Applications of Polynomial Eigenvalue Decomposition to Multichannel Broadband Signal Processing
Part 1: Background

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Part I: Background

2 Polynomial Matrics: What are they & where to the arise?

1. Overview
2. What are polynomial matrices & where do they occur?
   2.1 basic definition;
   2.2 common occurences:
      - MIMO communications: matrices of transfer functions
      - filter banks: polyphase analysis and synthesis matrices
      - array processing/statistics: space-time covariance matrices
3. Polynomial Matrix Basic Operations and Properties
4. 'Standard' Multichannel Broadband Processing
5. Polynomial matrix formulation of broadband problems
6. Summary

2.1 What is a Polynomial Matrix?

- A polynomial matrix is a polynomial with matrix-valued coefficients [14, 19], e.g.:

$$\boldsymbol{A}(z) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} z^{-1} + \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} z^{-2} \; ; \tag{1}$$

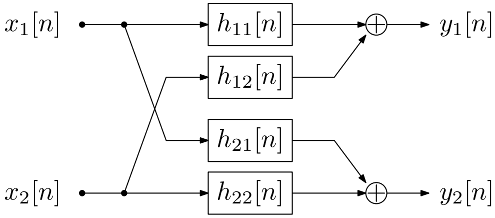
- a polynomial matrix can equivalently be understood a matrix with polynomial entries, i.e.

$$\boldsymbol{A}(z) = \begin{bmatrix} 1 + z^{-1} - z^{-2} & -1 + z^{-1} + 2z^{-2} \\ -1 + z^{-1} + z^{-2} & 2 - z^{-1} - z^{-2} \end{bmatrix} \; ; \tag{2}$$

- more generally, we will be looking at matrices of (analytic) functions.

Where Do Polynomial Matrices Arise?

- A multiple-input multiple-output (MIMO) system could be made up of a number of finite impulse response (FIR) channels:

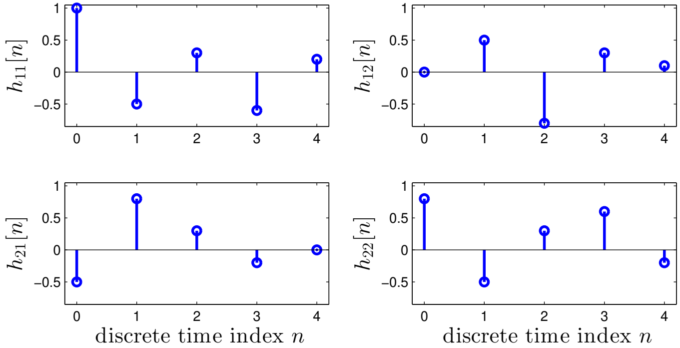


- writing this as a matrix of impulse responses:

$$\mathbf{H}[n] = \begin{bmatrix} h_{11}[n] & h_{12}[n] \\ h_{21}[n] & h_{22}[n] \end{bmatrix} . \tag{3}$$

Transfer Function of a MIMO System

- Example for MIMO matrix $\mathbf{H}[n]$ of impulse responses:

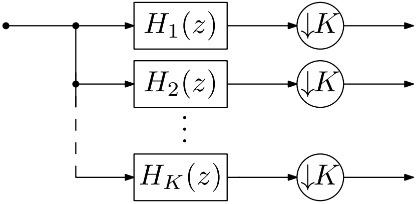


- the transfer function of this MIMO system is a polynomial matrix [21, 31]:

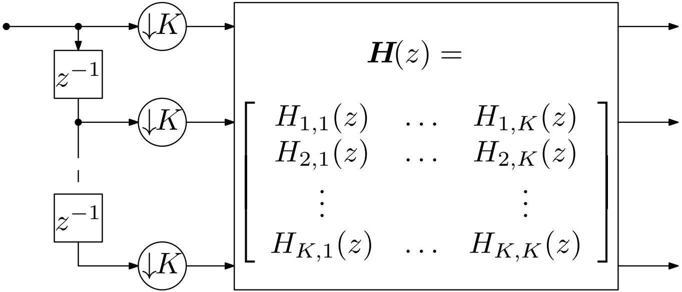
$$\boldsymbol{H}(z) = \sum_{n=-\infty}^{\infty} \mathbf{H}[n] z^{-1} \qquad \text{or} \qquad \boldsymbol{H}(z) \bullet\!\!-\!\!\circ\, \mathbf{H}[n] \tag{4}$$

Analysis Filter Bank

- Critically decimated K-channel analysis filter bank [32, 33, 13]:



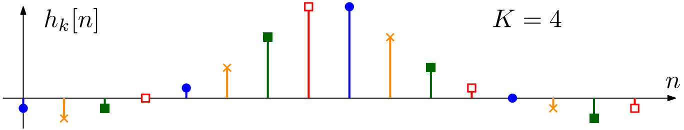
- equivalent polyphase representation:



Polyphase Analysis Matrix

- With the K-fold polyphase decomposition of the analysis filters

$$H_k(z) = \sum_{n=1}^{K} H_{k,n}(z^K) z^{-n+1} \tag{5}$$

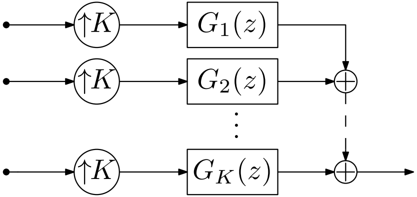


- the polyphase analysis matrix is a polynomial matrix [33, 35]:

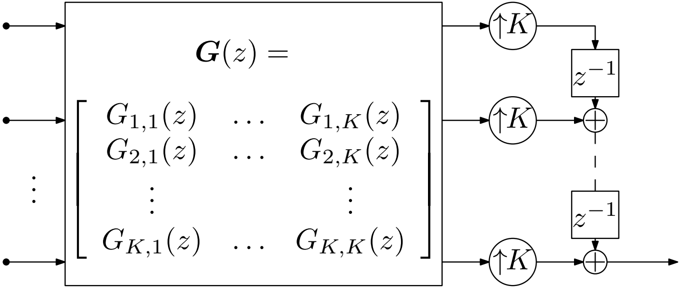
$$\boldsymbol{H}(z) = \begin{bmatrix} H_{1,1}(z) & H_{1,2}(z) & \dots & H_{1,K}(z) \\ H_{2,1}(z) & H_{2,2}(z) & \dots & H_{2,K}(z) \\ \vdots & \vdots & \ddots & \vdots \\ H_{K,1}(z) & H_{K,2}(z) & \dots & H_{K,K}(z) \end{bmatrix} \tag{6}$$

Synthesis Filter Bank

- Critically decimated K-channel synthesis filter bank:



- equivalent polyphase representation [11, 13, 33]:



Polyphase Synthesis Matrix

- Analoguous to analysis filter bank, the synthesis filters $G_k(z)$ can be split into K polyphase components, creating a polyphse synthesis matrix

$$\boldsymbol{G}(z) = \begin{bmatrix} G_{1,1}(z) & G_{1,2}(z) & \dots & G_{1,K}(z) \\ G_{2,1}(z) & G_{2,2}(z) & \dots & G_{2,K}(z) \\ \vdots & \vdots & \ddots & \vdots \\ G_{K,1}(z) & G_{K,2}(z) & \dots & G_{K,K}(z) \end{bmatrix} ; \quad (7)$$

- operating analysis and synthesis back-to-back, perfect reconstruction is achieved if

$$\boldsymbol{G}(z)\boldsymbol{H}(z) = \mathbf{I} ; \quad (8)$$

- i.e. for perfect reconstruction, the polyphase analysis matrix must be invertible:

$$\boldsymbol{G}(z) = \boldsymbol{H}^{-1}(z) . \quad (9)$$

Space-Time Covariance Matrix

- Measurements obtained from M sensors are collected in a vector $\mathbf{x}[n] \in \mathbb{C}^M$:

$$\mathbf{x}^{\mathrm{T}}[n] = [x_1[n] \;\; x_2[n] \;\; \ldots \;\; x_M[n]] \; ; \tag{10}$$

- with the expectation operator $\mathcal{E}\{\cdot\}$, the spatial correlation is captured by $\mathbf{R} = \mathcal{E}\left\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n]\right\}$;
- for spatial and temporal correlation, we require a space-time covariance matrix [20, 23, 33, 39, 40, 45, 42]:

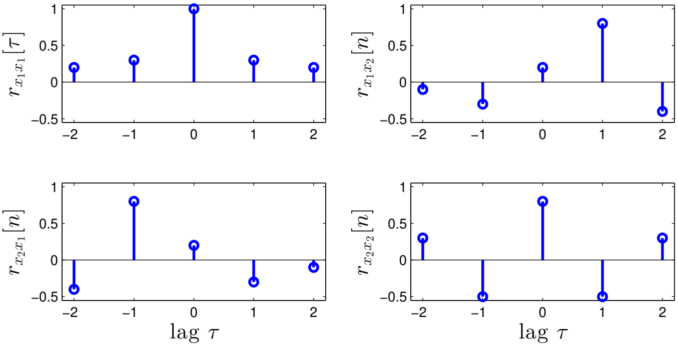
$$\mathbf{R}[\tau] = \mathcal{E}\left\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\right\} \; ; \tag{11}$$

- this space-time covariance matrix contains auto- and cross-correlation terms, e.g. for $M = 2$

$$\mathbf{R}[\tau] = \left[\begin{array}{cc} \mathcal{E}\{x_1[n]x_1^*[n-\tau]\} & \mathcal{E}\{x_1[n]x_2^*[n-\tau]\} \\ \mathcal{E}\{x_2[n]x_1^*[n-\tau]\} & \mathcal{E}\{x_2[n]x_2^*[n-\tau]\} \end{array} \right] . \tag{12}$$

Cross-Spectral Density Matrix

- example for a space-time covariance matrix $\mathbf{R}[\tau] \in \mathbb{R}^{2\times 2}$:



- the cross-spectral density (CSD) matrix contains (Laurent) polynomials:

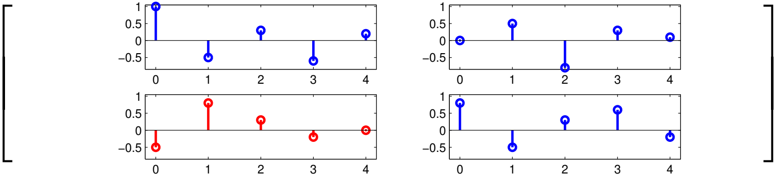
$$\boldsymbol{R}(z) = \sum_{\tau} \mathbf{R}[\tau] z^{-\tau} \qquad \text{or short} \qquad \boldsymbol{R}(z) \bullet\!\!-\!\!\circ\, \mathbf{R}[\tau] \, . \qquad (13)$$

3 Polynomial Matrix Basic Operations and Properties

3.1 Parahermitian Operator

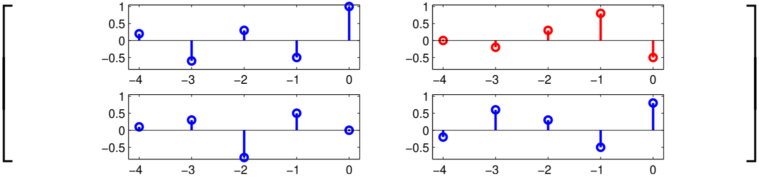
- A parahermitian operation is indicated by $\{\cdot\}^{\mathrm{P}}$, and compared to the Hermitian transposition of a matrix additionally performs a time-reversal;
- example:

$$\boldsymbol{A}(z) =$$



- parahermitian $\boldsymbol{A}^{\mathrm{P}}(z) = \{\boldsymbol{A}(1/z^*)\}^{\mathrm{H}}$:

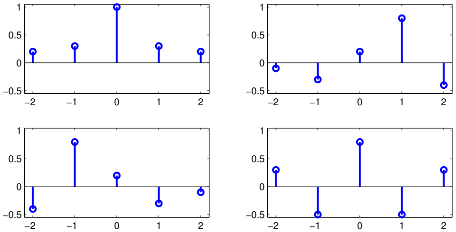
$$\boldsymbol{A}^{\mathrm{P}}(z) =$$



3.2 Parahermitian Property

- A polynomial matrix $\boldsymbol{R}(z)$ is parahermitian if $\boldsymbol{R}^{\mathrm{P}}(z) = \boldsymbol{R}^{\mathrm{H}}(1/z^*) = \boldsymbol{R}(z)$;
- this is an extension of the symmetric (if $\mathbf{R} \in \mathbb{R}$) or or Hermitian (if $\mathbf{R} \in \mathbb{C}$) property to the polynomial case:
  transposition, complex conjugation and time reversal (in any order) do not alter a parahermitian $\boldsymbol{R}(z)$;
- any CSD matrix is parahermitian;
- example:

$$\boldsymbol{R}(z) = \left[\quad \right] = \boldsymbol{R}^{\mathrm{P}}(z) \quad .$$



Paraunitary Matrices

- Recall that $\mathbf{A} \in \mathbb{C}$ (or $\mathbf{A} \in \mathbb{R}$) is a unitary (or orthonormal) matrix if $\mathbf{A}\mathbf{A}^{\mathrm{H}} = \mathbf{A}^{\mathrm{H}}\mathbf{A} = \mathbf{I}$;
- in the polynomial case, $\mathbf{A}(z)$ is paraunitary if

$$\boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z) = \boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{A}(z) = \mathbf{I} \; ; \tag{14}$$

- therefore, if $\boldsymbol{A}(z)$ is paraunitary, then the polynomial matrix inverse is simple:

$$\boldsymbol{A}^{-1}(z) = \boldsymbol{A}^{\mathrm{P}}(z) \; ; \tag{15}$$

- example: polyphase analysis or synthesis matrices of perfectly reconstructing (or lossless) filter banks are usually paraunitary.

3.3 Matrix-Valued Polynomials and Power Series

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- A power series $a(z)$ arises as the z-transform

$$a(z) = \sum_n a[n] z^{-n} \qquad \text{or} \quad \text{short} \qquad a(z) \bullet\!\!-\!\!\circ\, a[n] \,, \tag{16}$$

- for $a(z)$ to exist as a power series, $a[n]$ must be
  causal: $a[n] = 0 \; \forall n < 0$;
  absolutely convergent: $\sum_n |a[n]| < \infty$
- absolute convergence implies that $a[n]$ decays at least as fast as an exponential function;
- a polynomial is a power series, but of finite length;
- polynomials or power series can form the entries of a matrix $\boldsymbol{A}(z)$.

Example of a Power Series

- For the geometric series

$$a[n] = \begin{cases} 0, & n < 0 \\ (\frac{1}{2})^n, & n \geq 0 \end{cases} \quad (17)$$

we have

$$\sum_n |a[n]| = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \quad = 2 < \infty \; ; \quad (18)$$

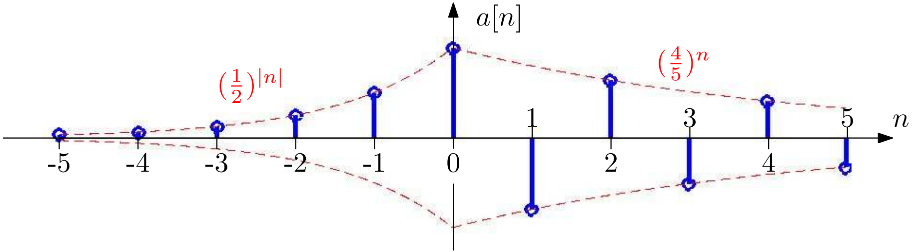
- therefore $a[n]$ is an absolutely convergent power series, and $a(z)$ exists as an analytic function;
- here, for $a(z)$:

$$a(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \dots \quad = \frac{1}{1 - \frac{1}{2}z^{-1}} \; ; \quad (19)$$

- this looks like the transfer function of a causal infinite impulse response (IIR) filter.

Laurent Series and Laurent Polynomials

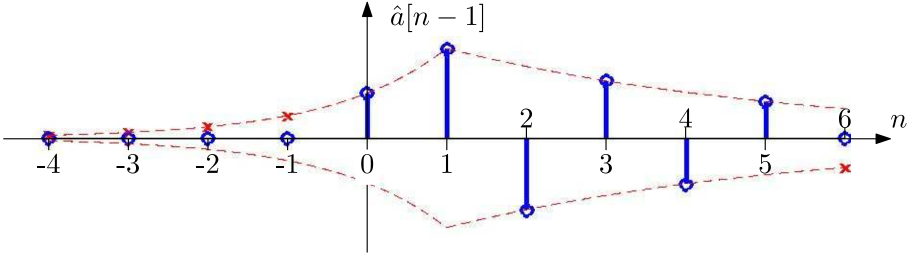
- A Laurent series $a[n]$ is potentially infinite, but can include non-negative terms for both $n \geq 0$ and $n < 0$;
- for $a(z) \bullet\!\!-\!\!\circ\, a[n]$ to exist, $a[n]$ needs to decay at least exponentially in both positive and negative time direction [1];



- if it possesses finite support, $a(z)$ is a Laurent polynomial.

Analyticity and Polynomial Approximation

- Absolute convergence of $a[n]$ implies analyticity of $a(z) \bullet\!\!-\!\!\circ\, a[n]$;
- the best approximation of an infinite order, analytic $a(z)$ in the least squares sense is by truncation (power series $\longrightarrow$ polynomial) [9, 10, 42];
- likewise, a Laurent series can be approximated by a polynomial through truncation ($\longrightarrow$ Laurent polynomial) and an appropriate delay ($\longrightarrow$ polymomial) [45];



- hence polynomials can typically approximate any general analytic function well, and arbitrarily closely.

Arithmetic Operations — Attempt of Gaussian Elimination

- System of polynomial equations:

$$\begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{bmatrix} \cdot \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} B_1(z) \\ B_2(z) \end{bmatrix} \quad (20)$$

- modification of 2nd row (provided no division by spectral zeros):

$$\begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{11}(z) & \frac{A_{11}(z)}{A_{21}(z)} A_{22}(z) \end{bmatrix} \cdot \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} B_1(z) \\ \frac{A_{11}(z)}{A_{21}(z)} B_2(z) \end{bmatrix} \quad (21)$$

- upper triangular form by subtracting 1st row from 2nd:

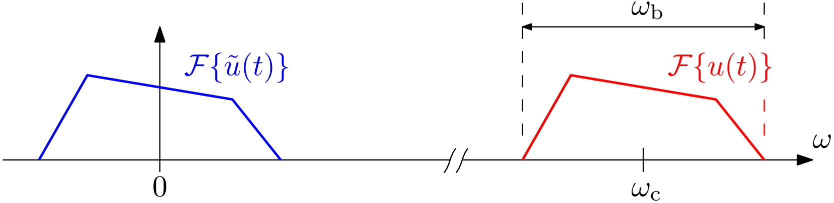
$$\begin{bmatrix} A_{11}(z) & A_{12}(z) \\ 0 & \frac{A_{11}(z)A_{22}(z) - A_{12}(z)A_{21}(z)}{A_{21}(z)} \end{bmatrix} \cdot \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} B_1(z) \\ \bar{B}_2(z) \end{bmatrix} \quad (22)$$

- we end up with rational functions; through delay and truncation, these can be arbitrarily closely approximated by polynomials.

4. 'Standard' Multichannel Broadband Processing

4.1 Narrowband vs Broadband

- Assume as source a bandpass signals $u(t)$ of finite bandwidth ω_b and with centre frequency ω_c:



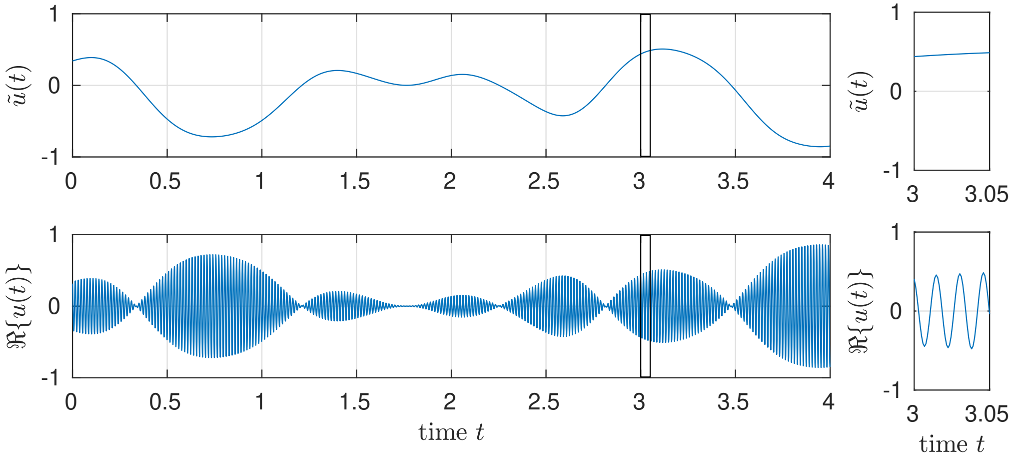
- using a baseband representation

$$u(t) = \tilde{u}(t) \cdot \mathrm{e}^{\mathrm{j}\omega_c t} \, , \tag{23}$$

with $\tilde{u}(t)$ the baseband signal.

Narrowband Assumption

- Narrowband: propagation delay across the array must be small w.r.t. any changes in the baseband signal $\tilde{u}(t)$ (or of the envelope of $u(t)$);



Received Narrowband Array Signal

- An array receives a single modulated bandpass signal $u(t)$:

$$\mathbf{x}(t) = \begin{bmatrix} u(t-\tau_1) \\ \vdots \\ u(t-\tau_M) \end{bmatrix} = \begin{bmatrix} \tilde{u}(t-\tau_1) \\ \vdots \\ \tilde{u}(t-\tau_M) \end{bmatrix} \cdot \begin{bmatrix} \mathrm{e}^{\mathrm{j}\omega_\mathrm{c}(t-\tau_1)} \\ \vdots \\ \mathrm{e}^{\mathrm{j}\omega_\mathrm{c}(t-\tau_M)} \end{bmatrix}$$

$$\approx \tilde{u}(t-\tau_1)\mathrm{e}^{\mathrm{j}\omega_\mathrm{c}t} \begin{bmatrix} \mathrm{e}^{-\mathrm{j}\omega_\mathrm{c}\tau_1} \\ \vdots \\ \mathrm{e}^{-\mathrm{j}\omega_\mathrm{c}\tau_M} \end{bmatrix} = \tilde{u}(t-\tau_1)\mathrm{e}^{\mathrm{j}\omega_\mathrm{c}t}\mathbf{s}_{\vartheta,\omega_\mathrm{c}}$$

- after sampling: $\mathbf{x}[n] = \tilde{u}[n] \cdot \mathrm{e}^{\mathrm{j}\omega_\mathrm{c}\tau_1} \cdot \mathbf{s}_{\vartheta_\ell,\Omega_\mathrm{c}}$;
- for the covariance matrix:

$$\mathbf{R} = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^\mathrm{H}[n]\} = \mathcal{E}\{\tilde{u}[n]\tilde{u}^*[n]\}\,\mathbf{s}_{\vartheta,\Omega_\mathrm{c}}\mathbf{s}^\mathrm{H}_{\vartheta_\ell,\Omega_\mathrm{c}} = \sigma^2\mathbf{s}_{\vartheta,\Omega_\mathrm{c}}\mathbf{s}^\mathrm{H}_{\vartheta,\Omega_\mathrm{c}}.$$

Narrowband Assumption — Limits

- For L independent source signals, $\mathcal{E}\{\tilde{u}_\ell[n]\tilde{u}_k^*[n]\} = 0$ for $\ell \neq k$; therefore in the noise-free case:

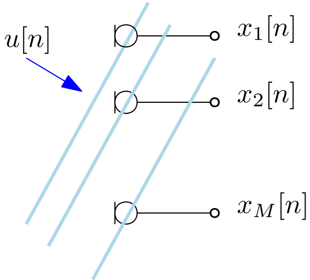
$$\mathbf{R} = \sum_{\ell=1}^{L} \sigma_\ell^2 \mathbf{s}_{\vartheta_\ell,\Omega_c} \mathbf{s}_{\vartheta_\ell,\Omega_c}^{\mathrm{H}} \; ; \tag{24}$$

- this matrix has rank L as long as the steering vectors $\mathbf{s}_{\vartheta_\ell,\Omega_c}$ are linearly independent;
- when is the narrowband assumption violated?
  - rank$\{\mathbf{R}\} > L$ [50];
  - signals at opposite ends of the array are no longer fully correlated [7];
  - can be tied to performance of processing [8, 24, 25, 36];
  - rule of thumb: fractional bandwidth ω_b/ω_c exceeds 5%;
  - "it's broadband when you need a tap delay line" (John McWhirter).

Broadband Case

- In the broadband case, the signal $u[n]$ experiences propagation delays, which need to be explicitly accounted:

$$\mathbf{x}[n] = u[n] * \begin{bmatrix} f_{\tau_1}[n] \\ \vdots \\ f_{\tau_M}[n] \end{bmatrix} = u[n] * \mathbf{a}[n] \qquad (25)$$

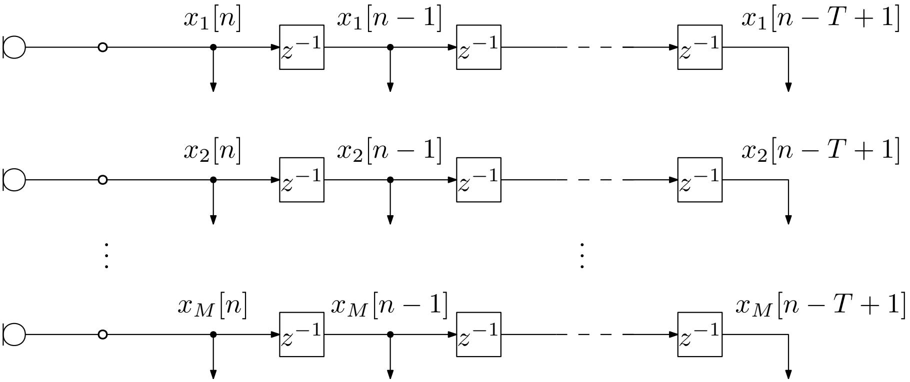


- $\mathbf{a}[n]$ is a broadband steering vector [2, 3, 23, 38];
- e.g. coherent combining requires explicit (fractional) delay filters; phase shifts are insufficient;
- broadband nature requires a space-time covariance matrix

$$\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\} = r_{uu}[\tau] * \mathbf{a}[\tau] * \mathbf{a}^{\mathrm{H}}[-\tau] \, . \qquad (26)$$

4.2 Tap Delay Line Processing

- A window of T samples can be used for filtering each channel [6, 34, 23]:



- e.g. implementation of fractional delay filters for coherent signal alignment [18, 27].

Tap Delay Line and Covariance Matrix

- Spatio-temporal data vector:

$$\boldsymbol{\chi}[n] = \left[\mathbf{x}^{\mathrm{T}}[n],\ \ \mathbf{x}^{\mathrm{T}}[n-1],\ \ldots\ \mathbf{x}^{\mathrm{T}}[n-T+1]\right]^{\mathrm{T}} \ ; \tag{27}$$

- the associated covariance matrix

$$\mathbf{R}_{\boldsymbol{\chi}} = \mathcal{E}\{\boldsymbol{\chi}[n]\boldsymbol{\chi}^{\mathrm{H}}[n]\} = \begin{bmatrix} \mathbf{R}[0] & \mathbf{R}[1] & \ldots & \mathbf{R}[T-1] \\ \mathbf{R}[-1] & \mathbf{R}[0] & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{R}[1-T] & \ldots & \ldots & \mathbf{R}[0] \end{bmatrix} \tag{28}$$

contains samples of the space-time covariance $\mathbf{R}[\tau]$.

TDL Processing and Challenges

- The selection of T should exceed the coherence time;
- otherwise signal correlations are missed, and the 2nd order statistics may be insufficiently characterised [23];
- with $\mathbf{R}_{\chi}$, we have mixed time and spatial domains;
- eigenvalue decomposition, with partitioning into signal-plus-noise and noise-only subspaces:

$$\mathbf{R}_{\chi} = [\mathbf{Q}_{\mathrm{s}} \; \mathbf{Q}_{\mathrm{n}}] \begin{bmatrix} \mathbf{\Lambda}_{\mathrm{s}} & \\ & \mathbf{\Lambda}_{\mathrm{n}} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_{\mathrm{s}}^{\mathrm{H}} \\ \mathbf{Q}_{\mathrm{n}}^{\mathrm{H}} \end{bmatrix} \tag{29}$$

- the number of dominant eigenvalues cannot be used for source enumeration;
- in case of a single source, $\mathbf{Q}_{\mathrm{s}}$ does not represent a steering vector of that source [23].

4.3 Processing in DFT Bins

- DFT matrix $\mathbf{W}$ applied to each tap delay line yields

$$\boldsymbol{\xi}[n] = (\mathbf{W} \odot \mathbf{I}_T)\boldsymbol{\chi}[n] \; ; \tag{30}$$

- covariance matrix:

$$\mathbf{R}_{\boldsymbol{\xi}} = (\mathbf{W} \odot \mathbf{I}_T)\mathbf{R}_{\boldsymbol{\chi}}(\mathbf{W} \odot \mathbf{I}_T)^{\mathrm{H}} \; ; \tag{31}$$

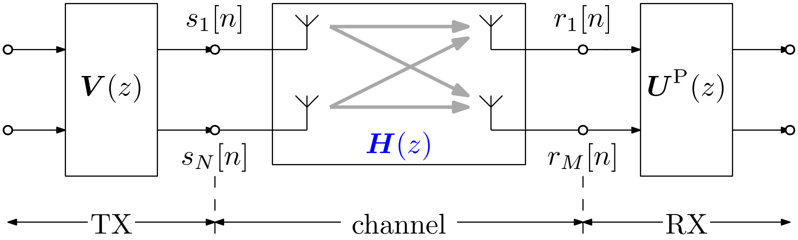
- this matrix is dense; regardless, often only a block-diagonal component is considered, and processing is performed independently in frequency-bins;
- independent DFT-bin processing is inexpensive, but ignores spectral coherence [43] and is suboptimal;
- cross-terms between bins can be introduced to achieve time-domain optimality [17, 28, 12, 49, 47, 48] but increase computational cost.

5. Polynomial Matrix Formulation of Broadband Problems

1. Overview
2. What are polynomial matrices & where do they occur?
3. Basic properties and operations
4. 'Standard' Multichannel Broadband Processing
5. Polynomial Matrix Formulation of Broadband Problems
   - 5.1 MIMO system decoupling
   - 5.2 broadband steering vector
   - 5.3 linearly constrained minimum variance beamforming
6. Summary

5.1 MIMO System Decoupling

- Aim: spatially decouple a channel by appropriate precoding and equalisation;



- narrowband case — SVD [15]:

$$\mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{H}} \; ;$$

- spatial decoupling leads to optimality in various senses [37];

- broadband case [5, 44]:

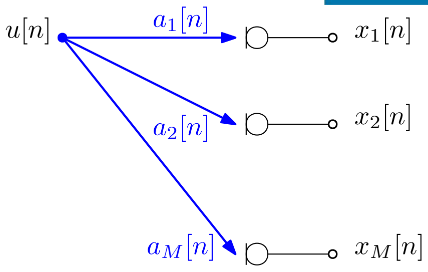
$$\boldsymbol{H}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\boldsymbol{V}^{\mathrm{P}}(z) \; ;$$

- diagonalisation for all values of z (or all values on the unit circle) [20, 22, 31, 30].

5.2 Broadband Steering Vector

- Assume an array of M sensors, and a single source $u[n]$:

$$\mathbf{x}[n] = \begin{bmatrix} a_1[n] \\ \vdots \\ a_M[n] \end{bmatrix} * u[n] \; ;$$

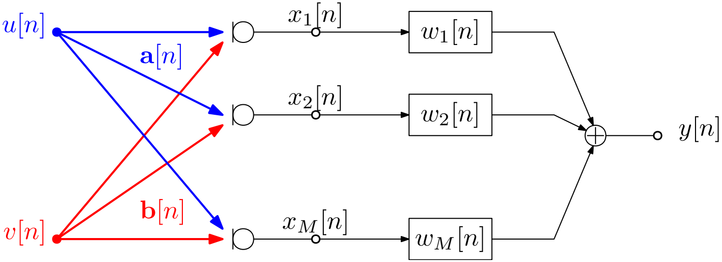


- $\mathbf{a}[n]$ $\bullet\!\!-\!\!\circ$ $\boldsymbol{a}(z)$ is a broadband steering vector;
- it can contain fractional delay filters [18] or general transfer functions;
- set of filters operating on the array signals:

$$\boldsymbol{w}^{\mathrm{P}}(z) = [w_1(z),\ w_2(z),\ \ldots\ w_M(z)]\ . \tag{32}$$

Simplistic Beamforming

- Filtering to coherently combine $u[n]$ and to suppress $v[n]$:



- we want $\boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{a}(z) = 1$ and $\boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{b}(z) = 0$;
- narrowband case:

$$\mathbf{w} = \begin{bmatrix} \mathbf{a}^{\mathrm{H}} \\ \mathbf{b}^{\mathrm{H}} \end{bmatrix}^{\dagger} \begin{bmatrix} 1 \\ 0 \end{bmatrix} ;$$

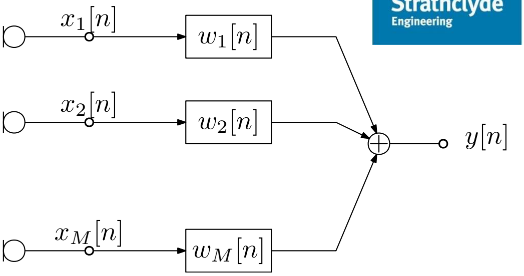
- broadband case:

$$\boldsymbol{w}(z) = \begin{bmatrix} \mathbf{a}^{\mathrm{P}}(z) \\ \mathbf{b}^{\mathrm{P}}(z) \end{bmatrix}^{\dagger} \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

5.3 Linearly Constrained Minimum Variance Beamforming

- To address unknown interferers, we want to minimize the output power subject to constraints (e.g. in look direction):

$$\begin{bmatrix} \mathbf{a}^{\mathrm{P}}(z) \\ \mathbf{b}^{\mathrm{P}}(z) \end{bmatrix} \boldsymbol{w}(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ;$$



- narrowband case [16]:

$$\min_{\mathbf{w}} \mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w} \qquad \text{s.t.} \;\; \mathbf{C}\mathbf{w} = \mathbf{f} \; ;$$

- broadband case [38]:

$$\min_{\boldsymbol{w}(z)} \oint_{|z|=1} \boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{R}(z)\boldsymbol{w}(z)\frac{\mathrm{d}z}{z}$$

$$\text{s.t.} \;\; \boldsymbol{C}(z)\boldsymbol{w}(z) = \boldsymbol{f}(z) \; .$$

LCMV Solution

- Narrowband formulation [16]:

$$\min_{\mathbf{w}} \mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w} \qquad \text{s.t.} \;\; \mathbf{C}\mathbf{w} = \mathbf{f} \; ;$$

- narrowband solution:
  Capon beamformer [29]:

$$\mathbf{w}_{\mathsf{opt}} = \mathbf{R}^{-1}\mathbf{C}^{\mathrm{H}}\{\mathbf{C}\mathbf{R}^{-1}\mathbf{C}^{\mathrm{H}}\}^{-1}\mathbf{f} \; ;$$

- broadband formulation [38]:

$$\min_{\boldsymbol{w}(z)} \oint_{|z|=1} \boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{R}(z)\boldsymbol{w}(z)\frac{\mathrm{d}z}{z}$$
$$\text{s.t.} \;\; \boldsymbol{C}(z)\boldsymbol{w}(z) = \boldsymbol{f}(z) \; .$$

- broadband solution:
  Capon equivalent [38, 4]:

$$\boldsymbol{w}_{\mathsf{opt}}(z) = \boldsymbol{R}^{-1}(z)\boldsymbol{C}^{\mathrm{P}}(z) \cdot$$
$$\{\boldsymbol{C}(z)\boldsymbol{R}^{-1}(z)\mathbf{C}^{\mathrm{P}}(z)\}^{-1}\boldsymbol{f}(z) \; .$$

6. Summary

- "Polynomial matrices" is simplistic for what potentially are Laurent series; absolute convergence implies analyticity, and arbitrarily close approximations can be obtained by polynomials of sufficient order;
- operations and properties:

	real-valued	complex-valued	polynomial
transposition	$\mathbf{A}^{\mathrm{T}}$	$\mathbf{A}^{\mathrm{H}} = (\mathbf{A}^{\mathrm{T}})^*$	$\boldsymbol{A}^{\mathrm{P}}(z) = \{\boldsymbol{A}(1/z^*)\}^{\mathrm{H}}$
energy preservation	orthonormal $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$	unitary $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{H}}$	para-unitary $\boldsymbol{A}^{-1}(z) = \boldsymbol{A}^{\mathrm{P}}(z)$
structure	symmetric $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$	Hermitian $\mathbf{A}^{\mathrm{H}} = \mathbf{A}$	para-Hermitian $\boldsymbol{A}^{\mathrm{P}}(z) = \boldsymbol{A}(z)$

- using polynomial notation, broadband formulations generally just extend from the narrowband case;
- to access solutions to polynomial matrix formulations, the eigenvalue decomposition of a parahermitian $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ\, \mathbf{R}[\tau]$ will be key;
- such an EVD must provide a diagonalisation for every value of z or for every lag τ [20, 26, 39, 46, 41].

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