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Strong Convergence of an Explicit Numerical Approximation for n-Dimensional Superlinear SDEs with Positive Solutions

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Abstract

For a stochastic differential equation (SDE) with a unique positive solution, a rational numerical method is expected to be structure preserving. However, most existing methods are not, as far as we know. Some characteristic of the SDE models including the multi-dimension and super-linearity make it even more challenging. In this work, we fill the gap by proposing an explicit numerical method which is not only structure preserving but also cost effective. The strong convergence framework is set up by moment convergence analysis. We use the Lotka-Volterra system to elaborate our theory, nevertheless, the method works for a wide range of multi-dimensional superlinear SDE models.

Keywords: Stochastic differential equation, structure preserving numerical method, n-dimensional superlinear Lotka-Volterra model, strong convergence

1 Introduction

In recent years, stochastic differential equations (SDEs) have been intensively employed in a variety of areas including biology, finance and engineering (e.g. [2, 4–6, 15, 18]). For example, some authors studied an n-dimensional Lotka-Volterra population model

$$dx(t) = \mathrm{diag}(x_1(t), \cdots, x_n(t))[(b + Ax(t))dt + \sigma dB(t)], \tag{1.1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T$ is the populations of the n species at time t, and the parameters $b = (b_1, \cdots, b_n)^T \in \mathbb{R}^n, A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}, \sigma = (\sigma_1, \cdots, \sigma_n)^T \in \mathbb{R}^n$ and $B(t)$ is a scalar Brownian motion. In particular, Bahar and Mao [4] examined a generalised version of (1.1) taking into account the time delay. They discovered the key role an external randomness plays in the dynamics of the population systems.

Since the exact solution of (1.1) cannot be written in an explicit form, such analytical investigation is a good way of realising its asymptotic properties. Besides, numerical simulation has also been attracting more attention. It is known that many stochastic differential equation (SDE) models are with unique positive solutions. For instance, financial models including the Heston stochastic volatility

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model [10], the stochastic Ait-Sahalia-type interest rate model [1] and the Chan-Karolyi-Longstaff-Sanders model [8], as well as biological models such as (1.1). And in practice, we hope the Monte Carlo simulations preserve the structures of the exact solutions. Therefore, a positivity preserving numerical algorithm is much in demand.

Such numerical methods have been discussed by some authors, while most are implicit [3, 16]. Compared with them, the explicit methods are more desirable due to its low computational cost, etc. Kahl et al. [12] examined how stochastic integration schemes can be constructed to preserve positivity or boundedness of the true solution. Milstein-type methods are found to be the method-of-choice with respect to both efficiency and preservation of structural properties. Recently, Chen et al. [9] constructed a positivity preserving scheme for the 1-dimensional stochastic SIS model

$$dz(t) = z(t)(\eta_1 - \eta_2 z(t))dt + \sigma z(t)(N - z(t))dW(t),$$

where $W(t)$ is a scalar Brownian motion, by combining a Lamperti-type transformation with an explicit truncation method and obtained a convergence rate of order 1. Yang and Huang [22] suggested a modified method for the same epidemic model by combining the logarithmic transformation and the EM method and also showed the first-order rate in L_p convergence. Cai et al. [7] worked on a more general version involving a square-root diffusion term. The positivity preserving method proposed therein is also capable of coping with the mean-reverting square-root process.

However, such explicit methods are hardly applicable to SDE (1.1) due to its multi-dimension and super-linearity. For explict methods that are suitable for superlinear SDEs, one may recall various modified Euler-Maruyama (EM) methods. Examples include the tamed EM [11], the truncated EM (TEM) [19], the tamed Milstein EM [21], and the adaptive EM [13], among others. It is a challenging but meaningful topic to establish the positivity preserving theory by taking advantages of such existing methods.

Recently, by modifying the truncated EM method, Mao et al. [20] have worked out a positivity preserving numerical method for (1.1). However, due to the technical limit, the method does not work for a big portion of model known as the Lotka–Volterra cooperation model. The key issue is that the technique used therein has limited the pth moment boundedness of the numerical solution to be considered only under a weak condition. In this work, we will develop a novel methodology to address this issue. Firstly, a modified numerical scheme is formulated. Given the new numerical format, the continuous-sample approximation considered in [19, 20] is no longer available. Hence the Itô formula does not work and instead the binomial formula will make great contribution. The strong convergence is finally constructed by moment convergence analysis.

Since there are a range of SDE models in different fields that are with positive solutions, the investigation of the numerical methods that preserve their structures is becoming quite necessary and meaningful in practice. The method developed in this paper has delivered an advanced way for this interesting topic.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t)$ be a scalar Brownian motion defined on the probability space. Let $\mathbb{I}_\Omega$ be the indicator function of Ω. We denote by $\mathbb{R}^n_+$ the positive cone in $\mathbb{R}^n$, that is, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0 \text{ for } 1 \leqslant i \leqslant n\}$. For any real number y, denote by $\lfloor y \rfloor$ the greatest integer no more than y, known as the floor function. If M is a vector or matrix, its transpose is denoted by M^T. If $M \in \mathbb{R}^n$, $|M|$ is the Euclidean norm. If M is a matrix, we let $|M| := \sqrt{\text{trace}(M^T M)}$ be its trace norm and $\|M\| := \sup\{|Mx| : |x| = 1\}$ its operator norm. If M is an $n \times n$ symmetric matrix, denote by $\lambda^+_{\max}(M)$ the largest eigenvalue of M in $\mathbb{R}^n_+$, namely,

$$\lambda^+_{\max}(M) = \sup_{z\in\mathbb{R}^n_+, |z|=1} z^T M z.$$

Compared it to the largest eigenvalue $\lambda_{\max}(M)$ with

$$\lambda_{\max}(M) = \sup_{z\in\mathbb{R}^n, |z|=1} z^T M z,$$

it is obvious that

$$\lambda^+_{\max}(M) \leqslant \lambda_{\max}(M)$$

always holds. In addition, similar to $\lambda_{\max}(M)$, we have

$$z^T M z \leqslant \lambda^+_{\max}(M)|z|^2 \quad \text{for } z \in \mathbb{R}^n_+.$$

For more properties of $\lambda^+_{\max}(\cdot)$, please refer to [18, pp. 379-381].

In this paper, we choose some initial value $x(0) \in \mathbb{R}^n_+$, two real values $T > 0$ and $\theta \geqslant 2$, and an integer $q \geqslant 2$. Denote by C some positive values dependent on T, q and $x(0)$ but independent of the value $j > 0$ and the stepsize Δ to be introduced shortly.

Let us first look at some known results on the exact solution to model (1.1) before moving on to the numerical section.

Assumption 2.1. *There exist positive values $\gamma_1, \gamma_2, \cdots, \gamma_n$ for*

$$-\lambda := \lambda^+_{max}\big(\bar{\Gamma}A + A^T\bar{\Gamma}\big) \leqslant 0,$$

where $\bar{\Gamma} := \mathrm{diag}(\gamma_1, \cdots, \gamma_n)$.

Lemma 2.2. *([4]). Under Assumption 2.1, there is a unique global solution $x(t) \in \mathbb{R}^n_+$ for $t \geqslant 0$ almost surely.*

Given any $j > |x(0)| \vee 1/x_i(0)$ for $1 \leqslant i \leqslant n$, define a stopping time

$$\zeta_j = \inf\{t \geqslant 0 : |x(t)| \geqslant j \text{ or } x_i(t) \leqslant 1/j \text{ for some } i\}. \tag{2.1}$$

Lemma 2.3. *Under Assumption 2.1,*

$$\mathbb{P}(\zeta_j \leqslant T) \leqslant C\big[\big(j - n\log j + n\log n\big) \wedge \big(j^{-1} + \log j + \log n\big)\big]^{-1}.$$

This result follows from the proof of [4, Theorem 2.1, p. 384].

Lemma 2.4. *Let Assumption 2.1 hold. The solution $x(t)$ of SDE (1.1) has the property*

$$\mathbb{E}\big(\sup_{0\leqslant t\leqslant T} |x(t)|^{2q}\big) < \infty.$$

Proof. We let $\Gamma = (\gamma_1, \cdots, \gamma_n)$ and $V(x) = \Gamma x$ for $x \in \mathbb{R}^n_+$. By the Itô formula,

$$dV(x(t)) = LV(x(t))dt + x^T(t)\bar{\Gamma}\sigma dB(t),$$

where

$$LV(x) = x^T\bar{\Gamma}(b + Ax) = x^T\bar{\Gamma}b + \frac{1}{2}x^T(\bar{\Gamma}A + A^T\bar{\Gamma})x \leqslant x^T\bar{\Gamma}b - \frac{1}{2}\lambda|x|^2 \leqslant x^T\bar{\Gamma}b$$

under Assumption 2.1. Applying Itô's formula once again yields

$$\begin{aligned} d(\Gamma x(t))^{2q} &\leqslant \big[2q(\Gamma x(t))^{2q-1}x^T(t)\bar{\Gamma}b + q(2q-1)(\Gamma x(t))^{2q-2}(x^T(t)\bar{\Gamma}\sigma)^2\big]dt \\ &\quad + 2q(\Gamma x(t))^{2q-1}x^T(t)\bar{\Gamma}\sigma dB(t) \\ &\leqslant C|x(t)|^{2q}dt + 2q(\Gamma x(t))^{2q-1}x^T(t)\bar{\Gamma}\sigma dB(t). \end{aligned}$$

Hence

$$\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}(\Gamma x(t))^{2q}\big)\leqslant(\Gamma x(0))^{2q}+C\int_0^T\mathbb{E}\big(\sup_{0\leqslant u\leqslant t}|x(u)|^{2q}\big)dt$$
$$+2q\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}\Big|\int_0^t(\Gamma x(s))^{2q-1}x^T(s)\bar{\Gamma}\sigma dB(s)\Big|\Big).$$

This together with the fundamental inequality

$$(\Gamma z)^{2q}\geqslant\underline{\gamma}^{2q}|z|^{2q},\quad\forall z\in\mathbb{R}^n_+,\tag{2.2}$$

where $\underline{\gamma}=\min_{1\leqslant i\leqslant n}\gamma_i$, suggests that

$$\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}|x(t)|^{2q}\big)\leqslant\underline{\gamma}^{-2q}\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}(\Gamma x(t))^{2q}\big)$$
$$\leqslant C+C\int_0^T\mathbb{E}\big(\sup_{0\leqslant u\leqslant t}|x(u)|^{2q}\big)dt+2q\underline{\gamma}^{-2q}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}\Big|\int_0^t(\Gamma x(s))^{2q-1}x^T(s)\bar{\Gamma}\sigma dB(s)\Big|\Big).$$

Then the Burkholder–Davis–Gundy inequality and the Young inequality are applied to compute

$$\begin{aligned}&\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}|x(t)|^{2q}\big)\\&\leqslant C+C\int_0^T\mathbb{E}\big(\sup_{0\leqslant u\leqslant t}|x(u)|^{2q}\big)dt+8\sqrt{2}q\underline{\gamma}^{-2q}|\Gamma|^{2q}|\sigma|\mathbb{E}\Big(\int_0^T|x(t)|^{4q}dt\Big)^{1/2}\\&\leqslant C+C\int_0^T\mathbb{E}\big(\sup_{0\leqslant u\leqslant t}|x(u)|^{2q}\big)dt+8\sqrt{2}q\underline{\gamma}^{-2q}|\Gamma|^{2q}|\sigma|\mathbb{E}\Big[\Big(\sup_{0\leqslant t\leqslant T}|x(t)|^{2q}\Big)\int_0^T|x(t)|^{2q}dt\Big]^{1/2}\\&\leqslant C+C\int_0^T\mathbb{E}\big(\sup_{0\leqslant u\leqslant t}|x(u)|^{2q}\big)dt+\frac{1}{2}\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}|x(t)|^{2q}\big)+C\mathbb{E}\int_0^T|x(t)|^{2q}dt,\end{aligned}$$

and therefore

$$\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}|x(t)|^{2q}\big)\leqslant C+C\int_0^T\mathbb{E}\big(\sup_{0\leqslant u\leqslant t}|x(u)|^{2q}\big)dt.$$

Finally, the Gronwall inequality yields

$$\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}|x(t)|^{2q}\big)\leqslant C$$

as required. □

Remark 2.5. *Recall that the boundedness of the* 1*st moment of* $x(t)$ *has been considered in [4]. While in Lemma 2.4, a better result of the* $2q$*-th moment boundedness has been derived.*

3 A Structure Preserving Scheme

To set up the numerical scheme, we first define functions f_1, f_2 and $g:\mathbb{R}^n_+\to\mathbb{R}^n$ by

$$f_1(x)=(b_1x_1,\cdots,b_nx_n)^T,\quad f_2(x)=\mathrm{diag}(x_1,\cdots,x_n)Ax.$$

and

$$g(x)=(\sigma_1x_1,\cdots,\sigma_nx_n)^T.$$

Also setting

$$f:=f_1+f_2,$$

we can rewrite (1.1) as

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{3.1}$$

in $\mathbb{R}^n_+$.

Now we explain how we set up the explicit numerical method that is able to cope with the superlinear term, and meanwhile guarantee positivity, based on the well-known truncated EM method [19]. Firstly, find a strictly increasing continuous function $\delta : [1, \infty) \to \mathbb{R}_+$ for $\delta(s) \to \infty$ as $s \to \infty$, in addition, for any $s \geqslant 1$,

$$\sup_{x\in\mathbb{R}^n, |x|\leqslant s} \frac{|f_2(x)|}{1+|x|} \leqslant \delta(s).$$

We then have its inverse function $\delta^{-1} : [\delta(1), \infty) \to \mathbb{R}_+$. Now for some $K \geqslant 1 \vee \delta(1)$, find a strictly decreasing function $h : (0, 1] \to [\delta(1), \infty)$ for

$$h(\Delta)\Delta^{1/2} \leqslant K \quad \text{for } \Delta \in (0, 1] \quad \text{and} \quad \lim_{\Delta\to 0} h(\Delta) = \infty. \tag{3.2}$$

The candidates for $\delta(\cdot)$ and $h(\cdot)$ are various. For instance,

$$h(\Delta) = K\Delta^{-p} \quad \text{for } p \in (0, 1/2] \quad \text{and} \quad \delta(s) = \|A\|s.$$

Given a stepsize $\Delta \in (0, 1]$, now define two mappings $\chi_\Delta : \mathbb{R}^n \to \{z \in \mathbb{R}^n : |z| \leqslant \delta^{-1}(h(\Delta))\}$ by

$$\chi_\Delta(z) = (|z| \wedge \delta^{-1}(h(\Delta)))\frac{z}{|z|},$$

assuming $z/|z| = 0$ at $z = 0$ and $\Lambda_\Delta : \mathbb{R}^n \to \mathbb{R}^n_+$ by

$$\Lambda_\Delta(z) = (z_1 \vee \Delta, \cdots, z_n \vee \Delta)^T.$$

Note that

$$|\Lambda_\Delta(z)| \leqslant \sqrt{(|z_1| \vee \Delta)^2 + \cdots + (|z_n| \vee \Delta)^2} \leqslant \sqrt{|z|^2 + n\Delta^2}$$

We are now ready to establish the positivity preserving truncated Euler-Maruyama (PPTEM) method to approximate the true solution $x(t)$ of SDE (1.1) in the form of (3.1). Given any stepsize $\Delta \in (0, 1]$, the PPTEM approximation $\{X_k, k = 0, 1, 2, \cdots\}$ is formulated by

$$\begin{cases} \bar{X}_0 = X_0 = x(0), \\ \bar{X}_{k+1} = X_k + f(X_k)\Delta + g(X_k)\Delta B_k, \\ X_{k+1} = \Lambda_\Delta(\chi_\Delta(\bar{X}_{k+1})), \end{cases} \tag{3.3}$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Let $\bar{X}_{k,i}$ and $X_{k,i}$ be the ith elements of $\bar{X}_k$ and X_k respectively. This scheme is inspired by Li et al. [17]. One may notice that given the current numerical format (3.3), we are unable to form the corresponding continuous-time continuous-sample approximation as in [19, 20].

It is useful to see

$$|X_k| = |\Lambda_\Delta(\chi_\Delta(\bar{X}_k))| \leqslant \sqrt{[\delta^{-1}(h(\Delta))]^2 + n\Delta^2} \leqslant \sqrt{[\delta^{-1}(Ch(\Delta))]^2} = \delta^{-1}(Ch(\Delta)).$$

In addition,

$$|f_1(X_k)| \leqslant |b||X_k|, \quad |f_2(X_k)| \leqslant Ch(\Delta)(1 + |X_k|) \quad \text{and} \quad |g(X_k)| \leqslant |\sigma||X_k|. \tag{3.4}$$

Lastly, we expand $\bar{X}_k$ and X_k to the complete timeline by

$$\bar{X}(t) := \bar{X}_k \quad \text{and} \quad X(t) := X_k \quad \text{for } t \in [t_k, t_{k+1}). \tag{3.5}$$

4 Finite-Time L^θ-convergence

To study the strong convergence, we first consider the following necessary result.

Lemma 4.1. *Under Assumption 2.1, the PPTEM solution $X(t)$ has the property*

$$\sup_{0<\Delta\leqslant 1}\sup_{0\leqslant t\leqslant T}\mathbb{E}|X(t)|^{2q}\leqslant C.$$

Proof. For any integer $k\geqslant 0$,

$$\begin{aligned}(\Gamma\bar{X}_{k+1})^2&=[\Gamma\big(X_k+f(X_k)\Delta+g(X_k)\Delta B_k\big)]^2\\&=(\Gamma X_k)^2+(\Gamma f(X_k))^2\Delta^2+(\Gamma g(X_k))^2(\Delta B_k)^2+2\Gamma X_k\Gamma f(X_k)\Delta\\&\quad+2\Gamma X_k\Gamma g(X_k)\Delta B_k+2\Gamma f(X_k)\Gamma g(X_k)\Delta\Delta B_k.\end{aligned}$$

Therefore,

$$(1+(\Gamma\bar{X}_{k+1})^2)^q=(1+(\Gamma X_k)^2)^q(1+M(X_k))^q,\tag{4.1}$$

where

$$\begin{aligned}M(X_k)=(1+(\Gamma X_k)^2)^{-1}\big[&(\Gamma f(X_k))^2\Delta^2+(\Gamma g(X_k))^2(\Delta B_k)^2+2\Gamma X_k\Gamma f(X_k)\Delta\\&+2\Gamma X_k\Gamma g(X_k)\Delta B_k+2\Gamma f(X_k)\Gamma g(X_k)\Delta\Delta B_k\big].\end{aligned}$$

Recall the binomial formula

$$(1+z)^q=\sum_{j=0}^{q}\frac{q!}{j!(q-j)!}z^j,\quad z\in\mathbb{R}.$$

Making use of this, we get from (4.1) that

$$\begin{aligned}&\mathbb{E}\big((1+(\Gamma\bar{X}_{k+1})^2)^q|\mathcal{F}_{t_k}\big)\\&=(1+(\Gamma X_k)^2)^q\sum_{j=0}^{q}\frac{q!}{j!(q-j)!}\mathbb{E}\big(M^j(X_k)|\mathcal{F}_{t_k}\big)\\&=(1+(\Gamma X_k)^2)^q\Big(1+q\mathbb{E}\big(M(X_k)|\mathcal{F}_{t_k}\big)+\sum_{j=2}^{q}\frac{q!}{j!(q-j)!}\mathbb{E}\big(M^j(X_k)|\mathcal{F}_{t_k}\big)\Big).\end{aligned}\tag{4.2}$$

Bearing in mind

$$\mathbb{E}\big((\Delta B_k)^{2m-1}|\mathcal{F}_{t_k}\big)=0\quad\text{and}\quad\mathbb{E}\big((\Delta B_k)^{2m}|\mathcal{F}_{t_k}\big)=(2m-1)!!\Delta^m,\quad\forall m\geqslant 1,\tag{4.3}$$

we first consider

$$\begin{aligned}&\mathbb{E}\big(M(X_k)|\mathcal{F}_{t_k}\big)\\&=(1+(\Gamma X_k)^2)^{-1}\big[(\Gamma f(X_k))^2\Delta^2+(\Gamma g(X_k))^2\Delta+2\Gamma X_k\Gamma f_1(X_k)\Delta+2\Gamma X_k\Gamma f_2(X_k)\Delta\big]\\&\leqslant(1+(\Gamma X_k)^2)^{-1}\big[2(\Gamma f_1(X_k))^2\Delta^2+2(\Gamma f_2(X_k))^2\Delta^2+(\Gamma g(X_k))^2\Delta+2\Gamma X_k\Gamma f_1(X_k)\Delta\\&+2\Gamma X_kX_k^T\bar{\Gamma}AX_k\Delta\big]\\&\leqslant(1+(\Gamma X_k)^2)^{-1}\big[2(\Gamma f_1(X_k))^2\Delta^2+2|\Gamma|^2|f_2(X_k)|^2\Delta^2+(\Gamma g(X_k))^2\Delta+2\Gamma X_k\Gamma f_1(X_k)\Delta\\&+\Gamma X_kX_k^T(\bar{\Gamma}A+A^T\bar{\Gamma})X_k\Delta\big]\\&\leqslant(1+(\Gamma X_k)^2)^{-1}\big[2(\Gamma f_1(X_k))^2\Delta^2+Ch^2(\Delta)(1+|X_k|)^2\Delta^2+(\Gamma g(X_k))^2\Delta\\&+2\Gamma X_k\Gamma f_1(X_k)\Delta-\lambda\Gamma X_k|X_k|^2\Delta\big].\end{aligned}$$

Assumption 2.1 immediately indicates

$$\mathbb{E}\big(M(X_k)|\mathcal{F}_{t_k}\big)\leqslant C(\Delta^2+h^2(\Delta)\Delta^2+\Delta)\leqslant C\Delta.$$

Next compute

$$\begin{aligned}
&\sum_{j=2}^{q}\frac{q!}{j!(q-j)!}\mathbb{E}\big(M^j(X_k)|\mathcal{F}_{t_k}\big)\\
&=\sum_{j=2}^{q}\frac{q!}{j!(q-j)!}(1+(\Gamma X_k)^2)^{-j}\mathbb{E}\Big[\Big((\Gamma f(X_k))^2\Delta^2+(\Gamma g(X_k))^2(\Delta B_k)^2+2\Gamma X_k\Gamma f(X_k)\Delta\\
&+2\Gamma X_k\Gamma g(X_k)\Delta B_k+2\Gamma f(X_k)\Gamma g(X_k)\Delta\Delta B_k\Big)^j\big|\mathcal{F}_{t_k}\Big]\\
&\leqslant\sum_{j=2}^{q}\frac{5^j q!}{j!(q-j)!}(1+(\Gamma X_k)^2)^{-j}\mathbb{E}\Big[\Big((\Gamma f(X_k))^{2j}\Delta^{2j}+(\Gamma g(X_k))^{2j}(\Delta B_k)^{2j}\\
&+2^j|\Gamma X_k\Gamma f(X_k)|^j\Delta^j\Big)\big|\mathcal{F}_{t_k}\Big]\\
&+\sum_{j=1}^{\lfloor q/2\rfloor}\frac{5^{2j}q!}{(2j)!(q-2j)!}(1+(\Gamma X_k)^2)^{-2j}\mathbb{E}\Big[\Big(2^{2j}(\Gamma X_k\Gamma g(X_k))^{2j}(\Delta B_k)^{2j}\\
&+2^{2j}(\Gamma f(X_k)\Gamma g(X_k))^{2j}\Delta^{2j}(\Delta B_k)^{2j}\Big)\big|\mathcal{F}_{t_k}\Big],
\end{aligned}$$

where in the last step the summation is split into two parts to facilitate the use of (4.3). Therefore,

$$\begin{aligned}
&\sum_{j=2}^{q}\frac{q!}{j!(q-j)!}\mathbb{E}\big(M^j(X_k)|\mathcal{F}_{t_k}\big)\\
&\leqslant C\sum_{j=2}^{q}(1+(\Gamma X_k)^2)^{-j}\Big((\Gamma f_1(X_k))^{2j}\Delta^{2j}+(\Gamma f_2(X_k))^{2j}\Delta^{2j}+(\Gamma g(X_k))^{2j}\Delta^j\\
&+|\Gamma X_k\Gamma f_1(X_k)|^j\Delta^j+|\Gamma X_k\Gamma f_2(X_k)|^j\Delta^j\Big)\\
&+C\sum_{j=1}^{\lfloor q/2\rfloor}(1+(\Gamma X_k)^2)^{-2j}\Big((\Gamma X_k\Gamma g(X_k))^{2j}\Delta^j+(\Gamma f_1(X_k)\Gamma g(X_k))^{2j}\Delta^{3j}\\
&+(\Gamma f_2(X_k)\Gamma g(X_k))^{2j}\Delta^{3j}\Big)\\
&\leqslant C\sum_{j=2}^{q}(1+(\Gamma X_k)^2)^{-j}\Big((\Gamma f_1(X_k))^{2j}\Delta^{2j}+Ch^{2j}(\Delta)(1+|X_k|)^{2j}\Delta^{2j}+(\Gamma g(X_k))^{2j}\Delta^j\\
&+|\Gamma X_k\Gamma f_1(X_k)|^j\Delta^j+Ch^j(\Delta)|X_k|^j(1+|X_k|)^j\Delta^j\Big)\\
&+C\sum_{j=1}^{\lfloor q/2\rfloor}(1+(\Gamma X_k)^2)^{-2j}\Big((\Gamma X_k\Gamma g(X_k))^{2j}\Delta^j+(\Gamma f_1(X_k)\Gamma g(X_k))^{2j}\Delta^{3j}\\
&+Ch^{2j}(\Delta)|X_k|^{2j}(1+|X_k|)^{2j}\Delta^{3j}\Big)\\
&\leqslant C\sum_{j=2}^{q}(\Delta^{2j}+h^{2j}(\Delta)\Delta^{2j}+\Delta^j+h^j(\Delta)\Delta^j)+C\sum_{j=1}^{\lfloor q/2\rfloor}(\Delta^j+\Delta^{3j}+h^{2j}(\Delta)\Delta^{3j}).
\end{aligned}$$

As a result, we see

$$\sum_{j=2}^{q}\frac{q!}{j!(q-j)!}\mathbb{E}\big(M^j(X_k)|\mathcal{F}_{t_k}\big)\leqslant C\Delta.$$

Substituting them back to (4.2) leads to

$$\mathbb{E}\big((1+(\Gamma\bar{X}_{k+1})^2)^q|\mathcal{F}_{t_k}\big)\leqslant(1+(\Gamma X_k)^2)^q(1+C\Delta).\tag{4.4}$$

With this, we have

$$\begin{aligned}
&\mathbb{E}(1+(\Gamma X_k)^2)^q = \mathbb{E}\big[\mathbb{E}\big((1+(\Gamma X_k)^2)^q|\mathcal{F}_{t_{k-1}}\big)\big]\\
&\leqslant \mathbb{E}\big[\mathbb{E}\big((1+(\Gamma(\bar{X}_{k,1}\vee\Delta,\cdots,\bar{X}_{k,n}\vee\Delta)^T)^2)^q|\mathcal{F}_{t_{k-1}}\big)\big]\\
&\leqslant \mathbb{E}\Big[\mathbb{E}\Big(\Big(1+\big(\Gamma\bar{X}_k+\Delta\sum_{i=1}^n\gamma_i\big)^2\Big)^q|\mathcal{F}_{t_{k-1}}\Big)\Big]\\
&= \mathbb{E}\Big[\mathbb{E}\Big(\Big(1+(\Gamma\bar{X}_k)^2+2\Gamma\bar{X}_k\Delta\sum_{i=1}^n\gamma_i+\Delta^2\big(\sum_{i=1}^n\gamma_i\big)^2\Big)^q|\mathcal{F}_{t_{k-1}}\Big)\Big]\\
&= \mathbb{E}\Big[\mathbb{E}\Big((1+(\Gamma\bar{X}_k)^2)^q\Big(1+\frac{2\Gamma\bar{X}_k\Delta\sum_{i=1}^n\gamma_i+\Delta^2\big(\sum_{i=1}^n\gamma_i\big)^2}{1+(\Gamma\bar{X}_k)^2}\Big)^q|\mathcal{F}_{t_{k-1}}\Big)\Big]\\
&\leqslant \mathbb{E}\Big[\mathbb{E}\Big((1+(\Gamma\bar{X}_k)^2)^q\Big(1+\frac{2\Delta\sum_{i=1}^n\gamma_i}{1/|\Gamma\bar{X}_k|+|\Gamma\bar{X}_k|}+\Delta^2\big(\sum_{i=1}^n\gamma_i\big)^2\Big)^q|\mathcal{F}_{t_{k-1}}\Big)\Big]\\
&\leqslant \Big(1+\Delta\sum_{i=1}^n\gamma_i+\Delta^2\big(\sum_{i=1}^n\gamma_i\big)^2\Big)^q\mathbb{E}\big[\mathbb{E}\big((1+(\Gamma\bar{X}_k)^2)^q|\mathcal{F}_{t_{k-1}}\big)\big]\\
&\leqslant \Big(1+\Delta\sum_{i=1}^n\gamma_i+\Delta^2\big(\sum_{i=1}^n\gamma_i\big)^2\Big)^q(1+C\Delta)\mathbb{E}(1+(\Gamma X_{k-1})^2)^q.
\end{aligned}$$

By iteration,

$$\begin{aligned}
&\mathbb{E}(1+(\Gamma X_k)^2)^q\\
&\leqslant \Big(1+\Delta\sum_{i=1}^n\gamma_i+\Delta^2\big(\sum_{i=1}^n\gamma_i\big)^2\Big)^{qk}(1+C\Delta)^k\mathbb{E}(1+(\Gamma X_0)^2)^q\\
&\leqslant \exp\Big(qk\Delta\sum_{i=1}^n\gamma_i+qk\Delta^2\big(\sum_{i=1}^n\gamma_i\big)^2+Ck\Delta\Big)(1+(\Gamma x(0))^2)^q\\
&= e^{CT}(1+(\Gamma x(0))^2)^q.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sup_{0<\Delta\leqslant1}\sup_{0\leqslant t\leqslant T}\mathbb{E}|X(t)|^{2q} &\leqslant \underline{\gamma}^{-2q}\sup_{0<\Delta\leqslant1}\sup_{0\leqslant t\leqslant T}\mathbb{E}(\Gamma X(t))^{2q}\\
&= \underline{\gamma}^{-2q}\sup_{0<\Delta\leqslant1}\sup_{0\leqslant k\Delta\leqslant T}\mathbb{E}(\Gamma X_k)^{2q}\leqslant C,
\end{aligned}$$

where (2.2) was used once again. This completes the proof. □

It is also straightforward to show

$$\sup_{0<\Delta\leqslant1}\sup_{0\leqslant t\leqslant T}\mathbb{E}|\bar{X}(t)|^{2q}\leqslant C$$

in view of (4.4).

For some $j>|x(0)|\vee 1/x_i(0)$ for $1\leqslant i\leqslant n$, now consider the SDE

$$du(t)=f_j(u(t))dt+g_j(u(t))dB(t) \tag{4.5}$$

for $t\geqslant 0$ with initial value $u(0)=x(0)$, where

$$f_j(u)=f(\bar{\Lambda}_j(\bar{\chi}_j(u)))\quad\text{and}\quad g_j(u)=g(\bar{\Lambda}_j(\bar{\chi}_j(u)))\quad\text{for } u\in\mathbb{R}^n,$$

with two mappings $\bar{\chi}_j:\mathbb{R}^n\to\{u\in\mathbb{R}^n:|u|\leqslant j+\frac{1}{2j}\}$ by

$$\bar{\chi}_j(u)=\Big(|u|\wedge\big(j+\frac{1}{2j}\big)\Big)\frac{u}{|u|}$$

and $\bar{\Lambda}_j : \mathbb{R}^n \to \mathbb{R}^n_+$ by

$$\bar{\Lambda}_j(u) = \Big(u_1 \vee \frac{1}{2j}, \cdots, u_n \vee \frac{1}{2j}\Big)^T.$$

Clearly, they are analogues of $\chi_\Delta(\cdot)$ and $\Lambda_\Delta(\cdot)$ defined in Section 3 respectively. And moreover, it is easy to see

$$|\bar{\Lambda}_j(\bar{\chi}_j(u))| \leqslant \sqrt{\big(j + \frac{1}{2j}\big)^2 + \frac{n}{4j^2}}.$$

Obviously, SDE (4.5) is with global Lipschitz coefficients $f_j(\cdot)$ and $g_j(\cdot)$. So SDE (4.5) has a unique global positive solution on $t \geqslant 0$.

Given stepsize $\Delta \in (0, 1]$, the EM scheme applied to (4.5) with the approximation $\{U_k, k = 0, 1, 2, \cdots\}$ is formulated by

$$\begin{cases} U_0 = x(0), \\ U_{k+1} = U_k + f_j(U_k)\Delta + g_j(U_k)\Delta B_k \end{cases} \tag{4.6}$$

for $k = 0, 1, \cdots$. Then define the continuous-time extension $U(t)$ for $t \geqslant 0$ by

$$U(t) = U_k, \quad \forall t \in [t_k, t_{k+1})$$

and the Itô process

$$\hat{U}(t) = x(0) + \int_0^t f_j(U(s))ds + \int_0^t g_j(U(s))dB(s).$$

It is well-known (e.g. [14, 17]) that

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T} |\hat{U}(t) - u(t)|^\theta\Big) \leqslant C_j\Delta^{\theta/2}, \tag{4.7}$$

where C_j is a constant dependent on j and T but independent of Δ.

Notice that

$$x(t \wedge \zeta_j) = u(t \wedge \zeta_j) \quad \text{a.s. } \forall t \geqslant 0.$$

This along with (4.7) indicates

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T} |\hat{U}(t \wedge \zeta_j) - x(t \wedge \zeta_j)|^\theta\Big) \leqslant C_j\Delta^{\theta/2}, \quad \forall \Delta \in (0, 1]. \tag{4.8}$$

We have one more lemma before showing our main results.

Lemma 4.2. *Find an integer $z \geqslant 3 \vee \theta$ such that*

$$\Big(\frac{2z}{2z-1}\Big)^\theta (T+1)^{\theta/2z} \leqslant 2. \tag{4.9}$$

Then

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T} |\hat{U}(t) - U(t)|^\theta\Big) \leqslant C_j\Delta^{\theta(z-1)/2z}.$$

Proof. Let d be the integer part of T/Δ. Applying the Hölder inequality leads to

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T} |\hat{U}(t) - U(t)|^\theta\Big) \\
&\leqslant \mathbb{E}\Big(\max_{0\leqslant l\leqslant d} \sup_{t_l\leqslant t\leqslant t_{l+1}} |f_j(U_l)(t - t_l) + g_j(U_l)(B(t) - B(t_l))|^\theta\Big) \\
&\leqslant C_j\Delta^\theta + C_j\mathbb{E}\Big(\max_{0\leqslant l\leqslant d} \sup_{t_l\leqslant t\leqslant t_{l+1}} |B(t) - B(t_l)|^\theta\Big) \\
&\leqslant C_j\Delta^\theta + C_j\Big[\mathbb{E}\Big(\max_{0\leqslant l\leqslant d} \sup_{t_l\leqslant t\leqslant t_{l+1}} |B(t) - B(t_l)|^{2z}\Big)\Big]^{\theta/2z}.
\end{aligned}$$

It then follows from the Doob martingale inequality that

$$\begin{aligned}
&\mathbb{E}\Big(\max_{0\leqslant l\leqslant d}\sup_{t_l\leqslant t\leqslant t_{l+1}}|B(t)-B(t_l)|^{2z}\Big)\\
&\leqslant\sum_{l=0}^{d}\mathbb{E}\Big(\sup_{t_l\leqslant t\leqslant t_{l+1}}|B(t)-B(t_l)|^{2z}\Big)\leqslant\Big(\frac{2z}{2z-1}\Big)^{2z}\sum_{l=0}^{d}\mathbb{E}|B(t_{l+1})-B(t_l)|^{2z}\\
&\leqslant\Big(\frac{2z}{2z-1}\Big)^{2z}\sum_{l=0}^{d}(2z-1)!!\Delta^{z}\leqslant\Big(\frac{2z}{2z-1}\Big)^{2z}(2z-1)!!(T+1)\Delta^{z-1}.
\end{aligned}$$

Plugging this back yields

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|U(t)-\hat{U}(t)|^{\theta}\Big)\\
&\leqslant C_j\Delta^{\theta}+C_j\Big[\Big(\frac{2z}{2z-1}\Big)^{2z}(2z-1)!!(T+1)\Delta^{z-1}\Big]^{\theta/2z}\\
&\leqslant C_j\Delta^{\theta}+C_jz^{\theta/2}\Delta^{\theta(z-1)/2z},
\end{aligned}$$

where the inequality

$$[(2z-1)!!]^{1/z}\leqslant\frac{1}{z}\sum_{i=1}^{z}(2i-1)=z$$

is used. The required assertion thus follows. □

Theorem 4.3. *Let Assumption 2.1 hold. Recall that $x(t)$ is the solution to SDE (1.1) and $X(t)$ is the PPTEM solution. For any $\theta\geqslant 2$,*

$$\lim_{\Delta\to 0}\mathbb{E}|X(T)-x(T)|^{\theta}=0.$$

Proof. Let $\epsilon>0$ be arbitrary. From Lemma 2.3, there is a $j=j(\epsilon)$ sufficiently large for

$$\mathbb{P}(\zeta_j\leqslant T)\leqslant\epsilon/2.$$

That is,

$$\mathbb{P}(\Omega_1)\geqslant 1-\epsilon/2, \tag{4.10}$$

where

$$\Omega_1=\{\zeta_j>T\}=\Big\{|x(t)|<j\text{ and }x_i(t)>1/j\text{ for all }1\leqslant i\leqslant n\text{ and }0\leqslant t\leqslant T\Big\}.$$

With this j, define

$$\Omega_2=\Big\{\sup_{0\leqslant t\leqslant T}|\hat{U}(t\wedge\zeta_j)-x(t\wedge\zeta_j)|<\frac{1}{2j}\Big\}.$$

Applying the Chebyshev inequality to (4.8) gives

$$\mathbb{P}(\Omega_2^c)=\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T}|\hat{U}(t\wedge\zeta_j)-x(t\wedge\zeta_j)|\geqslant\frac{1}{2j}\Big)\leqslant C_j\Delta^{\theta/2},\quad\forall\Delta\in(0,1].$$

One can hence find a $\Delta^*=\Delta^*(\epsilon)\in(0,1]$ so small that

$$\mathbb{P}(\Omega_2)\geqslant 1-\epsilon/2,\quad\forall\Delta\in(0,\Delta^*]. \tag{4.11}$$

Now let $\Omega_0=\Omega_1\cap\Omega_2$, where

$$\mathbb{P}(\Omega_0)\geqslant 1-\epsilon,\quad\forall\Delta\in(0,\Delta^*] \tag{4.12}$$

in view of (4.10) and (4.11). For any $\omega \in \Omega_0$ and $\Delta \in (0, \Delta^*]$, we see $\zeta_j > T$, and thus

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}|U(t)| &\leqslant \sup_{0\leqslant t\leqslant T}|\hat{U}(t)| \\ &\leqslant \sup_{0\leqslant t\leqslant T}|x(t)| + \sup_{0\leqslant t\leqslant T}|\hat{U}(t) - x(t)| \\ &< j + \frac{1}{2j} \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\inf_{0\leqslant t\leqslant T}U_i(t) &\geqslant \inf_{0\leqslant t\leqslant T}\hat{U}_i(t) \\ &\geqslant \inf_{0\leqslant t\leqslant T}x_i(t) - \sup_{0\leqslant t\leqslant T}|x_i(t) - \hat{U}_i(t)| \\ &> \inf_{0\leqslant t\leqslant T}x_i(t) - \sup_{0\leqslant t\leqslant T}|x(t) - \hat{U}(t)| \\ &> \frac{1}{j} - \frac{1}{2j} = \frac{1}{2j}. \end{aligned} \tag{4.14}$$

Now choose $\Delta_0 \in (0, \Delta^*]$ so small that

$$\delta^{-1}(h(\Delta_0)) \geqslant j + \frac{1}{2j} \quad \text{and} \quad \Delta_0 \leqslant \frac{1}{2j}.$$

For any $\omega \in \Omega_0$ and $\Delta \in (0, \Delta_0]$, it follows from (4.13) and (4.14) that

$$U(t) = X(t) = \bar{X}(t), \quad \forall t \in [0, T] \tag{4.15}$$

by comparing (4.6) with (3.3). This is to be used shortly.

We are now ready to show the required assertion. For any $\Delta \in (0, \Delta_0]$, choose an integer $q_0 > 2 \vee \theta/2$ and compute

$$\begin{aligned}&\mathbb{E}|X(T) - x(T)|^\theta \\ &= \mathbb{E}\big(|X(T) - x(T)|^\theta \mathbb{I}_{\Omega_0}\big) + \mathbb{E}\big(|X(T) - x(T)|^\theta \mathbb{I}_{\Omega_0^c}\big) \\ &\leqslant \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|X(t) - x(t)|^\theta \mathbb{I}_{\Omega_0}\Big) + \big(\mathbb{P}(\Omega_0^c)\big)^{1-\theta/2q_0}\big(\mathbb{E}|X(T) - x(T)|^{2q_0}\big)^{\theta/2q_0}. \end{aligned} \tag{4.16}$$

Making use of (4.15), (4.8) as well as Lemma 4.2 for any integer $z \geqslant 3 \vee \theta$ under (4.9), we derive

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|X(t) - x(t)|^\theta \mathbb{I}_{\Omega_0}\Big) \\ &= \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|U(t) - x(t)|^\theta \mathbb{I}_{\Omega_0}\Big) \\ &\leqslant \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|U(t) - \hat{U}(t)|^\theta\Big) + \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|\hat{U}(t) - x(t)|^\theta \mathbb{I}_{\Omega_1}\Big) \\ &\leqslant \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|U(t) - \hat{U}(t)|^\theta\Big) + \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|\hat{U}(t\wedge\zeta_j) - x(t\wedge\zeta_j)|^\theta\Big) \\ &\leqslant C_j\Delta^{\theta/3}, \quad \forall \Delta \in (0, \Delta_0]. \end{aligned}$$

In addition, Lemmas 2.4 and 4.1 yield

$$\big(\mathbb{E}|X(T) - x(T)|^{2q_0}\big)^{\theta/2q_0} \leqslant 2^\theta\big(\mathbb{E}|X(T)|^{2q_0} + \mathbb{E}|x(T)|^{2q_0}\big)^{\theta/2q_0} \leqslant C.$$

Substituting them to (4.16) finally leads to

$$\mathbb{E}|X(T) - x(T)|^\theta \leqslant C_j\Delta^{\theta/3} + C\epsilon^{1-\theta/2q_0}, \quad \forall \Delta \in (0, \Delta_0].$$

And thus

$$\lim_{\Delta\to 0}\mathbb{E}|X(T) - x(T)|^\theta \leqslant C\epsilon^{1-\theta/2q_0}.$$

The desired assertion follows, given that ϵ is arbitrary. □

5 Numerical Results

This section is made up of two parts. In the first part, we will illustrate the advantages of our PPTEM method in preserving the positivity of the solution to SDE (3.1) by comparing the performances of the PPTEM with the EM and the TEM method. In the second part, we will compare the mean square errors of the PPTEM among different stepsizes to demonstrate the strong convergence, as well as to make a prediction on the convergence rate.

Consider the following two-dimensional Lotka-Volterra cooperation system

$$\begin{aligned} dx(t) &= \mathrm{diag}(x_1(t), x_2(t))[b + Ax(t)dt + \sigma dB(t)] \\ &= [f_1(x(t)) + f_2(x(t))]dt + g(x(t))dB(t) \end{aligned} \tag{5.1}$$

in $\mathbb{R}^2_+$, where

$$b = \begin{pmatrix} 30 \\ 15 \end{pmatrix}, A = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 \\ 7 \end{pmatrix}.$$

In addition, set $f(x) = g(x) = 0$ for $x \notin \mathbb{R}^2_+$.

Define $\delta : [1, \infty) \to \mathbb{R}_+$ by

$$\delta(s) = 3.5s$$

which ensures

$$\sup_{x \in \mathbb{R}^n, |x| \leqslant s} \frac{|f_2(x)|}{1 + |x|} \leqslant \|A\| s < 3.5s = \delta(s),$$

and hence the inverse function

$$\delta^{-1}(s) = \frac{2}{7}s.$$

Let $h : (0, 1] \to [\delta(1), \infty)$ be

$$h(\Delta) = 10^5 \Delta^{-1/2}.$$

Hence

$$\delta^{-1}(h(\Delta)) = \frac{2 \times 10^5}{7} \Delta^{-1/2}.$$

The mapping $\Lambda_\Delta(\chi_\Delta(x)) : \mathbb{R}^2 \to \left\{ x \in \mathbb{R}^2_+ : |x| \leqslant \sqrt{\frac{4 \times 10^{10}}{49} \Delta^{-1} + 2\Delta^2} \right\}$ is of the form

$$\Lambda_\Delta(\chi_\Delta(x)) = \left(\left(x_1 \wedge \frac{2 \times 10^5}{7} \Delta^{-1/2} \frac{x_1}{|x|} \right) \vee \Delta, \left(x_2 \wedge \frac{2 \times 10^5}{7} \Delta^{-1/2} \frac{x_2}{|x|} \right) \vee \Delta \right) \quad \text{for } x \in \mathbb{R}^2.$$

Then our PPTEM solution $X(t)$ is determined by first forming the discrete time approximation $\{X_k, k = 0, 1, 2, \cdots\}$ via (3.3), and then expanding it to the complete timeline by (3.5). From Theorem 4.3,

$$\lim_{\Delta \to 0} \mathbb{E}|X(T) - x(T)|^\theta = 0 \tag{5.2}$$

for any $T > 0$ and $\theta \geqslant 2$ under Assumption 2.1.

In order to verify Assumption 2.1, let us first cite a useful result from Mao [18].

Lemma 5.1. *([18]) For a symmetric matrix $D = (d_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$,*

$$\lambda^+_{\max}(D) \leqslant \max_{1 \leqslant i \leqslant n} \left(d_{ii} + \sum_{j \neq i} (0 \vee d_{ij}) \right).$$

So $\lambda^+_{\max}(D) \leqslant 0$ if

$$d_{ii} \leqslant - \sum_{j \neq i} (0 \vee d_{ij}).$$

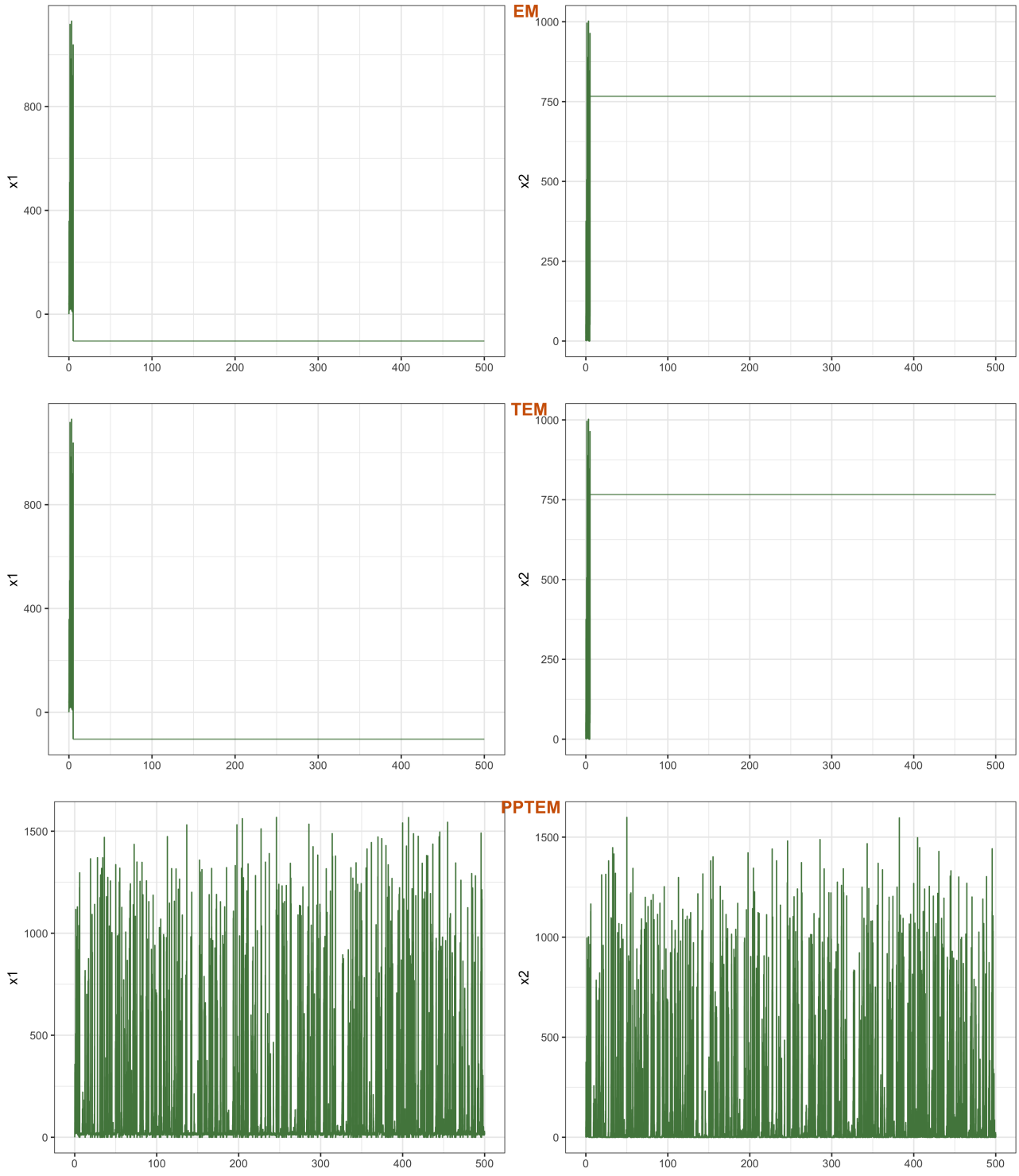

Figure 1: The sample paths for SDE (5.1) using the EM, TEM and the PPTEM for a duration of 500, each with a stepsize $\Delta = 10^{-3}$ and the initial value $x(0) = (2, 1)^T$.

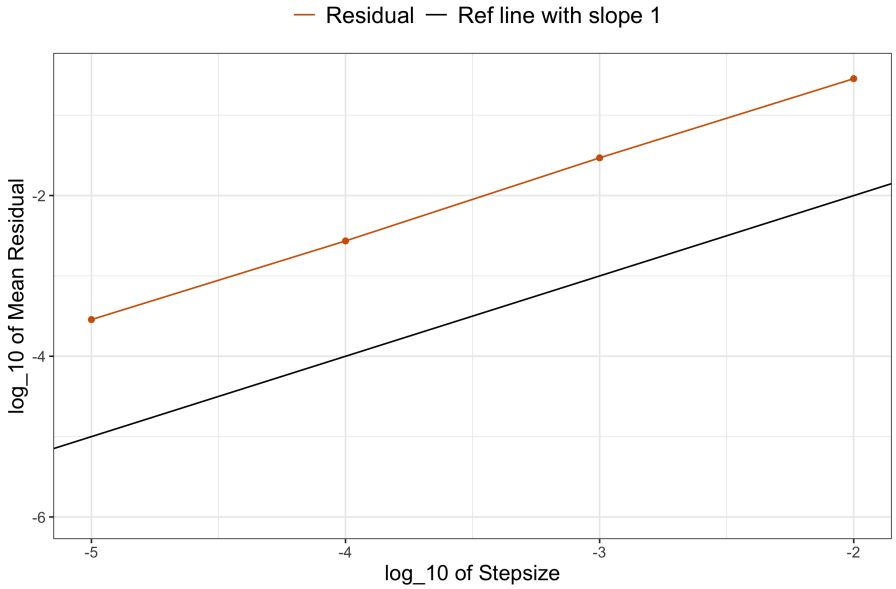

Figure 2: The mean residuals of the PPTEM with stepsize $\Delta = 10^{-5}, 10^{-4}, 10^{-3}$ and 10^{-2}, given the initial data $x(0) = (2,1)^T$.

From Assumption 2.1, we need to find two positive numbers γ_1 and γ_2 such that $-\lambda \leqslant 0$. Next compute

$$\begin{aligned}\bar{\Gamma}A + A^T\bar{\Gamma} &= \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \\ &= \begin{pmatrix} -4\gamma_1 & 2\gamma_1 + \gamma_2 \\ 2\gamma_1 + \gamma_2 & -2\gamma_2 \end{pmatrix}.\end{aligned}$$

From Lemma 5.1,

$$-\lambda = \lambda^+_{max}\big(\bar{\Gamma}A + A^T\bar{\Gamma}\big) \leqslant 0$$

if

$$\begin{cases} -4\gamma_1 \leqslant -(2\gamma_1 + \gamma_2) \\ -2\gamma_2 \leqslant -(2\gamma_1 + \gamma_2), \end{cases} \quad \text{i.e.} \quad \begin{cases} \gamma_2 \leqslant 2\gamma_1 \\ 2\gamma_1 \leqslant \gamma_2. \end{cases}$$

Here we simply let $\gamma_1 = 1$ and $\gamma_2 = 2$ to ensure Assumption 2.1 holds, and therefore our PPTEM is a strong convergent approximation to SDE (5.1).

The performances of the EM, TEM and PPTEM are compared in Figure 1. The stepsize $\Delta = 10^{-3}$ and the total duration $T = 500$. It is not surprising that both the EM and the truncated EM paths collapse to negative halfway through. Since this conflicts with the fact that the solution to SDE (5.1) is positive, we then conclude that they are not representing the true system behaviours. By contrast, the exponential EM simulation fluctuates within a certain band, preventing the values from getting negative for all time. This experimental result clearly reveals the key advantage of our new method.

Furthermore, we would also like to illustrate the reliability of the PPTEM through our strong convergence result (5.2) with, let's say, $\theta = 2$. Firstly, a TEM simulation $\{W_k, k = 1, \cdots, 10^6\}$ is generated with the stepsize $\Delta = 10^{-6}$ for a duration $T = 1$. Based on the existing strong convergence result of TEM, this is assumed as one reliable sample path of the true solution to SDE (5.1), given that the stepsize is rather small. Next, we develop PPTEM paths for stepsize $\Delta = 10^{-5}, 10^{-4}, 10^{-3}$ and 10^{-2} respectively, again for a duration of $T = 1$. Considering the computational cost, we do the procedure for 100 times. Consequently, the 2nd moment in (5.2) can be approximated by

$$\frac{1}{100}\sum_{r=1}^{100}\left[\left(X^r_{10^5,1} - W^r_{10^6,1}\right)^2 + \left(X^r_{10^5,2} - W^r_{10^6,2}\right)^2\right]$$

when $\Delta = 10^{-5}$,

$$\frac{1}{100}\sum_{r=1}^{100}\left[\left(X^r_{10^4,1} - W^r_{10^6,1}\right)^2 + \left(X^r_{10^4,2} - W^r_{10^6,2}\right)^2\right]$$

when $\Delta = 10^{-4}$, etc.

The results are shown in Figure 2. Clearly, as the stepsize gets smaller, the PPTEM captures the true solution more precisely. In addition, it suggests a convergence rate of order 1/2 to the true solution. This will be formally investigated in the future.

6 Summary

A rational numerical method is expected to be structure preserving, so it does reflect the real dynamics of the SDE system. This work concerns about an explicit structure preserving numerical algorithm for the multi-dimensional superlinear SDE models. We interpret our theory using the classical stochastic Lotka-Volterra model, though the scheme can be naturally applied to a bunch of popular SDE models widely used in finance, biology and epidemiology. The strong convergence framework is then constructed by moment convergence analysis. Finally, the advantages of the method are demonstrated by the numerical experiment.

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