}(a_1,\ldots,a_{N_1},\gamma_{N_1+1}-\kappa,\ldots,\gamma_N-\kappa) - Q$ is a non-singular M-matrix. Therefore, we can choose $\kappa_i = \kappa$ for all $i \in \mathbb{S}_2$. Rule 2 hence holds. Then for sufficiently large κ, $\eta_i \approx \sum_{j=1}^{N_1}(A_1^{-1})_{ij}$ for $i \in \mathbb{S}_1$, and $\eta_i \approx 1/(2\kappa)$ for $i \in \mathbb{S}_2$. As a result, $D_1 \approx D_b$ and $D_2 \approx \frac{1}{\kappa}(\hat{D}\min_{i\in\mathbb{S}_2} c_i/D_q - \Delta^*\max_{i\in\mathbb{S}_2} d_i)$. Since D_b, $\hat{D}$ and D_d are positive, Rule 3 could be satisfied. Certainly, in application, we need to make use of the special forms of f and g to take κ_i wisely.

4. Stabilisation results

4.1. The upper bound of observation duration

Now, we introduce a method on how to determine the value of τ^*, the upper bound of τ. Let $\eta_M = \max_{i\in\mathbb{S}_2}\eta_i$ and set a domain $\mathcal{E} = (0, D_1/(K\eta_M))$. Define three functions on $\mathcal{E}$ by $\varphi(\varepsilon) = \frac{1}{K\eta_M}\left(\frac{D_1-K\eta_M\varepsilon}{\varphi_1(\varepsilon)} \wedge \frac{D_2}{\varphi_2(\varepsilon)}\right)$, $\varphi_1(\varepsilon) = \frac{D_1}{\eta_M} + 2K + 2K_1 + (1+\Delta^*)K_2 + \frac{K_5+\Delta^*K_6}{\varepsilon}$ and $\varphi_2(\varepsilon) = \frac{D_2}{\eta_M} + 2K_3 + \frac{2(1+p\Delta^*)K_4}{p+1} + \frac{K_7+\Delta^*K_8}{\varepsilon}$. It is clear that φ is a positive continuous function and $|\varphi(\varepsilon)| \le 1/K$. When ε tends to $D_1/(K\eta_M)$ or 0, $\varphi(\varepsilon)$ goes to zero. Therefore, there exists a number $\varepsilon^* \in \mathcal{E}$ such that $\varphi(\varepsilon^*) = \max_{\varepsilon\in\mathcal{E}}\varphi(\varepsilon)$. Then the upper bound of observation duration is given by $\tau^* = \varphi(\varepsilon^*) = \max_{\varepsilon\in\mathcal{E}}\varphi(\varepsilon)$. It will also be very useful later that $\tau < \tau^* \le 1/K$. From now on, we always have $\tau < \tau^*$.

4.2. Lyapunov functional

The main method to study stability in this paper is the technique of Lyapunov functional. For this purpose, we define $x_t = \{x(t+\theta)| -\tau-\Delta \le \theta \le 0\}$ for $t \ge 0$. For x_t to be well defined for $t \in [0, \tau+\Delta]$, we set $x(\theta) = \xi(-\Delta)$ for $\theta \in [-\tau-\Delta, -\Delta)$. The Lyapunov functional will be

$$V(x_t,t,r(t)) = U(x(t),r(t)) + \int_{-\tau}^0\int_{t+s}^t H_1(v)\mathrm{d}v\mathrm{d}s$$

for any $t \ge 0$, where $H_1(t) = \varpi_1^*|x(t)|^2 + \varpi_2^*|x(t-\delta(t))|^2 + \varpi_3^*|x(t)|^{p+1} + \varpi_4^*|x(t-\delta(t))|^{p+1}$. Here, ϖ_1^*, ϖ_2^*, ϖ_3^*, ϖ_4^* are positive constants to be determined later.

By the generalised Itô formula and the fundamental theory of calculus, we can show that $V(x_t,t,r(t))$ is in fact an Itô process on $t \ge 0$ with its Itô differential

$$\mathrm{d}V(x_t,t,r(t)) = \mathcal{L}V(x_t,t,r(t))\mathrm{d}t + \mathrm{d}M(t), \tag{4.1}$$

where

$$\begin{aligned} \mathcal{L}V(x_t,t,r(t)) =& LU(x(t),x(t-\delta(t)),t,r(t)) + \bar{U}(t) \\ &+ \tau H_1(t) - \int_{t-\tau}^t H_1(v)\mathrm{d}v \end{aligned} \tag{4.2}$$

and $M(t)$ is a continuous local martingale vanishing at $t = 0$. Here $\bar{U}(t) = U_x(x(t),r(t))(u(x(t_\tau),t,r(t)) - u(x(t),t,r(t)))$, and the explicit form of $M(t)$ is of no use in this paper so we omit it here, but it can be found in Theorem 1.45 in Mao and Yuan (2006). Also from (4.2), we have to estimate $\bar{U}(t)$.

Lemma 5. *Under Assumption 4 and Rules 1, 2, 3, we have*

$$E|\bar{U}(t)| \le \int_{t-\tau}^t EH_2(v)\mathrm{d}v + \Lambda_5E|x(t)|^2 + \Lambda_6E|x(t)|^{p+1}$$

for any $t \ge 0$, where $H_2(t) = \Lambda_1|x(t)|^2 + \Lambda_2|x(t-\delta(t))|^2 + \Lambda_3|x(t)|^{p+1} + \Lambda_4|x(t-\delta(t))|^{p+1}$ with $\Lambda_1 = \frac{K\eta_M}{1-K\tau}\left(K_1 + \frac{K_5}{\varepsilon^*}\right)$, $\Lambda_2 = \frac{K\eta_M}{1-K\tau}\left(K_2 + \frac{K_6}{\varepsilon^*}\right)$, $\Lambda_3 = \frac{K\eta_M}{1-K\tau}\left(\frac{2pK_3}{p+1} + \frac{K_7}{\varepsilon^*}\right)$, $\Lambda_4 = \frac{K\eta_M}{1-K\tau}\left(\frac{2pK_4}{p+1} + \frac{K_8}{\varepsilon^*}\right)$, and $\Lambda_5 = \frac{K\eta_M}{1-K\tau}((2K+K_1+K_2)\tau + \varepsilon^*)$, $\Lambda_6 = \frac{K\eta_M}{1-K\tau}\frac{2(K_3+K_4)\tau}{p+1}$.

Proof. For any fixed $t \geq 0$, we can find a non-negative integer k such that $k\tau \leq t < (k+1)\tau$. Hence $x(t_\tau) = x(k\tau)$, and also $x(v_\tau) = x(k\tau)$ for any $k\tau \leq v \leq t$. Then it is easy to derive from (3.6) that

$$\begin{aligned}|\bar{U}(t)| \leq& \big(2\eta_{r(t)}|x(t)| + (p+1)\hat{\mu}\hat{\eta}_{r(t)}|x(t)|^p 1_{\{r(t)\in\mathbb{S}_1\}}\big)\\ &\times K|x(t)-x(k\tau)|1_{\{r(t)\in\mathbb{S}_2\}}\\ \leq& 2K\eta_M|x(t)||x(t)-x(k\tau)|. \end{aligned} \tag{4.3}$$

Using (2.4) and (3.7) and letting $M_k = \int_{k\tau}^t g(x(v), x(v-\delta(v)), v, r(v))dW(v)$, compute

$$\begin{aligned}&|x(t)||x(t)-x(k\tau)|\\ \leq& \frac{\varepsilon^*}{2}|x(t)|^2 + \frac{|M_k|^2}{2\varepsilon^*} + \int_{k\tau}^t |x(t)|(K|x(k\tau)| + K_1|x(v)|\\ &+ K_2|x(v-\delta(v))| + K_3|x(v)|^p + K_4|x(v-\delta(v))|^p)dv\\ \leq& \frac{(2K+K_1+K_2)\tau+\varepsilon^*}{2}|x(t)|^2 + \frac{|M_k|^2}{2\varepsilon^*}\\ &+ \frac{(K_3+K_4)\tau}{p+1}|x(t)|^{p+1} + K\tau|x(t)||x(t)-x(k\tau)|\\ &+ \int_{k\tau}^t\Big(\frac{K_1}{2}|x(v)|^2 + \frac{K_2}{2}|x(v-\delta(v))|^2\\ &+ \frac{pK_3}{p+1}|x(v)|^{p+1} + \frac{pK_3}{p+1}|x(v-\delta(v))|^{p+1}\Big)dv.\end{aligned}$$

Since $\tau < 1/K$, we further have

$$\begin{aligned}&|x(t)||x(t)-x(k\tau)|\\ \leq& \frac{1}{1-K\tau}\int_{k\tau}^t\Big(\frac{K_1}{2}|x(v)|^2 + \frac{K_2}{2}|x(v-\delta(v))|^2\\ &+ \frac{pK_3}{p+1}|x(v)|^{p+1} + \frac{pK_4}{p+1}|x(v-\delta(v))|^{p+1}\Big)dv\\ &+ \frac{1}{1-K\tau}\Big(\frac{|M_k|^2}{2\varepsilon^*} + \frac{(2K+K_1+K_2)\tau+\varepsilon^*}{2}|x(t)|^2\\ &+ \frac{(K_3+K_4)\tau}{p+1}|x(t)|^{p+1}\Big).\end{aligned}$$

We can substitute this into (4.3) and then take expectations on both sides to obtain

$$\begin{aligned}E|\bar{U}(t)| \leq& \frac{K\eta_M}{1-K\tau}\int_{k\tau}^t (K_1E|x(v)|^2 + K_2E|x(v-\delta(v))|^2\\ &+ \frac{2pK_3}{p+1}E|x(v)|^{p+1} + \frac{2pK_4}{p+1}E|x(v-\delta(v))|^{p+1}\\ &+ \frac{1}{\varepsilon^*}E|g(x(v), x(v-\delta(v)), v, r(v))|^2)dv\\ &+ \Lambda_5E|x(t)|^2 + \Lambda_6E|x(t)|^{p+1}.\end{aligned}$$

Then the required assertion follows if we use (2.5). □

4.3. Exponential stabilisation

In this part, we demonstrate that the original unstable system (2.1) can be stabilised by the feedback control designed in this paper in the sense of mean square exponential stability. For this purpose, the following remark is helpful.

Remark 4. Let $\eta_{\max} = \max_{i\in\mathbb{S}}\eta_i$ and $\hat{\eta}_M = \max_{i\in\mathbb{S}_1}\hat{\eta}_i$. Define six functions on $[0, 1/\tau)$ by $\varpi_j(\lambda) = \Lambda_j/(1-\lambda\tau)$ $(j = 1,2,3,4)$, and $\Phi_1(\lambda) = 1 - \Lambda_5 - \varpi_1(\lambda)\tau - (2\beta_1 + \varpi_2(\lambda)\tau)\Delta^*e^{\lambda\tau} - \eta_{\max}\lambda$, $\Phi_2(\lambda) = (1-(p-1)\beta_2)\hat{\mu} - \Lambda_6 - \varpi_3(\lambda)\tau - (2\beta_2\hat{\mu} + 2\beta_3 + \varpi_4(\lambda)\tau)\Delta^*e^{\lambda\tau} - \hat{\eta}_M\hat{\mu}\lambda$.

It is easy to see that all $\varpi_j(\cdot)$ are positive increasing functions and tend to infinity when $\lambda \to 1/\tau$. This observation implies that the decreasing function $\Phi_1(\cdot)$ goes to negative infinity when λ approaches its right bound. Next, compute $\Phi_1(0) = D_1 - \Lambda_5 - (\Lambda_1 + \Delta^*\Lambda_2)\tau$. Recalling the determination of τ^* in Section 4.1, we obtain that $\Phi_1(0) > 0$. Consequently, there exists a unique solution $\lambda_1^* \in (0, 1/\tau)$ such that $\Phi_1(\lambda) = 0$. The same analysis applying to $\Phi_2(\lambda)$ yields that there is a unique solution $\lambda_2^* \in (0, 1/\tau)$ so that $\Phi_2(\lambda) = 0$. Then $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ are non-negative for any $\lambda \in [0, \lambda^*]$, where $\lambda^* = \lambda_1^* \wedge \lambda_2^*$.

Theorem 2. *Under Assumption 4, let the control function $u(x,t,i)$ satisfy Rules 1, 2, 3. Then the solution of the controlled system (3.1) has the property that*

$$\limsup_{t\to\infty}\frac{1}{t}\log\big(E|x(t)|^2\big) \leq -\lambda^*. \tag{4.4}$$

Proof. We firstly choose the parameters set in the Lyapunov functional as $\varpi_j^* = \varpi_j(\lambda^*)$ $(j = 1,2,3,4)$, which are all positive from discussions in Remark 4. Applying the generalised Itô formula and using (4.1), we obtain that for any $t \geq 0$ (if necessary, using the procedure of stopping times since $EV(x_t, t, r(t)) < \infty$ and $E|\mathcal{L}V(x_t, t, r(t))| < \infty$)

$$\begin{aligned}&e^{\lambda^* t}EV(x_t, t, r(t)) - V(x_0, 0, r_0)\\ \leq& \int_0^t e^{\lambda^* s}(\lambda^*EV(x_s, s, r(s)) + E\mathcal{L}V(x_s, s, r(s)))ds.\end{aligned} \tag{4.5}$$

By Lemmas 4 and 5, we derive from (4.2) that

$$E\mathcal{L}V(x_s, s, r(s)) \leq EJ_1(s) - \int_{s-\tau}^s E(H_1(v) - H_2(v))dv,$$

where $J_1(s) = -(1-\varpi_1^*\tau-\Lambda_5)|x(s)|^2 + (2\beta_1+\varpi_2^*\tau)|x(s-\delta(s))|^2 - ((1-(p-1)\beta_2)\hat{\mu} - \varpi_3^*\tau - \Lambda_6)|x(s)|^{p+1} + (2\beta_2\hat{\mu} + 2\beta_3 + \varpi_4^*\tau)|x(s-\delta(s))|^{p+1}$. Recalling the definition of $V(x_s, s, r(s))$ and using the fact that $\int_{-\tau}^0\int_{t+s}^t \phi(v)dvds \leq \tau\int_{t-\tau}^t\phi(v)dv$ for any non-negative integrable function ϕ, we then have

$$\begin{aligned}&\lambda^*EV(x_s, s, r(s)) + E\mathcal{L}V(x_s, s, r(s))\\ \leq& EJ_2(s) - \int_{s-\tau}^s E((1-\lambda^*\tau)H_1(v) - H_2(v))dv,\end{aligned} \tag{4.6}$$

where $J_2(s) = \lambda^*U(x(s), r(s)) + J_1(s)$. Noting that for $j = 1,2,3,4$, $(1-\lambda^*\tau)\varpi_j^* - \Lambda_j = 0$, and so $(1-\lambda^*\tau)H_1(v) = H_2(v)$, substituting (4.6) into (4.5) shows that

$$e^{\lambda^* t}EV(x_t, t, r(t)) \leq V(x_0, 0, r_0) + E\int_0^t e^{\lambda^* s}J_2(s)ds. \tag{4.7}$$

Since $U(x,t) \leq \eta_{\max}|x|^2 + \hat{\eta}_M\hat{\mu}|x|^{p+1}$, we have

$$\begin{aligned}&\int_0^t e^{\lambda^* s}J_2(s)ds\\ \leq& -(1-\varpi_1^*\tau - \Lambda_5 - \eta_{\max}\lambda^*)\int_0^t e^{\lambda^* s}|x(s)|^2ds\\ &+ (2\beta_1 + \varpi_2^*\tau)\int_0^t e^{\lambda^* s}|x(s-\delta(s))|^2ds - ((1-\varpi_3^*\tau\\ &- (p-1)\beta_2)\hat{\mu} - \Lambda_6 - \hat{\eta}_M\hat{\mu}\lambda^*)\int_0^t e^{\lambda^* s}|x(s)|^{p+1}ds\\ &+ (2\beta_2\hat{\mu} + 2\beta_3 + \varpi_4^*\tau)\int_0^t e^{\lambda^* s}|x(s-\delta(s))|^{p+1}ds.\end{aligned}$$

Using Lemma 1 and $e^{\lambda^* s} \leq e^{\lambda^*\Delta}e^{\lambda^*(s-\delta(s))}$, we further have

$$\begin{aligned}\int_0^t e^{\lambda^* s}J_2(s)ds \leq& C_2 - \Phi_1(\lambda^*)E\int_0^t e^{\lambda^* s}|x(s)|^2ds\\ &- \Phi_2(\lambda^*)E\int_0^t e^{\lambda^* s}|x(s)|^{p+1}ds,\end{aligned}$$

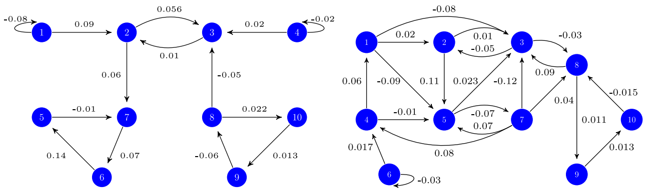

Fig. 1. The neuron network connections at free mode (left) and busy mode (right).

where $C_2 = e^{\lambda^* \tau} \Delta^* \tau((2\beta_1 + \varpi_2^* \tau)\|\xi\|^2 + (2\beta_2 \hat{\mu} + 2\beta_3 + \varpi_4^* \tau)\|\xi\|^{p+1})$. From Remark 4, we see that they are all non-negative. Therefore, we obtain from (4.7) that $\eta_{\min} e^{\lambda^* t} E|x(t)|^2 \le C_2 + V(x_0, 0, r_0)$, where $\eta_{\min} = \min_{i \in \mathbb{S}} \eta_i$. Letting $t \to \infty$ give the desired assertion (4.4). The proof is hence complete. □

5. An application to neural networks

Consider a stochastic delay neural network with 10 neurons perturbed by a scalar Brownian motion $W(t)$, operating in two modes, busy and free. In the jth neuron, it obeys the Hopfield model in free mode $\mathrm{d}x_j(t) = \Big(-L_j x_j(t) + \sum_{k=1}^{10} \Pi_{jk} \vartheta_k(x_k(t - \delta(t)))\Big)$ $\mathrm{d}t + \sigma x_j(t)\mathrm{d}W(t)$, while in busy mode, it could be described by the Cohen–Grossberg neuron network $\mathrm{d}x_j(t) = -\Gamma x_j(t)\big(P(x_j^2(t) - \varrho) - \sum_{k=1}^{10} \tilde{\Pi}_{jk} \tilde{\vartheta}_k(x_k(t - \delta(t)))\big)\mathrm{d}t + \tilde{\sigma} x_j^2(t - \delta(t))\mathrm{d}W(t)$. Here Π_{jk} and $\tilde{\Pi}_{jk}$ stand for the connection weight from neuron k to neuron j in free mode and busy mode, respectively, $L_j = \sum_{k=1}^{10} |\Pi_{jk}|$, $\vartheta_j(x_j) = \rho(1 - e^{-x_j})/(1 + e^{-x_j})$ and $\tilde{\vartheta}_j(x_j) = \tilde{\rho}(e^{x_j} - e^{-x_j})/(e^{x_j} + e^{-x_j})$ are the transfer functions, $\delta(t) = \sum_{k=0}^{\infty}(0.2(t - k)1_{[k,k+0.5)}(t) + (0.2 - 0.2(t - k))1_{[k+0.5,k+1)}(t))$ is the system time lag. For more information about these two types of neuron network, we cite (Blythe, Mao, & Liao, 2001; Wang, Shu, Fang, & Liu, 2006; Ye, Michel, & Wang, 1995) for references.

This neuron network switches from one mode into the other according to a Markov chain $r(t)$ on the state space $\mathbb{S} = \{1, 2\}$ (1 for free mode, 2 for busy mode) with transition rate matrix $Q = [-8, 8; 1, -1]$. The network parameters are given as $\varrho = 0.15$, $\rho = 0.3$, $\tilde{\rho} = 0.15$, $\Gamma = 3$, $P = 2.5$, $\sigma = 0.3$, $\tilde{\sigma} = 0.1$. The connection weight Π_{jk} and $\tilde{\Pi}_{jk}$ can be obtained from the network connection graphs in Fig. 1.

Let $x = (x_1, \dots, x_{10})^{\mathrm{T}}$, $L = \mathrm{diag}(L_1, \dots, L_{10})$, $\vartheta(x) = (\vartheta_1(x_1), \dots, \vartheta_{10}(x_{10}))^{\mathrm{T}}$, $\tilde{\vartheta}(x) = (\tilde{\vartheta}_1(x_1), \dots, \tilde{\vartheta}_{10}(x_{10}))^{\mathrm{T}}$, $\mathcal{P} = (\varrho, \dots, \varrho)^{\mathrm{T}}$, $\Pi = \big(\Pi_{jk}\big)_{10\times 10}$, $\tilde{\Pi} = (\tilde{\Pi}_{jk})_{10\times 10}$. Then rewrite the network into a general form of hybrid SDDE as

$$
\begin{aligned}
\mathrm{d}x(t) =& f(x(t), x(t - \delta(t)), r(t))\mathrm{d}t \\
& + g(x(t), x(t - \delta(t)), r(t))\mathrm{d}W(t).
\end{aligned} \tag{5.1}
$$

Here $g(x, y, 1) = \sigma x$, $g(x, y, 2) = \tilde{\sigma} y^2$, $f(x, y, 1) = -Lx + \Pi \vartheta(y)$, $f(x, y, 2) = -\Gamma \mathrm{diag}(x)(P(x^2 - \mathcal{P}) - \tilde{\Pi}\tilde{\vartheta}(y))$. It is easy to see that Assumption 1 holds with $\Delta^* = 1.25$ and Assumption 2 is satisfied with $K_1 = 1.151$, $K_2 = 0.0861$, $K_3 = 7.526$, $K_4 = 0.026$, $K_5 = 0.09$, $K_6 = 0$, $K_7 = 0$, $K_8 = 0.01$, $p = 3$. Next selecting $q = 8 \ge 2p$, we derive that $\tilde{a}_1 = 0.4071$, $\tilde{A} = 3.9295$, $\tilde{b}_1 = 0.043$, $\tilde{\gamma}_2 = 1.125$, $\tilde{b}_2 = 0.052$, $\tilde{c}_2 = 0.7448$, $\tilde{d}_2 = 0.035$. Then Assumption 3 is satisfied. Moreover, $\tilde{\eta}_1 = 0.2544$, $\tilde{\mu} = 1.963$, $\tilde{D} = 0.8851$ and $\tilde{\mu}\tilde{D} - q(q - 2 + (p + 1)\Delta^*)\tilde{d}_M/(q + p - 1) = 1.4294$. Until now, all the conditions in Theorem 1 are fulfilled. Thus, neuron network (5.1) has a unique global solution.

Then we want to design a state feedback control $u(x, i)$ based on discrete-time observations at $0, \tau, 2\tau, \dots$ to stabilise neuron network (5.1). The controlled network then becomes

$$
\begin{aligned}
\mathrm{d}x(t) =& (f(x(t), x(t - \delta(t)), r(t)) + u(x(t_\tau), r(t)))\mathrm{d}t \\
& + g(x(t), x(t - \delta(t)), r(t))\mathrm{d}W(t).
\end{aligned} \tag{5.2}
$$

Before that, we observe that $\mathbb{S}$ can be divided into two parts, $\mathbb{S}_1 = \{1\}$ and $\mathbb{S}_2 = \{2\}$ (Hopfield structure and Cohen–Grossberg structure, respectively). Through calculation, we obtain that for $i \in \mathbb{S}_1$, $a_1 = 0.047$, $b_1 = 0.043$, $\hat{a}_1 = 0.1371$, $\hat{b}_1 = 0.0431$, and $A_1 = 7.9059$ $\hat{A} = 7.4518$, which are non-singular M-matrices. It is then easy to derive that $D_b = 0.9864$, $\hat{D} = 0.974$. While for $i \in \mathbb{S}_2$, we get $\gamma_2 = 1.125$, $b_2 = 0.052$, $c_2 = 0.7448$, $d_2 = 0.005$, $D_d = 0.7375$. As a result, Assumption 4 holds.

Then we choose $\kappa_2 = 2$ and design the control function as follows: for any $x \in \mathbb{R}^{10}$, $u(x, 1) = 0$, and $u(x, 2) = -2x$ if $x| \le R_2$, $u(x, 2) = -2(2R_2/|x| - 1)x$ if $R_2 < |x| \le 2R_2$, $u(x, 2) = 0$ if $|x| > 2R_2$. Here $R_2 = 2.8968$. Consequently, Rule 1 is satisfied with $K = 2$. It is straightforward to derive that $a_2 = -0.875$. We then see that Rule 2 is true with $A = [7.9058, -8; -1, 2.75]$ being a non-singular M-matrix. Compute $\hat{\mu} = 0.8512$, $(\eta_1, \eta_2)^{\mathrm{T}} = (0.7823, 0.6481)^{\mathrm{T}}$, $\hat{\eta}_1 = 0.1142$. It is then easy to obtain that $D_1 = 0.9157$, $D_2 = 0.8243$. Rule 3 is hence fulfilled. Up to now, we have verified all the conditions in Theorem 2. Then we conclude that controlled system (5.2) is exponentially stable in mean square if $\tau < \tau^* = 0.0387$.

6. Conclusion

In this paper, we have designed the discrete-time state feedback control in a bounded state area to stabilise a kind of structured hybrid SDDEs with more general time delay. Not only could more general stochastic systems be covered, but also the control could be less costly. The conditions given on the original system and the control function were less restrictive and could also be verified easily in practice, in particular comparing with Assumption 6 in Mei et al. (2020) or Lemma 4.3 in Shi et al. (2022). For convenience, we only divided the system into two proper subsystems, which satisfied the Khasminskii-type structure and the generalised Khasminskii-type structure, respectively. But owing to mathematical restriction, we could not impose any control in the former subsystem. Our future work will be devoted to this problem.

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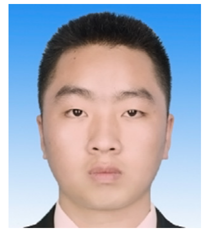

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