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On ordering of β-description trees ☆

Sumin Huang [a], Sergey Kitaev [b,*]

[a] School of Mathematical Sciences, Xiamen University, Xiamen 361005, PR China
[b] Department of Mathematics and Statistics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, United Kingdom

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ABSTRACT

Tutte introduced planar maps in the 1960s in connection with what later became the celebrated Four-Color Theorem. A planar map is an embedding of a planar graph in the plane. Description trees, in particular, β-description trees, were introduced by Cori, Jacquard and Schaeffer in 1997, and they give a powerful tool to study planar maps.
In this paper we introduce a relation on β-description trees and conjecture that this relation is a total order. Towards solving this conjecture, we provide an embedding of $\beta(a,b)$-trees into $\beta(a-t,b+t)$-trees for $t \le a \le b+t$, which is a far-reaching generalization of an unpublished result of Claesson, Kitaev and Steingrímsson on embedding of $\beta(1,0)$-trees into $\beta(0,1)$-trees that gives a combinatorial proof of the fact that the number of rooted nonseparable planar maps with $n+1$ edges is more than the number of bicubic planar maps with $3n$ edges.

1. Introduction

A *planar map* is a connected multigraph with a given embedding in the sphere. A planar map is *rooted* if one of its edges, called the root, is directed. In the 1960s, Tutte founded the enumeration theory of planar maps, motivated by what later became the Four-Color Theorem, in a series of papers [6–9]. In [8], Tutte determined that the number of rooted nonseparable planar maps (that is, rooted planar maps without cut vertices) with $n+1$ edges is given by

$$\frac{4(3n)!}{n!(2n+2)!}. \tag{1}$$

This result was also given by Brown [1]. Moreover, the number of *bicubic maps* (that is, bipartite 3-regular rooted planar maps) with $3n$ edges was given by Tutte [8]:

$$\frac{3\cdot 2^{n-1}(2n)!}{n!(n+2)!}. \tag{2}$$

Description trees, in particular, β-description trees, were introduced by Cori, Jacquard and Schaeffer [4] in 1997 as a general framework related to planar maps. A *rooted plane tree* is either a single node (the elementary tree) or is a node (the root) connected to an ordered sequence of plane trees. A rooted plane tree is *labeled* if each node is assigned a nonnegative integer as its label. Let v

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* Corresponding author.
E-mail addresses: sumin2019@sina.com (S. Huang), sergey.kitaev@strath.ac.uk (S. Kitaev).

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be a node of a rooted plane tree T, then the label of v in T is denoted by $l_T(v)$, the degree of v (that is, the number of nodes adjacent to v) is denoted by $d_T(v)$, and the sum of its children's labels by $s_T(v)$.

A $\beta(a,b)$-*tree* T is a labeled rooted plane tree in which the following conditions, referred to as the $\beta(a,b)$-*rules*, are satisfied for each node v:

(1) If v is a leaf, then $l_T(v)=a$.
(2) If v is the root, then $l_T(v)=s_T(v)+b$.
(3) If v is neither a leaf nor the root, then $a \le l_T(v) \le s_T(v)+b$.

There are four known bijections between $\beta(a,b)$-trees with n edges and corresponding families of planar maps [2,4,5]:

$\beta(1,0)$-trees ⇔ rooted nonseparable planar maps with $n+1$ edges;

$\beta(0,1)$-trees ⇔ bicubic planar maps with $3n$ edges;

$\beta(1,1)$-trees ⇔ 3-connected cubic planar maps with $3n+3$ edges;

$\beta(2,2)$-trees ⇔ cubic nonseparable planar maps with $3n$ edges.

Comparing formulas (1) and (2), for $n \ge 2$, we see that the number of rooted nonseparable planar maps with $n+1$ edges is less than the number of bicubic planar maps with $3n$ edges. To explain this fact combinatorially, Claesson, Kitaev and Steingrímsson, in an unpublished work [3], provided an embedding of $\beta(1,0)$-trees into $\beta(0,1)$-trees. In this paper, we generalize this result by introducing Algorithm A that embeds $\beta(a,b)$-trees into $\beta(a-t,b+t)$-trees, for $t \le a \le b+t$; the result of Claesson et al. is obtained by letting $a=t=1$ and $b=0$ in our embedding. Moreover, our embedding is a step towards solving the conjecture on the relation on $\beta(a,b)$-trees to be introduced next.

Let $N(a,b,n)$ be the number of all $\beta(a,b)$-trees with n edges. We say that $\beta(a,b) \preceq \beta(c,d)$ if $N(a,b,n) \le N(c,d,n)$ for all $n \ge n_0$ where n_0 is a natural number.

Clearly, the relation is reflexive and transitive. We state the following conjectures regarding the relation, where Conjecture 2 implies Conjecture 1. Recall that a partial order is a reflexive, anti-symmetric and transitive relation, where the anti-symmetry in our case means that if $\beta(a,b) \preceq \beta(c,d)$ and $\beta(c,d) \preceq \beta(a,b)$ then $a=c$ and $b=d$.

Conjecture 1. *The relation on β-description trees is a partial order.*

Conjecture 2. *The relation on β-description trees is a total order.*

Towards confirming the conjectures, in Section 3 we prove the following theorem.

Theorem 1. *We have the following facts about the relation on $\beta(a,b)$-trees:*

(1) *If $n \ge 2$, then $N(a,b,n) < N(a,b+t,n)$ for any nonnegative integers a,b,t, so $\beta(a,b) \preceq \beta(a,b+t)$.*
(2) *If $n \ge 3$, then $N(a,b,n) < N(a+t,b,n)$ for any nonnegative integers a,b,t, so $\beta(a,b) \preceq \beta(a+t,b)$.*
(3) *If $n \ge 2$ and $t \le a \le b+t$, then $N(a,b,n) < N(a-t,b+t,n)$ for any nonnegative integers a,b,t, so $\beta(a,b) \preceq \beta(a-t,b+t)$.*
(4) *If $n \ge 2$ and $a > b$, then $N(a,b,n) < N(b,a,n)$ for any nonnegative integers a,b, so $\beta(a,b) \preceq \beta(b,a)$ for $a > b$.*
(5) *If $n \ge 2$, then $N(a,b,n) < N(0,a+b,n)$ for any nonnegative integers a,b, so $\beta(a,b) \preceq \beta(0,a+b)$.*

Fig. 1 presents known comparisons for $0 \le a,b \le 3$ based on Theorem 1; the pattern can be continued for other values of a and b. However, for the moment, we cannot compare, for example, $\beta(a,b)$-trees and $\beta(a-2,b+1)$-trees with the smallest unknown example being comparing $\beta(2,0)$-trees and $\beta(0,1)$-trees. We cannot also compare $\beta(a,b)$-trees and $\beta(a-1,b+1)$-trees for $a > b+1$ with the smallest unknown example being comparing $\beta(2,0)$-trees and $\beta(1,1)$-trees.

2. Embedding $\beta(a,b)$-trees into $\beta(a-t,b+t)$-trees

2.1. Modified trees

Before introducing Algorithm A, we give some required notations. Assume that T is a rooted plane tree. We say that a node u is *above* another node v if u is on the path from v to the root of T. If a node v is not above a node u, and vice versa, then we say that v is *to the left* of u in T, or u is *to the right* of v in T, if v is found first in the pre-order traversal of T. For example, in the tree to the left in Fig. 2, u_1 is to the left of u_2 and u'. For any node v in T, the R-*path* of v is the path $v=v_0,v_1,\dots,v_l$ such that v_{i+1} is the rightmost child of v_i and v_l is a leaf of T. We call v_l the R-*leaf* of v. In particular, if v is a leaf, then the R-leaf of v is v itself.

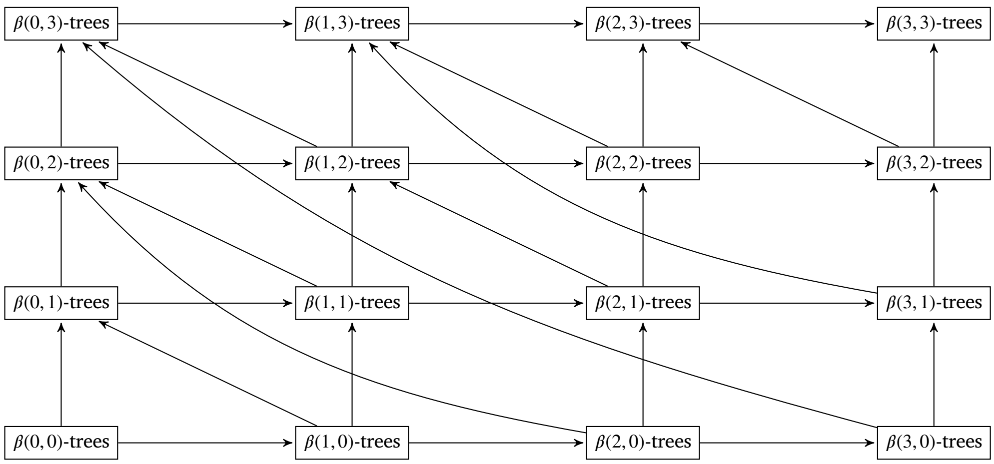

Fig. 1. Known comparisons on β-description trees for $0 \le a, b \le 3$. Arrows indicate which objects are being embedded.

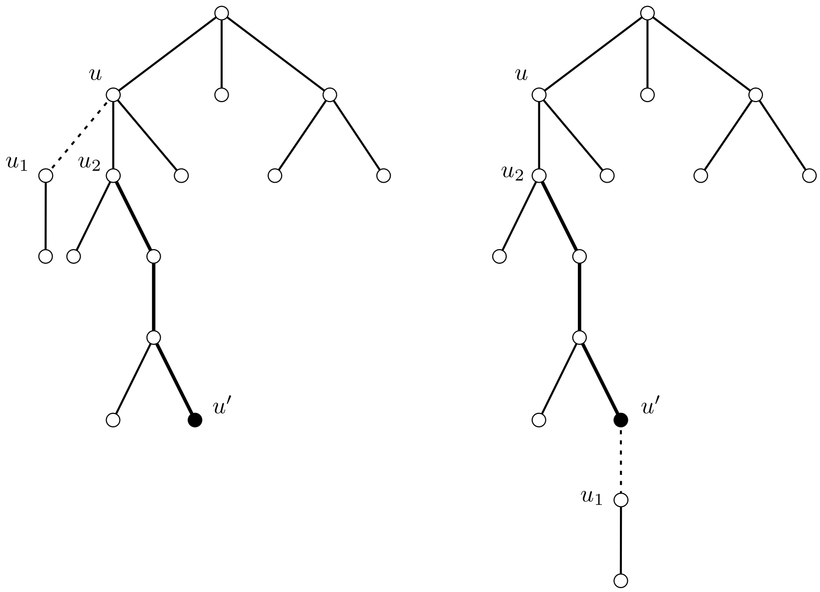

Fig. 2. A rooted plane tree T and its modified tree T' with respect to u. The R-path of u_2 consists of the thick edges. u' is the R-leaf of u_2 in T and the mark in T'. It can be noted that u is the first node on the path from u' to the root of T' such that u' is not on the R-path of u.

For a node u with at least three children $u_1, \ldots, u_k$ from left to right, the *modified tree* of T with respect to u is a tree obtained from T by deleting uu_1 and adding the edge between u_1 and the R-leaf of u_2. We say the R-leaf of u_2 is the *mark* in the modified tree (see Fig. 2 for an illustration).

Conversely, assume that T' is the modified tree of T with respect to u. Suppose that the mark of T' is u' and the unique child of u' is u_1. Since u has at least three children in T, the rightmost child of u is not u_2. Then u is the first node on the path from u' to the root of T', such that u' is not on the R-path of u. T can be obtained from T' by deleting $u'u_1$ and adding the edge uu_1. We have proved the following lemma.

Lemma 1. *T can be determined uniquely from a given modified tree T' with a mark u'.*

2.2. Algorithm A

Suppose T is a $\beta(a, b)$-tree and S is the set of nodes in T with at least three children. Assume that the children of a node $u \in S$ are $u_1, \ldots, u_s$ ($s \ge 3$). Then we denote the set consisting of all $(s-2)$ R-leaves of $u_2, \ldots, u_{s-1}$ by S_u. None of the leaves in $\bigcup_{u \in S} S_u$ are on the same path to the root of T, and hence the leaves in this set can be ordered from left to right. Also, it should be noted that S_u does not contain the R-leaf of u_s and that of u, which implies that $S_u \cap S_{u'} = \emptyset$ for $u \ne u'$. Provided $S \ne \emptyset$, we introduce a function $\mathcal{O}_T$

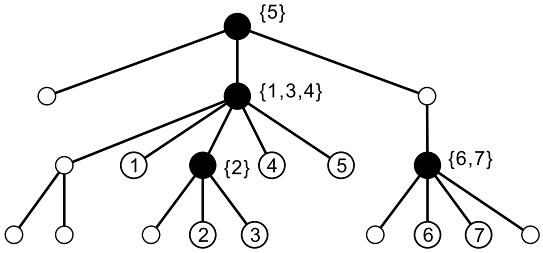

Fig. 3. S consists of the black nodes and the R-leaves in $\bigcup_{u\in S} S_u$ are ordered from left to right. Then, the set of all pre-images of each node in S for the function $\mathcal{O}_T$ is given next to the respective node.

from $\{1,2,\ldots,|\bigcup_{u\in S} S_u|\}$ to S. For any $1 \le k \le |\bigcup_{u\in S} S_u|$, if the k-th R-leaf $v \in \bigcup_{u\in S} S_u$ belongs to S_u, then we set $\mathcal{O}_T(k) := u$ (see Fig. 3 for an illustration). In general, the function $\mathcal{O}_T(k)$ is not injective.

We now present Algorithm A producing a $\beta(a-t, b+t)$-tree from a $\beta(a,b)$-tree. In Theorem 2 we will show that the outcome of the algorithm is well-defined and unique, and in Fig. 4 we will illustrate the work of Algorithm A on an example.

Algorithm A: *Turning a* $\beta(a,b)$*-tree into a* $\beta(a-t,b+t)$*-tree.*
Input: A $\beta(a,b)$-tree T with n edges ($t \le a \le b+t$).
Output: A $\beta(a-t,b+t)$-tree T' with n edges.

Steps of the algorithm:
Step 1. Set $i := 0$ and $T_0 := T$.
Step 2. If each node in T_i has at most two children, then go to Step 6. Otherwise, assume that u is the node $\mathcal{O}_T(i+1)$.[1]
Step 3. Let the children of u in T_i be $u_1,\ldots,u_s$ from left to right ($s \ge 3$). Let T_{i+1} be the modified tree of the underlying tree of T_i (that is obtained by erasing all labels in the tree T_i) with respect to u. Suppose that the mark of T_{i+1} is u'.
Step 4. Set the labels of T_{i+1} as follows. Firstly, $l_{T_{i+1}}(u_1) := l_{T_i}(u_1)$. Then, for each node v on the path from u' to u_2, $l_{T_{i+1}}(v) := l_{T_i}(v) - a + b + t + l_{T_i}(u_1)$. In particular, we have $l_{T_{i+1}}(u') = l_{T_{i+1}}(u_1) + b + t$. Furthermore, for each node v on the path from u to the root of T_{i+1}, $l_{T_{i+1}}(v) := l_{T_i}(v) - a + b + t$. Finally, for any other node v, $l_{T_{i+1}}(v) := l_{T_i}(v)$.
Step 5. Set $i := i+1$ and go back to Step 2.
Step 6. Set $T' := T_i$. Then set $l_{T'}(v) := l_{T_i}(v) - t$ for each non-root node v and $l_{T'}(r) := s_{T'}(r) + b + t$ for the root r of T'. Return T'.

Theorem 2. *Algorithm A produces, in an injective way, a* $\beta(a-t,b+t)$*-tree.*

Proof. Firstly, we prove that T' is a $\beta(a-t,b+t)$-tree. Note that T_{i+1} is the modified tree of the underlying tree of T_i in Step 3. Let $M_0 = \emptyset$ and $M_{i+1} = M_i \cup \{u'\}$, where u' is the mark of T_{i+1}.

Claim 1. *In* T_i*, each node* v *in* M_i *has a single child and it satisfies* $l_{T_i}(v) = s_{T_i}(v) + b + t$*. Each node not in* M_i *satisfies the* $\beta(a,b)$*-rules.*

We prove Claim 1 by induction on i. It is clear that Claim 1 holds for $i = 0$ since $T_0 = T$ is a $\beta(a,b)$-tree and $M_0 = \emptyset$ (nothing to check for node in M_i). Suppose that Claim 1 holds for T_i and consider T_{i+1}.

Since only the nodes above u_1 may have different labels in T_i and T_{i+1}, any node v in M_i but not above u_1 and the child of v (v has a single child by the induction hypothesis) have the same labels in T_i and T_{i+1}. So for any $v \in M_i$ not above u_1, $l_{T_{i+1}}(v) = l_{T_i}(v) = s_{T_i}(v) + b + t = s_{T_{i+1}}(v) + b + t$.

Also, for $v \in M_i$, assume that v is above u_1, then both v and its unique child v' are either on the path from u' to u_2, or on the path from u to the root. Indeed, v cannot be u because u has at least two children in T_{i+1}. Then $l_{T_{i+1}}(v) - l_{T_i}(v) = l_{T_{i+1}}(v') - l_{T_i}(v') = s_{T_{i+1}}(v) - s_{T_i}(v)$, which implies $l_{T_{i+1}}(v) = s_{T_{i+1}(v)} + b + t$ holds.

Besides, recall that u' is the mark of T_{i+1} and u_1 is the unique child of u'. So $l_{T_{i+1}}(u') = l_{T_{i+1}}(u_1) + b + t = s_{T_{i+1}}(u_1) + b + t$, as claimed.

In the following, we only consider these nodes v not in M_{i+1}. For each node v on the path from u' to u_2, if $v \ne u'$, then we have

$$\begin{aligned} l_{T_{i+1}}(v) =& l_{T_i}(v) - a + b + t + l_{T_i}(u_1) \\ \ge& l_{T_i}(v) + l_{T_i}(u_1) \\ \ge& a, \end{aligned}$$

where the first inequality holds because $a \le b + t$ and the second inequality holds because $l_{T_i}(u_1) \ge a$. On the other hand,

$$l_{T_{i+1}}(v) = l_{T_i}(v) - a + b + t + l_{T_i}(u_1)$$

[1] In fact, in Step 2, any node u from the set S (of nodes in T with at least three children) can be selected. However, we select the node $u = \mathcal{O}_T(i+1)$ to simplify the proof of injectivity of Algorithm A given below.

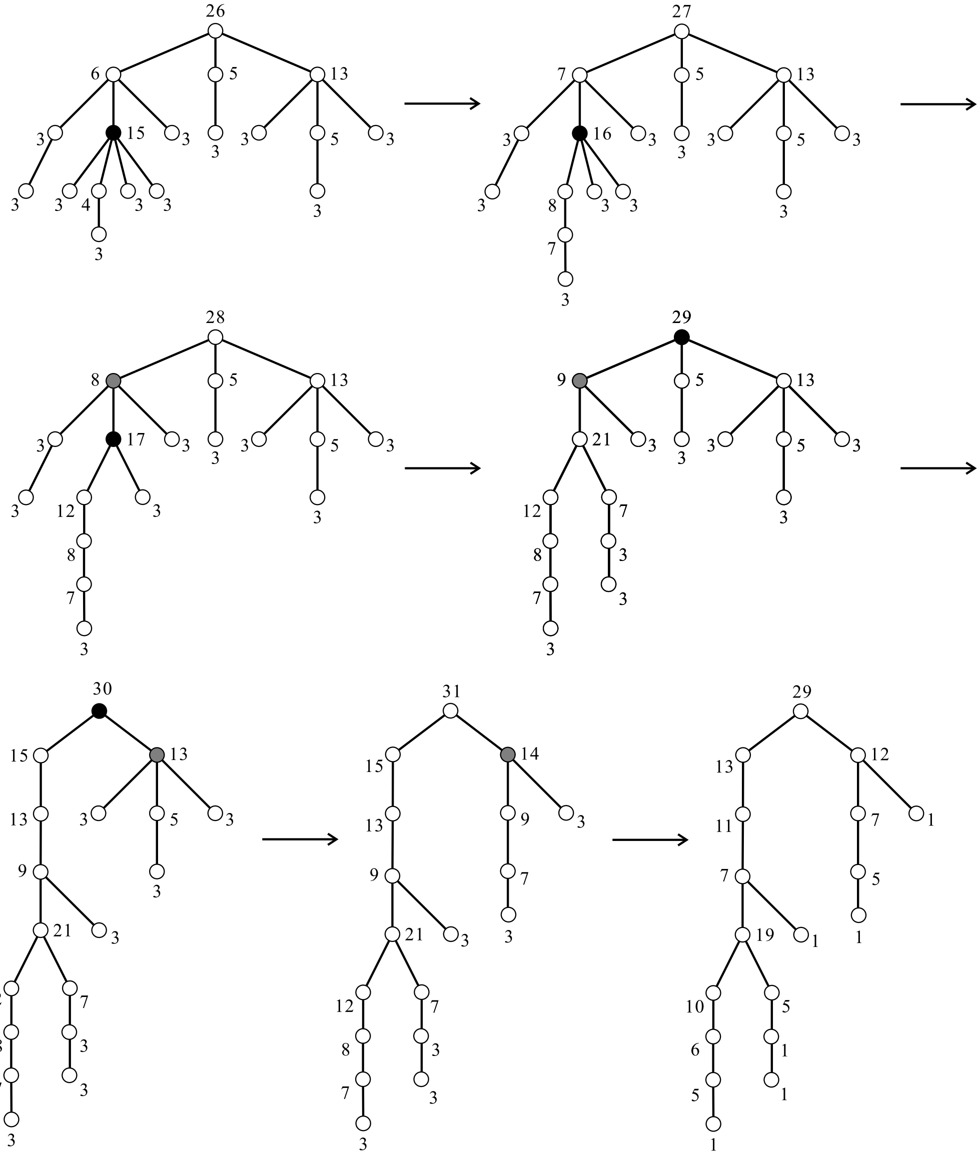

Fig. 4. Turning a $\beta(3,2)$-tree into a $\beta(1,4)$-tree. The black and grey nodes are the nodes u chosen in different T_i's.

$$
\begin{aligned}
&\leq s_{T_i}(v) + b - a + b + t + l_{T_i}(u_1)\\
&= s_{T_{i+1}}(v) + b,
\end{aligned}
$$

where the first inequality holds because of the induction hypothesis and the last equality holds because all children of v not on the path from u' to u_2 have the same labels in T_i and T_{i+1}.

Hence, each node on the path from u' to u_2, except u', satisfies the $\beta(a,b)$-rules. By similar arguments, each node on the path from u to the root of T_{i+1} satisfies the $\beta(a,b)$-rules. Moreover, any other node neither in M_{i+1} nor on the path from u' to the root of T_{i+1} satisfies the $\beta(a,b)$-rules by the induction hypothesis. Claim 1 is proved.

Now we shall focus on T' in Step 6. Note that we go to Step 6 if and only if each node in T_i has at most two children. According to Step 6, the root r satisfies the $\beta(a-t,b+t)$-rules. By Claim 1, for each node v in M_i, we have $l_{T'}(v) = l_{T_i}(v) - t = s_{T_i}(v) + b = s_{T'}(v) + b + t$, which implies that v satisfies the $\beta(a-t,b+t)$-rules. Also, each leaf has label $a-t$. Moreover, for each node v not in M_i, if v is neither a leaf nor a root, then we have $a - t \leq l_{T_i}(v) - t = l_{T'}(v)$ and

$$
\begin{aligned}
l_{T'}(v) =& l_{T_i}(v) - t\\
&\leq s_{T_i}(v) + b - t
\end{aligned}
$$

$$
\begin{aligned}
&= s_{T'}(v) + (d_{T'}(v) - 1)\cdot t + b - t \\
&\leq s_{T'}(v) + b + t,
\end{aligned}
$$

where the last inequality holds because $d_{T'}(v) \leq 3$. Hence, T' is a $\beta(a-t, b+t)$-tree.

Injectivity. It is clear that Step 6 is injective. So it suffices to confirm that T_k can be determined from a given T_{k+1} uniquely for $0 \leq k \leq i-1$. By Claim 1, M_{k+1} consists of all nodes in T_{k+1} that do not satisfy the $\beta(a,b)$-rules. So M_{k+1} can be determined from T_{k+1} uniquely. In particular, $|M_{k+1}| = k+1$.

Note that to construct T_{k+1}, for $0 \leq k \leq i-1$, we repeated k times Step 2, and $\mathcal{O}_T(k+1)$ is a node chosen in Step 2. Thus, the underlying tree of T_{k+1} is the modified tree of that of T_k with respect to $\mathcal{O}_T(k+1)$.

Suppose that m_{k+1} is the mark of T_{k+1}. Then m_{k+1} must be the R-leaf of a child of $\mathcal{O}_T(k+1)$ in T and $\{m_{k+1}\} = M_{k+1} \backslash M_k$. For any $m_s \in M_k$ $(1 \leq s \leq k)$, m_s must be the R-leaf of a child of $\mathcal{O}_T(s)$, and by definition of the function $\mathcal{O}_T$, m_s is to the left of m_{k+1} in T. By definition of a modified tree, during the process $T_0 \to T_1 \to \cdots \to T_k$, m_s is always to the left of m_{k+1}. Hence, m_{k+1} is still the R-leaf of a child of $\mathcal{O}_T(k+1)$ in T_k. Furthermore, there exists no m_s above m_{k+1} in T_{k+1} because T_{k+1} is obtained from T_k by adding a subtree to m_{k+1}. Therefore, m_{k+1} is the rightmost node among all nodes in M_{k+1} with no other m_s above it, so that we can determine where the mark of T_{k+1} is.

By Lemma 1, the underlying tree of T_k can be determined uniquely from T_{k+1}. Moreover, based on Step 4, the labels of nodes in T_k can also be determined uniquely from T_{k+1}. Hence, T_k can be determined uniquely from a given T_{k+1} for $0 \leq k \leq i-1$, which implies the injectivity of Algorithm A. □

3. Proof of Theorem 1

(1) Clearly, it is sufficient to consider the case of $t = 1$. Assume that T is a $\beta(a,b)$-tree. Then each node in T satisfies the $\beta(a,b)$-rules. Note that any node except for the root r of T also satisfies the $\beta(a,b+1)$-rules. Replace $l_T(r)$ by $s_T(r)+b+1$ in T to obtain a $\beta(a,b+1)$-tree in an injective way. Hence, $N(a,b,n) \leq N(a,b+1,n)$. Clearly, turning a $\beta(a,b)$-tree into a $\beta(a,b+1)$-tree is not surjective for $n \geq 2$, so then $N(a,b,n) < N(a,b+1,n)$.

(2) Clearly, it is sufficient to consider the case of $t = 1$. Assume that T is a $\beta(a,b)$-tree and T' is the tree obtained from T by adding 1 to each node's label. Clearly, any node except for the root r of T' satisfies the $\beta(a+1,b)$-rules. Adjusting the label of r in T' so that it also satisfies the $\beta(a+1,b)$-rules, we obtain a $\beta(a+1,b)$-tree in an injective way. Hence, $N(a,b,n) \leq N(a+1,b,n)$. It is easy to see that turning a $\beta(a,b)$-tree into a $\beta(a+1,b)$-tree is not surjective for $n \geq 3$, so then $N(a,b,n) < N(a+1,b,n)$.

(3) This is a direct consequence of Theorem 2 and the following observation. Let T_0 be a $\beta(a-t,b+t)$-tree with at least two edges that satisfies the following conditions:
   (a) each node in T_0 has at most one child;
   (b) each node in T_0 except for the root and its unique child has label $a-t$;
   (c) the root of T_0 has label $a+2b+t$ and the child of the root has label $a+b$.

   Then T_0 is not in the image of Algorithm A. Indeed, suppose T_0' is the pre-image of T_0 under Algorithm A. Because each node in T_0 has at most one child, Algorithm A applied to T_0' never implemented steps 2–5, so T_0' can be obtained from T_0 by adding t to the label of each node different from the root. Then $l_{T_0'}(u) = a+b+t > a+b = l_{T_0'}(v)+b$, where u is the son of the root and v is that of u, and T_0' is not a $\beta(a,b)$-tree; contradiction. So Algorithm A is not surjective when $n \geq 2$ hence $N(a,b,n) < N(a-t,b+t,n)$.

(4) Setting $t = a-b$ in (3), we obtain that if $a > b$ then $N(a,b,n) < N(b,a,n)$.

(5) Setting $t = a$ in (3), we obtain that $N(a,b,n) < N(0,a+b,n)$.

4. Concluding remarks

In this paper, we defined a reflexive and transitive relation on β-description trees. We conjecture (Conjecture 1) that the relation is a partial order. Moreover, we conjecture (Conjecture 2) that the relation is a total order. If Conjecture 1 is true, settling Conjecture 2 will require arguing, for $(a,b) \neq (c,d)$, why if $N(a,b,i) \leq N(c,d,i)$, for $i \in I$, and $N(a,b,j) \leq N(c,d,j)$, for $j \in J$, then one of I and J is finite or empty. Regardless of settling Conjectures 1 and 2, we are in need of introducing new embeddings/algorithms, similar to our embedding of $\beta(a,b)$-trees into $\beta(a-t,b+t)$-trees, to be able to compare $\beta(a,b)$-trees and $\beta(c,d)$-trees for any a,b,c,d. We finish this paper with stating another conjecture about the relation.

Conjecture 3. $\beta(a,b) \preceq \beta(a-1,b+1)$ *for any* $a \geq 1$ *and* $b \geq 0$.

Note that Algorithm A confirms Conjecture 3 in the case of $a \leq b+1$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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