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On semi-transitive orientability of split graphs

Sergey Kitaev [a,*], Artem Pyatkin [b,c]

[a] *Department of Mathematics and Statistics, University of Strathclyde, 26 Richmond Street, Glasgow G1, 1XH, United Kingdom*
[b] *Sobolev Institute of Mathematics, Koptyug ave, 4, Novosibirsk, 630090, Russia*
[c] *Novosibirsk State University, Pirogova str. 2, Novosibirsk, 630090, Russia*

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A B S T R A C T

A directed graph is semi-transitive if and only if it is acyclic and for any directed path $u_1 \to u_2 \to \cdots \to u_t$, $t \geq 2$, either there is no edge from u_1 to u_t or all edges $u_i \to u_j$ exist for $1 \leq i < j \leq t$. An undirected graph is semi-transitive if it admits a semi-transitive orientation. Recognizing semi-transitive orientability of a graph is an NP-complete problem. A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. Semi-transitive orientability of split graphs was recently studied in a series of papers in the literature. The main result in this paper is proving that recognition of semi-transitive orientability of split graphs can be done in a polynomial time.

1. Introduction

A directed graph is semi-transitive if and only if it is acyclic and for any directed path $u_1 \to u_2 \to \cdots \to u_t$, $t \geq 2$, either there is no edge from u_1 to u_t or all edges $u_i \to u_j$ exist for $1 \leq i < j \leq t$. An undirected graph is *semi-transitive* if it all its edges can be directed in such a way that the obtained orientation (directed graph) would be semi-transitive. The notion of a semi-transitive orientation was introduced in [7] to characterize *word-representable graphs* that were introduced in [10]. These graphs generalize several important and well-studied classes of graphs such as 3-colorable graphs, circle graphs, subcubic graphs and comparability graphs; see a survey [11] or a book [14] for more details.

A graph $G = (V, E)$ is called a *split graph* if its vertex set V can be partitioned into two parts $I \cup C$ such that C induces a clique, and I induces an independent set [5]. We assume that a clique in a split graph is inclusion-wise maximal, i.e. none of the vertices from I is adjacent to all vertices of C. Throughout the paper, given some split graph, we always assume C to be the clique and I to be the independent set. The study of split graphs attracted much attention in the literature (e.g. see [3] and references therein). Some split graphs are semi-transitive, others are not. Semi-transitive orientability (equivalently, word-representability) of split graphs was recently studied in the papers [2,8,9,13]. Interestingly, split graphs were instrumental in [2] to solve a 10 year old open problem in the theory of word-representable graphs asking whether gluing two word-representable graphs in a clique results in a word-representable graph. Namely, it was shown in [2], using certain split graphs, that gluing two word-representable graphs in any clique of size at least 2 may, or may not, result in a word-representable graph.

In general, recognizing if a given graph is semi-transitive is an NP-complete problem [14, Section 4.2.2]. The main result in this paper is showing that for the class of split graphs this problem is polynomially solvable. This result gives another motivation for our study. The class of split graphs is a subclass of *chordal graphs* (a chordal graph is a graph in which all cycles on four or more vertices have a chord, i.e. an edge that is not part of the cycle but connects

* Corresponding author.
E-mail addresses: sergey.kitaev@strath.ac.uk (S. Kitaev), artem@math.nsc.ru (A. Pyatkin).

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two vertices of the cycle). Till now, no graph problem is known that is NP-hard for chordal graphs but polynomially solvable for split graphs. So, if the problem of recognition of semi-transitivity would be NP-hard for chordal graphs (which is the open problem now), then it will be the first example of such graph problem.

Thus, our main focus in the current paper is the following problem.

Problem 1. Given a split graph G, is it semi-transitive?

Problem 1 was studied in [2,8,13]. In particular, a characterization of semi-transitive split graphs in terms of minimal forbidden induced subgraphs for $|C| \leq 5$ follows from [2,13]. Moreover, in [13] the following structural property of semi-transitive split graphs was proved. By $N(v)$ we denote the neighbourhood of a vertex v. Clearly, if $I \cup C$ is the bipartition of a split graph, then $N(v) \subset C$ for each $v \in I$. Also, for any two integers $a \leq b$, we let $[a,b] = \{a, a+1, \dots, b\}$.

Theorem 1 *([13]). A split graph G is semi-transitive if and only if the vertices of C can be labeled from 1 to $k = |C|$ in such a way that for each $v, u \in I$:*

(1) either $N(v) = [a,b]$ for $a \leq b$ or $N(v) = [1,a] \cup [b,k]$ for $a < b$.
(2) If $N(u) = [a_1, b_1]$ and $N(v) = [1, a_2] \cup [b_2, k]$, for $a_1 \leq b_1$, $a_2 < b_2$, then $a_1 > a_2$ or $b_1 < b_2$.
(3) If $N(u) = [1, a_1] \cup [b_1, k]$ and $N(v) = [1, a_2] \cup [b_2, k]$, for $a_1 < b_1$ and $a_2 < b_2$, then $a_2 < b_1$ and $a_1 < b_2$.

Note that Theorem 1 is a convenient, for our purposes, reformulation of Theorem 15 in [13]. Indeed, once a proper labeling of a split graph is found, to create a semi-transitive orientation of the graph,

- there is a unique way to orient edges in C and between $v \in I$ and $N(v) = [1,a] \cup [b,k]$ (if there are any such vertices), and
- there are two choices to orient edges between $v \in I$ and $N(v) = [a,b]$ (if there are any such vertices), namely, v can be a source (all edges incident with v are directed away from v) or a sink (all edges incident with v are directed towards v).

Determining semi-transitivity of split graphs is closely related to the well-known circular ones property of $(0,1)$-matrices. A $(0,1)$-matrix has the *consecutive ones* property (for columns) if after some permutation of its rows in all columns the ones are consecutive. A $(0,1)$-matrix has the *circular ones* property (for columns) if after some permutation of its rows in all columns either ones or zeroes are consecutive. Note that if zeroes are consecutive then ones are “almost consecutive” in the sense that they are allowed to wrap around from the bottom of a column to its top.

Note that from the algorithmic point of view, searching for permutations of rows giving a consecutive ones property and giving a circular ones property is equivalent, as follows from the following lemma:

Lemma 2 *([15]). Let M be a $(0,1)$-matrix. Denote by M_1 the matrix obtained from M by the inversion (changing 0 s by 1 s and vice versa) of all columns having 1 in the first row. Then M has the circular ones property if and only if M_1 has the consecutive ones property.*

As for the consecutive ones property, the first polynomial algorithm for determining whether a $(0,1)$-matrix has it was given in [6]. In [1] a more general algorithm based on the concept of PQ-trees was presented (in particular, it allows to find all possible permutations of rows providing consecutive ones orderings for all columns); it solves the decision problem in time $\mathcal{O}(m+n+f)$ where m, n, and f are the number of rows, columns and ones in M, respectively. Moreover, it was shown there that the problem of finding out whether a $(0,1)$-matrix has a circular ones property can also be solved in time $\mathcal{O}(m+n+f)$. In [12] an algorithm finding all possible permutations providing circular ones orderings for all columns in linear time is given. In [4] a linear time algorithm for checking the isomorphism of any two $(0,1)$-matrices having circular one property was constructed; this solves the graph isomorphism problem in linear time for several graph classes.

Given a split graph G with $C = \{u_1, \dots, u_k\}$ and $I = \{v_1, \dots, v_t\}$, consider a $(0,1)$-matrix $M(G)$ with k rows and t columns where $m_{ij} = 1$ if and only if u_i is adjacent to v_j. Then, clearly, any labeling of the vertices in C defines a permutation of the rows of the matrix $M(G)$. Moreover, such a labeling satisfies condition (1) of Theorem 1 if and only if the corresponding permutation provides a circular ones property. The first attempts of translating conditions (2)–(3) of Theorem 1 into the languages of matrices were made in [8,9], but the condition stated there was erroneous (see details about this right after Theorem 3 below). The correct statement is as follows:

Theorem 3. *A split graph G is semi-transitive if and only if the rows of matrix $M(G)$ can be permuted in such a way that:*

(i) The obtained matrix has a circular ones property in each column.
(ii) If a column of the obtained matrix has the form $1^a0^b1^c$ where $a+b+c = k$ and $a, b, c \geq 1$ then no other column may contain ones in all positions from a to $a+b+1$.

Indeed, it is easy to see that the property (i) of Theorem 3 is equivalent to the condition (1) of Theorem 1. In order to show that the property (ii) of Theorem 3 is equivalent to the conditions (2)–(3) of Theorem 1 just note that a column of the form $1^a0^b1^c$ means that the corresponding vertex $v \in I$ has $N(v) = [1,a] \cup [a+b+1,k]$. Therefore, if some other column has ones in all positions from a to $a+b+1$, then the corresponding vertex $u \in I$ has either

- $N(u) = [a_1, b_1]$ with $a_1 \leq a$ and $b_1 \geq a+b+1$ which contradicts (2), or
- $N(u) = [1, a_1] \cup [b_1, k]$ with $a_1 \geq a+b+1$ which contradicts (3), or
- $N(u) = [1, a_1] \cup [b_1, k]$ with $b_1 \leq a$ which also contradicts (3).

Note that in [8,9] instead of the requirement of "ones in all positions from a to $a+b+1$", there was the requirement of ones just in positions a and $a+b+1$, which does not work, for example, for the columns 1001 and 1101 with $k=4$, $a=1$ and $b=2$ (these columns meant to give a non-semi-transitive split graph, while the graph is indeed semi-transitive).

So, a natural attempt to solve Problem 1 would be as follows: consider all permutations of rows providing circular ones ordering using the result from [12] and check whether they satisfy property (ii) of Theorem 3. Unfortunately, this approach does not provide a polynomial-time algorithm for Problem 1. Indeed, although the result from [12] allows to find each permutation with circular ones property in linear time, the total number of such permutations may be exponential, while the number of permutations satisfying property (ii) of Theorem 3 among them could be small.

Example 1. Consider the following matrix M with k rows and $k+2$ columns.

$$M=\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 1 & \dots & 0 & 0 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 & 1 \end{pmatrix}$$

Since each column contains either one 1, or one 0, every permutation of its rows has circular ones property, i.e. there are $k!$ such permutations. On the other hand, only those permutations where 0's in the last two columns are in consecutive rows satisfy property (ii) of Theorem 3 (rows 1 and k are also considered as consecutive here); hence, there are only $2k$ such permutations.

In order to apply the circular ones permutations techniques for solving Problem 1 we need a deeper study of the structure of semi-transitive split graphs provided in the following section. Namely, in Section 2 we introduce an auxiliary Problem 2 of finding a bijection of a set into a cycle that maps the family of given subsets into a family connected subgraphs on the cycle and prove that Problem 1 is equivalent to it. Then we reduce Problem 2 to the problem of determining (0, 1)-matrices with the circular ones property.

2. Polynomial solvability of Problem 1

Denote by $V(C_k)$ the vertex set of a cycle graph C_k. Consider the following auxiliary problem.

Problem 2. Given a set A of cardinality k and a multiset $\{A_1,\dots,A_t \mid A_i \subset A,\ 1\le i\le t\}$ of proper and non-empty subsets of A, is there a bijection $F: A \longrightarrow V(C_k)$ such that for all $i,j\in[1,t]$ the subgraph induced by $F(A_i\cap A_j)$ is connected (the empty subgraph is considered to be connected)?

Note that in Problem 2, the subgraphs induced by each $F(A_i)$ must be connected (this corresponds to the case when $j=i$). The following theorem shows the equivalence of Problems 1 and 2.

Theorem 4. *Problem 1 is polynomially solvable if and only if Problem 2 is polynomially solvable.*

Proof. Assume that Problem 2 is polynomially solvable. Consider an arbitrary instance of Problem 1 with the graph $G=(I\cup C,E)$ where $I=\{v_1,\dots,v_t\}$, $C=\{u_1,\dots,u_k\}$. Construct a corresponding instance of Problem 2 as follows. Let $A=C$ and put $A_i=N(v_i)$ for all $i=1,\dots,t$. Let us prove the following claim.

Claim. *Graph G has a semi-transitive orientation if and only if the set A has an appropriate bijection.*

Indeed, if G satisfies the conditions of Theorem 1 then let the labeling of the vertices in C define the bijection F (the order of the vertices in the cycle graph C_k). Then (1) ensures that each subgraph induced by $F(A_i)$ is connected since it is a path from a to b that either goes through 1 or not. Consider arbitrary $v_i,v_j\in C$. If $N(v_i)=[a_1,b_1]$ and $N(v_j)=[a_2,b_2]$ then $F(A_i\cap A_j)$ induces either an empty subgraph or a path from $\max\{a_1,a_2\}$ to $\min\{b_1,b_2\}$; anyway, it is connected. Let $N(v_i)=[a_1,b_1]$ and $N(v_j)=[1,a_2]\cup[b_2,k]$. By (2), either $a_1>a_2$ or $b_1<b_2$ or both. Then, respectively, $F(A_i\cap A_j)$ induces either a path from b_2 to b_1, or a path from a_1 to a_2, or the empty subgraph. Finally, let $N(v_i)=[1,a_1]\cup[b_1,k]$ and $N(v_j)=[1,a_2]\cup[b_2,k]$. Then by (3), $a_2<b_1$ and $a_1<b_2$. Hence, $F(A_i\cap A_j)$ induces a path $b,b+1,\dots,k,1,2,\dots,a$ where $a=\min\{a_1,a_2\}$ and $b=\max\{b_1,b_2\}$. So, the subgraph induced by $F(A_i\cap A_j)$ is connected in C_k for all $i,j\in[1,t]$.

Assume now that there is a bijection from A into C_k such that all intersections of the subsets induce connected subgraphs. Label C in the order of the cycle graph C_k starting from an arbitrary vertex. Since each subgraph induced by $F(A_i)$ is connected, the corresponding $N(v_i)$ is either an interval $[a,b]$ or the union of two intervals $[1,a]\cup[b,k]$ for some $a<b$, i.e. (1) holds. If $N(v_i)=[a_1,b_1]$ and $N(v_j)=[1,a_2]\cup[b_2,k]$ but $a_1\le a_2$ and $b_1\ge b_2$ then $F(A_i\cap A_j)$ induces a union of two paths $a_1,\dots,a_2$ and $b_1,\dots,b_2$ which is disconnected, a contradiction. So, (2) holds. Assume that $N(v_i)=[1,a_1]\cup[b_1,k]$ and $N(v_j)=[1,a_2]\cup[b_2,k]$. If $a_2\ge b_1$ then $F(A_i\cap A_j)$ induces the union of three paths $1,\dots,a_1$; $b_1,\dots,a_2$; and $b_2,\dots,k$, while if $a_1\ge b_2$ then $F(A_i\cap A_j)$ induces the union of paths $1,\dots,a_2$; $b_2,\dots,a_1$; and $b_1,\dots,k$. In both cases, the subgraph is clearly disconnected; because of this contradiction, (3) must be true.

By the claim, applying an algorithm solving the constructed instance of Problem 2 gives a solution of the initial instance of Problem 1.

If Problem 1 is polynomially solvable then the reduction from an instance of Problem 2 to Problem 1 is made in a similar way: put $C=A$, $I=\{v_1,\dots,v_t\}$, and $N(v_i)=A_i$. Then the same claim provides the polynomial solvability of Problem 2. $\square$

In order to solve Problem 2, we use the above-mentioned results [1,12] on the polynomial solvability of checking the circular ones property of $(0,1)$-matrices. Clearly, $\mathcal{O}(m+n+f)$ can be bounded from above by $\mathcal{O}(mn)$, which is the input size of the consecutive ones problem in general. Note that a split graph is defined by the neighbourhood sets of the vertices in I. Hence, the size of the input of Problem 1 is $\mathcal{O}(tk)$. Recall that here, and in what follows, $|I|=t$ and $|C|=k$. We can now prove the main result of the paper.

Theorem 5. *Problem 1 can be solved in time $\mathcal{O}(t^2k)$.*

Proof. Given an instance of Problem 1, first construct an equivalent instance of Problem 2 as shown in the proof of Theorem 4. Clearly, it takes time $\mathcal{O}(tk)$. Let $A=\{a_1,\dots,a_k\}$. Put $m=k$ and $n=(t^2+t)/2$. Construct the $(0,1)$-matrix M of size $m\times n$ as follows. For convenience, let the columns be indexed by the pairs (i,j) where $i,j\in[1,t]$ and $i\leq j$ (note that including the case $i=j$ is crucial here). Let each row of M correspond to an element of the set A and each column (i,j) be a characteristic vector of the subset $A_i\cap A_j$ (i.e. $m_{s,(i,j)}=1$ if and only if $a_s\in A_i\cap A_j$). Then, clearly, finding the appropriate bijection F is equivalent to finding a permutation of the rows that provides a circular ones ordering for all columns of the matrix M. By the result from [1] mentioned above, the latter can be done in time $\mathcal{O}(mn)=\mathcal{O}(t^2k)$. □

Remark 6. Note that in practice many columns can contain no, or just one, 1; such columns can be omitted since they have the circular ones property after any permutation of rows. This can make the algorithm faster, although in the worst case, we have the bound of $\mathcal{O}(t^2k)$.

Declaration of competing interest

No conflict of interest exists.

Data availability

No data was used for the research described in the article.

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