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Explicit approximation of the invariant measure for SDDEs with the nonlinear diffusion term

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Abstract

To our knowledge, the existing measure approximation theory requires the diffusion term of the stochastic delay differential equations (SDDEs) to be globally Lipschitz continuous. Our work is to develop a new explicit numerical method for SDDEs with the nonlinear diffusion term and establish the measure approximation theory. Precisely, we construct a function-valued explicit truncated Euler-Maruyama segment process (TEMSP) and prove that it admits a unique ergodic numerical invariant measure. We also prove that the numerical invariant measure converges to the underlying one of SDDE in the Fortet-Mourier distance. Finally, we give an example and numerical simulations to support our theory.

Keywords: Stochastic delay differential equations, truncated Euler-Maruyama segment proces, stability in distribution, numerical invariant measure

1 Introduction

Since time delays are omnipresent and entrenched in real systems, delay differential equations have a wide range of emerging and existing applications in, for instance, physics, biology, medical sciences, automatic control systems; see, e.g., [18, 31, 32, 35]. On the other hands, systems in the real word are always subject to environmental noise. Stochastic delay differential equations (SDDEs) have become more and more popular mathematical models for many real systems; see e.g. [23, 28, 30] and references therein. Asymptotic stability is one of the most important topics in the study of SDDEs. There are two fundamental categories: (ASE) asymptotic stability of an equilibrium state; (ASD) asymptotic stability in distribution. ASE is to study whether the solutions of a given SDDE system will tend to the equilibrium state (e.g., 0 as in most papers) in moment or in probability; while ASD is to study whether the probability distributions of the solutions of the given SDDE system will converge to a probability distribution, known as stationary distribution or invariant measure. There is an intensive literature on ASE (see, e.g., [17, 18, 23, 30] and many others). The literature on ASD is much less than ASE but has been growing quickly for the past 10 years (see, e.g., [4–7, 19]). The reason why there are fewer papers on ASD than ASE is because the mathematics involved is much more complicated than that used for the study of ASE but certainly not because ASD is less important. In fact, it is inappropriate to study ASE for many SDDE systems in the real world but more appropriate to study ASD. For example, for many population systems under random environment, the stochastic permanence is a more desired control objective than the extinction (see, e.g., [1, 2, 13]). In this situation it is useful to investigate whether or not the probability distribution of the solutions will converge to a probability distribution (i.e., ASD), but not to zero (i.e., ASE) (see, e.g., [13, 24, 37]). The two stability categories can also be illustrated by the control of Covid-19. There are essentially two control strategies: one is to suppress infected to 0 but the other is to live with Covid-19. The former is to stabilise the infected to 0 with probability 1 (i.e., ASE), while the latter is to stabilise the distribution of the infected to a stationary distribution (i.e., ASD). More details on ASD and the related ergodic theory can be found in, e.g., [11, 12, 39, 40].

Although there is a theory on the existence and uniqueness of the invariant measure of the segment solution process to an SDDE, there is so far no way to obtain the theoretical cumulative probability distribution (CPD) of the invariant measure. It is therefore significant to establish numerical methods to approximate the CPD. For the past ten years, several numerical schemes have been proposed to approximate the CPD of the invariant measure of a stochastic differential equation (SDE) (see, e.g., [20–22, 27, 29, 34] and references therein), where the invariant measure is distributed on the finite-dimensional space $\mathbb{R}^d$. However, the invariant measure of an SDDE is distributed on the infinite-dimensional space $C([-\tau, 0;\ \mathbb{R}^d)$. It is hence much harder to approximate its CPD numerically. Nevertheless, some progress has been made recently in this direction under the condition that the diffusion coefficient of the underlying infinite-dimensional system is globally Lipschitz continuous. We would like to mention [8, 14, 15] for the study of stochastic partial differential equations and [3, 33] for SDDEs. On the other hand, the global Lipschitz condition is very restrictive and most of SDDE models in applications do not satisfy it (see, e.g., [1, 2, 9, 16]). It therefore becomes necessary and urgent to design numerical methods to approximate the invariant measure of an SDDE whose coefficients are only locally Lipschitz continuous. From the point of computational cost, it is more desired that the numerical methods are explicit. This is our main aim in this paper.

Consider an SDDE described by

$$\mathrm{d}x(t) = f(x(t), x(t-\tau))\mathrm{d}t + g(x(t), x(t-\tau))\mathrm{d}W(t), \quad t > 0 \tag{1}$$

with an initial data $x(\theta) = \xi(\theta)$, $\theta \in [-\tau, 0]$, where $\tau > 0$, $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable, $W(t)$ is an m-dimensional Brownian motion on a probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$ with a right-continuous complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and ξ is an $\mathcal{F}_0$-measurable, continuous function-valued random variable from $[-\tau,\ 0]$ to $\mathbb{R}^d$. Let $\{x_t\}_{t\geq 0}$ be the segment process, where $x_t(\theta) := x(t+\theta)$ for $\theta \in [-\tau, 0]$. As we know, approximation of the invariant measure in the infinite horizon is quite challenging for nonlinear SDDEs with non-globally Lipschitz diffusion coefficient. We mainly face the following difficulties:

- Mathematically speaking, the time-homogenous Markov property of the segment process of SDDEs plays a crucial role in investigating the ergodicity. In the numerical case, how to construct a continuous function-valued explicit numerical time-homogenous Markov process $\{Y^{\xi,\triangle}_{t_k}\}_{k\geq 0}$ for approximating the invariant measure?
- Generally, the tightness of measures is often used to derive the existence of an invariant measure. In the infinite-dimensional space, $\sup_{k\geq 0} \mathbb{E}\|Y^{\xi,\triangle}_{t_k}\|^2 < \infty$ fails to imply the tightness of correspding measures $\{\mu^{\xi,\triangle}_{t_k}\}_{k\geq 0}$ since the relative compactness does not follows from the boundedness. The existence of numerical invariant measures therefore needs to be proved carefully.
- The super-linear diffusion coefficient makes it difficult to obtain the attraction of second moment of $\{Y^{\xi,\triangle}_{t_k}\}_{k\geq 0}$, i.e.,

  $$\mathbb{E}\|Y^{\xi,\triangle}_{t_k} - Y^{\zeta,\triangle}_{t_k}\|^2 \leq \phi(t_k)\mathbb{E}\|\xi - \zeta\|^2,$$

  where $\phi : [0,\infty) \to \mathbb{R}_+$ satisfies $\lim_{t\to\infty} \phi(t) = 0$. Hence we need to explore the attraction in probability or in distribution and further discuss the uniqueness of numerical invariant measures.

To investigate the measure approximation of nonlinear SDDEs with non-globally Lipschitz diffusion coefficient, we and our coauthors in [36] first constructed an explicit truncated EM method and proved numerical solutions strongly convergent to the exact ones in the finite horizon. Furthermore, in current paper we improve the truncated EM method given by [36] and borrowing the linear interpolation go a further step to design the continuous function-valued numerical time-homogenous Markov process $\{Y_{t_k}^{\xi,\triangle}\}_{k\geq 0}$ (see (8)–(11) for details), called TEMSP. By the strong convergence of numerical solutions ($\mathbb{R}^d$-valued) in [36] we obtain the weak convergence of $\{Y_{t_k}^{\xi,\triangle}\}_{k\geq 0}$ (continuous function-valued) in the finite horizon (see Lemma 4.7 for details). Moreover, taking advantage of TEMSP, we yield the uniform boundedness and the attraction of $\{Y_{t_k}^{\xi,\triangle}\}_{k\geq 0}$ in probability (see Proposition 3 and Proposition 4 for details). Making use of above propositions we reveal that the sequence of numerical measures $\{\mu_{t_k}^{\xi,\triangle}\}_{k\geq 0}$ is Cauchy in the Fortet-Mourier distance d_Ξ defined by (2). Together with the completeness of $(\mathcal{P}(C), d_\Xi)$ (see Page 4) we prove the existence of numerical invariant measure. Furthermore, the uniqueness of numerical invariant measures follows from the attraction of $\{Y_{t_k}^{\xi,\triangle}\}_{k\geq 0}$ in probability. Finally, utilizing weak convergence of $\{Y_{t_k}^{\xi,\triangle}\}_{k\geq 0}$ we prove the numerical invariant measure converges to the underlying one in d_Ξ as the step size tends to zero.

The rest of this paper is arranged as follows. Section 2 introduces some notations and cites some used results. Section 3 proposes the truncated EM linear interpolation scheme and states the main results. Section 4 gives the proofs in details. Section 5 gives an example and numerical simulations to illustrate our results. Finally, Section 6 concludes this paper.

2 Preliminaries

In the beginning of this section, we introduce some notations. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$ and the trace norm in $\mathbb{R}^{d\times m}$, and by $\langle\cdot,\cdot\rangle$ the inner product in $\mathbb{R}^d$. For real numbers a and b, let $a\vee b=\max\{a,b\}$ and $a\wedge b=\min\{a,b\}$, respectively. Let $\lfloor a\rfloor$ be the integer part of the real number a. Let $\mathbf{1}_A(x)$ be the indicator function of the set A. Let $\mathbb{R}_+=[0,\infty)$. Denote by $C:=C([-\tau,\ 0];\ \ \mathbb{R}^d)$ the family of continuous functions X from $[-\tau,\ 0]$ to $\mathbb{R}^d$ with the supremum norm $\|X\|=\sup_{-\tau\leq\theta\leq 0}|X(\theta)|$. For $M>0$, define $B(M):=\{X\in C:\|X\|\leq M\}$ and $B^c(M)$ is its complementary in C. For $p\geq 2$, denote by $\mathcal{C}^p_{\mathcal{F}_0}:=\mathcal{C}^p_{\mathcal{F}_0}([-\tau,\ 0];\ \ \mathbb{R}^d)$ the family of $\mathcal{F}_0$-measurable, C-valued random variables such that $\sup_{-\tau\leq\theta\leq 0}\mathbb{E}|\eta(\theta)|^p<\infty$. For $M>0$ and $p\geq 2$, define $\mathbb{B}(M,p):=\{\xi\in\mathcal{C}^p_{\mathcal{F}_0}:\mathbb{E}\|\xi\|\leq M\}$ and $\mathbb{B}^c(M,p)$ is its complementary in $\mathcal{C}^p_{\mathcal{F}_0}$. Denote by $\hat{C}(\mathbb{R}^d\times\mathbb{R}^d;\ \ \mathbb{R}_+)$ the family of continuous functions from $\mathbb{R}^d\times\mathbb{R}^d$ to $\mathbb{R}_+$ satisfying $V(x,x)=0$ for all $x\in\mathbb{R}^d$. Throughout this work, L denotes a generic positive constant which may take different values at different appearances.

Let $\mathfrak{B}(C)$ be the Borel algebra of C and $\mathcal{P}(C):=\mathcal{P}(C,\mathfrak{B}(C))$ the family of probability measures on $(C,\mathfrak{B}(C))$. Define the Fortet-Mourier distance d_Ξ

on $\mathcal{P}(C)$ [10, p.8.2] as below,

$$d_{\Xi}(P_1, P_2) = \sup_{\Psi\in\Xi}\Big|\int_C \Psi(X)P_1(\mathrm{d}X) - \int_C \Psi(X)P_2(\mathrm{d}X)\Big|, \quad \forall P_1, P_2 \in \mathcal{P}(C), \tag{2}$$

where Ξ is the test functional space

$$\Xi := \Big\{\Psi : C \to \mathbb{R} \ \Big|\ |\Psi(X_1) - \Psi(X_2)| \le \|X_1 - X_2\| \text{ and } \sup_{X\in C}|\Psi(X)| \le 1\Big\}. \tag{3}$$

Remark 1 It is useful to point out that $(\mathcal{P}(C), d_{\Xi})$ is a complete metric space (see [10, Corollary 10.5] for details). In addition, the metric d_{Ξ} is equivalent to the Wasserstein distance below

$$\mathbb{W}(P_1, P_2) = \inf_{\pi\in\Pi(P_1,P_2)}\int_{C\times C}(1\wedge\|X_1 - X_2\|)\pi(\mathrm{d}X_1, \mathrm{d}X_2),$$

where $P_1,\ P_2 \in \mathcal{P}(C)$ and $\Pi(P_1, P_2)$ is their collection of couplings (see [38, Chapter 6] for details).

We impose the following hypotheses.

(H1) For any $R > 0$, there exists a positive constant ℓ_R such that

$$|f(x, y) - f(\bar{x}, \bar{y})| \vee |g(x, y) - g(\bar{x}, \bar{y})| \le \ell_R(|x - \bar{x}| + |y - \bar{y}|)$$

for any $x,\ \bar{x},\ y,\ \bar{y} \in \mathbb{R}^d$ with $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \le R$.

(H2) There exist nonnegative constants $\alpha \ge 2,\ a_1,\ a_2,\ a_3$ with $a_2 > a_3$ such that

$$\big\langle 2x, f(x, y)\big\rangle + |g(x, y)|^2 \le a_1 - a_2|x|^\alpha + a_3|y|^\alpha$$

for any $x,\ y \in \mathbb{R}^d$.

(H3) There exist nonnegative constants $b_1,\ b_2,\ b_3,\ b_4$ with $b_1 > b_2,\ b_3 > b_4$, and a function $V(\cdot,\cdot) \in \hat{C}(\mathbb{R}^d \times \mathbb{R}^d,\ \mathbb{R}_+)$ such that

$$\begin{aligned} &2\big\langle x - \bar{x}, f(x, y) - f(\bar{x}, \bar{y})\big\rangle + |g(x, y) - g(\bar{x}, \bar{y})|^2 \\ &\le - b_1|x - \bar{x}|^2 + b_2|y - \bar{y}|^2 - b_3V(x, \bar{x}) + b_4V(y, \bar{y}) \end{aligned}$$

for any $x,\ \bar{x},\ y,\ \bar{y} \in \mathbb{R}^d$.

It should be pointed out that under (H1) and (H2), SDDE (1) with the initial data $\xi \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$ has a unique global solution $x^{\xi}(t)$ for $t \ge -\tau$ (see [25, Theorem 2.4] and [26, p.278, Theorem 7.12] for details). Let $\{x^{\xi}_t\}_{t\ge0}$ be the corresponding segment process, where $x^{\xi}_t(\theta) := x^{\xi}(t + \theta)$ for $\theta \in [-\tau, 0]$. For any $\xi \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$ and $t \geq 0$, denote by μ^{ξ}_t the probability measure generated by x^{ξ}_t, namely,

$$\mu^{\xi}_t(A) = \mathbb{P}\big\{\omega \in \Omega : x^{\xi}_t \in A\big\}, \quad \forall A \in \mathfrak{B}(C). \tag{4}$$

Let us cite an ergodicity result [39, Theorem 3.6] to close this section.

Lemma 2.1 [39, Theorem 3.6] Suppose that (H1)–(H3) hold. Then the segment process $\{x^{\xi}_t\}_{t\geq 0}$ of (1) with the initial data $\xi \in C$ is asymptotically stable in distribution and admits a unique invariant measure $\pi(\cdot) \in \mathcal{P}(C)$. Namely,

$$\lim_{t\to\infty} d_{\Xi}\big(\mu^{\xi}_t(\cdot), \pi(\cdot)\big) = 0, \quad \forall\, \xi \in C.$$

Remark 2 For the initial data $\xi \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$, making use of the Chapman-Kolmogorov identity [30, p.18-19], the result of Lemma 2.1 still holds.

3 Main results

In this section, we focus on constructing an appropriate explicit scheme and give the main results on the existence and convergence of the numerical invariant measure. Due to (H1), we may choose a strictly increasing continuous function $\Phi : [1,\infty) \to \mathbb{R}_+$ such that $\Phi(R) \to \infty$ as $R \to \infty$ and

$$\sup_{|x|\vee|\bar{x}|\vee|y|\vee|\bar{y}|\leq R} \frac{|f(x,y) - f(\bar{x},\bar{y})|}{|x-\bar{x}| + 1\wedge|y-\bar{y}|} + \frac{|g(x,y) - g(\bar{x},\bar{y})|^2}{(|x-\bar{x}| + 1\wedge|y-\bar{y}|)^2} \leq \Phi(R) \tag{5}$$

for any $R \geq 1$. Define a truncation mapping $\Gamma^{\triangle}_{\Phi,\nu} : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\Gamma^{\triangle}_{\Phi,\nu}(x) = \Big(|x| \wedge \Phi^{-1}\big(K\triangle^{-\nu}\big)\Big)\frac{x}{|x|}, \tag{6}$$

where Φ^{-1} is the inverse function of Φ, $x/|x| = \mathbf{0}$ if $x = \mathbf{0} \in \mathbb{R}^d$, $\nu \in (0, 1/3]$, and

$$K := 1 \vee \Phi(1) \vee \mathsf{f}(\mathbf{0},\mathbf{0}) \vee \mathsf{g}(\mathbf{0},\mathbf{0})^2. \tag{7}$$

We may suppose without loss of generality that $\triangle = \tau/N \in (0,1)$ with some integer $N > \tau$. Let $t_k = k\triangle$ for $k \geq -N$. Then for any $\xi \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$, we define the truncated EM scheme of SDDE (1) by

$$\begin{cases} \breve{u}^{\xi,\triangle}(t_k) = \xi(t_k), \quad k = -N, \dots, 0, \\ u^{\xi,\triangle}(t_k) = \Gamma^{\triangle}_{\Phi,\nu}(\breve{u}^{\xi,\triangle}(t_k)), \quad k = -N, \dots, 0, 1, \dots, \\ \breve{u}^{\xi,\triangle}(t_{k+1}) = u^{\xi,\triangle}(t_k) + f(u^{\xi,\triangle}(t_k), u^{\xi,\triangle}(t_{k-N}))\triangle \\ \qquad\qquad + g(u^{\xi,\triangle}(t_k), u^{\xi,\triangle}(t_{k-N}))\triangle W_k, \quad k = 0, 1, \dots, \end{cases} \tag{8}$$

where $\triangle W_k = W(t_{k+1}) - W(t_k)$. Define a piecewise constant process by

$$u^{\xi,\triangle}(t) = u^{\xi,\triangle}(t_k), \quad t \in [t_k, t_{k+1}), \ k \geq -N, \tag{9}$$

and a piecewise linear continuous process by

$$\begin{cases} y^{\xi,\triangle}(t) = \dfrac{t_{k+1}-t}{\triangle} u^{\xi,\triangle}(t_k) + \dfrac{t-t_k}{\triangle} u^{\xi,\triangle}(t_{k+1}), & t \in [t_k, t_{k+1}], \ k \geq 0, \\ y^{\xi,\triangle}(t) = \Gamma^{\triangle}_{\Phi,\nu}(\xi(t)), \quad t \in [-\tau, 0]. \end{cases} \tag{10}$$

For any $k \geq 0$, let

$$Y^{\xi,\triangle}_{t_k}(\theta) = y^{\xi,\triangle}(t_k + \theta), \quad \text{for } \theta \in [-\tau, 0]. \tag{11}$$

We call $\{Y^{\xi,\triangle}_{t_k}\}_{k\geq 0}$ the *truncated EM segment process (TEMSP)*.

Remark 3 In fact, Scheme (8) extended from the one in [36]. So by virtue of [36] the results also hold for (8). Precisely, under Assumptions (H1) and (H2), for any $\xi \in C$, $u^{\xi,\triangle}(\cdot)$ defined by (9) converges strongly to the exact solution $x^{\xi}(\cdot)$ of (1) in any finite horizon, and reproduces the exponentially stability in infinite horizon when $a_1 = 0$, $\alpha = 2$ given in Assumption (H2). More generally, for any initial data $\xi \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$, by the techniques in [36] and Theorem 1 of this paper, these results still hold.

It is well known that the Markov property plays a crucial role in investigating the ergodicity. Since $\{u^{\xi,\triangle}(t)\}_{t\geq -\tau}$ defined by (9) is not Markovian, it fails to be used directly to approximate the underlying invariant measure. Using the similar argument as [3, Lemma 5.1], we obtain the following result.

Lemma 3.1 Suppose that (H1) and (H2) hold. Then for any $\xi \in C$ and $\triangle \in (0,1)$, TEMSP $\{Y^{\xi,\triangle}_{t_k}\}_{k\geq 0}$ defined by (11) is a time-homogenous Markov process, that is, for any $A \in \mathfrak{B}(C)$, $0 \leq i < k$, and $\eta \in C$,

$$\mathbb{P}\big(Y^{\xi,\triangle}_{t_k} \in A | \mathcal{F}_{t_i}\big) = \mathbb{P}\big(Y^{\xi,\triangle}_{t_k} \in A | Y^{\xi,\triangle}_{t_i}\big),$$
$$\mathbb{P}\big(Y^{\xi,\triangle}_{t_k} \in A | Y^{\xi,\triangle}_{t_i} = \eta\big) = \mathbb{P}\big(Y^{\eta,\triangle}_{t_{k-i}} \in A\big).$$

For any $\xi \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$, $\triangle \in (0,1)$, and $k \geq 0$, define

$$\mu^{\xi,\triangle}_{t_k}(A) = \mathbb{P}\big\{\omega \in \Omega : Y^{\xi,\triangle}_{t_k} \in A\big\}, \quad \forall A \in \mathfrak{B}(C). \tag{12}$$

It is worth to point out that for any $M > 0$, there exists a $\triangle^*_M \in (0,1)$ sufficiently small such that

$$M \leq \Phi^{-1}\big(K(\triangle^*_M)^{-\nu}\big), \tag{13}$$

which implies that $\Gamma^{\triangle}_{\Phi,\nu}(x) = x$ for any $x \in \mathbb{R}^d$ with $|x| \le M$ and $\triangle \in (0, \triangle^*_M]$. Clearly, in view of (H2), there exists a $\hat{\triangle}_1 \in (0,1)$ sufficiently small such that

$$a_2 - 6K^2\hat{\triangle}_1^{1-2\nu} > a_3. \tag{14}$$

In view of (H3), there exists a $\hat{\triangle}_2 \in (0,1)$ sufficiently small such that

$$b_1 - 4K^2\hat{\triangle}_2^{1-2\nu} > b_2. \tag{15}$$

In what follows, we state our main results in this paper.

Theorem 1 *Suppose that* (H1)–(H3) *hold. Let $\hat{\triangle} = \hat{\triangle}_1 \wedge \hat{\triangle}_2$. Then for any $\xi \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$ and $\triangle \in (0, \hat{\triangle}]$,* TEMSP $\{Y^{\xi,\triangle}_{t_k}\}_{k\ge 0}$ *defined by* (11) *is asymptotically stable in distribution and admits a unique numerical invariant measure $\pi^{\triangle}(\cdot) \in \mathcal{P}(C)$ satisfying*

$$\lim_{t_k\to\infty} d_{\Xi}\big(\mu^{\xi,\triangle}_{t_k}(\cdot), \pi^{\triangle}(\cdot)\big) = 0, \quad \textit{uniformly in } \ \triangle \in (0, \hat{\triangle}]. \tag{16}$$

Moreover, for any $M > 0$, this convergence is also uniform for $\xi \in \mathbb{B}(M, \alpha)$.

Theorem 2 *Suppose that* (H1)–(H3) *hold. Then*

$$\lim_{\triangle\to 0} d_{\Xi}\big(\pi(\cdot), \pi^{\triangle}(\cdot)\big) = 0,$$

where $\pi(\cdot)$ and $\pi^{\triangle}(\cdot)$ are the underlying invariant measure and the numerical one, respectively.

4 Proofs of theorems

Since the proofs of Theorem 1–2 are rather technical, we prepare several notations and lemmas, and then complete the proofs. For any $\xi \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$, for short, we write

$$F^{\xi,\triangle}_t := f(u^{\xi,\triangle}(t), u^{\xi,\triangle}(t-\tau)), \ G^{\xi,\triangle}_t := g(u^{\xi,\triangle}(t), u^{\xi,\triangle}(t-\tau)) \quad \forall\, t \ge 0, \tag{17}$$

where $u^{\xi,\triangle}(t)$ is defined by (9). It follows from (5), (6), and (8) that for any $\xi, \zeta \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$ and $t \ge 0$,

$$\begin{aligned} &\big|F^{\xi,\triangle}_t - F^{\zeta,\triangle}_t\big| \\ \le&\Phi\Big(\Phi^{-1}(K\triangle^{-\nu})\Big)\Big(|u^{\xi,\triangle}(t) - u^{\zeta,\triangle}(t)| + 1 \wedge |u^{\xi,\triangle}(t-\tau) - u^{\zeta,\triangle}(t-\tau)|\Big) \\ \le&K\triangle^{-\nu}\Big(|u^{\xi,\triangle}(t) - u^{\zeta,\triangle}(t)| + |u^{\xi,\triangle}(t-\tau) - u^{\zeta,\triangle}(t-\tau)|\Big), \end{aligned} \tag{18}$$

This, along with (7), implies that

$$\big|F_t^{\xi,\triangle}\big| \le K\triangle^{-\nu}\big(1+|u^{\xi,\triangle}(t)|+|u^{\xi,\triangle}(t-\tau)|\big). \tag{19}$$

Similarly,

$$\big|G_t^{\xi,\triangle}\big| \le K^{\frac{1}{2}}\triangle^{-\frac{\nu}{2}}\big(1+|u^{\xi,\triangle}(t)|+|u^{\xi,\triangle}(t-\tau)|\big). \tag{20}$$

For convenience, we define an auxiliary process

$$\begin{cases} z^{\xi,\triangle}(t) = u^{\xi,\triangle}(t_k) + \int_{t_k}^t F_s^{\xi,\triangle}\mathrm{d}s + \int_{t_k}^t G_s^{\xi,\triangle}\mathrm{d}W(s), & \forall\, t\in[t_k,t_{k+1}),\\ z^{\xi,\triangle}(t) = \Gamma^{\triangle}_{\Phi,\nu}(\xi(t)), \quad \forall\, t\in[-\tau,0]. \end{cases} \tag{21}$$

To show the uniform boundedness of the norm of TEMSP $\{Y_{t_k}^{\xi,\triangle}\}_{k\ge0}$ in probability we begin with the moment analysis of the numerical solutions.

Lemma 4.1 Suppose that (H1) and (H2) hold. Then for any $M>0$,

$$\sup_{\triangle\in(0,\hat{\triangle}_1]}\ \sup_{k\ge -N}\ \sup_{\xi\in B(M)} \mathbb{E}|u^{\xi,\triangle}(t_k)|^2 \le L_1, \tag{22}$$

where L_1 is a constant dependent on $\hat{\triangle}_1$ and M. Moreover, there exists a constant $\bar{a}=\bar{a}(\hat{\triangle}_1)\in(0,1]$ such that for any $\triangle\in(0,\hat{\triangle}_1]$, $k\ge0$, and $\xi\in B(M)$,

$$\triangle\sum_{i=0}^{k} e^{\bar{a}t_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i)|^\alpha \le L_2(1+e^{\bar{a}t_{k+1}}), \tag{23}$$

where L_2 is a constant dependent on $\hat{\triangle}_1$ and M.

Proof Fix an $M>0$ and let $\xi\in B(M)$. For any integer $i\ge0$, it follows from (H2), (8) and (19) that

$$\begin{aligned}
&\mathbb{E}\big(|u^{\xi,\triangle}(t_{i+1})|^2|\mathcal{F}_{t_i}\big) \le \mathbb{E}\big(|\breve{u}^{\xi,\triangle}(t_{i+1})|^2|\mathcal{F}_{t_i}\big)\\
=&\mathbb{E}\big(|u^{\xi,\triangle}(t_i)+F_{t_i}^{\xi,\triangle}\triangle+G_{t_i}^{\xi,\triangle}\triangle W_i|^2|\mathcal{F}_{t_i}\big)\\
=&|u^{\xi,\triangle}(t_i)|^2+2\langle u^{\xi,\triangle}(t_i),F_{t_i}^{\xi,\triangle}\rangle\triangle+|G_{t_i}^{\xi,\triangle}|^2\triangle+|F_{t_i}^{\xi,\triangle}|^2\triangle^2\\
\le&|u^{\xi,\triangle}(t_i)|^2+a_1\triangle-a_2|u^{\xi,\triangle}(t_i)|^\alpha\triangle+a_3|u^{\xi,\triangle}(t_{i-N})|^\alpha\triangle\\
&+K^2(1+|u^{\xi,\triangle}(t_i)|+|u^{\xi,\triangle}(t_{i-N})|)^2\triangle^{2-2\nu}\\
\le&|u^{\xi,\triangle}(t_i)|^2+(a_1+9K^2\triangle^{1-2\nu})\triangle-(a_2-3K^2\triangle^{1-2\nu})|u^{\xi,\triangle}(t_i)|^\alpha\triangle\\
&+(a_3+3K^2\triangle^{1-2\nu})|u^{\xi,\triangle}(t_{i-N})|^\alpha\triangle,
\end{aligned} \tag{24}$$

where the last inequality uses $|u^{\xi,\triangle}(t_i)|^2\le 1+|u^{\xi,\triangle}(t_i)|^\alpha$. An application of Lagrange's mean value theorem derives that for any $a>0$, there exists a $\varsigma\in(t_i,t_{i+1})$ such that $e^{at_{i+1}}-e^{at_i}=e^{a\varsigma}a\triangle$, which implies $e^{at_{i+1}}\le e^{at_i}+e^{at_{i+1}}a\triangle$. Taking expectations in (24) and using the above inequality and $\triangle \in (0,1)$, $\nu \in (0,1/3]$, we arrive at

$$\begin{aligned}
e^{at_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_{i+1})|^2 \leq{}& e^{at_{i+1}}\mathbb{E}\Big(\mathbb{E}\big(|\breve{u}^{\xi,\triangle}(t_{i+1})|^2|\mathcal{F}_{t_i}\big)\Big)\\
\leq&(e^{at_i}+e^{at_{i+1}}a\triangle)\mathbb{E}|u^{\xi,\triangle}(t_i)|^2+e^{at_{i+1}}(a_1+9K^2)\triangle\\
&-(a_2-3K^2\triangle^{1-2\nu})e^{at_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i)|^\alpha\triangle\\
&+(a_3+3K^2\triangle^{1-2\nu})e^{at_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_{i-N})|^\alpha\triangle\\
\leq& e^{at_i}\mathbb{E}|u^{\xi,\triangle}(t_i)|^2+e^{at_{i+1}}(a_1+9K^2+a)\triangle\\
&-(a_2-3K^2\triangle^{1-2\nu}-a)e^{at_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i)|^\alpha\triangle\\
&+(a_3+3K^2\triangle^{1-2\nu})e^{at_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_{i-N})|^\alpha\triangle.
\end{aligned}$$

Summing the above inequality on both sides from 0 to k derives

$$\begin{aligned}
&e^{at_{k+1}}\mathbb{E}|u^{\xi,\triangle}(t_{k+1})|^2\\
\leq&|u^{\xi,\triangle}(0)|^2+(a_1+9K^2+a)\triangle\sum_{i=0}^{k}e^{at_{i+1}}\\
&-(a_2-3K^2\triangle^{1-2\nu}-a)\triangle\sum_{i=0}^{k}e^{at_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i)|^\alpha\\
&+(a_3+3K^2\triangle^{1-2\nu})\triangle\sum_{i=0}^{k}e^{at_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_{i-N})|^\alpha\\
\leq&\|\xi\|^2+(a_1+9K^2+a)\triangle\frac{e^{at_{k+2}}-e^{a\triangle}}{e^{a\triangle}-1}\\
&-(a_2-3K^2\triangle^{1-2\nu}-a)\triangle\sum_{i=0}^{k}e^{at_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i)|^\alpha+(a_3+3K^2\triangle^{1-2\nu})\tau e^{a\tau}\|\xi\|^\alpha\\
&+(a_3+3K^2\triangle^{1-2\nu})e^{a\tau}\triangle\sum_{i=0}^{k}e^{at_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i)|^\alpha.
\end{aligned}\tag{25}$$

By virtue of (14) we further choose an $\bar{a}\in(0,1]$ sufficiently small such that

$$c:=a_2-3K^2\hat{\triangle}_1^{1-2\nu}-\bar{a}-(a_3+3K^2\hat{\triangle}_1^{1-2\nu})e^{\bar{a}\tau}>0.\tag{26}$$

Taking $a=\bar{a}$ in (25) yields that for any $\triangle\in(0,\hat{\triangle}_1]$

$$\begin{aligned}
e^{\bar{a}t_{k+1}}\mathbb{E}|u^{\xi,\triangle}(t_{k+1}|^2\leq&\|\xi\|^2+(a_3+3K^2)\tau e^{\bar{a}\tau}\|\xi\|^\alpha+(a_1+9K^2+\bar{a})\frac{e^{\bar{a}t_{k+2}}}{\bar{a}}\\
&-c\triangle\sum_{i=0}^{k}e^{\bar{a}t_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i)|^\alpha.
\end{aligned}\tag{27}$$

A direct computation derives

$$\mathbb{E}|u^{\xi,\triangle}(t_{k+1})|^2\leq M^2+(a_3+3K^2)\tau e^{\bar{a}\tau}M^\alpha+(a_1+9K^2+\bar{a})\frac{e^{\bar{a}\triangle}}{\bar{a}},$$

which implies that (22) holds. Moreover, the other desired assertion (23) follows from (27) directly. □

Lemma 4.2 Suppose that (H1) and (H2) hold. Then for any $\triangle \in (0, \hat{\triangle}_1]$ and $M > 0$,

$$\sup_{t\geq 0} \sup_{\xi\in B(M)} \mathbb{E}\big|z^{\xi,\triangle}(t) - u^{\xi,\triangle}(t)\big|^2 \leq L_3 \triangle^{1-\nu}, \tag{28}$$

where L_3 is a constant dependent on $\hat{\triangle}_1$ and M.

Proof Fix an $M > 0$. For any $\xi \in B(M)$ and $t \in [t_k, t_{k+1})$ with $k \geq 0$, using (19)–(21), and (22), we derive

$$\begin{aligned}
&\mathbb{E}\big|z^{\xi,\triangle}(t) - u^{\xi,\triangle}(t)\big|^2 \\
\leq& 2\mathbb{E}\big|F^{\xi,\triangle}_{t_k}\big|^2 \triangle^2 + 2\mathbb{E}\big(|G^{\xi,\triangle}_{t_k}|^2 |W(t) - W(t_k)|^2\big) \\
\leq& 2K^2\mathbb{E}\big(1 + |u^{\xi,\triangle}(t_k)| + |u^{\xi,\triangle}(t_{k-N})|\big)^2 \triangle^{2-2\nu} \\
&\quad + 2K\mathbb{E}\big(1 + |u^{\xi,\triangle}(t_k)| + |u^{\xi,\triangle}(t_{k-N})|\big)^2 \triangle^{1-\nu} \\
\leq& 6K(K+1)\big(1 + \mathbb{E}|u^{\xi,\triangle}(t_k)|^2 + \mathbb{E}|u^{\xi,\triangle}(t_{k-N})|^2\big)\triangle^{1-\nu}.
\end{aligned}$$

Then the desired assertion follows. □

Lemma 4.3 Suppose that (H1) and (H2) hold. Then for any $M > 0$, $\varepsilon > 0$, $T > 0$, there exists a $\triangle_3 = \triangle_3(M, \varepsilon, T \in (0, \hat{\triangle}_1 \wedge \triangle^*_M]$ such that

$$\sup_{\triangle\in(0,\triangle_3]} \sup_{k\geq -N} \sup_{\xi\in B(M)} \mathbb{P}\Big\{ \sup_{s\in[t_k,t_k+T]} |z^{\xi,\triangle}(s)| > \Phi^{-1}(K\triangle_3^{-\nu})\Big\} < \varepsilon. \tag{29}$$

Proof Our analysis uses a localization procedure. For any $\eta \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$, $\bar{\triangle} \in (0, 1)$, $\triangle \in (0, \bar{\triangle}]$ and $k \geq -N$, define

$$\beta^{\eta,\triangle}_{\bar{\triangle},k} = \inf\Big\{ s \geq t_k : |z^{\eta,\triangle}(s)| > \Phi^{-1}(K\bar{\triangle}^{-\nu})\Big\}. \tag{30}$$

Fix an $M > 0$. Let $\xi \in B(M)$, $\triangle_3 \in (0, \hat{\triangle}_1 \wedge \triangle^*_M]$, $\triangle \in (0, \triangle_3], k \geq 0$. We should point out that for any $t \in [t_k, \beta^{\xi,\triangle}_{\triangle_3,k}]$,

$$z^{\xi,\triangle}(t) = u^{\xi,\triangle}(t_k) + \int_{t_k}^{t} F^{\xi,\triangle}_s \mathrm{d}s + \int_{t_k}^{t} G^{\xi,\triangle}_s \mathrm{d}W(s). \tag{31}$$

Using the Itô formula, we obtain from (22) and (31) that for any $T > 0$,

$$\begin{aligned}
&\mathbb{E}\Big(e^{\bar{a}((t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k})} |z^{\xi,\triangle}((t_k + T) \wedge \beta^{\xi,\triangle}_{\triangle_3,k})|^2\Big) \\
=& e^{\bar{a}t_k}\mathbb{E}|u^{\xi,\triangle}(t_k)|^2 + \mathbb{E}\int_{t_k}^{(t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}} e^{\bar{a}s}\Big(\bar{a}|z^{\xi,\triangle}(s)|^2 + 2\langle z^{\xi,\triangle}(s), F^{\xi,\triangle}_s\rangle \\
&+ |G^{\xi,\triangle}_s|^2\Big)\mathrm{d}s \leq L_1 e^{\bar{a}t_k} + I_1 + I_2 + I_3,
\end{aligned} \tag{32}$$

where $\bar{a} \in (0, 1]$ is given by Lemma 4.1, and

$$I_1 := \mathbb{E}\int_{t_k}^{(t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}} e^{\bar{a}s}\Big(\bar{a}|u^{\xi,\triangle}(s)|^2 + 2\langle u^{\xi,\triangle}(s), F^{\xi,\triangle}_s\rangle + |G^{\xi,\triangle}_s|^2\Big)\mathrm{d}s,$$

$$I_2 := \mathbb{E}\int_{t_k}^{(t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}} \bar{a}e^{\bar{a}s}\big(|z^{\xi,\triangle}(s)|^2-|u^{\xi,\triangle}(s)|^2\big)\mathrm{d}s,$$
$$I_3 := \mathbb{E}\int_{t_k}^{(t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}} 2e^{\bar{a}s}\langle z^{\xi,\triangle}(s)-u^{\xi,\triangle}(s),F^{\xi,\triangle}_s\rangle\mathrm{d}s.$$

We start with estimating I_1. Using (H2) and together with (9) leads to

$$\begin{aligned}I_1 \leq&\mathbb{E}\int_{t_k}^{(t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}} e^{\bar{a}s}\Big(\bar{a}|u^{\xi,\triangle}(s)|^2+a_1-a_2|u^{\xi,\triangle}(s)|^\alpha+a_3|u^{\xi,\triangle}(s-\tau)|^\alpha\Big)\mathrm{d}s\\ \leq&(\bar{a}+a_1)Te^{\bar{a}(t_k+T)}+(\bar{a}-a_2+a_3e^{\bar{a}\tau})\mathbb{E}\int_{t_k}^{(t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}} e^{\bar{a}s}|u^{\xi,\triangle}(s)|^\alpha\mathrm{d}s\\ &+a_3e^{\bar{a}\tau}\mathbb{E}\int_{t_k-\tau}^{t_k} e^{\bar{a}s}|u^{\xi,\triangle}(s)|^\alpha\mathrm{d}s\\ \leq&(\bar{a}+a_1)Te^{\bar{a}(t_k+T)}+(\bar{a}-a_2+a_3e^{\bar{a}\tau})\mathbb{E}\int_{t_k}^{(t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}} e^{\bar{a}s}|u^{\xi,\triangle}(s)|^\alpha\mathrm{d}s\\ &+a_3e^{\bar{a}\tau}\triangle\sum_{i=k-N}^{k-1} e^{\bar{a}t_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i)|^\alpha.\end{aligned}$$

It is straightforward to see from (26) that $\bar{a}-a_2+a_3e^{\bar{a}\tau}<0$. This, along with (23) implies that

$$I_1 \leq (\bar{a}+a_1)Te^{\bar{a}(t_k+T)}+a_3e^{\bar{a}\tau}L_2(1+e^{\bar{a}t_k}) \leq R_1e^{\bar{a}(t_k+T)}, \tag{33}$$

where $R_1 := (1+a_1)T+2a_3e^\tau L_2$. Next we aim to estimate I_2. According to (22) and (28) yields

$$\begin{aligned}I_2 \leq&\mathbb{E}\int_{t_k}^{(t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}} \bar{a}e^{\bar{a}s}\big(2|z^{\xi,\triangle}(s)-u^{\xi,\triangle}(s)|^2+2|u^{\xi,\triangle}(s)|^2-|u^{\xi,\triangle}(s)|^2\big)\mathrm{d}s\\ \leq&\bar{a}e^{\bar{a}(t_k+T)}\int_{t_k}^{t_k+T}\big(2\mathbb{E}|z^{\xi,\triangle}(s)-u^{\xi,\triangle}(s)|^2+\mathbb{E}|u^{\xi,\triangle}(s)|^2\big)\mathrm{d}s \leq R_2e^{\bar{a}(t_k+T)},\end{aligned} \tag{34}$$

where $R_2 := T(2L_3+L_1)$. Finally, by virtue of (19), (22), (28), $\nu\in(0,1/3]$, and then using the Hölder inequality, we get

$$\begin{aligned}I_3 \leq&2e^{\bar{a}(t_k+T)}\int_{t_k}^{t_k+T}\mathbb{E}\big(|z^{\xi,\triangle}(s)-u^{\xi,\triangle}(s)||F^{\xi,\triangle}_s|\big)\mathrm{d}s\\ \leq&2e^{\bar{a}(t_k+T)}K\triangle^{-\nu}\int_{t_k}^{t_k+T}\Big(\mathbb{E}|z^{\xi,\triangle}(s)-u^{\xi,\triangle}(s)|^2\mathbb{E}\big(1+|u^{\xi,\triangle}(s)|\\ &+|u^{\xi,\triangle}(s-\tau)|\big)^2\Big)^{\frac{1}{2}}\mathrm{d}s\\ \leq&2\sqrt{3}KT(L_3(1+2L_1)\triangle^{1-3\nu})^{1/2}e^{\bar{a}(t_k+T)} \leq R_3e^{\bar{a}(t_k+T)},\end{aligned} \tag{35}$$

where $R_3 := 2\sqrt{3}KT(L_3(1+2L_1))^{1/2}$. Plugging (33)–(35) back into (32) gives

$$\begin{aligned}e^{\bar{a}t_k}\mathbb{E}\big|z^{\xi,\triangle}\big((t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}\big)\big|^2 \leq&\mathbb{E}\Big(e^{\bar{a}((t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k})}|z^{\xi,\triangle}((t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k})|^2\Big)\\ \leq&Re^{\bar{a}(t_k+T)},\end{aligned}$$

where $R := L_1 + R_1 + R_2 + R_3$. This, along with $\bar{a} \in (0,1]$ implies

$$\mathbb{E}\big|z^{\xi,\triangle}\big((t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k}\big)\big|^2 \le Re^T. \tag{36}$$

For any $\varepsilon > 0$, choose a $\triangle_3 = \triangle_3(M,\varepsilon,T) \in (0, \hat{\triangle}_1 \wedge \triangle^*_M]$ sufficiently small such that

$$Re^T < \varepsilon(\Phi^{-1}(K\triangle_3^{-\nu}))^2. \tag{37}$$

According to (36) and (37) concludes that

$$\begin{aligned}
&\sup_{\triangle\in(0,\triangle_3]}\sup_{k\ge 0}\sup_{\xi\in B(M)} \mathbb{P}\Big\{\beta^{\xi,\triangle}_{\triangle_3,k} < t_k+T\Big\}\\
\le&\sup_{\triangle\in(0,\triangle_3]}\sup_{k\ge 0}\sup_{\xi\in B(M)} \frac{\mathbb{E}|z^{\xi,\triangle}((t_k+T)\wedge\beta^{\xi,\triangle}_{\triangle_3,k})|^2}{(\Phi^{-1}(K\triangle_3^{-\nu}))^2}\\
\le&\sup_{\triangle\in(0,\triangle_3]}\sup_{k\ge 0}\sup_{\xi\in B(M)} \frac{Re^T}{(\Phi^{-1}(K\triangle_3^{-\nu}))^2} < \varepsilon.
\end{aligned}$$

Note that $\{z^{\xi,\triangle}(t)\}_{t\ge 0}$ is right-continuous and left-limit, and

$$\lim_{t\uparrow t_k} |z^{\xi,\triangle}(t)| = |\breve{u}(t_k)| \ge |u(t_k)| = |z^{\xi,\triangle}(t_k)|, \quad \forall k \ge 0.$$

This implies that for any $\triangle \in (0,\triangle_3],\ \ k \ge 0$, and $\xi \in B(M)$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{s\in[t_k,t_k+T]} |z^{\xi,\triangle}(s)| > \Phi^{-1}(K\triangle_3^{-\nu})\Big\}\\
=&\mathbb{P}\Big\{|z^{\xi,\triangle}(s)| > \Phi^{-1}(K\triangle_3^{-\nu}), \exists s \in [t_k, t_k+T)\Big\} = \mathbb{P}\Big\{\beta^{\xi,\triangle}_{\triangle_3,k} < t_k+T\Big\} < \varepsilon.
\end{aligned} \tag{38}$$

Therefore (29) is characterized by (13) and (38). □

Thanks for the above lemmas, we go a further step to analyze the uniform boundedness of the norm of TEMSP $\{Y^{\xi,\triangle}_{t_k}\}_{k\ge 0}$ in probability.

Proposition 3 *Suppose that* (H1) *and* (H2) *hold. Then for any $M > 0$, $\varepsilon > 0$, there exists a $\Lambda^* = \Lambda^*(\hat{\triangle}_1, M, \varepsilon) > M$ such that*

$$\sup_{\triangle\in(0,\hat{\triangle}_1]}\sup_{k\ge 0}\sup_{\xi\in B(M)} \mathbb{P}\Big\{\|Y^{\xi,\triangle}_{t_k}\| > \Lambda^*\Big\} < \varepsilon. \tag{39}$$

Proof Fix an $M > 0$. For any $\varepsilon > 0$, making use of Lemma 4.3, there exists a $\triangle_3 = \triangle_3(M,\varepsilon,\tau) \in (0, \hat{\triangle}_1 \wedge \triangle^*_M]$ such that

$$\sup_{\triangle\in(0,\triangle_3]}\sup_{k\ge -N}\sup_{\xi\in B(M)} \mathbb{P}\Big\{\sup_{t\in[t_k,t_k+\tau]} |z^{\xi,\triangle}(t)| > \Phi^{-1}\big(K\triangle_3^{-\nu}\big)\Big\} < \varepsilon. \tag{40}$$

It follows from (10), (11), and (21) that for any $\triangle \in (0,\triangle_3]$, $k \ge 0$, and $\xi \in B(M)$,

$$\|Y^{\xi,\triangle}_{t_k}\| = \sup_{t\in[t_k-\tau,t_k]} |y^{\xi,\triangle}(t)| \le M \vee \sup_{k-N\le i\le k} |u^{\xi,\triangle}(t_i)| \le M \vee \sup_{t\in[t_k-\tau,t_k]} |z^{\xi,\triangle}(t)|.$$

Letting $\Lambda^* = \Phi^{-1}\big(K\triangle_3^{-\nu}\big)$. Combining the above inequality with (40) and using $\Lambda^* \ge M$ we derive

$$\sup_{\triangle\in(0,\triangle_3]}\sup_{k\ge 0}\sup_{\xi\in B(M)} \mathbb{P}\Big\{\|Y^{\xi,\triangle}_{t_k}\| > \Lambda^*\Big\}$$

$$\leq \sup_{\triangle\in(0,\triangle_3]}\sup_{k\geq 0}\sup_{\xi\in B(M)}\mathbb{P}\Big\{\sup_{t\in[t_k-\tau,t_k]}|z^{\xi,\triangle}(t)|>\Phi^{-1}\big(K\triangle_3^{-\nu}\big)\Big\}<\varepsilon. \tag{41}$$

It follows from the truncation property in (8)–(10) that

$$\sup_{\triangle\in[\triangle_3,\hat{\triangle}_1]}\sup_{k\geq 0}\sup_{\xi\in B(M)}\|Y_{t_k}^{\xi,\triangle}\|\leq M\vee\sup_{\triangle\in[\triangle_3,\hat{\triangle}_1]}\Phi^{-1}\big(K\triangle^{-\nu}\big)\leq\Lambda^*.$$

This, along with (41) implies

$$\begin{aligned}&\sup_{\triangle\in(0,\hat{\triangle}_1]}\sup_{k\geq 0}\sup_{\xi\in B(M)}\mathbb{P}\big\{\|Y_{t_k}^{\xi,\triangle}\|>\Lambda^*\big\}\\ \leq&\sup_{\triangle\in(0,\triangle_3]}\sup_{k\geq 0}\sup_{\xi\in B(M)}\mathbb{P}\big\{\|Y_{t_k}^{\xi,\triangle}\|>\Lambda^*\big\}+\sup_{\triangle\in[\triangle_3,\hat{\triangle}_1]}\sup_{k\geq 0}\sup_{\xi\in B(M)}\mathbb{P}\big\{\|Y_{t_k}^{\xi,\triangle}\|>\Lambda^*\big\}\\ <&\varepsilon.\end{aligned}$$

The proof is complete. □

Furthermore, we study the attraction of TEMSP $\{Y_{t_k}^{\xi,\triangle}\}_{k\geq 0}$ in probability. The strategy is as follows: we first prove the continuity of the sample path $y^{\xi,\triangle}(t)$ with respect to t and the continuity of $\{y^{\xi,\triangle}(t_k)\}_{k\geq 0}$ (equivalent with $\{u^{\xi,\triangle}(t_k)\}_{k\geq 0}$) with respect to the initial data ξ; Next making use of above results and the fact

$$\begin{aligned}&|y^{\xi,\triangle}(t_k+\theta)-y^{\zeta,\triangle}(t_k+\theta)|\\ \leq&|y^{\xi,\triangle}(t_k+\theta)-y^{\xi,\triangle}(t_k)|+|y^{\xi,\triangle}(t_k)-y^{\zeta,\triangle}(t_k)|\\ &+|y^{\zeta,\triangle}(t_k)-y^{\zeta,\triangle}(t_k+\theta)|,\quad\forall\theta\in[-\tau,0],\end{aligned}$$

we estimate $\sup_{\theta\in[-\tau,0]}|y^{\xi,\triangle}(t_k+\theta)-y^{\zeta,\triangle}(t_k+\theta)|$ in probability and obtain the attraction of $\{Y_{t_k}^{\xi,\triangle}\}_{k\geq 0}$ in probability.

Lemma 4.4 Suppose that (H1) and (H2) hold. Then for any $M>0$, $\varepsilon_1>0$, $\varepsilon_2>0$, there exists a $\triangle_4=\triangle_4(M,\varepsilon_1,\varepsilon_2)\in(0,\triangle_3]$ such that

$$\sup_{\triangle\in(0,\triangle_4]}\sup_{k\geq 0}\sup_{\xi\in B(M)}\mathbb{P}\Big\{\sup_{t\in[t_k,t_k+\tau]}|y^{\xi,\triangle}(t)-z^{\xi,\triangle}(t)|\geq\varepsilon_2\Big\}<\varepsilon_1, \tag{42}$$

where $\triangle_3=\triangle_3(M,\varepsilon_1/2,2\tau)$ is given by Lemma 4.3.

Proof Fix an $M>0$. For any $\varepsilon_1>0$, recalling (30) and applying Lemma 4.3, there is a $\triangle_3=\triangle_3(M,\varepsilon_1/2,2\tau)\in(0,\hat{\triangle}_1\wedge\triangle_M^*]$ such that

$$\sup_{\triangle\in(0,\triangle_3]}\sup_{k\geq -N}\sup_{\xi\in B(M)}\mathbb{P}\Big\{\beta_{\triangle_3,k}^{\xi,\triangle}<t_k+2\tau\Big\}<\frac{\varepsilon_1}{2}. \tag{43}$$

Let $\triangle\in(0,\triangle_3]$, $k\geq -N$, $\xi\in B(M)$. It follows from (10) and (31) that

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\in[t_k+\tau,t_k+2\tau]}\big(\big|y^{\xi,\triangle}(t)-z^{\xi,\triangle}(t)\big|^4\mathbf{1}_{\{\beta_{\triangle_3,k}^{\xi,\triangle}\geq t_k+2\tau\}}\big)\Big)\\ =&\mathbb{E}\Big(\sup_{i\in\{k+N,\cdots,k+2N-1\}}\sup_{t_i\leq t\leq t_{i+1}}\big(\big|\frac{t_{i+1}-t}{\triangle}u^{\xi,\triangle}(t_i)+\frac{t-t_i}{\triangle}u^{\xi,\triangle}(t_{i+1})\end{aligned}$$

$$
\begin{aligned}
&- u^{\xi,\triangle}(t_i) - F^{\xi,\triangle}_{t_i}(t-t_i) - G^{\xi,\triangle}_{t_i}\big(W(t)-W(t_i)\big)\big|^4 \mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k}\ge t_k+2\tau\}}\big)\Big)\\
=&\mathbb{E}\Big(\sup_{i\in\{k+N,\cdots,k+2N-1\}}\sup_{t_i\le t\le t_{i+1}}\big(\big|\frac{t-t_i}{\triangle}\big(u^{\xi,\triangle}(t_{i+1})-u^{\xi,\triangle}(t_i)\big)\\
&- F^{\xi,\triangle}_{t_i}(t-t_i) - G^{\xi,\triangle}_{t_i}\big(W(t)-W(t_i)\big)\big|^4 \mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k}\ge t_k+2\tau\}}\big)\Big).
\end{aligned}
$$

By virtue of (8) we obtain

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[t_k+\tau,t_k+2\tau]}\big(\big|y^{\xi,\triangle}(t)-z^{\xi,\triangle}(t)\big|^4\mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k}\ge t_k+2\tau\}}\big)\Big)\\
=&\mathbb{E}\Big(\sup_{i\in\{k+N,\cdots,k+2N-1\}}\sup_{t_i\le t\le t_{i+1}}\big(\big|\frac{t-t_i}{\triangle}\big(F^{\xi,\triangle}_{t_i}\triangle+G^{\xi,\triangle}_{t_i}\triangle W_i\big)\\
&- F^{\xi,\triangle}_{t_i}(t-t_i) - G^{\xi,\triangle}_{t_i}\big(W(t)-W(t_i)\big)\big|^4 \mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k}\ge t_k+2\tau\}}\big)\Big)\\
\le&8\mathbb{E}\Big(\sup_{i\in\{k+N,\cdots,k+2N-1\}}\sup_{t_i\le t\le t_{i+1}}\big(\big|\frac{t-t_i}{\triangle}G^{\xi,\triangle}_{t_i}\triangle W_i\big|^4\mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k}\ge t_k+2\tau\}}\big)\Big)\\
&+8\mathbb{E}\Big(\sup_{i\in\{k+N,\cdots,k+2N-1\}}\sup_{t_i\le t\le t_{i+1}}\big(\big|G^{\xi,\triangle}_{t_i}(W(t)-W(t_i))\big|^4\mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k}\ge t_k+2\tau\}}\big)\Big).
\end{aligned}
\tag{44}
$$

In view of (H1), there exists a constant $L>0$ sufficiently large such that

$$
\sup_{|x|\vee|y|\le\Phi^{-1}(K\triangle_3^{-\nu})}|g(x,y)|\le L.
$$

Inserting this into (44) and using the Doob martingale inequality implies

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[t_k+\tau,t_k+2\tau]}\big(\big|y^{\xi,\triangle}(t)-z^{\xi,\triangle}(t)\big|^4\mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k}\ge t_k+2\tau\}}\big)\Big)\\
\le&8L^4\mathbb{E}\Big(\sup_{i\in\{k+N,\cdots,k+2N-1\}}\big|\triangle W_i\big|^4\Big)\\
&+8L^4\mathbb{E}\Big(\sup_{i\in\{k+N,\cdots,k+2N-1\}}\sup_{t_i\le t\le t_{i+1}}\big|W(t)-W(t_i)\big|^4\Big)\\
\le&8L^4\sum_{i=k+N}^{k+2N-1}\mathbb{E}\Big(\big|\triangle W_i\big|^4+\sup_{t_i\le t\le t_{i+1}}\big|W(t)-W(t_i)\big|^4\Big)\le\tilde{L}N\triangle^2\le\tilde{L}\tau\triangle,
\end{aligned}
$$

where $\tilde{L}$ is a constant. For any $\varepsilon_2>0$, choose a $\triangle_4=\triangle_4(M,\varepsilon_1,\varepsilon_2)\in(0,\triangle_3]$ sufficiently small such that

$$
\frac{\tilde{L}\tau\triangle_4}{\varepsilon_2^4}<\frac{\varepsilon_1}{2}.
$$

An application of Chebyshev's inequality arrives at that for any $\triangle\in(0,\triangle_4]$,

$$
\begin{aligned}
&\mathbb{P}\Big\{\beta^{\xi,\triangle}_{\triangle_3,k}\ge t_k+2\tau,\ \sup_{t\in[t_k+\tau,t_k+2\tau]}\big|y^{\xi,\triangle}(t)-z^{\xi,\triangle}(t)\big|\ge\varepsilon_2\Big\}\\
\le&\frac{1}{\varepsilon_2^4}\mathbb{E}\Big(\sup_{t\in[t_k+\tau,t_k+2\tau]}\big(\big|y^{\xi,\triangle}(t)-z^{\xi,\triangle}(t)\big|^4\mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k}\ge t_k+2\tau\}}\big)\Big)\le\frac{\tilde{L}\tau\triangle_4}{\varepsilon_2^4}<\frac{\varepsilon_1}{2}.
\end{aligned}
$$

This, combining with (43) implies that the required assertion (42) follows. □

Lemma 4.5 Suppose that (H1) and (H2) hold. Then for any $M > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists a positive integer $j^* = j^*(M, \varepsilon_1, \varepsilon_2, \triangle_4)$ such that

$$\sup_{\triangle \in (0,\triangle_4]} \sup_{k \geq 0} \sup_{\xi \in B(M)} \mathbb{P}\Big\{ \sup_{\substack{|s_1 - s_2| \leq \tau/j^* \\ s_1, s_2 \in [t_k, t_k+\tau]}} |y^{\xi,\triangle}(s_1) - y^{\xi,\triangle}(s_2)| \geq \varepsilon_2 \Big\} < \varepsilon_1,$$

where $\triangle_4 = \triangle_4(M, \varepsilon_1/2, \varepsilon_2/3)$ is given by Lemma 4.4.

Proof Fix an $M > 0$. For any $\varepsilon_1 > 0$, recalling (30) and applying Lemma 4.3, there is a $\triangle_3 = \triangle_3(M, \varepsilon_1/4, 2\tau) \in (0, \hat{\triangle}_1 \wedge \triangle_M^*]$ such that

$$\sup_{\triangle \in (0,\triangle_3]} \sup_{k \geq -N} \sup_{\xi \in B(M)} \mathbb{P}\Big\{ \beta^{\xi,\triangle}_{\triangle_3,k} < t_k + 2\tau \Big\} < \frac{\varepsilon_1}{4}. \tag{45}$$

Let $\triangle \in (0, \triangle_3]$, $k \geq -N$, $\xi \in B(M)$. For any integer $j^* \geq 1$, define $t_k^{j^*,j} = t_k + j\tau/j^*$, $j = 0, \cdots, j^*$. According to (31) and the Burkholder-Davis-Gundy inequality, we derive that for any $j \in \{0, \cdots, j^* - 1\}$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t \in [t_{k+N}^{j^*,j}, t_{k+N}^{j^*,j+1}]} \big(|z^{\xi,\triangle}(t \wedge \beta^{\xi,\triangle}_{\triangle_3,k}) - z^{\xi,\triangle}(t_{k+N}^{j^*,j} \wedge \beta^{\xi,\triangle}_{\triangle_3,k})|^4 \big) \Big) \\
=&\mathbb{E}\Big(\sup_{t \in [t_{k+N}^{j^*,j}, t_{k+N}^{j^*,j+1}]} \big(\Big| \int_{t_{k+N}^{j^*,j} \wedge \beta^{\xi,\triangle}_{\triangle_3,k}}^{t \wedge \beta^{\xi,\triangle}_{\triangle_3,k}} F_h^{\xi,\triangle} \mathrm{d}h + \int_{t_{k+N}^{j^*,j} \wedge \beta^{\xi,\triangle}_{\triangle_3,k}}^{t \wedge \beta^{\xi,\triangle}_{\triangle_3,k}} G_h^{\xi,\triangle} \mathrm{d}W(h) \Big|^4 \big) \Big) \\
\leq&8\mathbb{E}\Big(\sup_{t \in [t_{k+N}^{j^*,j}, t_{k+N}^{j^*,j+1}]} \big(\Big| \int_{t_{k+N}^{j^*,j}}^{t} F_h^{\xi,\triangle} \mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k} \geq h\}} \mathrm{d}h \Big|^4 \\
&+ \Big| \int_{t_{k+N}^{j^*,j}}^{t} G_h^{\xi,\triangle} \mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k} \geq h\}} \mathrm{d}W(h) \Big|^4 \big) \Big) \\
\leq&8\mathbb{E}\Big(\int_{t_{k+N}^{j^*,j}}^{t_{k+N}^{j^*,j+1}} |F_h^{\xi,\triangle}| \mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k} \geq h\}} \mathrm{d}h \Big)^4 + \frac{2^{21}}{3^6} \mathbb{E}\Big(\int_{t_{k+N}^{j^*,j}}^{t_{k+N}^{j^*,j+1}} |G_h^{\xi,\triangle}|^2 \mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k} \geq h\}} \mathrm{d}h \Big)^2.
\end{aligned} \tag{46}$$

By virtue of (H1) there exists a constant $L > 0$ such that

$$\sup_{|x| \vee |y| \leq \Phi^{-1}(K\triangle_3^{-\nu})} \big(|f(x,y)| \vee |g(x,y)| \big) \leq L. \tag{47}$$

Inserting this into (46) implies

$$\mathbb{E}\Big(\sup_{t \in [t_{k+N}^{j^*,j}, t_{k+N}^{j^*,j+1}]} \big(|z^{\xi,\triangle}(t \wedge \beta^{\xi,\triangle}_{\triangle_3,k}) - z^{\xi,\triangle}(t_{k+N}^{j^*,j} \wedge \beta^{\xi,\triangle}_{\triangle_3,k})|^4 \big) \Big) \leq \frac{R_4}{(j^*)^2}, \tag{48}$$

where $R_4 := 8L^4\tau^4 + 2^{21}L^4\tau^2/3^6$. For any $\varepsilon_2 > 0$, choose $j^* \geq 1 \vee (4 \cdot 9^4 R_4/(\varepsilon_1 \varepsilon_2^4))$. The fundamental theory of calculus shows that

$$\begin{aligned}
&\mathbb{P}\Big\{ \beta^{\xi,\triangle}_{\triangle_3,k} \geq t_k + 2\tau, \sup_{\substack{|s_1 - s_2| \leq \tau/j^* \\ s_1, s_2 \in [t_k+\tau, t_k+2\tau]}} |z^{\xi,\triangle}(s_1) - z^{\xi,\triangle}(s_2)| \geq \frac{\varepsilon_2}{3} \Big\} \\
\leq&\mathbb{P}\Big\{ \beta^{\xi,\triangle}_{\triangle_3,k} \geq t_k + 2\tau,\ 3 \max_{0 \leq j \leq j^*-1} \sup_{t \in [t_{k+N}^{j^*,j}, t_{k+N}^{j^*,j+1}]} |z^{\xi,\triangle}(t) - z^{\xi,\triangle}(t_{k+N}^{j^*,j})| \geq \frac{\varepsilon_2}{3} \Big\}
\end{aligned}$$

$$\leq \sum_{j=0}^{j^*-1} \mathbb{P}\Big\{\beta^{\xi,\triangle}_{\triangle_3,k} \geq t_k + 2\tau, \sup_{t\in[t^{j^*,j}_{k+N}, t^{j_1,j+1}_{k+N}]} |z^{\xi,\triangle}(t) - z^{\xi,\triangle}(t^{j^*,j}_{k+N})| \geq \frac{\varepsilon_2}{9}\Big\}.$$

Using the Chebyshev inequality implies

$$\mathbb{P}\Big\{\beta^{\xi,\triangle}_{\triangle_3,k} \geq t_k + 2\tau, \sup_{\substack{|s_1-s_2|\leq\tau/j^*\\ s_1,s_2\in[t_k+\tau,t_k+2\tau]}} |z^{\xi,\triangle}(s_1) - z^{\xi,\triangle}(s_2)| \geq \frac{\varepsilon_2}{3}\Big\}$$

$$\leq \frac{9^4}{\varepsilon_2^4} \sum_{j=0}^{j^*-1} \mathbb{E}\Big(\sup_{t\in[t^{j^*,j}_{k+N}, t^{j^*,j+1}_{k+N}]} \big(|z^{\xi,\triangle}(t) - z^{\xi,\triangle}(t^{j^*,j}_{k+N})|^4 \mathbf{1}_{\{\beta^{\xi,\triangle}_{\triangle_3,k}\geq t_k+2\tau\}}\big)\Big)$$

$$\leq \frac{9^4}{\varepsilon_2^4} \sum_{j=0}^{j^*-1} \mathbb{E}\Big(\sup_{t\in[t^{j^*,j}_{k+N}, t^{j^*,j+1}_{k+N}]} \big(|z^{\xi,\triangle}(t\wedge\beta^{\xi,\triangle}_{\triangle_3,k}) - z^{\xi,\triangle}(t^{j^*,j}_{k+N}\wedge\beta^{\xi,\triangle}_{\triangle_3,k})|^4\big)\Big).$$

This, together with (48) implies that

$$\mathbb{P}\Big\{\beta^{\xi,\triangle}_{\triangle_3,k} \geq t_k + 2\tau, \sup_{\substack{|s_1-s_2|\leq\tau/j^*\\ s_1,s_2\in[t_k+\tau,t_k+2\tau]}} |z^{\xi,\triangle}(s_1) - z^{\xi,\triangle}(s_2)| \geq \frac{\varepsilon_2}{3}\Big\} \leq \frac{9^4 R_4}{\varepsilon_2^4 j^*} \leq \frac{\varepsilon_1}{4}.$$

Combining the above inequality with (45) we arrive at

$$\sup_{\triangle\in(0,\triangle_3]} \sup_{k\geq 0} \sup_{\xi\in B(M)} \mathbb{P}\Big\{\sup_{\substack{|s_1-s_2|\leq\tau/j^*\\ s_1,s_2\in[t_k,t_k+\tau]}} |z^{\xi,\triangle}(s_1) - z^{\xi,\triangle}(s_2)| \geq \frac{\varepsilon_2}{3}\Big\} < \frac{\varepsilon_1}{2}. \tag{49}$$

In view of Lemma 4.4, there exists a $\triangle_4 = \triangle_4(M, \varepsilon_1/2, \varepsilon_2/3) \in (0, \triangle_3]$ such that

$$\sup_{\triangle\in(0,\triangle_4]} \sup_{k\geq 0} \sup_{\xi\in B(M)} \mathbb{P}\Big\{\sup_{t\in[t_k,t_k+\tau]} |y^{\xi,\triangle}(t) - z^{\xi,\triangle}(t)| \geq \frac{\varepsilon_2}{3}\Big\} < \frac{\varepsilon_1}{2}. \tag{50}$$

It is straightforward to see from (49) and (50) that

$$\sup_{\triangle\in(0,\triangle_4]} \sup_{k\geq 0} \sup_{\xi\in B(M)} \mathbb{P}\Big\{\sup_{\substack{|s_1-s_2|\leq\tau/j^*\\ s_1,s_2\in[t_k,t_k+\tau]}} \big|y^{\xi,\triangle}(s_1) - y^{\xi,\triangle}(s_2)\big| \geq \varepsilon_2\Big\}$$

$$\leq \sup_{\triangle\in(0,\triangle_4]} \sup_{k\geq 0} \sup_{\xi\in B(M)} \mathbb{P}\Big\{2\sup_{t\in[t_k,t_k+\tau]} \big|y^{\xi,\triangle}(t) - z^{\xi,\triangle}(t)\big| \geq \frac{2\varepsilon_2}{3}\Big\}$$

$$+ \sup_{\triangle\in(0,\triangle_4]} \sup_{k\geq 0} \sup_{\xi\in B(M)} \mathbb{P}\Big\{\sup_{\substack{|s_1-s_2|\leq\tau/j^*\\ s_1,s_2\in[t_k,t_k+\tau]}} \big|z^{\xi,\triangle}(s_1) - z^{\xi,\triangle}(s_2)\big| \geq \frac{\varepsilon_2}{3}\Big\} < \varepsilon_1.$$

The proof is complete. □

Lemma 4.6 Suppose that (H3) holds. Then there is a $\bar{\lambda} = \bar{\lambda}(\hat{\triangle}_2) \in (0, 1]$ such that for any $M > 0$, $\triangle \in (0, \hat{\triangle}_2]$, and $k \geq 0$,

$$\sup_{\xi,\zeta\in B(M)} \mathbb{E}|u^{\xi,\triangle}(t_k) - u^{\zeta,\triangle}(t_k)|^2 \leq L_M e^{-\bar{\lambda}t_k},$$

where

$$L_M := 4M^2(1 + (b_2 + 2K^2)\tau e^{\tau}) + b_4\tau e^{\tau} \sup_{|x|\vee|y|\leq M} V(x, y).$$

Proof Fix an $M > 0$. For any $\triangle \in (0, \hat{\triangle}_2]$, $i \geq 0$, and $\xi, \zeta \in B(M)$, one observes from (8) that

$$\begin{aligned}
&|\breve{u}^{\xi,\triangle}(t_{i+1}) - \breve{u}^{\zeta,\triangle}(t_{i+1})|^2 \\
=&|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2 + 2\langle u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i), F^{\xi,\triangle}_{t_i} - F^{\zeta,\triangle}_{t_i}\rangle\triangle \\
&+ |(G^{\xi,\triangle}_{t_i} - G^{\zeta,\triangle}_{t_i})\triangle W_i|^2 + |F^{\xi,\triangle}_{t_i} - F^{\zeta,\triangle}_{t_i}|^2\triangle^2 \\
&+ 2\langle u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i), (G^{\xi,\triangle}_{t_i} - G^{\zeta,\triangle}_{t_i})\triangle W_i\rangle \\
&+ 2\langle F^{\xi,\triangle}_{t_i} - F^{\zeta,\triangle}_{t_i}, (G^{\xi,\triangle}_{t_i} - G^{\zeta,\triangle}_{t_i})\triangle W_i\rangle\triangle.
\end{aligned}$$

Taking expectations in both sides of the above inequality , and by using (H3), and (18), leads to

$$\begin{aligned}
&\mathbb{E}|\breve{u}^{\xi,\triangle}(t_{i+1}) - \breve{u}^{\zeta,\triangle}(t_{i+1})|^2 \\
=&\mathbb{E}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2 + 2\mathbb{E}\langle u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i), F^{\xi,\triangle}_{t_i} - F^{\zeta,\triangle}_{t_i}\rangle\triangle \\
&+ \mathbb{E}|(G^{\xi,\triangle}_{t_i} - G^{\zeta,\triangle}_{t_i})|^2\triangle + \mathbb{E}|F^{\xi,\triangle}_{t_i} - F^{\zeta,\triangle}_{t_i}|^2\triangle^2 \\
\leq&\mathbb{E}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2 - b_1\mathbb{E}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2\triangle \\
&+ b_2\mathbb{E}|u^{\xi,\triangle}(t_{i-N}) - u^{\zeta,\triangle}(t_{i-N})|^2\triangle - b_3\mathbb{E}V\big(u^{\xi,\triangle}(t_i), u^{\zeta,\triangle}(t_i)\big)\triangle \\
&+ b_4\mathbb{E}V\big(u^{\xi,\triangle}(t_{i-N}), u^{\zeta,\triangle}(t_{i-N})\big)\triangle \\
&+ K^2\mathbb{E}\big(|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)| + |u^{\xi,\triangle}(t_{i-N}) - u^{\zeta,\triangle}(t_{i-N})|\big)^2\triangle^{2-2\nu} \\
\leq&\mathbb{E}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2 - (b_1 - 2K^2\triangle^{1-2\nu})\mathbb{E}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2\triangle \\
&+ (b_2 + 2K^2\triangle^{1-2\nu})\mathbb{E}|u^{\xi,\triangle}(t_{i-N}) - u^{\zeta,\triangle}(t_{i-N})|^2\triangle \\
&- b_3\mathbb{E}V\big(u^{\xi,\triangle}(t_i), u^{\zeta,\triangle}(t_i)\big)\triangle + b_4\mathbb{E}V\big(u^{\xi,\triangle}(t_{i-N}), u^{\zeta,\triangle}(t_{i-N})\big)\triangle. \qquad (51)
\end{aligned}$$

According to the Lipschitz continuity of the truncation mapping $\Gamma^{\triangle}_{\Phi,\nu}$ (cf. [22, (7.21)]) we arrive at

$$\big|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)\big|^2 \leq \big|\breve{u}^{\xi,\triangle}(t_i) - \breve{u}^{\zeta,\triangle}(t_i)\big|^2. \qquad (52)$$

Making use of inequality $e^{\lambda t_{i+1}} - e^{\lambda t_i} \leq e^{\lambda t_{i+1}}\lambda\triangle$ for any $\lambda \in (0, 1]$, we obtain from (51) and (52) that

$$\begin{aligned}
&e^{\lambda t_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_{i+1}) - u^{\zeta,\triangle}(t_{i+1})|^2 \leq e^{\lambda t_{i+1}}\mathbb{E}|\breve{u}^{\xi,\triangle}(t_{i+1}) - \breve{u}^{\zeta,\triangle}(t_{i+1})|^2 \\
\leq&e^{\lambda t_i}\mathbb{E}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2 - (b_1 - 2K^2\triangle^{1-2\nu} - \lambda)e^{\lambda t_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2\triangle \\
&+ (b_2 + 2K^2\triangle^{1-2\nu})e^{\lambda t_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_{i-N}) - u^{\zeta,\triangle}(t_{i-N})|^2\triangle \\
&- b_3e^{\lambda t_{i+1}}\mathbb{E}V\big(u^{\xi,\triangle}(t_i), u^{\zeta,\triangle}(t_i)\big)\triangle + b_4e^{\lambda t_{i+1}}\mathbb{E}V\big(u^{\xi,\triangle}(t_{i-N}), u^{\zeta,\triangle}(t_{i-N})\big)\triangle. \qquad (53)
\end{aligned}$$

For any $k \geq 0$, summing (53) from $i = 0$ to k, and together with (52) derives

$$\begin{aligned}
&e^{\lambda t_{k+1}}\mathbb{E}|u^{\xi,\triangle}(t_{k+1}) - u^{\zeta,\triangle}(t_{k+1})|^2 \\
\leq&|u^{\xi,\triangle}(0) - u^{\zeta,\triangle}(0)|^2 - \big(b_1 - 2K^2\triangle^{1-2\nu} - \lambda\big)\sum_{i=0}^{k} e^{\lambda t_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2\triangle \\
&+ (b_2 + 2K^2\triangle^{1-2\nu})\sum_{i=0}^{k} e^{\lambda t_{i+1}}\mathbb{E}|u^{\xi,\triangle}(t_{i-N}) - u^{\zeta,\triangle}(t_{i-N})|^2\triangle
\end{aligned}$$

$$
\begin{aligned}
&- b_3 \sum_{i=0}^{k} e^{\lambda t_{i+1}} \mathbb{E} V\big(u^{\xi,\triangle}(t_i), u^{\zeta,\triangle}(t_i)\big)\triangle \\
&+ b_4 \sum_{i=0}^{k} e^{\lambda t_{i+1}} \mathbb{E} V\big(u^{\xi,\triangle}(t_{i-N}), u^{\zeta,\triangle}(t_{i-N})\big)\triangle \\
\le& L_M - \big(b_1 - 2K^2\triangle^{1-2\nu} - \lambda - (b_2 + 2K^2\triangle^{1-2\nu})e^{\lambda\tau}\big) \\
&\times \sum_{i=0}^{k} e^{\lambda t_{i+1}} \mathbb{E}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)|^2\triangle \\
&- \big(b_3 - b_4 e^{\lambda\tau}\big) \sum_{i=0}^{k} e^{\lambda t_{i+1}} \mathbb{E} V\big(u^{\xi,\triangle}(t_i), u^{\zeta,\triangle}(t_i)\big)\triangle. \qquad (54)
\end{aligned}
$$

Making use of (15) and $b_3 > b_4$, we choose a $\bar{\lambda} = \bar{\lambda}(\hat{\triangle}_2) \in (0, 1]$ sufficiently small such that

$$
b_1 - 2K^2\hat{\triangle}_2^{1-2\nu} - \bar{\lambda} - (b_2 + 2K^2\hat{\triangle}_2^{1-2\nu})e^{\bar{\lambda}\tau} \ge 0 \quad \text{and} \quad b_3 - b_4 e^{\bar{\lambda}\tau} \ge 0.
$$

Taking $\lambda = \bar{\lambda}$ in (54) implies that for any $\triangle \in (0, \hat{\triangle}_2]$, $k \ge 0$, and $\xi, \zeta \in B(M)$,

$$
\mathbb{E}|u^{\xi,\triangle}(t_{k+1}) - u^{\zeta,\triangle}(t_{k+1})|^2 \le L_M e^{-\bar{\lambda} t_{k+1}}.
$$

The proof is therefore complete. □

Now we formulate the key proposition, which plays an important role in the analysis of the existence and uniqueness of numerical invariant measures.

Proposition 4 *Suppose that* (H1)–(H3) *hold. Let* $\hat{\triangle} = \hat{\triangle}_1 \wedge \hat{\triangle}_2$. *Then for any* $M > 0$, $\varepsilon > 0$, *there exists a* $\bar{T} = \bar{T}(\hat{\triangle}, M, \varepsilon) > \tau$ *such that for any* $\triangle \in (0, \hat{\triangle}]$ *and* $k\triangle \ge \bar{T}$,

$$
\sup_{\xi,\zeta \in B(M)} \mathbb{P}\Big\{\|Y^{\xi,\triangle}_{t_k} - Y^{\zeta,\triangle}_{t_k}\| \ge \varepsilon\Big\} < \varepsilon. \qquad (55)
$$

Proof Let $\hat{\triangle} = \hat{\triangle}_1 \wedge \hat{\triangle}_2$. For any $M > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, by virtue of Lemma 4.5 there exist $\triangle_4 = \triangle_4(M, \varepsilon_1/6, \varepsilon_2/9) \in (0, \hat{\triangle} \wedge \triangle^*_M]$ and $j^* = j^*(M, \varepsilon_1/3, \varepsilon_2/3, \triangle_4) \ge 1$ such that

$$
\sup_{\triangle \in (0,\triangle_4]} \sup_{k \ge 0} \sup_{\xi \in B(M)} \mathbb{P}\Big\{ \sup_{\substack{|s_1 - s_2| \le \tau/j^* \\ s_1, s_2 \in [t_k, t_k+\tau]}} \big|y^{\xi,\triangle}(s_1) - y^{\xi,\triangle}(s_2)\big| \ge \frac{\varepsilon_2}{3}\Big\} < \frac{\varepsilon_1}{3}. \qquad (56)
$$

It follows from (10) that for any $\triangle \in (0, \triangle_4]$ and $k \ge N$,

$$
\begin{aligned}
&\sup_{\xi,\zeta \in B(M)} \sup_{t \in [t_k - \tau, t_k]} \mathbb{P}\Big\{|y^{\xi,\triangle}(t) - y^{\zeta,\triangle}(t)| \ge \frac{\varepsilon_2}{3}\Big\} \\
\le& \sup_{\xi,\zeta \in B(M)} \sup_{i \in \{k-N,\cdots,k-1\}} \sup_{t \in [t_i, t_{i+1}]} \mathbb{P}\Big\{\frac{t_{i+1} - t}{\triangle}|u^{\xi,\triangle}(t_i) - u^{\zeta,\triangle}(t_i)| \ge \frac{\varepsilon_2}{6}\Big\} \\
&+ \sup_{\xi,\zeta \in B(M)} \sup_{i \in \{k-N,\cdots,k-1\}} \sup_{t \in [t_i, t_{i+1}]} \mathbb{P}\Big\{\frac{t - t_i}{\triangle}|u^{\xi,\triangle}(t_{i+1}) - u^{\zeta,\triangle}(t_{i+1})| \ge \frac{\varepsilon_2}{6}\Big\}
\end{aligned}
$$

$$\leq 2 \sup_{\xi,\zeta\in B(M)} \sup_{i\in\{k-N,\cdots,k\}} \mathbb{P}\Big\{|u^{\xi,\triangle}(t_i)-u^{\zeta,\triangle}(t_i)|\geq \frac{\varepsilon_2}{6}\Big\}. \tag{57}$$

It follows from the Chebyshev inequality and Lemma 4.6 that

$$\begin{aligned}
&\sup_{i\in\{k-N,\cdots,k\}} \sup_{\xi,\zeta\in B(M)} \mathbb{P}\Big\{|u^{\xi,\triangle}(t_i)-u^{\zeta,\triangle}(t_i)|\geq \frac{\varepsilon_2}{6}\Big\}\\
&\leq \frac{36}{\varepsilon_2^2} \sup_{i\in\{k-N,\cdots,k\}} \sup_{\xi,\zeta\in B(M)} \mathbb{E}|u^{\xi,\triangle}(t_i)-u^{\zeta,\triangle}(t_i)|^2 \leq \frac{36L_M e^{-\bar{\lambda}(t_k-\tau)}}{\varepsilon_2^2}.
\end{aligned} \tag{58}$$

Choose a $T_1 = T_1(M,\varepsilon_1,\varepsilon_2,\triangle_4,j^*) > \tau$ sufficiently large such that

$$\frac{36L_M e^{-\bar{\lambda}(T_1-\tau)}}{\varepsilon_2^2} < \frac{\varepsilon_1}{6j^*}. \tag{59}$$

Inserting (58) and (59) into (57) implies

$$\sup_{\triangle\in(0,\triangle_4]} \sup_{k\triangle\geq T_1} \sup_{\xi,\zeta\in B(M)} \sup_{t\in[t_k-\tau,t_k]} \mathbb{P}\Big\{|y^{\xi,\triangle}(t)-y^{\zeta,\triangle}(t)|\geq \frac{\varepsilon_2}{3}\Big\} < \frac{\varepsilon_1}{3j^*}. \tag{60}$$

According to (56) and (60) yields

$$\begin{aligned}
&\sup_{\triangle\in(0,\triangle_4]} \sup_{k\triangle\geq T_1} \sup_{\xi,\zeta\in B(M)} \mathbb{P}\Big\{\|Y_{t_k}^{\xi,\triangle}-Y_{t_k}^{\zeta,\triangle}\|\geq \varepsilon_2\Big\}\\
&= \sup_{\triangle\in(0,\triangle_4]} \sup_{k\triangle\geq T_1} \sup_{\xi,\zeta\in B(M)} \mathbb{P}\Big\{\sup_{0\leq j\leq j^*-1} \sup_{t\in[t_{k-N}^{j^*,j},t_{k-N}^{j^*,j+1}]} |y^{\xi,\triangle}(t)-y^{\zeta,\triangle}(t)|\geq \varepsilon_2\Big\}\\
&\leq 2 \sup_{\triangle\in(0,\triangle_4]} \sup_{k\triangle\geq T_1} \sup_{\xi\in B(M)} \mathbb{P}\Big\{\sup_{\substack{|s_1-s_2|\leq\tau/j^*\\ s_1,s_2\in[t_k-\tau,t_k]}} |y^{\xi,\triangle}(s_1)-y^{\xi,\triangle}(s_2)|\geq \frac{\varepsilon_2}{3}\Big\}\\
&\quad + j^* \sup_{\triangle\in(0,\triangle_4]} \sup_{k\triangle\geq T_1} \sup_{\xi,\zeta\in B(M)} \sup_{t\in[t_k-\tau,t_k]} \mathbb{P}\Big\{|y^{\xi,\triangle}(t)-y^{\zeta,\triangle}(t)|\geq \frac{\varepsilon_2}{3}\Big\} < \varepsilon_1.
\end{aligned} \tag{61}$$

where $t_k^{j^*,j} = t_k + j\tau/j^*$. On the other hand, for any $\triangle\in[\triangle_4,\ \hat{\triangle}]$ and $k\geq N$, it follows from (10) that

$$\begin{aligned}
&\sup_{\xi,\zeta\in B(M)} \mathbb{P}\Big\{\|Y_{t_k}^{\xi,\triangle}-Y_{t_k}^{\zeta,\triangle}\|\geq \varepsilon_2\Big\}\\
&= \sup_{\xi,\zeta\in B(M)} \mathbb{P}\Big\{\sup_{i\in\{k-N,\cdots,k-1\}} \sup_{t\in[t_i,t_{i+1}]} |y^{\xi,\triangle}(t)-y^{\zeta,\triangle}(t)|\geq \varepsilon_2\Big\}\\
&\leq \sum_{i=k-N}^{k-1} \sup_{\xi,\zeta\in B(M)} \mathbb{P}\Big\{\sup_{t\in[t_i,t_{i+1}]} \frac{t_{i+1}-t}{\triangle}|u^{\xi,\triangle}(t_i)-u^{\zeta,\triangle}(t_i)|\geq \frac{\varepsilon_2}{2}\Big\}\\
&\quad + \sum_{i=k-N}^{k-1} \sup_{\xi,\zeta\in B(M)} \mathbb{P}\Big\{\sup_{t\in[t_i,t_{i+1}]} \frac{t-t_i}{\triangle}|u^{\xi,\triangle}(t_{i+1})-u^{\zeta,\triangle}(t_{i+1})|\geq \frac{\varepsilon_2}{2}\Big\}\\
&\leq 2N \sup_{i\geq k-N} \sup_{\xi,\zeta\in B(M)} \mathbb{P}\Big\{|u^{\xi,\triangle}(t_i)-u^{\zeta,\triangle}(t_i)|\geq \frac{\varepsilon_2}{2}\Big\}.
\end{aligned}$$

Choose a $T_2 = T_2(M,\varepsilon_1,\varepsilon_2,\triangle_4) > \tau$ sufficiently large such that

$$\frac{8\tau L_M e^{-\bar{\lambda}(T_2-\tau)}}{\varepsilon_2^2\triangle_4} < \varepsilon_1.$$

Making use of the Chebyshev inequality and Lemma 4.6 we arrive at

$$
\begin{aligned}
&\sup_{\triangle\in[\triangle_4,\hat{\triangle}]}\sup_{k\triangle\geq T_2}\sup_{\xi,\zeta\in B(M)}\mathbb{P}\Big\{\|Y^{\xi,\triangle}_{t_k}-Y^{\zeta,\triangle}_{t_k}\|\geq\varepsilon_2\Big\}\\
\leq&\frac{8N}{\varepsilon_2^2}\sup_{i\geq k-N}\sup_{\xi,\zeta\in B(M)}\mathbb{E}|u^{\xi,\triangle}(t_i)-u^{\zeta,\triangle}(t_i)|^2\leq\frac{8\tau L_M e^{-\bar{\lambda}(T_2-\tau)}}{\varepsilon_2^2\triangle_4}<\varepsilon_1. \qquad (62)
\end{aligned}
$$

Let $\bar{T}=T_1\vee T_2$ and $\varepsilon_1=\varepsilon_2=\varepsilon$. According to (61), and (62) yields

$$
\begin{aligned}
&\sup_{\triangle\in(0,\hat{\triangle}]}\sup_{k\triangle\geq\bar{T}}\sup_{\xi,\zeta\in B(M)}\mathbb{P}\{\|Y^{\xi,\triangle}_{t_k}-Y^{\zeta,\triangle}_{t_k}\|\geq\varepsilon\}\\
\leq&\max\Big\{\sup_{\triangle\in(0,\triangle_4]}\sup_{k\triangle\geq T_1}\sup_{\xi,\zeta\in B(M)}\mathbb{P}\{\|Y^{\xi,\triangle}_{t_k}-Y^{\zeta,\triangle}_{t_k}\|\geq\varepsilon_2\},\\
&\sup_{\triangle\in[\triangle_4,\hat{\triangle}]}\sup_{k\triangle\geq T_2}\sup_{\xi,\zeta\in B(M)}\mathbb{P}\{\|Y^{\xi,\triangle}_{t_k}-Y^{\zeta,\triangle}_{t_k}\|\geq\varepsilon_2\}\Big\}<\varepsilon
\end{aligned}
$$

as required. □

Proof of Theorem 1 Since the proof is rather technical, we divide it into two steps.

Step 1. Firstly, choose a special initial data $\xi=\mathbf{0}$. For any $\varepsilon\in(0,1)$, in view of (12) and Proposition 3, there exists a positive constant $\Lambda^*=\Lambda^*(\hat{\triangle},0,\varepsilon/8)$ such that

$$
\sup_{\triangle\in(0,\hat{\triangle}]}\sup_{i\geq 0}\mu^{\mathbf{0},\triangle}_{t_i}\big(B^c(\Lambda^*)\big)=\sup_{\triangle\in(0,\hat{\triangle}]}\sup_{i\geq 0}\mathbb{P}\Big\{\|Y^{\mathbf{0},\triangle}_{t_i}\|>\Lambda^*\Big\}<\frac{\varepsilon}{8}. \qquad (63)
$$

By virtue of Proposition 4, there exists a $\bar{T}=\bar{T}(\hat{\triangle},\Lambda^*,\varepsilon/4)>\tau$ such that for any $\triangle\in(0,\hat{\triangle}]$ and $k\triangle\geq\bar{T}$,

$$
\sup_{X\in B(\Lambda^*)}\mathbb{P}\Big\{\|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|\geq\frac{\varepsilon}{4}\Big\}<\frac{\varepsilon}{4}. \qquad (64)
$$

For any $\triangle\in(0,\hat{\triangle}]$, $t_k\geq\bar{T}$, and $i\geq 0$, recalling definitions (2) and (12), we obtain

$$
\begin{aligned}
&d_{\Xi}\big(\mu^{\mathbf{0},\triangle}_{t_{k+i}}(\cdot),\mu^{\mathbf{0},\triangle}_{t_k}(\cdot)\big)=\sup_{\Psi\in\Xi}\big|\mathbb{E}\Psi(Y^{\mathbf{0},\triangle}_{t_{k+i}})-\mathbb{E}\Psi(Y^{\mathbf{0},\triangle}_{t_k})\big|\\
=&\sup_{\Psi\in\Xi}\big|\mathbb{E}\big(\mathbb{E}\big(\Psi(Y^{\mathbf{0},\triangle}_{t_{k+i}})|\mathcal{F}_{t_i}\big)\big)-\mathbb{E}\Psi(Y^{\mathbf{0},\triangle}_{t_k})\big|\\
=&\sup_{\Psi\in\Xi}\big|\mathbb{E}\big(\mathbb{E}\Psi(Y^{X,\triangle}_{t_k})|_{X=Y^{\mathbf{0},\triangle}_{t_i}}\big)-\mathbb{E}\Psi(Y^{\mathbf{0},\triangle}_{t_k})\big|\\
\leq&\sup_{\Psi\in\Xi}\int_C\mathbb{E}\big|\Psi(Y^{X,\triangle}_{t_k})-\Psi(Y^{\mathbf{0},\triangle}_{t_k})\big|\mu^{\mathbf{0},\triangle}_{t_i}(\mathrm{d}X)\\
\leq&\int_C\mathbb{E}(2\wedge\|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|)\mu^{\mathbf{0},\triangle}_{t_i}(\mathrm{d}X).
\end{aligned}
$$

Making use of (63) and (64) yields

$$\begin{aligned} d_{\Xi}\big(\mu^{\mathbf{0},\triangle}_{t_{k+i}}(\cdot),\mu^{\mathbf{0},\triangle}_{t_k}(\cdot)\big) \leq & \int_{B(\Lambda^*)} \mathbb{E}(2\wedge \|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|)\mu^{\mathbf{0},\triangle}_{t_i}(\mathrm{d}X)+\frac{\varepsilon}{4} \\ \leq & 2\int_{B(\Lambda^*)} \mathbb{P}\{\|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|\geq \frac{\varepsilon}{4}\}\mu^{\mathbf{0},\triangle}_{t_i}(\mathrm{d}X)+\frac{\varepsilon}{4}+\frac{\varepsilon}{4} \\ \leq & 2\sup_{X\in B(\Lambda^*)} \mathbb{P}\Big\{\|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|\geq \frac{\varepsilon}{4}\Big\}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}<\varepsilon. \end{aligned} \tag{65}$$

This implies that the measure sequence $\{\mu^{\mathbf{0},\triangle}_{t_k}(\cdot)\}_{k\geq 0}$ is uniformly Cauchy. Since $(\mathcal{P}(C),d_{\Xi})$ is complete (see [10, Corollary 10.5]), there exists a unique probability measure $\pi^{\triangle}(\cdot)$ such that

$$\lim_{t_k\to\infty} d_{\Xi}\big(\mu^{\mathbf{0},\triangle}_{t_k}(\cdot),\pi^{\triangle}(\cdot)\big)=0, \quad \text{uniformly in } \triangle\in(0,\hat{\triangle}]. \tag{66}$$

Step 2. Let $\xi\in\mathcal{C}^{\alpha}_{\mathcal{F}_0}$. It is straightforward to see that $\|\xi\|<\infty$, $\mathbb{P}-$a.s. Hence for any $\varepsilon>0$ there exists an $M>0$ sufficiently large such that

$$P_{\xi}(B^c(M))=\mathbb{P}\Big\{\|\xi\|>M\Big\}<\frac{\varepsilon}{8}, \tag{67}$$

where

$$P_{\xi}(A):=\mathbb{P}\Big\{\omega\in\Omega:\xi(\omega)\in A\Big\}, \quad \forall\ A\in\mathfrak{B}(C).$$

By Proposition 4 there exists a $\bar{T}_1=\bar{T}_1(\hat{\triangle},M,\varepsilon/4)>\tau$ sufficiently large such that for any $\triangle\in(0,\hat{\triangle}]$ and $k\triangle\geq\bar{T}_1$,

$$\sup_{X\in B(M)} \mathbb{P}\Big\{\|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|\geq\frac{\varepsilon}{4}\Big\}<\frac{\varepsilon}{4}. \tag{68}$$

For any $\triangle\in(0,\hat{\triangle}]$ and $k\triangle\geq\bar{T}_1$, using (67) and (68) implies

$$\begin{aligned} & d_{\Xi}\big(\mu^{\xi,\triangle}_{t_k}(\cdot),\mu^{\mathbf{0},\triangle}_{t_k}(\cdot)\big)\leq \mathbb{E}\Big(\mathbb{E}\big(2\wedge\|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|\big)\big|_{\{X=\xi\}}\Big) \\ =&\int_{B(M)} \mathbb{E}\big(2\wedge\|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|\big)P_{\xi}(\mathrm{d}X) \\ &+\int_{B^c(M)} \mathbb{E}\big(2\wedge\|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|\big)P_{\xi}(\mathrm{d}X) \\ \leq & 2\int_{B(M)} \mathbb{P}\{\|Y^{X,\triangle}_{t_k}-Y^{\mathbf{0},\triangle}_{t_k}\|\geq\frac{\varepsilon}{4}\}P_{\xi}(\mathrm{d}X)+\frac{\varepsilon}{4}+2P_{\xi}(B^c(M))<\varepsilon. \end{aligned}$$

Therefore, for any $\xi \in \mathcal{C}^{\alpha}_{\mathcal{F}_0}$,

$$\lim_{t_k\to\infty} d_{\Xi}\big(\mu^{\xi,\triangle}_{t_k}(\cdot), \mu^{\mathbf{0},\triangle}_{t_k}(\cdot)\big) = 0, \quad \text{uniformly} \ \ \text{in } \triangle \in (0, \hat{\triangle}]. \tag{69}$$

It follows from (66) and (69) that

$$\begin{aligned}
&\lim_{t_k\to\infty} d_{\Xi}(\mu^{\xi,\triangle}_{t_k}(\cdot), \pi^{\triangle}(\cdot))\\
\le&\lim_{t_k\to\infty} d_{\Xi}(\mu^{\mathbf{0},\triangle}_{t_k}(\cdot), \pi^{\triangle}(\cdot)) + \lim_{t_k\to\infty}\big(\mu^{\xi,\triangle}_{t_k}(\cdot), \mu^{\mathbf{0},\triangle}_{t_k}(\cdot)\big)\\
=&0, \quad \text{uniformly} \ \ \text{in } \triangle \in (0, \hat{\triangle}].
\end{aligned}$$

The required assertion (16) follows. By the similar way as Step 2, for any $M > 0$, we may also prove that the convergence in (16) is also uniform for the initial data $\xi \in \mathbb{B}(M, \alpha)$. The proof is complete. □

Next, we give the convergence between the numerical segment process $Y^{\mathbf{0},\triangle}_{t_k}$ and the exact one $x^{\mathbf{0}}_{t_k}$.

Lemma 4.7 Suppose that (H1) and (H2) hold. Then for any $\varepsilon > 0$, $T > 0$, there exists a $\triangle_5 = \triangle_5(\varepsilon, T) \in (0, \hat{\triangle}_1]$ such that

$$\sup_{\triangle\in(0,\triangle_5]}\ \sup_{0\le k\triangle\le T} \mathbb{P}\Big\{\|Y^{\mathbf{0},\triangle}_{t_k} - x^{\mathbf{0}}_{t_k}\| \ge \varepsilon\Big\} < \varepsilon.$$

Proof **Proof.** Without loss of generality, for any $\varepsilon > 0$, $T > \tau$, by Lemma 4.3 there is a $\triangle_3 = \triangle_3(0, \varepsilon/4, T) \in (0, \hat{\triangle}_1]$ such that

$$\sup_{\triangle\in(0,\triangle_3]} \mathbb{P}\Big\{\beta^{\mathbf{0},\triangle}_{\triangle_3,-N} < T\Big\} < \frac{\varepsilon}{4}, \tag{70}$$

where $\beta^{\mathbf{0},\triangle}_{\triangle_3,-N}$ is given by (30). In view of Lemma 4.4, choose a $\triangle_4 = \triangle_4(0, \varepsilon/8, \varepsilon/2) \in (0, \triangle_3]$ sufficiently small such that

$$\sup_{\triangle\in(0,\triangle_4]}\ \sup_{k\ge0} \mathbb{P}\Big\{\sup_{t\in[t_k,t_k+\tau]} |y^{\mathbf{0},\triangle}(t) - z^{\mathbf{0},\triangle}(t)| \ge \frac{\varepsilon}{2}\Big\} < \frac{\varepsilon}{8}. \tag{71}$$

For any positive constant ℓ, define

$$\delta^{\mathbf{0}}_{\ell} = \inf\Big\{t \ge -\tau : |x^{\mathbf{0}}(t)| > \ell\Big\}.$$

By virtue of [36, Theorem 2.1] we have

$$\mathbb{P}\Big\{\delta^{\mathbf{0}}_{\ell} < T\Big\} \le \frac{L}{\ell^2},$$

Choose an ℓ sufficiently large such that

$$\mathbb{P}\Big\{\delta^{\mathbf{0}}_{\ell} < T\Big\} < \frac{\varepsilon}{4}. \tag{72}$$

Let $\gamma^{\triangle_3,\triangle}_{\mathbf{0},\ell} := \delta^{\mathbf{0}}_{\ell} \wedge \beta^{\mathbf{0},\triangle}_{\triangle_3,-N}$. By the similar way as [36, Theorem 3.3], there exists a $\triangle_5 = \triangle_5(\varepsilon, T) \in (0, \triangle_4]$ such that

$$\sup_{\triangle\in(0,\triangle_5]} \mathbb{E}\Big(\sup_{0\le t\le T} |z^{\mathbf{0},\triangle}(t) - x^{\mathbf{0}}(t)|^2 \mathbf{1}_{\{\gamma^{\triangle_3,\triangle}_{\mathbf{0},\ell} \ge T\}}\Big) < \frac{\varepsilon^3}{16}.$$

This, along with the Chebyshev inequality, implies that for any $\triangle \in (0, \triangle_5]$,

$$\begin{aligned}
&\sup_{\tau \le k\triangle \le T} \mathbb{P}\Big\{\gamma_{\mathbf{0},\ell}^{\triangle_3,\triangle} \ge T, \sup_{t\in[t_k-\tau,t_k]} |z^{\mathbf{0},\triangle}(t) - x^{\mathbf{0}}(t)| \ge \frac{\varepsilon}{2}\Big\} \\
\le& \mathbb{P}\Big\{\gamma_{\mathbf{0},\ell}^{\triangle_3,\triangle} \ge T, \sup_{0\le t\le T} |z^{\mathbf{0},\triangle}(t) - x^{\mathbf{0}}(t)| \ge \frac{\varepsilon}{2}\Big\} \\
\le& \frac{4}{\varepsilon^2}\mathbb{E}\Big(\sup_{0\le t\le T} |z^{\mathbf{0},\triangle}(t) - x^{\mathbf{0}}(t)|^2 \mathbf{1}_{\{\gamma_{\mathbf{0},\ell}^{\triangle_3,\triangle} \ge T\}}\Big) < \frac{\varepsilon}{4}.
\end{aligned} \tag{73}$$

It follows from (71) and (73) that

$$\begin{aligned}
&\sup_{\triangle\in(0,\triangle_5]} \sup_{\tau\le k\triangle\le T} \mathbb{P}\Big\{\gamma_{\mathbf{0},\ell}^{\triangle_3,\triangle} \ge T, \|Y_{t_k}^{\mathbf{0},\triangle} - x_{t_k}^{\mathbf{0}}\| \ge \varepsilon\Big\} \\
\le& \sup_{\triangle\in(0,\triangle_5]} \sup_{\tau\le k\triangle\le T} \mathbb{P}\Big\{\sup_{t\in[t_k-\tau,t_k]} |y^{\mathbf{0},\triangle}(t) - z^{\mathbf{0},\triangle}(t)| \ge \frac{\varepsilon}{2}\Big\} \\
&+ \sup_{\triangle\in(0,\triangle_5]} \sup_{\tau\le k\triangle\le T} \mathbb{P}\Big\{\gamma_{\mathbf{0},\ell}^{\triangle_3,\triangle} \ge T, \sup_{t\in[t_k-\tau,t_k]} |z^{\mathbf{0},\triangle}(t) - x^{\mathbf{0}}(t)| \ge \frac{\varepsilon}{2}\Big\} < \frac{3\varepsilon}{8}.
\end{aligned} \tag{74}$$

Since for any $\triangle \in (0, \triangle_5]$,

$$y^{\mathbf{0},\triangle}(\theta) = \Gamma_{\Phi,\nu}^{\triangle}(0) = 0 = x^{\mathbf{0},\triangle}(\theta), \quad \theta \in [-\tau, 0],$$

it is obvious that

$$\begin{aligned}
&\sup_{0\le k\triangle\le\tau} \|Y_{t_k}^{\mathbf{0},\triangle} - x_{t_k}^{\mathbf{0},\triangle}\| = \sup_{-\tau\le t\le\tau} |y^{\mathbf{0},\triangle}(t) - x^{\mathbf{0},\triangle}(t)| \\
=& \sup_{0\le t\le\tau} |y^{\mathbf{0},\triangle}(t) - x^{\mathbf{0},\triangle}(t)| = \|Y_{\tau}^{\mathbf{0},\triangle} - x_{\tau}^{\mathbf{0},\triangle}\|.
\end{aligned}$$

According to (70), (72), and (74) yields

$$\begin{aligned}
&\sup_{\triangle\in(0,\triangle_5]} \sup_{0\le k\triangle\le T} \mathbb{P}\Big\{\|Y_{t_k}^{\mathbf{0},\triangle} - x_{t_k}^{\mathbf{0}}\| \ge \varepsilon\Big\} \\
=& \sup_{\triangle\in(0,\triangle_5]} \sup_{\tau\le k\triangle\le T} \mathbb{P}\Big\{\|Y_{t_k}^{\mathbf{0},\triangle} - x_{t_k}^{\mathbf{0}}\| \ge \varepsilon\Big\} \\
\le& \sup_{\triangle\in(0,\triangle_5]} \sup_{\tau\le k\triangle\le T} \mathbb{P}\Big\{\gamma_{\mathbf{0},\ell}^{\triangle_3,\triangle} \ge T, \|Y_{t_k}^{\mathbf{0},\triangle} - x_{t_k}^{\mathbf{0}}\| \ge \varepsilon\Big\} \\
&+ \sup_{\triangle\in(0,\triangle_5]} \mathbb{P}\Big\{\beta_{\triangle_3,-N}^{\mathbf{0},\triangle} < T\Big\} + \mathbb{P}\Big\{\delta_{\ell}^{\mathbf{0}} < T\Big\} < \varepsilon.
\end{aligned} \tag{75}$$

The proof is therefore complete. □

Proof of Theorem 2 For any $\varepsilon \in (0, 1)$, in view of Lemma 2.1 there exists a $T_1^* > 0$ such that for any $t \ge T_1^*$,

$$d_{\Xi}\big(\pi(\cdot), \mu_t^{\mathbf{0}}(\cdot)\big) < \frac{\varepsilon}{3}. \tag{76}$$

By virtue of Theorem 1 there exists a $T_2^* > \tau$ such that for any $\triangle \in (0, \hat{\triangle}]$ and $k\triangle \ge T_2^*$,

$$d_{\Xi}\big(\mu_{t_k}^{\mathbf{0},\triangle}(\cdot), \pi^{\triangle}(\cdot)\big) < \frac{\varepsilon}{3}. \tag{77}$$

Let $T^* := T_1^* \vee T_2^*$. By Lemma 4.7 there exists a $\triangle^* \in (0, \hat{\triangle}]$ such that for any $\triangle \in (0, \triangle^*]$ and $0 \le k\triangle \le T^* + 1$,

$$\mathbb{P}\Big\{\|x_{t_k}^{\mathbf{0}} - Y_{t_k}^{\mathbf{0},\triangle}\| \ge \frac{\varepsilon}{6}\Big\} < \frac{\varepsilon}{12}. \tag{78}$$

Furthermore, due to the definition of d_Ξ, and (78), we deduce that for any $\triangle \in (0, \triangle^*]$ and $0 \le k\triangle \le T^* + 1$,

$$\begin{aligned}
d_\Xi\big(\mu_{t_k}^{\mathbf{0}}(\cdot), \mu_{t_k}^{\mathbf{0},\triangle}(\cdot)\big) =& \sup_{\Psi\in\Xi} \mathbb{E}\big|\Psi(x_{t_k}^{\mathbf{0}}) - \Psi(Y_{t_k}^{\mathbf{0},\triangle})\big| \le \mathbb{E}\big(2 \wedge \|x_{t_k}^{\mathbf{0}} - Y_{t_k}^{\mathbf{0},\triangle}\|\big)\\
=&\mathbb{E}\big((2 \wedge \|x_{t_k}^{\mathbf{0}} - Y_{t_k}^{\mathbf{0},\triangle}\|)\mathbf{1}_{\{\|x_{t_k}^{\mathbf{0}} - Y_{t_k}^{\mathbf{0},\triangle}\| \ge \frac{\varepsilon}{6}\}}\big)\\
&+ \mathbb{E}\big((2 \wedge \|x_{t_k}^{\mathbf{0},\triangle} - Y_{t_k}^{\mathbf{0},\triangle}\|)\mathbf{1}_{\{\|x_{t_k}^{\mathbf{0}} - Y_{t_k}^{\mathbf{0},\triangle}\| < \frac{\varepsilon}{6}\}}\big)\\
\le&\frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.
\end{aligned} \tag{79}$$

For any $\triangle \in (0, \triangle^*]$, choose an integer k_0 such that $T^* \le k_0\triangle \le T^* + 1$. It follows from (76), (77), and (79) that

$$\begin{aligned}
d_\Xi(\pi(\cdot), \pi^\triangle(\cdot)) \le& d_\Xi\big(\pi(\cdot), \mu_{t_{k_0}}^{\mathbf{0}}(\cdot)\big) + d_\Xi(\mu_{t_{k_0}}^{\mathbf{0},\triangle}(\cdot), \pi^\triangle(\cdot)) + d_\Xi(\mu_{t_{k_0}}^{\mathbf{0}}(\cdot), \mu_{t_{k_0}}^{\mathbf{0},\triangle}(\cdot))\\
<&\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned} \tag{80}$$

The proof is therefore complete. $\square$

5 Numerical experiments

In this section we provide an example and numerical simulations to illustrate the efficiency of TEMSP (11).

Example 1 Consider the following nonlinear SDDE

$$\begin{cases} \mathrm{d}x_1(t) = (1 - x_1(t) - 3x_1^3(t))\mathrm{d}t + x_2^2(t-1)\mathrm{d}W_1(t)\\ \mathrm{d}x_2(t) = -(x_2(t) + 3x_2^3(t))\mathrm{d}t + x_1^2(t-1)\mathrm{d}W_2(t) \end{cases} \tag{81}$$

with different initial data

$$\begin{aligned}
\xi_1(\theta) &= (\xi_{11}(\theta), \xi_{12}(\theta))^T = (B_1(-\theta), B_2(-\theta))^T,\\
\xi_2(\theta) &= (\xi_{21}(\theta), \xi_{22}(\theta))^T = (2\theta, \theta + 1)^T,\\
\xi_3(\theta) &= (\xi_{31}(\theta), \xi_{32}(\theta))^T = (-3, 4)^T
\end{aligned}$$

for any $\theta \in [-\tau, 0]$, where $(B_1(\cdot), B_2(\cdot))$ is a two-dimensional Brownian motion, which is independent of $(W_1(\cdot), W_2(\cdot))$.

For any $R > 0$ and $x,\ \bar{x},\ y,\ \bar{y} \in \mathbb{R}$ with $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \le R$, we compute

$$\begin{aligned}
&|f(x, y) - f(\bar{x}, \bar{y})|\\
=&\big|(1 + 3x_1^2 + 3x_1\bar{x}_1 + 3\bar{x}_1^2)^2(x_1 - \bar{x}_1)^2 + (1 + 3x_2^2 + 3x_2\bar{x}_2 + 3\bar{x}_2^2)^2(x_2 - \bar{x}_2)^2\big|^{1/2}
\end{aligned}$$

$$\leq\Big|(1+9R^2)^2(x_1-\bar{x}_1)^2+(1+9R^2)^2(x_2-\bar{x}_2)^2\Big|^{1/2}=(1+9R^2)|x-\bar{x}|, \tag{82}$$

and

$$\begin{aligned}|g(x,y)-g(\bar{x},\bar{y})|=&\Big|(y_2+\bar{y}_2)^2(y_2-\bar{y}_2)^2+(y_1+\bar{y}_1)^2(y_1-\bar{y}_1)^2\Big|^{1/2}\\ \leq&2R|y-\bar{y}|\leq(2R\vee 4R^2)(1\wedge|y-\bar{y}|).\end{aligned} \tag{83}$$

It is straightforward to see that (H1) holds. In addition, it is easy to verify that for any $x,\ \bar{x},\ y,\ \bar{y}\in\mathbb{R}$,

$$\begin{aligned}&\big\langle 2x,f(x,y)\big\rangle+|g(x,y)|^2\\ =&2x_1-2|x|^2-6(x_1^4+x_2^4)+(y_1^4+y_2^4)\leq L-3|x|^4+|y|^4,\end{aligned}$$

which implies that (H2) holds with $a_2=3,a_3=1$. Furthermore,

$$\begin{aligned}&2\big\langle x-\bar{x},f(x,y)-f(\bar{x},\bar{y})\big\rangle+|g(x,y)-g(\bar{x},\bar{y})|^2\\ =&-2|x-\bar{x}|^2-6\big(x_1^2+x_1\bar{x}_1+\bar{x}_1^2\big)|x_1-\bar{x}_1|^2-6\big(x_2^2+x_2\bar{x}_2+\bar{x}_2^2\big)|x_2-\bar{x}_2|^2\\ &+(y_1+\bar{y}_1)^2(y_1-\bar{y}_1)^2+(y_2+\bar{y}_2)^2(y_2-\bar{y}_2)^2\\ \leq&-2|x-\bar{x}|^2-3(x_1+\bar{x}_1)^2(x_1-\bar{x}_1)^2-3(x_2+\bar{x}_2)^2(x_2-\bar{x}_2)^2\\ &+(y_1+\bar{y}_1)^2(y_1-\bar{y}_1)^2+(y_2+\bar{y}_2)^2(y_2-\bar{y}_2)^2,\end{aligned}$$

which implies that (H3) holds with $b_1=2,b_2=0$. Remark 2 allows us to conclude that the segment process $\{x_t^{\xi}\}_{t\geq 0}$ of (81) has a unique invariant measure $\pi(\cdot)\in\mathcal{P}(C)$.

According to (5), (82) and (83), we take $\Phi(R)=16R^4$ for all $R\geq 1$. Then,

$$\Phi^{-1}(R)=R^{1/4}/2,\quad \text{for } R\geq 16.$$

Let $\nu=1/100$. By virtue of (6) a direct computation yields that for any $\triangle\in(0,1)$,

$$\Gamma_{\Phi,\nu}^{\triangle}(x)=\Big(|x|\wedge\triangle^{-\frac{1}{400}}\Big)\frac{x}{|x|}.$$

Choose $\hat{\triangle}=10^{-3}$, and compute

$$1.1264=a_2-6K^2\hat{\triangle}^{1-2\nu}>a_3=1\ \text{ and }\ 0.8243=b_1-4K^2\hat{\triangle}^{1-2\nu}>b_2=0.$$

This, along with Theorem 1 and Theorem 2 implies that for any $\triangle\in(0,\hat{\triangle}]$, TEMLISP $\{Y_{t_k}^{\xi,\triangle}\}_{k\geq 0}$ defined by (11) is asymptotically stable in distribution and admits a unique numerical invariant measure $\pi^{\triangle}(\cdot)$ satisfying $\lim_{\triangle\to 0}d_{\Xi}(\pi(\cdot),\pi^{\triangle}(\cdot))=0$.

To test the efficiency of TEMSP (11), we carry out some numerical simulations using MATLAB. For each of the numerical experiments performed, the red dotted line, the blue long and short dash line, and the green line represent the sample means of $\{\Psi_l(Y_{t_k}^{\xi_i,\triangle_j})\}_{k\geq 0}$ by TEMSP starting from different initial data $\xi_1,\ \xi_2,\ \xi_3$, respectively. Figure 1 depicts the sample means of $\Psi_l(Y_{t_k}^{\xi_i,\triangle_j})$ $(j,l=1,2)$ with different initial data ξ_i $(i=1,2,3)$ in the interval $[0,10]$ for 2000 sample points and different step sizes $\triangle_1=10^{-3},\ \triangle_2=10^{-4}$, where test functionals $\Psi_1(\cdot)=\cos(\|\cdot\|)$ and $\Psi_2(\cdot)=2\wedge\|\cdot\|$. Figure 1 depicts that each $\mathbb{E}\Psi_l(Y_{t_k}^{\xi_i,\triangle_j})$ starting from different initial data tends to a constant as $t_k\to\infty$, which implies the existence of numerical invariant measure. Figure 2 displays empirical cumulative distribution functions (Empirical CDF) of $\Psi_l(Y_{10}^{\xi_i,\triangle_j})$ for 2000 sample points.

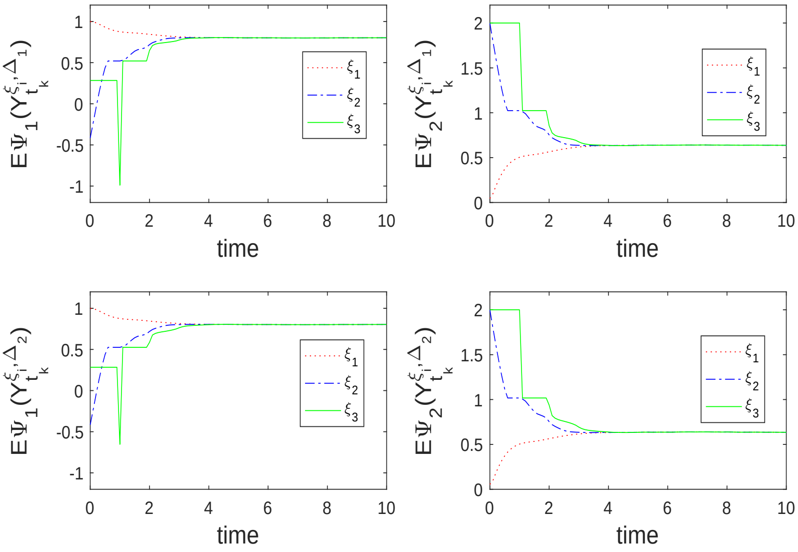

Fig. 1 Sample means of $\Psi_l(Y_{t_k}^{\xi_i,\triangle_j})$ $(i = 1, 2, 3;\ j, l = 1, 2)$ with different initial data ξ_i in the interval $[0, 10]$ for 2000 sample points and different step sizes $\triangle_1 = 10^{-3}$, $\triangle_2 = 10^{-4}$, and test functionals $\Psi_1(\cdot) = \cos(\|\cdot\|)$, $\Psi_2(\cdot) = 2 \wedge \|\cdot\|$.

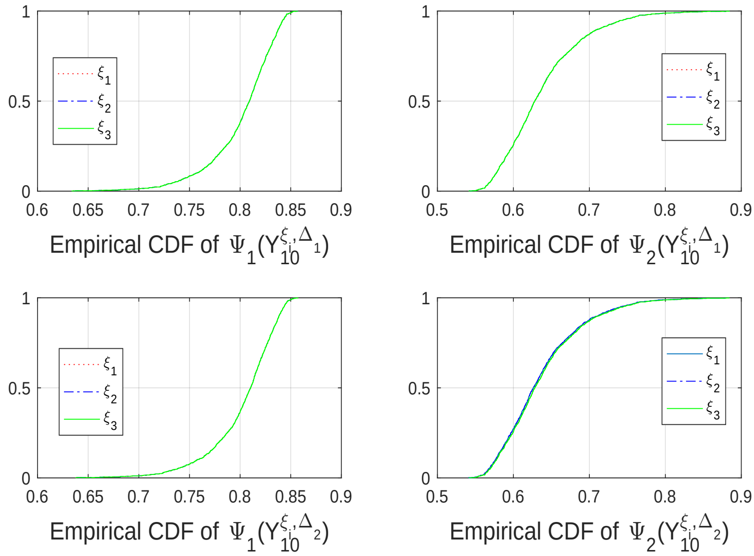

Fig. 2 Empirical cumulative distribution functions of $\Psi_l(Y_{10}^{\xi_i,\triangle_j})$ $(j, l = 1, 2)$ with different initial data ξ_i $(i = 1, 2, 3)$ for 2000 sample points and different step sizes $\triangle_j$ $(j = 1, 2)$, where step sizes $\triangle_1 = 10^{-3}$, $\triangle_2 = 10^{-4}$, and test functionals $\Psi_1(\cdot) = \cos(\|\cdot\|)$, $\Psi_2(\cdot) = 2 \wedge \|\cdot\|$.

6 Summary

We investigate the explicit approximation of the invariant measure for nonlinear SDDEs with non-globally Lipschitz diffusion coefficients. The appropriate numerical segment processes TEMSP are proposed. Since the mean square of the exact solutions may be not uniformly bounded and attracted, to overcome this difficulty, we take advantage of the linear structure of TEMSP to prove the uniform boundedness and attraction in probability. Finally we yield the existence of the unique numerical invariant measure, which converges to the exact one in the Fortet-Mourier distance.

Declarations

- **Ethics approval** Not applicable.
- **Availability of supporting data** All data or codes generated during the study are available from the corresponding author by request.
- **Conflict of interest/Competing interests** The authors declare no competing interests.
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