Applications of Polynomial Eigenvalue Decomposition to Multichannel Broadband Signal Processing, Part II: Eigenvalue Decomposition

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Part II: Eigenvalue Decomposition

2. Analytic Eigenvalue Decomposition

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2.1 Ordinary Eigenvalue Decomposition

- For a Hermitian matrix $\mathbf{R} = \mathbf{R}^{\mathrm{H}}$, we know that an eigenvalue decomposition (EVD) $\mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{H}}$ exists [27, 30];
- for eigenvalues $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_M\}$ and eigenvectors $\mathbf{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_M]$:

$$\mathbf{R}\mathbf{q}_m = \lambda_m \mathbf{q}_m$$

- eigenvalues $\lambda \in \mathbb{R}$;
- eigenvectors can be chosen as orthonormal, but may have an arbitary phase shift: $\mathbf{q}'_m = e^{j\varphi}\mathbf{q}_m$ is also an eigenvector;
- in case of an algebraic multiplicity C: $\lambda_m = \lambda_{m+1} = \cdots = \lambda_{m+C-1}$, only a C-dimensional subspace is defined, within which the eigenvectors can form an arbitrary orthonormal basis, with any unitary $\mathbf{V}$:

$$[\mathbf{q}'_m, \ \ldots \ \mathbf{q}'_{m+C-1}] = [\mathbf{q}_m, \ \ldots \ \mathbf{q}_{m+C-1}] \ \mathbf{V} \ . \tag{1}$$

2.2 Existence of an Analytic EVD on a Real Interval

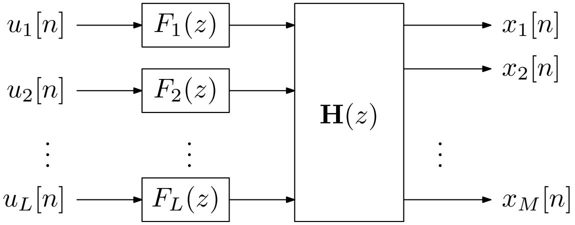
- A standard EVD can diagonalise $\boldsymbol{R}(z) \;\bullet\!\!-\!\!\circ\; \mathbf{R}[\tau]$ only for one specific value of z or of τ, respectively;
- Franz Rellich (1939, [50]) for a self-adjoint, analytic $\boldsymbol{R}(t) = \boldsymbol{R}^{\mathrm{H}}(t)$, $t \in \mathbb{R}$:

$$\boldsymbol{R}(t) = \boldsymbol{Q}(t)\boldsymbol{\Lambda}(t)\boldsymbol{Q}^{\mathrm{H}}(t) \; ;$$

- $\boldsymbol{Q}(t)$ and $\boldsymbol{\Lambda}(t)$ *can* be chosen analytic;
- similarly for an arbitrary (i.e. not necessarily Hermitian or square) analytic matrix, de Moor & Boyd (1989, [21]) and Bunse-Gerstner (1991, [7]) established an analytic SVD.

Analyticity of $\boldsymbol{R}(z)$

- The analyticity of $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ\, \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\}$ can be tied to a source model [47, 66]



- the innovation filters $F_\ell(z)$, $\ell = 1, \ldots, L$ describe the spectral shape of the L contributing source signals;
- a convolutive mixing system $\boldsymbol{H}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times N}$ models the transfer paths between the L sources and M sensors;
- if $F_\ell(z)$ and $\boldsymbol{H}(z)$ are stable and causal, then $\boldsymbol{R}(z) = \boldsymbol{H}(z)\boldsymbol{F}(z)\boldsymbol{F}^{\mathrm{P}}(z)\boldsymbol{H}^{\mathrm{P}}(z)$ is analytic.

Analytic EVD on the Unit Circle

- Analyticity of $\boldsymbol{R}(z)$ permits a restricted evaluation on the unit circle $z = \mathrm{e}^{\mathrm{j}\Omega}$;
- due to Rellich [50]:

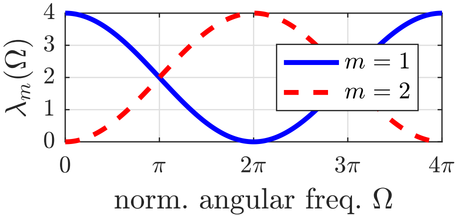
$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{Q}(\Omega)\ \boldsymbol{\Lambda}(\Omega)\ \boldsymbol{Q}^{\mathrm{H}}(\Omega)\ , \tag{2}$$

- unfortunately, while analytic in $\Omega \in \mathbb{R}$, $\boldsymbol{\Lambda}(\Omega)$ and $\boldsymbol{Q}(\Omega)$ can be $2\pi L$-periodic, with some $L \in \mathbb{Z}$ [67, 5];
- example [18, 55, 67]:

$$\boldsymbol{R}(z) = \frac{1}{2}\begin{bmatrix} 2 & 1+z^{-1} \\ 1+z & 2 \end{bmatrix}\ ,$$

$$\rightarrow \lambda_{1,2}(z) = 2 \pm (z^{\frac{1}{2}} + z^{-\frac{1}{2}})\ ,$$

$$\lambda_{1,2}(\mathrm{e}^{\mathrm{j}\Omega}) = 2 \pm \cos(\Omega/2)\ ;$$



- while $\cos(\Omega/2)$ is analytic in Ω, a fractional delay $z^{-\frac{1}{2}}$ is not analytic: its time domain equivalent decays too slowly [35].

Non-Existence of an Analytic EVD of $\boldsymbol{R}(z)$

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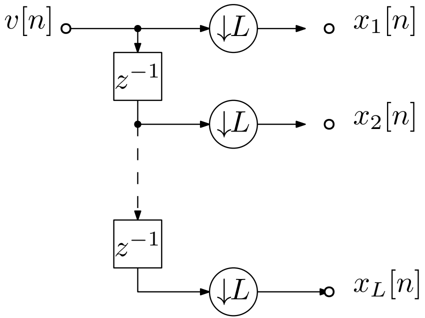
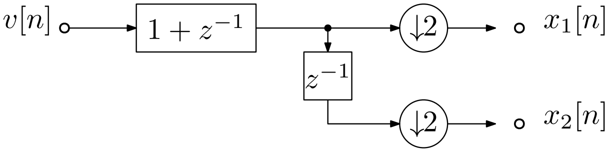
- The case $L > 1$ can be tied to multiplexing operation [67];
- assume $\mathbf{x}[n]$ is L-fold demultiplexed,
  $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ\ \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\}$;
- $\boldsymbol{R}(z)$ will be pseudo-circulant [58] with modulated eigenvalues [67, 5];
- $\boldsymbol{Q}(\Omega)$ and $\boldsymbol{\Lambda}(\Omega)$ will be $2\pi L$-periodic;
- as such, we can only find an analytic EVD if $\boldsymbol{R}(z)$ is L-fold oversampled [67]:

$$\boldsymbol{R}(z^L) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z)\ . \qquad (3)$$



Return to Example

- The previous example of $\boldsymbol{R}(z) = [2 \;\; 1+z^{-1}; 1+z \;\; 2]$ arises from the following arrangement with uncorrelated $v[n] \in \mathcal{N}(0,1)$:



- therefore we require oversampling by $L = 2$:

$$\boldsymbol{R}(z^2) = \begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix} \begin{bmatrix} z+2+z^{-1} & \\ & -z+2-z^{-1} \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 1 & -z^{-1} \end{bmatrix} ;$$

- if linked to block filtering, $\boldsymbol{R}(z)$ is pseudo-circulant [57], but this property may be obscured by paraunitary operations [67].

Analytic EVD of a Parahermitian Matrix

- For an analytic parahermitian matrix $\boldsymbol{R}(z)$, $z \in \mathbb{C}$, that is not connected to multiplexing, we can find [66, 67]

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z)\ \boldsymbol{\Lambda}(z)\ \boldsymbol{Q}^{\mathrm{P}}(z)\ , \tag{4}$$

  with analytic factors;
- $\boldsymbol{Q}(z) = [\boldsymbol{q}_1(z), \ldots, \boldsymbol{q}_M(z)]$ must be paraunitary [57, 59], such that

$$\boldsymbol{Q}(z)\boldsymbol{Q}^{\mathrm{P}}(z) = \boldsymbol{Q}^{\mathrm{P}}(z)\boldsymbol{Q}(z) = \mathbf{I}\ ; \tag{5}$$

- $\boldsymbol{\Lambda}(z) = \mathrm{diag}\{\lambda_1(z), \ldots, \lambda_M\}$ must be diagonal and parahermitian;
- the parahermitian property implies that on the unit circle, $\lambda(\mathrm{e}^{\mathrm{j}\Omega}) = \lambda(z)|_{z=\mathrm{e}^{\mathrm{j}\Omega}} \in \mathbb{R}$.

2.3 Properties: Uniqueness and Ambiguities

- For the analytic EVD [66, 67, 5]

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z) \cdot \boldsymbol{\Lambda}(z) \cdot \boldsymbol{Q}^{\mathrm{P}}(z) \; ; \qquad (6)$$

- the eigenvalues in $\boldsymbol{\Lambda}(z)$ are unique up to a permutation;
- if eigenvalues are distinct, then eigenvectors are unique up to an allpass filter $\psi_\ell(z)$;
- with $\boldsymbol{\Psi}(z) = \mathrm{diag}\{\psi_1(z), \ldots, \psi_M(z)\}$,

$$\begin{aligned} \boldsymbol{R}(z) &= \boldsymbol{Q}(z)\boldsymbol{\Psi}(z)\boldsymbol{\Lambda}(z)\boldsymbol{\Psi}^{\mathrm{P}}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \\ &= \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{\Psi}(z)\boldsymbol{\Psi}^{\mathrm{P}}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \\ &= \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \; ; \end{aligned}$$

- an analytic allpass $\psi_m(z)$ does not affect analyticity, but will affects the support of $\mathbf{Q}[n] \circ\!\!-\!\!\bullet \boldsymbol{Q}(z)$.

Properties: Support of EVD Factors

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- Given an arbitrary parahermitian $\boldsymbol{R}(z) \in \mathbb{C}^{2\times 2}$;
- eigenvalues $\gamma_{1,2}(z)$ can be directly computed in the z-domain as the roots of

$$\det\{\gamma(z)\mathbf{I} - \boldsymbol{R}(z)\} = \gamma^2(z) - T(z)\gamma(z) + D(z) = 0$$

- determinant $D(z) = \det\{\boldsymbol{R}(z)\}$ and trace $T(z) = \mathrm{trace}\{\boldsymbol{R}(z)\}$;
- this leads to

$$\gamma_{1,2}(z) = \frac{1}{2}T(z) \pm \frac{1}{2}\sqrt{T(z)T^{\mathrm{P}}(z) - 4D(z)}\ ; \qquad (7)$$

- awkward: $T(z)T^{\mathrm{P}}(z) - 4D(z) = S(z)S^{\mathrm{P}}(z)$ is parahermitian, but so must be the result of the square root.

Exact Calculation cont'd

- Maclaurin series: for every root of $S(z)$,

$$\sqrt{1-\beta z^{-1}} = \sum_{n=0}^{\infty} \xi_n \beta^n z^{-n} \tag{8}$$

$$\frac{1}{\sqrt{1-\alpha z^{-1}}} = \left(\sum_{n=0}^{\infty} \xi_n \alpha^n z^{-n} \right)^{-1} = \sum_{n=0}^{\infty} \chi_n \alpha^n z^{-n} \tag{9}$$

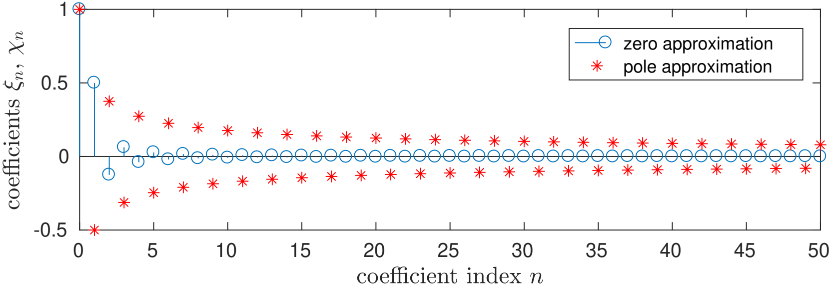
- with coefficients [6]

$$\xi_n = (-1)^n \binom{\frac{1}{2}}{n} = \frac{(-1)^n}{n!} \prod_{i=0}^{n-1} \left(\frac{1}{2} - i \right) \ , \tag{10}$$

$$\chi_n = (-1)^n \binom{-\frac{1}{2}}{n} = \frac{(-1)^{n-1}}{n!} \prod_{i=0}^{n-1} \left(\frac{1}{2} + i \right) \ . \tag{11}$$

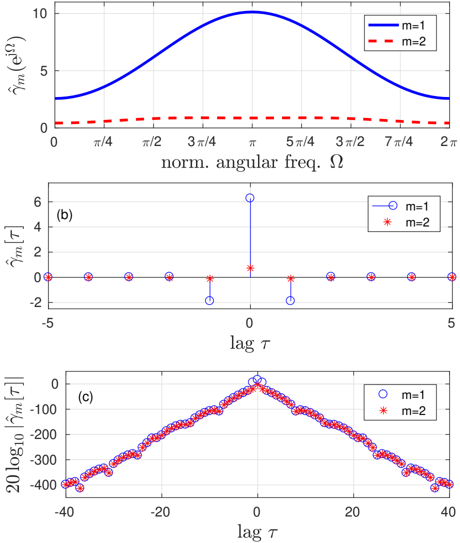
Maclaurin Series

- Coefficients ξ_n and χ_n for $n = 0 \ldots 50$ [66]:



- these coefficients additionally dampen a geometric series;
- only if $S(z)$ has double zeros (and double poles) is a polynomial (rational) solution possible;
- in general, the result are transcendental eigenvalues.

Numerical Example



- Example from Icart & Comon (2012, [29]):

$$\boldsymbol{R}(z) = \begin{bmatrix} 1 & 1 \\ 1 & -2z + 6 - 2z^{-1} \end{bmatrix}$$

- (top) solution on unit circle;
- (middle) coefficients of analytic eigenvalues;
- (bottom) decay of coefficients;
- solution generally can be transcendental, i.e. neither finite nor rational.

3. Polynomial Eigenvalue Decomposition

3.1 Polynomial EVD and Spectral Majorisation

- Polynomial EVD or McWhirter decomposition [37] of the CSD matrix

$$\boldsymbol{R}(z) \approx \boldsymbol{U}(z)\, \boldsymbol{\Gamma}(z)\, \boldsymbol{U}^{\mathrm{P}}(z) \qquad (12)$$

- with paraunitary, polynomial $\boldsymbol{U}(z)$, s.t. $\boldsymbol{U}(z)\boldsymbol{U}^{\mathrm{P}}(z) = \mathbf{I}$;
- diagonal Laurent polynomial matrix

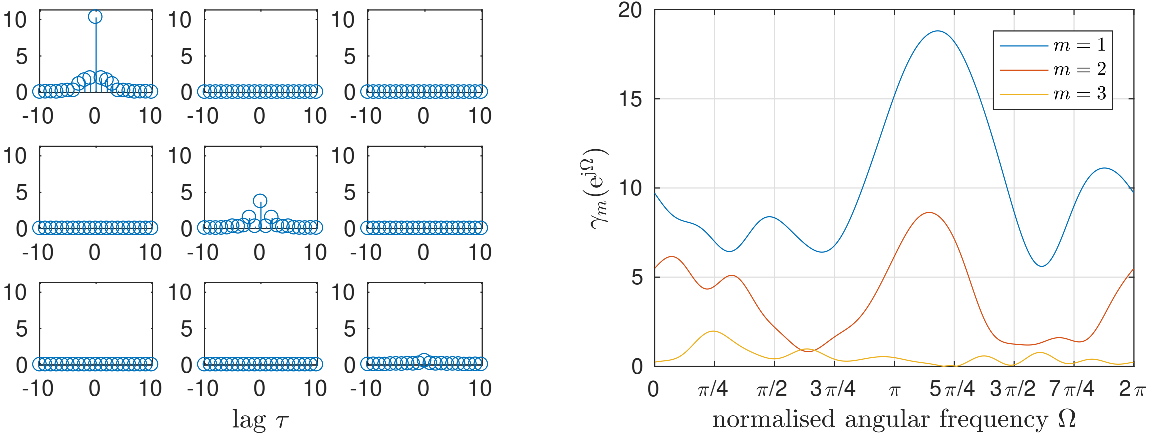
$$\boldsymbol{\Gamma}(z) = \mathrm{diag}\{\gamma_1(z),\ \ldots,\ \gamma_M(z)\}\ , \qquad (13)$$

- approximation sign due to restriction to polynomials [29];
- the eigenvalues are spectrally majorised [56], i.e. on the unit circle must satisfy

$$\gamma_m(\mathrm{e}^{\mathrm{j}\Omega}) \geq \gamma_{m+1}(\mathrm{e}^{\mathrm{j}\Omega})\ , \qquad \forall \Omega, \quad m = 1, \ldots (M-1)\ . \qquad (14)$$

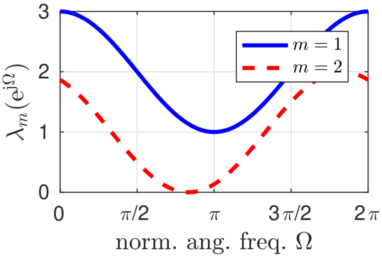
Polynomial Eigenvalues and Spectral Majorisation

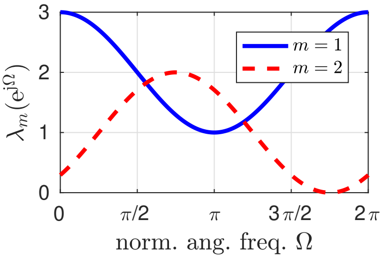
- Example for polynomial eigenvalues $\gamma_m[\tau] \circ\!\!-\!\!\bullet\, \gamma_m(e^{j\Omega})$ of a 3×3 matrix:



3.2 Relation to Analytic EVD

- If the analytic eigenvalues do not intersect on the unit circle, then analytic EVD and polynomial EVD (with sufficiently high order) are 'identical';
- the polynomial EVD has a strict ordering of eigenvalues;
- specific polynomial/analytic eigenvector solution may differ — recall the allpass ambiguity;





- if analytic eigenvalues intersect, then the solutions of analytic EVD and polynomial EVD differ;
- we explore by way of an example ...

3.3 Numerical Example

- We pick our own eigenvalues (order 2) and eigenvectors (order 1):

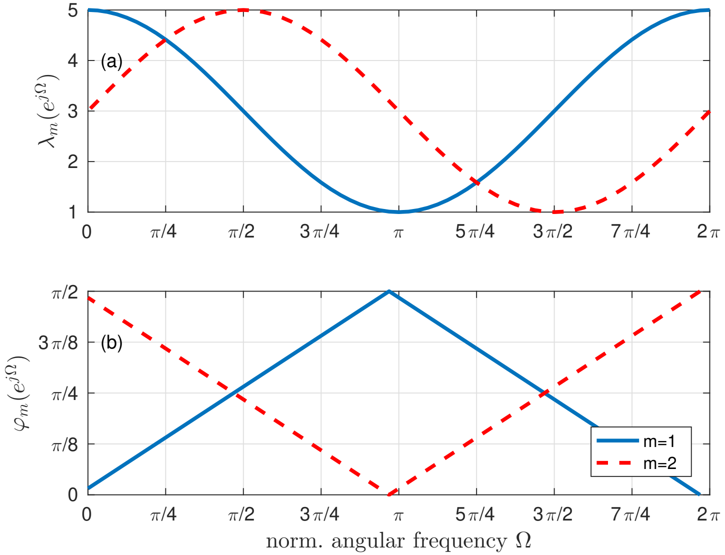
$$\boldsymbol{\Lambda}(z) = \begin{bmatrix} z + 3 + z^{-1} & \\ & -jz + 3 + jz^{-1} \end{bmatrix}$$

$$\boldsymbol{Q}(z) = [\boldsymbol{q}_1(z),\ \boldsymbol{q}_2(z)] \qquad \text{with} \qquad \boldsymbol{q}_{1,2}(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm z^{-1} \end{bmatrix} ;$$

- parahermitian matrix $\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z)$:

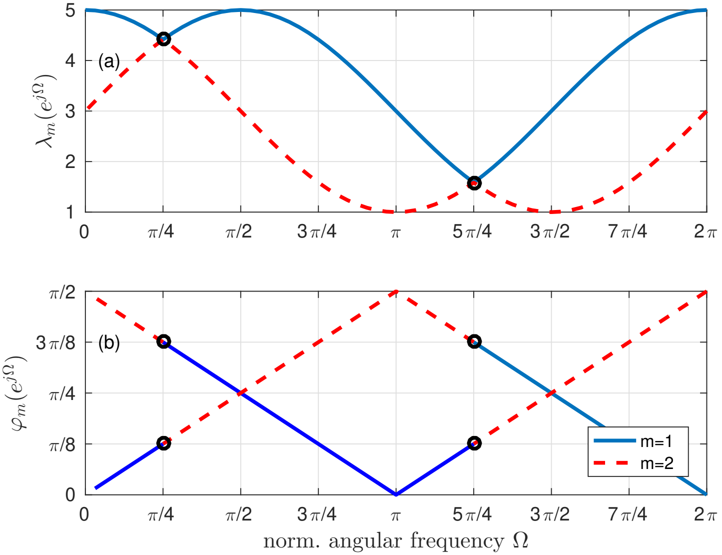
$$\boldsymbol{R}(z) = \begin{bmatrix} \frac{1-j}{2}z + 3 + \frac{1+j}{2}z^{-1} & \frac{1+j}{2}z^2 + \frac{1-j}{2} \\ \frac{1+j}{2} + \frac{1-j}{2}z^{-2} & \frac{1-j}{2}z + 3 + \frac{1+j}{2}z^{-1} \end{bmatrix} .$$

Numerical Example — Analytic Solution



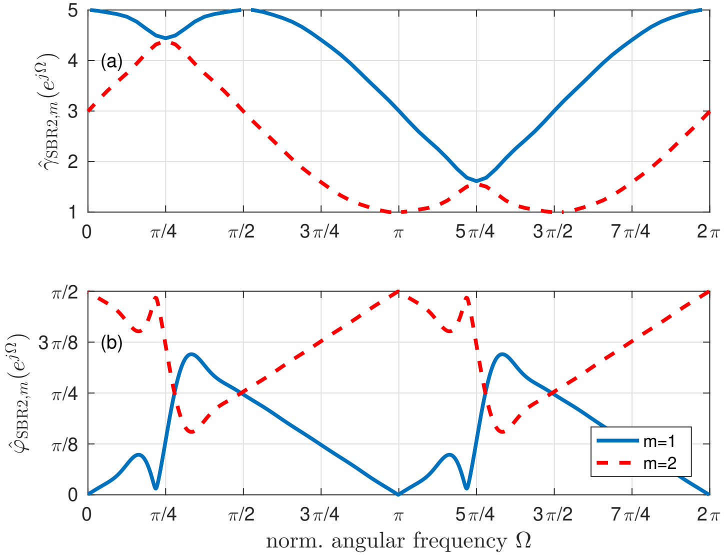
- Analytic (and therefore infinitely differentiable) eigenvalues $\lambda_m(\mathrm{e}^{\mathrm{j}\Omega})$;
- smooth Hermitian angles $\cos\varphi_m = |\boldsymbol{q}_1^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}0}) \cdot \boldsymbol{q}_m(\mathrm{e}^{\mathrm{j}\Omega})|$.

Numerical Example — Ideal Spectral Majorisation



- Analytic eigenvalues are permuted where they intersect;
- resulting spectrally majorised eigenvalues are piecewise analytic but not differentiable;
- corresponding eigenvectors are piecewise analytic but not continuous.

Numerical Example — PEVD Algorithmic Solution



- Using the SBR2 algorithm in [37] to approximate the McWhirter factorisation;
- trimming is applied to PEVD factors [13, 26, 14, 54];
- spectrally majorised eigenvalues $\mathbf{\Gamma}(z)$ of order 24;
- corresponding eigenvectors in $\boldsymbol{U}(z)$ of order 84.

4. Time Domain Algorithms

4.1 Iterative PEVD Approach

- Second order sequential best rotation (SBR2, McWhirter 2007, [37, 47, 49, 60]);
- iterative approach based on an elementary paraunitary operation:

$$\begin{aligned} \boldsymbol{S}^{(0)}(z) &= \boldsymbol{R}(z) \\ &\vdots \\ \boldsymbol{S}^{(i+1)}(z) &= \left\{\boldsymbol{H}^{(i+1)}(z)\right\}^{\mathrm{P}} \boldsymbol{S}^{(i)}(z)\boldsymbol{H}^{(i+1)}(z) \end{aligned}$$

- $\boldsymbol{H}^{(i)}(z)$ is an elementary paraunitary operation, which at the ith step eliminates the largest off-diagonal element in $\boldsymbol{s}^{(i-1)}(z)$;
- stop after I iterations:

$$\hat{\boldsymbol{\Gamma}}(z) = \boldsymbol{S}^{(I)}(z) \qquad , \qquad \hat{\boldsymbol{U}}(z) = \prod_{i=1}^{I} \boldsymbol{H}^{(i)}(z)$$

- sequential matrix diagonalisation (SMD) [10, 11, 8, 13, 12, 9, 19, 16, 15, 17, 48, 44] will follow the same scheme.

Elementary Paraunitary Operation

- An elementary paraunitary matrix [57] is defined as

$$\boldsymbol{H}^{(i)}(z) = \mathbf{I} - \mathbf{v}^{(i)}\mathbf{v}^{(i),\mathrm{H}} + z^{-1}\mathbf{v}^{(i)}\mathbf{v}^{(i),\mathrm{H}} \qquad , \quad \|\mathbf{v}^{(i)}\|_2 = 1$$

- we utilise a different definition:

$$\boldsymbol{H}^{(i)}(z) = \boldsymbol{D}^{(i)}(z)\mathbf{Q}^{(i)}$$

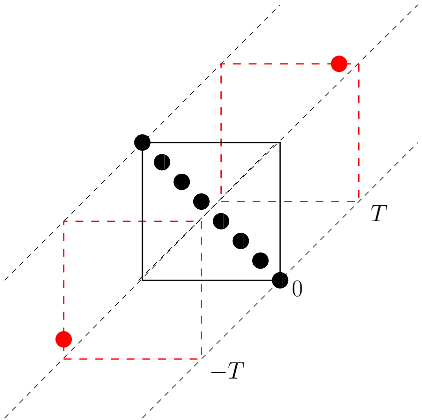
- $\boldsymbol{D}^{(i)}(z)$ is a delay matrix:

$$\boldsymbol{D}^{(i)}(z) = \text{diag}\{1 \ \ldots \ 1 \ z^{-\tau} \ 1 \ \ldots \ 1\}$$

- $\mathbf{Q}^{(i)}(z)$ is a Givens rotation.

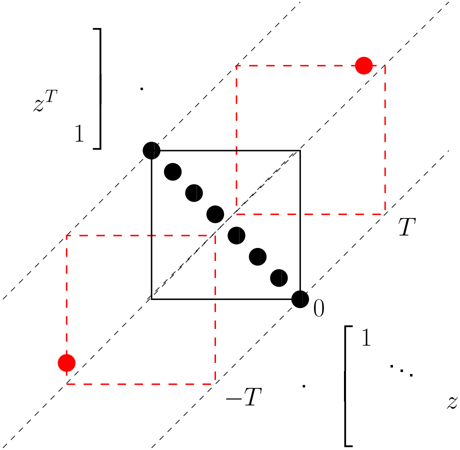
4.2 Sequential Best Rotation Algorithm (McWhirter [37])

- At iteration i, consider $\boldsymbol{S}^{(i-1)}(z) \circ\!\!-\!\!\bullet\, \mathbf{S}^{(i-1)}[\tau]$



4.2 Sequential Best Rotation Algorithm (McWhirter [37])

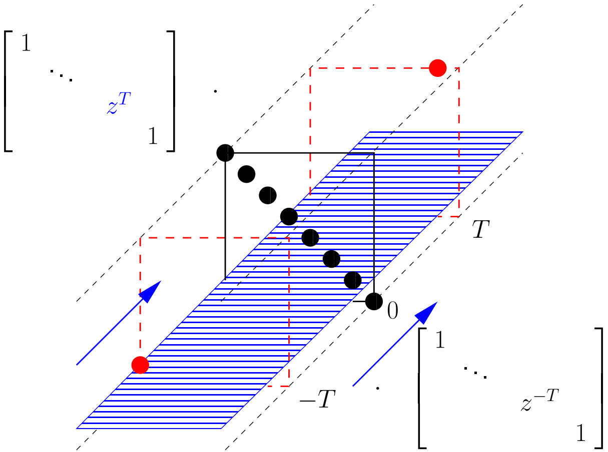
- $\tilde{\boldsymbol{D}}^{(i)}(z)\boldsymbol{S}^{(i-1)}(z)\boldsymbol{D}^{(i)}(z)$



$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & z^{T} & \\ & & & 1 \end{bmatrix} \cdot \qquad \cdot \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & z^{-T} & \\ & & & 1 \end{bmatrix}$$

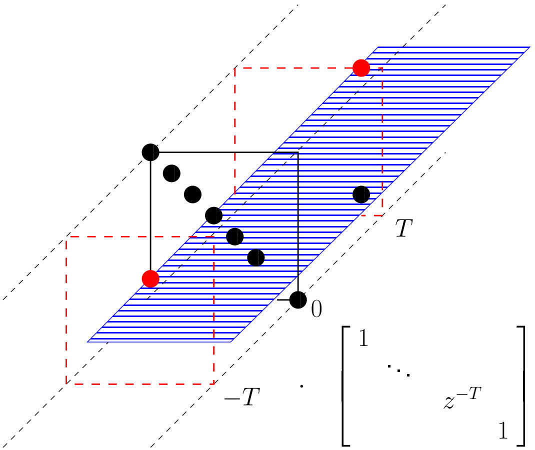
4.2 Sequential Best Rotation Algorithm (McWhirter [37])

- $\tilde{\boldsymbol{D}}^{(i)}(z)$ advances a row-slice of $\boldsymbol{S}^{(i-1)}(z)$ by T



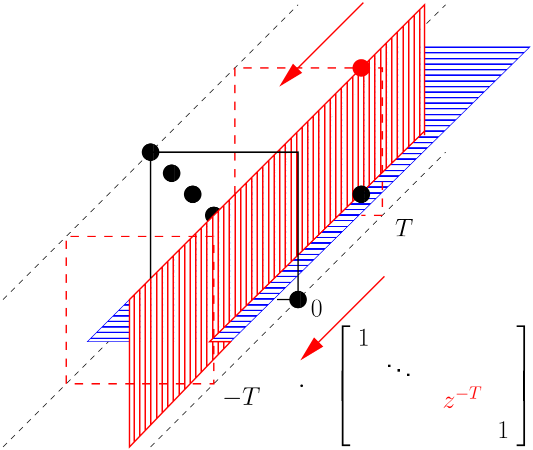
4.2 Sequential Best Rotation Algorithm (McWhirter [37])

- the off-diagonal element at $-T$ has now been translated to lag zero



$$\cdot \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & z^{-T} & \\ & & & 1 \end{bmatrix}$$

- $\mathbf{D}^{(i)}(z)$ delays a column-slice of $\boldsymbol{S}^{(i-1)}(z)$ by T



$$\cdot \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & z^{-T} & \\ & & & 1 \end{bmatrix}$$

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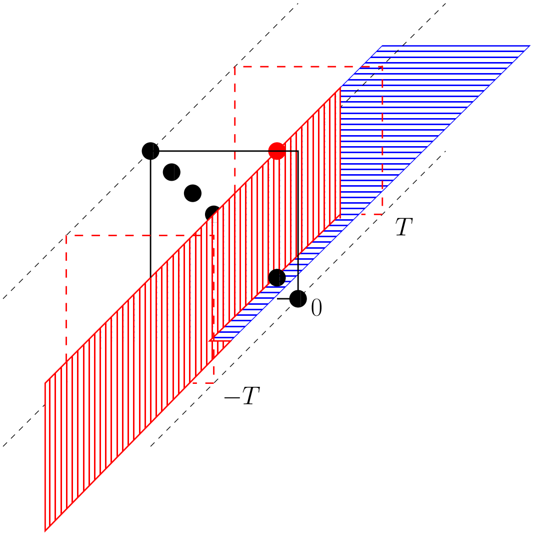
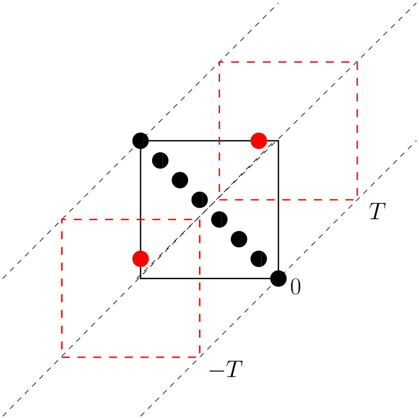


4.2 Sequential Best Rotation Algorithm (McWhirter [37])

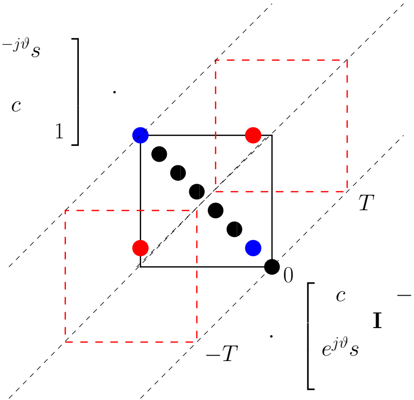
- the step $\tilde{\boldsymbol{D}}^{(i)}(z)\boldsymbol{S}^{(i-1)}(z)\boldsymbol{D}_{(i)}(z)$ has brought the largest off-diagonal elements to lag 0.



4.2 Sequential Best Rotation Algorithm (McWhirter [37])

- Jacobi step to eliminate largest off-diagonal elements by $\mathbf{Q}^{(i)}$

$$\begin{bmatrix} c & & e^{-j\vartheta}s & \\ & \mathbf{I} & & \\ -e^{j\vartheta}s & & c & \\ & & & 1 \end{bmatrix} \cdot$$

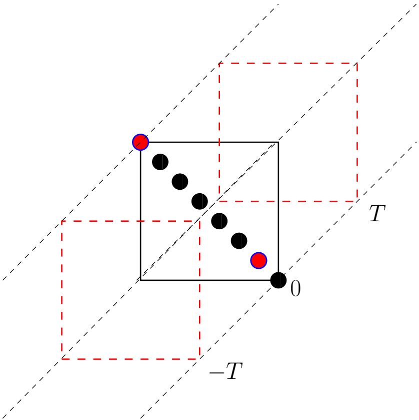


$$\cdot \begin{bmatrix} c & & -e^{-j\vartheta}s & \\ & \mathbf{I} & & \\ e^{j\vartheta}s & & c & \\ & & & 1 \end{bmatrix}$$

4.2 Sequential Best Rotation Algorithm (McWhirter [37])

- iteration i is completed, having performed

$$\boldsymbol{S}^{(i)}(z) = \mathbf{Q}^{(i)}\boldsymbol{D}^{(i)}(z)\boldsymbol{S}^{(i-1)}(z)\tilde{\boldsymbol{D}}^{(i)}(z)\tilde{\mathbf{Q}}^{(i)}(z)$$



SBR2 Outcome

- At the ith iteration, the zeroing of off-diagonal elements achieved during previous steps may be partially undone;
- the algorithm has proven convergence, transfering energy onto the main diagonal at every step (McWhirter 2007);
- after I iterations, we reach an approximate diagonalisation

$$\hat{\boldsymbol{\Gamma}}(z) = \boldsymbol{S}^{(L)}(z) = \hat{\boldsymbol{U}}^{\mathrm{P}}(z)\boldsymbol{R}(z)\hat{\boldsymbol{U}}(z)$$

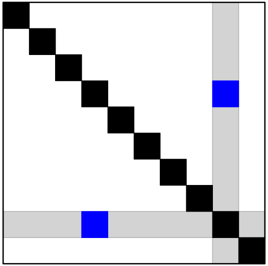
with

$$\hat{\boldsymbol{U}}(z) = \prod_{i=1}^{I} \boldsymbol{D}^{(i)}(z)\mathbf{Q}^{(i)}$$

- the factors may require trimming of trailing zeros or very small coefficients [26, 54, 13, 14, 32].

SBR2 — Givens Rotation

- A Givens rotation eliminates the maximum off-diagonal element once brought onto the lag-zero matrix;
- note I: in the lag-zero matrix, one column and one row are modified by the shift:



- note II: a Givens rotation only affects two columns and two rows in every matrix;
- Givens rotation is relatively low in computational cost!

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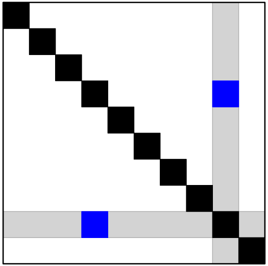
- Givens rotation is relatively low in computational cost!

4.3 Sequential Matrix Diagonalisation (SMD, [48])

- Main idea — the zero-lag matrix is diagonalised in every step;
- initialisation: diagonalise $\mathbf{R}[0]$ by EVD and apply modal matrix to all matrix coefficients $\longrightarrow \boldsymbol{S}^{(0)}$;
- at the ith step as in SBR2, the maximum element (or column with max. norm) is shifted to the lag-zero matrix:



- an EVD is used to re-diagonalise the zero-lag matrix;
- a full modal matrix is applied at all lags — more costly than SBR2.

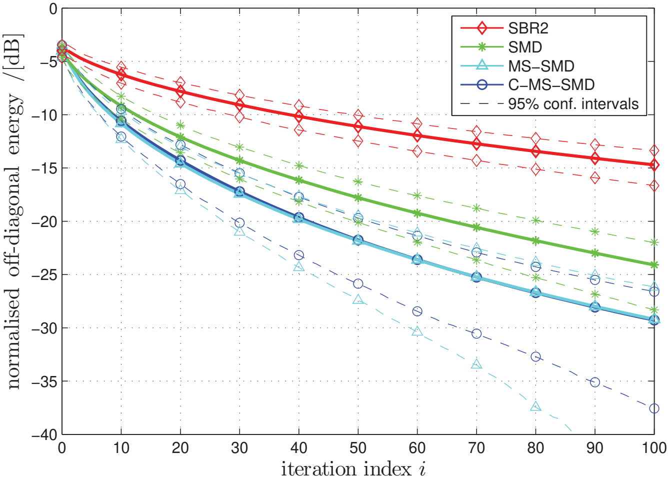
4.3 Sequential Matrix Diagonalisation (SMD, [48])

- Main idea — the zero-lag matrix is diagonalised in every step;
- initialisation: diagonalise $\mathbf{R}[0]$ by EVD and apply modal matrix to all matrix coefficients $\longrightarrow \boldsymbol{S}^{(0)}$;
- at the ith step as in SBR2, the maximum element (or column with max. norm) is shifted to the lag-zero matrix:

- an EVD is used to re-diagonalise the zero-lag matrix;
- a full modal matrix is applied at all lags — more costly than SBR2.

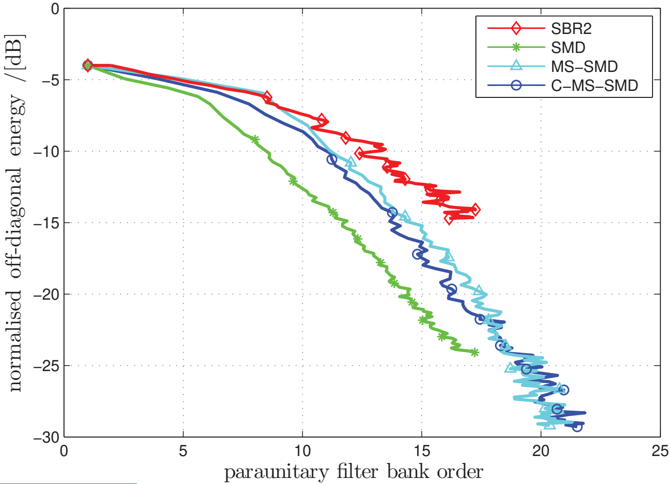
4.4 Comparison: SBR2/SMD Convergence

- Measuring the remaining normalised off-diagonal energy over an ensemble of space-time covariance matrices:



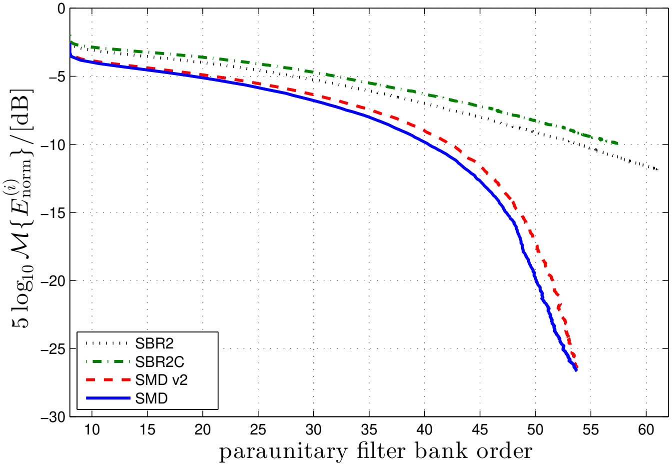
SBR2/SMD Application Cost 1

- Ensemble average of remaining off-diagonal energy vs. order of paraunitary filter banks to decompose 4x4 matrices of order 15:



SBR2/SMD Application Cost 2

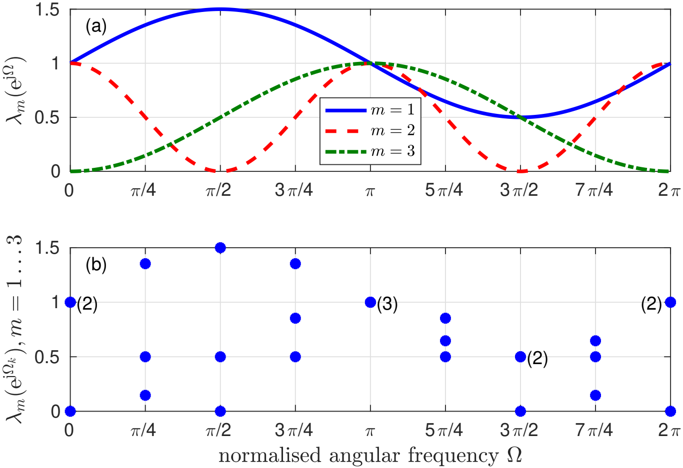
- Ensemble average of remaining off-diagonal energy vs. order of paraunitary filter banks to decompose 8x8 matrices of order 63:



5. DFT Domain Algorithms

5.1 Analytic Eigenvalue Extraction

- Idea for DFT-based algorithms: calculate an EVD in every DFT bin;



- spectral coherence must be re-established across bins;
- we exploit that the solution must be analytic, i.e. infinitely differentiable;
- we first extract eigenvalues, which are less volatile under perturbation [30];

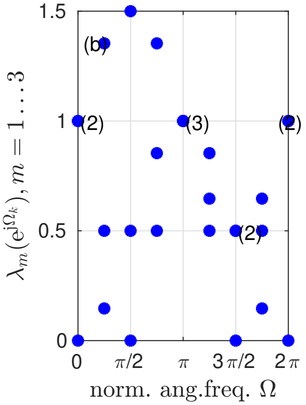
Analytic Eigenvalue Extraction Algorithm I

University of Strathclyde Engineering

- Bin-wise EVD yields:

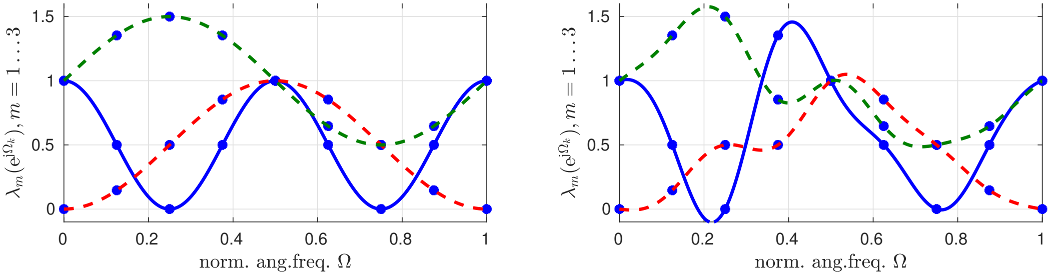
$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{Q}_k \boldsymbol{\Lambda}_k \mathbf{Q}_k^{\mathrm{H}} = \underbrace{\mathbf{Q}_k \boldsymbol{\Psi}_k \mathbf{P}_k}_{\boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}} \boldsymbol{\Lambda}_k \mathbf{P}_k}_{\boldsymbol{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}} \boldsymbol{\Psi}_k^{\mathrm{H}} \mathbf{Q}_k^{\mathrm{H}}}_{\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega_k})}$$

- $\mathbf{P}_k$ is a permutation matrix, since in the analytic EVD, eigenvalues can intersect and are not necessarily majorised;
- for distinct eigenvalues: $\boldsymbol{\Psi}_k$ is a diagonal matrix that accounts for the phase ambiguity of eigenvectors;
- in case of a C-fold algebraic multiplicity: $\boldsymbol{\Psi}_K$ is block diagonal, with a $C \times C$ unitary matrix accounting for eigenvectors forming an arbitrary basis within a C-dimensional subspace;
- a predecessor algorithm [55] can fail on this;



Analytic Eigenvalue Extraction Algorithm II

- To find the smoothest association of M functions across K frequency bins, we compare the power in the p-th derivative of a Dirichlet interpolation [65, 73, 76]:



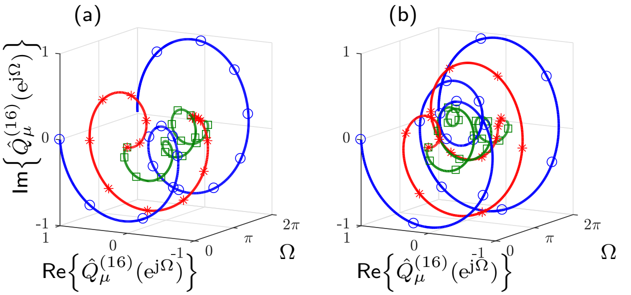
- for an exhaustive search, there would be $M!^{K-1}$ associations to check;
- a Viterbi-type scheme operates iteratively across bins [76], and only retains viable associations [71, 72];
- DFT length K can be increased until a criterion based on time-domain aliasing is met.

5.2 Analytic Eigenvector Extraction

- From the eigenvalue extraction, the correct association across frequency bins defines $\mathbf{P}_k$ [68]:

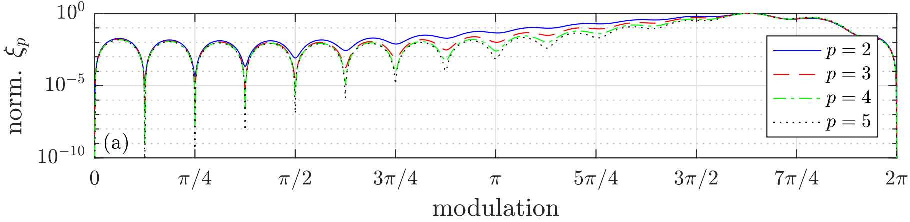
$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{Q}_k \boldsymbol{\Lambda}_k \mathbf{Q}_k^{\mathrm{H}} = \underbrace{\mathbf{Q}_k \boldsymbol{\Psi}_k \mathbf{P}_k}_{\boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}} \boldsymbol{\Lambda}_k \mathbf{P}_k}_{\boldsymbol{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega_k})} \cdot \underbrace{\mathbf{P}_k^{\mathrm{T}} \boldsymbol{\Psi}_k^{\mathrm{H}} \mathbf{Q}_k^{\mathrm{H}}}_{\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega_k})}$$

- for the eigenvalue extraction, it remains to find the correct phase adjustment $\boldsymbol{\Psi}_k$;
- again the smoothness of a Dirichlet interpolation can lead to the analytic solution;
- example: $M = 3$ components of one eigenvector with different $\boldsymbol{\Psi}_k$.



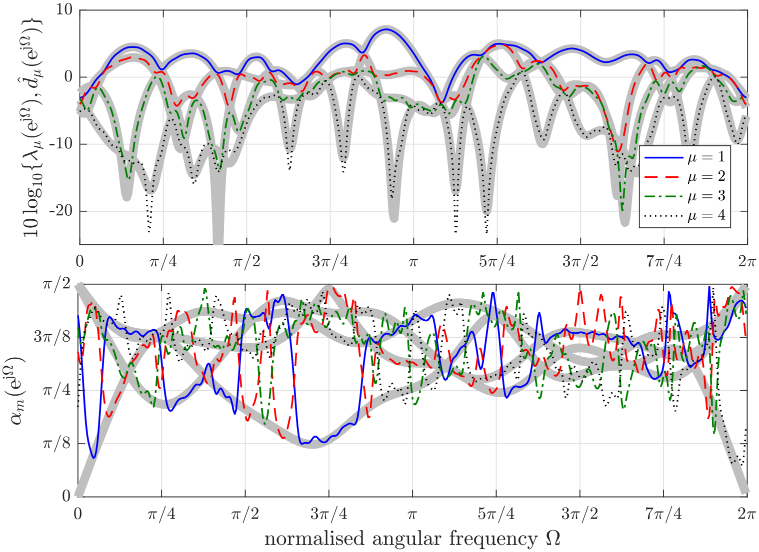
Analytic Eigenvector Extraction Algorithm

- The main task for the extraction of analytic eigenvectors is the adjustment of a smooth phase progression $\mathbf{\Psi}_k$ across bins;
- this problem is NP hard [51], but the specific smoothness cost function possesses — for a sufficient DFT size — stationary points that are approximately separated by a modulation [69];
- cut through cost function ξ_p for powers of different derivatives p:



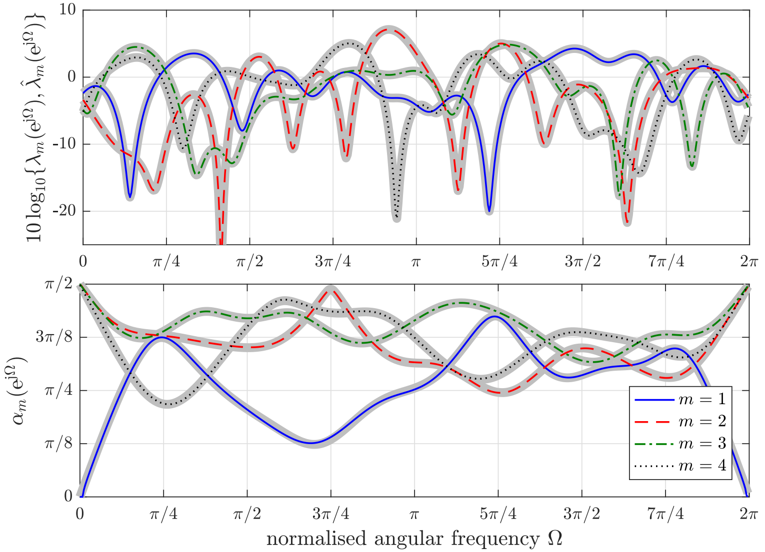
- an iterative algorithm is proven to converge [70], increasing the DFT length until a reconstruction error is minimised;

5.3 Comparison — SMD Algorithm Example



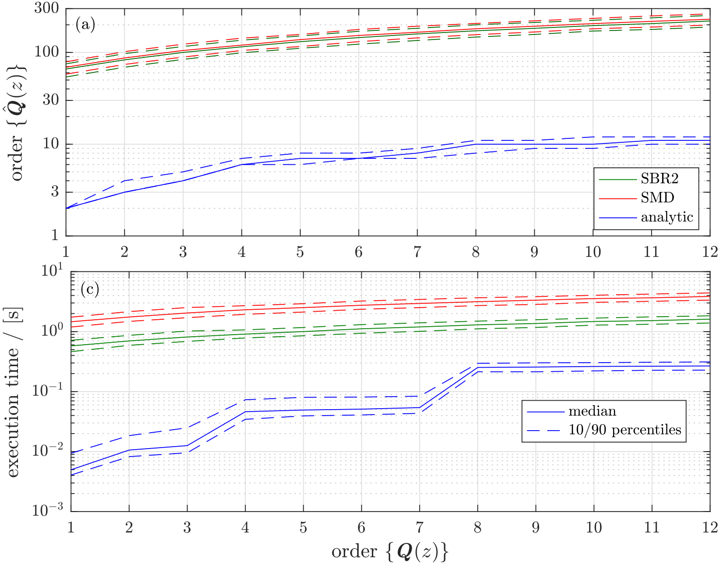
- $\boldsymbol{R}(z) : \mathbb{C} \to \mathbb{C}^{4\times4}$ of order 47;
- SMD algorithm [48] yields approximate spectral majorisation [38];
- Hermitian angles of eigenvectors to a reference vector indicate approximation of piecewise analytic functions.

Analytic EVD Extraction Example



- same matrix, but utilising analytic eigen-value [71] and -vector extraction [69];
- extracted analytic EVD factors are close to ground truth;
- lower order compared to SMD result.

Comparison — Ensemble Results



- ensemble results over matrices with different ground truth, and for various orders;
- above: application cost — the order of the extracted paraunitary matrices, required e.g. for a subspace projection;
- below: execution time of the algorithms.

6. Summary

- An analytic EVD exists in almost all cases — when data is not multiplexed;
- a polynomial EVD (PEVD) differs if eigenvalues intersect; it targets spectrally majorised eigenvalues;
- spectral majorisation in the EVD is desirable in coding or communications applications;
- the PEVD is supported by a number of well-established algorithms [20, 37, 47];
- analytic EVD algorithms are useful where low-order factors or low-perturbed subspace methods matter;
- if a space-time covariance matrix is estimated from limited data [22, 23, 24, 33], time- and DFT-domain algorithms target the same factors (within the allpass ambiguity of the eigenvectors) with probability one [34].
- algorithms find applications in coding [47, 74], angle of arrival estimation [2, 1, 28, 61], beamforming [3, 4, 42, 62], subspace detection [45, 46, 63], speech enhancement [43, 41], communications [39, 40, 53, 52] and others [64, 75];
- extension to other decompositions, such as e.g. SVD [25, 31, 36, 37, 70] or QRD [19, 25, 70].

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