On the existence and asymptotic stability of hybrid stochastic systems with neutral term and non-differentiable time delay

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Abstract

This paper focuses on a class of hybrid stochastic delay systems with neutral term (HSDSwNT), where the time delay may be non-differentiable and the coefficients may not satisfy the linear growth condition. Under the local Lipschitz condition and the Khasminskii-type condition, the existence and uniqueness as well as boundedness of the solution to the HSDSwNT are established. By virtue of the Lyapunov function method and the M-matrix theory, sufficient conditions on the asymptotic stability and general decay stability are also examined for HSDSwNT. Finally, two examples are given to show the effectiveness of our results.

Key words: Hybrid stochastic systems, Neutral term, Non-differentiable time delay, Asymptotic stability, General decay stability.

1 Introduction

Many stochastic systems may experience abrupt changes in their parameters and structures caused by phenomena such as changing subsystem interconnections and sudden environmental disturbances. The hybrid systems driven by continuous-time Markov chains have been developed to deal with this situation. One of the distinct features of such systems is that continuous dynamics are intertwined with discrete events. For the general theory of hybrid stochastic systems, we refer the reader to [1, 2]. Since many real-world systems often run for a long period of time, it is very necessary to study the asymptotic behavior of such systems. Due to existence of the Markovian switching, the asymptotic behavior of such hybrid system is completely different from that of each subsystem. Therefore, the stability of hybrid stochastic systems has received a great deal of

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attention (e.g., see [3]-[8]). On the other hand, in many real-world systems, time delay is ubiquitous and pervasive, and some stochastic systems do not only depend on present and past states but also involve derivatives with delays. Such systems have been referred to as hybrid stochastic systems with neutral term (HSSwNT). In the real world, there are many neutral delay differential systems. For example, [9] proposed a coupled system consisting of a mass spring damper (MSD) connected to the pendulum. In order to describe the delay of the displacement signal, the mathematical representation of the system is as follows

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m\ddot{y}(t-\tau_0) = 0 \tag{1.1}$$

where M, C and K are the mass, stiffness and damping of a MSD model, m is the mass of a pendulum, τ_0 is the constant delay, and $y(t)$, $\dot{y}(t)$ and $\ddot{y}(t)$ denote the position, velocity and acceleration of MSD at time t. Since time delay τ_0 is present in the highest derivative, equation (1.1) is called a neutral delay differential equation. Let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, equation (1.1) can be written as the following form

$$d[x(t) - Dx(t-\tau_0)] = Ax(t)dt$$

where $x(t) = (x_1(t), x_2(t))^\top$ and

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{m}{M} \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{C}{M} \end{pmatrix}.$$

This system has been applied to many important fields, such as computer-chip interface circuitries, distributed networks, population dynamics, chemical process control and so on. In addition, taking COVID-19 as an example, in the initial stage of the outbreak of COVID-19, it is generally believed that the number of people infected by the virus in the future only depends on the number of people infected by COVID-19 in the past week. However, as the infectious ability of the virus is increasing, the number of people infected with COVID-19 in the future depended not only on the number of people infected with COVID-19, but also on the infection rate of COVID-19. Therefore, HSSwNT can be used to describe the current COVID-19 model with rapid virus mutation. For the theoretical study of HSSwNT, please see Mao [1], Kolmanovskii et al. [14] and the references therein. More recent results on the stability of HSSwNT may be found in [15]-[22].

As we all know, time delay often causes instability of the control system. For example, if we choose the delay function $\bar{\delta}(t) = 0.75(1 + |\sin(t)|)$ in Example 5.1, then HSSwNT (5.2) is almost surely exponentially unstable. In fact, some literature have revealed that the stability of stochastic system depends on the size of time delay and established the delay-dependent stability criteria (see, e.g., [23, 24, 25, 26]). In particular, Fei et al.[27] first removed the restriction that the coefficient of the underlying hybrid stochastic delay differential equations (HSDDEs) is bounded by a linear function, and established the delay dependent criteria for highly nonlinear HSDDEs. Shen et al. [28] extended the delay-dependent stability results of [27] to the case of the neutral HSDDEs. Song et al. [29] further investigated the stability of the neutral HSDDEs and they extended the results in

[28] from neutral HSDDEs with a single delay to multiple delays. However, the time delay in above papers [23]-[29] is either constant or differentiable with its derivative less than 1. This condition is not a natural feature of stochastic delay systems in the real world. Obviously, the delay function $\bar{\delta}(t) = 0.75(1 + |\sin(t)|)$ we mentioned earlier is neither a constant nor differentiable. So, it is necessary to study stochastic systems with more general time delay. In 2022, Dong and Mao [30] studied a class of hybrid stochastic systems with time-varying delay

$$dx(t) = f(x(t), x(t - \tau(t)), r(t), t)dt + g(x(t), x(t - \tau(t)), r(t), t)dw(t), \tag{1.2}$$

where $\tau(t) \in [\lambda_1, \lambda_2]$ is a Borel measurable function which has the property (2.2). They first used the delay feedback control to stabilize the hybrid stochastic delay systems (1.2) with non-differentiable time delay. Recently, Dong et al.[31] extended the stabilization result of [30] to the case of hybrid stochastic delay systems with Lévy jump, while Hu et al. [32] investigated a class of highly nonlinear hybrid stochastic delay systems and established several new results on the asymptotic boundedness and stability for the hybrid stochastic delay systems.

Motivated by the above results, this paper examines stability results of the following hybrid stochastic delay systems with neutral term (HSDSwNT)

$$d[x(t) - D(x(t - \delta(t)), r(t))] = f(x(t), x(t - \delta(t)), t, r(t))dt + g(x(t), x(t - \delta(t)), t, r(t))dw(t). \tag{1.3}$$

Sufficient conditions are given that ensure the H_∞ and almost sure asymptotic stability of equation (1.3), even in the presence of non-differentiable delays. Since these two stabilities do not reveal the convergence rate, this paper also examines the general decay stability of HSDSwNT, which includes the exponential stability and the polynomial stability (also called as algebraic stability) (see [33]-[38]). Compared with the existing literature, the main contributions and challenges of this paper are highlighted as follows:

(i) The variable time delay $\delta(t)$ in the given HSDSwNT is now only required to satisfy much weaker conditions than those in most existing papers. In particular, it is no longer required to be differentiable. Coping with the much weaker conditions imposed on $\delta(t)$ presents a challenge. New mathematical techniques are developed to tackle the challenge.

(ii) In many existing papers, the boundedness is often imposed on $\delta(t)$, which excludes many interesting systems with unbounded delay, for example, the system described by the pantograph equations. It is another challenge to allow $\delta(t)$ to be unbounded, which arises difficulties in the analysis of the existence and uniqueness, H_∞ stability and almost sure asymptotic stability of the solution.

(iii) Moreover, the delay dependent stability is discussed, and the impacts of the time delay and the neutral term on the stability of equation (1.3) are revealed. By using the Lyapunov function method and the M-matrix theory, sufficient conditions on the asymptotic stability are also presented for HSDSwNT. At the same time, the moment and almost sure stability with general decay rate for HSDSwNT are also investigated, which includes both the exponential stability and the polynomial stability.

The paper is organized as follows. In Section 2, we introduce some hypotheses concerning Eq.(1.3) and establish the existence and uniqueness of solutions to it. In Sections 3 and 4, we will show the main results of our paper, where several useful criteria will be established on the almost sure asymptotic stability as well as the general decay stability for Eq.(1.3). To make our theory more applicable, we apply the M-matrix theory to get some sufficient criteria on these two types of stability in Section 5. Finally, we will discuss two examples to illustrate our theory.

2 Preliminaries and the global solution

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions. Let $w(t)$ be an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. For any $a, b \in R$, $a \vee b$ and $a \wedge b$ indicate the maximum value and the minimum value between a and b, respectively. Let $\bar{h} > 0$ and $\mathbb{C}([-\bar{h}, 0]; R^n)$ denote the family of continuous functions φ from $[-\bar{h}, 0] \to R^n$ with the norm $||\varphi|| = \sup_{-\bar{h}\leq\theta\leq 0} |\varphi(\theta)|$, where $|.|$ is the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by $A^\top$. If A is a matrix, its norm $||A||$ is defined by $||A|| = \sup\{|Ax| : |x| = 1\}$. Let $\mathcal{L}^2_{\mathcal{F}_0}([-\bar{h}, 0]; R^n)$ denote the family of all $\mathcal{F}_0$-measurable, $\mathbb{C}([-\bar{h}; 0], R^n)$-valued random variables $\varphi = \{\varphi(\theta) : -\bar{h} \leq \theta \leq 0\}$ such that $E||\varphi||^2 < \infty$. Let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2 \dots N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by:

$$\mathbb{P}(r(t+\Delta) = j | r(t) = i) = \begin{cases} \gamma_{ij}\Delta + \circ(\Delta), & if \quad i \neq j, \\ 1 + \gamma_{ij}\Delta + \circ(\Delta), & if \quad i = j. \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(.)$ is independent of the Brownian motion $w(.)$.

In order to deal with non-differentiable time delay, we give the following condition.

Assumption 2.1 *Assume that the delay function $\delta(t)$ is a Borel measurable function from R_+ to $[h, \infty)$, with the following properties*

$$-\bar{h} := \inf_{0\leq t<\infty} (t - \delta(t)) > -\infty \tag{2.1}$$

and

$$h_0 := \limsup_{\Delta\to 0} \Big(\sup_{s\geq -\bar{h}} \frac{m(Q_{s,\Delta})}{\Delta} \Big) < \infty \tag{2.2}$$

where h is a non-negative constant, $Q_{s,\Delta} = \{t \in R_+ : \ t - \delta(t) \in [s, s + \Delta)\}$ and $m(.)$ denotes the Lebesgue measure on R_+.

Remark 2.2 In the existing literature involving variable delay (e.g., [8, 27]), the delay function $\delta(t) : R_+ \to R_+$ is often either constant or differentiable with its derivative being bounded by $\hat{h} \in (0, 1)$. That is $\frac{d\delta(t)}{dt} \leq \hat{h} < 1$, $\forall t \geq 0$. However, these two conditions might not be a natural feature of stochastic delay systems in the real world. For example, piece-wise constant delays occur frequently in sampled-data controls but these are not differentiable.

Remark 2.3 In [19, 20, 21, 30], $\delta(t)$ is allowed to be nondifferentiable which is an important breakthrough. However, the boundedness imposed on $\delta(t)$ is also conservative, at least it excludes a class of stochastic differential equations with proportional delay, where the delay function $\delta(t) = (1-q)t$, $t \geq 0$ with $q \in (0,1)$. In this case, we provide Assumption 2.1 that the delay function $\delta(t)$ can be unbounded, so the existing results of [19, 20, 21, 30] are improved and generalized.

Remark 2.4 In practice, there are many delay functions that satisfy Assumption 2.1. Consider the delay function $\delta(t) = \theta(t + |\sin t|)$, $\theta \in (0, 0.5)$ as an example. It can be seen that $\delta(t) = \theta(t + |\sin t|)$ from R_+ to $[0, \infty)$ which obeys the Lipschitz condition $|\delta(t) - \delta(s)| \leq 2\theta(t-s)$ for any $0 \leq s < t < \infty$. In fact, we can conclude that it satisfies Assumption 2.1 with $\bar{h} = 0 > -\infty$ and $h_0 = 1/(1-2\theta) < \infty$. Obviously, we get that $\delta(t) \leq 2\theta t$ for all $t \geq 0$. Hence, $-\bar{h} = \inf_{0 \leq t < \infty}(t - \delta(t)) \geq (1-2\theta)\inf_{0 \leq t < \infty} t = 0$, namely $\bar{h} \leq 0$. Meantime, $-\bar{h} \leq 0 - \delta(0) = 0$, i.e., $\bar{h} \geq 0$. Therefore, we must have $\bar{h} = 0$. Next, we will show $h_0 = 1/(1-2\theta)$. Let $s \geq -\bar{h}$ and $\Delta \in (0,1)$ be arbitrary. Set $v = \inf\{t \in Q_{s,\Delta}\}$, we have $s \leq v - \delta(v) < s + \Delta$. If $t \geq v + \Delta/(1-2\theta)$, then we get

$$\begin{aligned} t - \delta(t) - s &\geq t - \delta(t) - |v - \delta(v)| \\ &\geq (1-2\theta)(t-v) \geq \Delta. \end{aligned}$$

Hence, we obtain $t \in Q_{s,\Delta}$. That is to say, we have shown $Q_{s,\Delta} \subset [v, v + \Delta/(1-2\theta))$ and $\frac{m(Q_{s,\Delta})}{\Delta} \leq 1/(1-2\theta)$. For arbitrary $s \geq -\bar{h}$ and $h \in (0,1)$, we see that Assumption 2.1 holds with $h_0 = 1/(1-2\theta)$. As another example, if $\delta(t)$ is differentiable and its derivative is bounded by $l \in (0,1)$, we can show that δ_t satisfies Assumption 2.1 with $h_0 = 1/(1-l)$.

Let us consider the nonlinear hybrid stochastic delay systems with neutral term

$$d[x(t) - D(x(t-\delta(t)), r(t))] = f(x(t), x(t-\delta(t)), t, r(t))dt + g(x(t), x(t-\delta(t)), t, r(t))dw(t) \quad (2.3)$$

with the initial data $\{x(t) : -\bar{h} \leq t \leq 0\} = \xi \in \mathcal{L}^2_{\mathcal{F}_0}([-\bar{h}, 0]; R^n)$, where the coefficients $f : R^n \times R^n \times R_+ \times S \to R^n$, $g : R^n \times R^n \times R_+ \times S \to R^{n \times m}$ and $D : R^n \times S \to R^n$ are Borel measurable functions. The following assumptions are imposed on the coefficients f, g and D.

Assumption 2.5 *For each integer $d \geq 1$, there exists a positive constant L_d such that*

$$|f(x_1, y_1, t, i) - f(x_2, y_2, t, i)| \vee |g(x_1, y_1, t, i) - g(x_2, y_2, t, i)| \leq L_d(|x_1 - x_2| + |y_1 - y_2|),$$

for those $x_1, y_1, x_2, y_2 \in R^n$ with $|x_1| \vee |y_1| \vee |x_2| \vee |y_2| \leq d$ and any $(t, i) \in R_+ \times S$. Moreover, we assume that $f(0, 0, t, i) = g(0, 0, t, i) = 0$ for all $(t, i) \in R_+ \times S$.

Assumption 2.6 *There exists a constant $k_i \in (0,1)$ such that*

$$|D(u, i) - D(v, i)| \leq k_i |u - v|,$$

for all $u, v \in R^n$ and $i \in S$. Moreover, we assume that $D(0, i) = 0$ for all $i \in S$ and let $k_0 = \max_{i \in S} k_i$.

It is known that Assumptions 2.1, 2.5 and 2.6 only guarantee that Eq.(1.3) has a unique maximal solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition in terms of the Lyapunov function. Let us denote by $C^{2,1}(R^n \times R_+ \times S; R_+)$ the family of all continuous non-negative functions $V(x,t,i)$ defined on $R^n \times R_+ \times S$ such that for each $i \in S$, they are continuously twice differentiable in x and once in t. For $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, we define the function $\mathcal{L}V : R^n \times R^n \times R_+ \times S \to R$ by

$$\begin{aligned}\mathcal{L}V(x,y,t,i) &= V_t(x - D(y,i),t,i) + V_x(x - D(y,i),t,i)f(x,y,t,i) \\ &\quad + \frac{1}{2}trace[g^\top(x,y,t,i)V_{xx}(x - D(y,i),t,i)g(x,y,t,i)] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} V(x - D(y,i),t,j), \end{aligned} \tag{2.4}$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t},\ V_x(x,t,i) = \Big(\frac{\partial V(x,t,i)}{\partial x_1}, \cdots, \frac{\partial V(x,t,i)}{\partial x_n}\Big),$$
$$V_{xx}(x,t,i) = \Big(\frac{\partial^2 V(x,t,i)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

Assumption 2.7 *(Khasminskii-type condition) There exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ and positive constants $c_1, c_2, \alpha_i, (i = 1,2,3), \gamma > 2$, such that*

$$c_1|x|^2 \le V(x,t,i) \le c_2|x|^2 \tag{2.5}$$

for $(x,t,i) \in R^n \times R_+ \times S$, and

$$\mathcal{L}V(x,y,t,i) \le \alpha_1(1 + |x|^2 + |y|^2) - \alpha_2|x|^\gamma + \alpha_3|y|^\gamma \tag{2.6}$$

for all $(x,y,t,i) \in R^n \times R^n \times R_+ \times S$.

Remark 2.8 This paper mainly concerns the Khaminskii-type condition, which is more general and covers the (one-sided) linear growth condition. In fact, in Assumption 2.7, when $V(x,t,i) = |x|^2$ and $\alpha_2 = \alpha_3 = 0$, we obtain the one-sided linear growth condition. It is well-known that the one-sided linear growth condition is more general than the classical linear growth condition (see, e.g., [1]).

Lemma 2.9 *([1]) Let $p \ge 1$ and $a, b \in R^n$. Then, for any $\delta \in (0,1)$,*

$$|a + b|^p \le \frac{|a|^p}{(1-\delta)^{p-1}} + \frac{|b|^p}{\delta^{p-1}}.$$

Lemma 2.10 *([30]) Let Assumption 2.1 holds. Let $T > T_0 \ge 0$ and $\Psi : [-\bar{h}, T - h] \to R_+$ be a continuous function. Then*

$$\int_{T_0}^{T} \Psi(t - \delta(t))dt \le h_0 \int_{T_0-\bar{h}}^{T-h} \Psi(t)dt. \tag{2.7}$$

Now, we give the existence and uniqueness results of the solution for equation (1.3). For convenience, we set

$$z(t) = x(t) - D(x(t-\delta(t)), r(t)),$$

which will be used throughout all the paper.

Theorem 2.11 *Let Assumptions 2.1, 2.5, 2.6 and 2.7 hold with $h > 0$. Then for any given initial data ξ, there is a unique global solution $x(t)$ to Eq.(1.3) on $t \in [-\bar{h}, \infty)$. Moreover, the solution has the properties that*

$$\sup_{0\le t\le T} \mathbb{E}|x(t)|^2 < \infty \quad \textit{and} \quad \mathbb{E}\int_0^T |x(s)|^\gamma ds < \infty \tag{2.8}$$

for all $T > 0$.

Proof. Since the coefficients of Eq.(1.3) are locally Lipschitz continuous, for any given initial data ξ, there is a maximal local solution $x(t)$ on $t \in [0, \sigma_e)$, where σ_e is the explosion time. To show that this solution is global, we only need to prove that $\sigma_e = \infty$ a.s. By Assumption 2.6, we have

$$\begin{aligned} |z(0)| &\le |x(0)| + |D(x(-\delta(0)), r(0))| \le |x(0)| + k_i|x(-\delta(0))| \\ &\le (1 + k_0)\|\xi\|. \end{aligned} \tag{2.9}$$

Noe, let $\bar{k}_0 > 0$ be sufficiently large for $(1+k_0)\|\xi\| < \bar{k}_0$. For each integer $k \ge \bar{k}_0$, define the stopping time $\tau_k = \inf\{t \in [0, \sigma_e) : |z(t)| \ge k\}$. Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \le \sigma_e$ a.s. Note if we can show that $\tau_\infty = \infty$ a.s., then $\sigma_e = \infty$ a.s.

We shall first show that $\tau_\infty > h$ a.s. For any $k \ge \bar{k}_0$ and $t \in [0, h]$, by the Itô formula (see e.g. [1]) and condition (2.6), we can show that,

$$\begin{aligned} \mathbb{E}V(z(\tau_k \wedge t), \tau_k \wedge t, r(\tau_k \wedge t)) &\le \mathbb{E}V(z(0), 0, r(0)) + \mathbb{E}\int_0^{\tau_k\wedge t} \Big(\alpha_1(1 + |x(s)|^2 \\ &+ |x(s-\delta(s))|^2) - \alpha_2|x(s)|^\gamma + \alpha_3|x(s-\delta(s))|^\gamma\Big)ds. \end{aligned} \tag{2.10}$$

By condition (2.5) and Assumption 2.6, we then get

$$\begin{aligned} c_1\mathbb{E}|z(\tau_k \wedge t)|^2 &\le \mathbb{E}V(z(0), 0, r(0)) + \mathbb{E}\int_0^{\tau_k\wedge t} \Big(\alpha_1(1 + |x(s-\delta(s))|^2) + \alpha_3|x(s-\delta(s))|^\gamma\Big)ds \\ &+ \alpha_1\mathbb{E}\int_0^{\tau_k\wedge t} |x(s)|^2 ds - \alpha_2\mathbb{E}\int_0^{\tau_k\wedge t} |x(s)|^\gamma ds \\ &\le H_1 + 2\alpha_1\mathbb{E}\int_0^{\tau_k\wedge t} |z(s)|^2 ds - \alpha_2\mathbb{E}\int_0^{\tau_k\wedge t} |x(s)|^\gamma ds \end{aligned} \tag{2.11}$$

where $H_1 = \mathbb{E}V(z(0), 0, r(0)) + \mathbb{E}\int_0^h \Big(\alpha_1(1 + 2k_0|x(s-\delta(s))|^2) + \alpha_3|x(s-\delta(s))|^\gamma\Big)ds$. Noting that $-\bar{h} \le t - h(t) \le 0$, we have

$$\begin{aligned} H_1 &\le c_2\mathbb{E}|z(0)|^2 + \mathbb{E}\int_0^h \Big(\alpha_1(1 + 2k_0|x(t-\delta(s))|^2) + \alpha_3|x(t-\delta(s))|^\gamma\Big)ds \\ &\le c_2(1 + k_0)^2\mathbb{E}||\xi||^2 + (\alpha_1 + 2\alpha_1 k_0\mathbb{E}\|\xi\|^2 + \alpha_3\mathbb{E}\|\xi\|^\gamma)h < \infty. \end{aligned}$$

Then it follows from (2.11) that $\mathbb{E}|z(\tau_k \wedge t)|^2 \leq \frac{H_1}{c_1} + \frac{2\alpha_1}{c_1}\mathbb{E}\int_0^t |z(\tau_k \wedge s)|^2 ds$. Hence, by the Gronwall inequality, we have

$$\mathbb{E}|z(\tau_k \wedge t)|^2 \leq \frac{H_1}{c_1} e^{\frac{2\alpha_1}{c_1}h}, \quad 0 \leq t \leq h \tag{2.12}$$

for any $k \geq \bar{k}_0$. In particular, $\mathbb{E}|z(\tau_k \wedge h)|^2 \leq \frac{H_1}{c_1} e^{\frac{2\alpha_1}{c_1}h}$, $\forall k \geq \bar{k}_0$. This implies $k^2\mathbb{P}(\tau_k \leq h) \leq \frac{H_1}{c_1} e^{\frac{2\alpha_1}{c_1}h}$. Letting $k \to \infty$, we hence obtain that $\mathbb{P}(\tau_\infty \leq h) = 0$, namely $\mathbb{P}(\tau_\infty > h) = 1$. Letting $k \to \infty$ in (2.12) yields

$$\mathbb{E}|z(t)|^2 \leq \frac{H_1}{c_1} e^{\frac{2\alpha_1}{c_1}h} \tag{2.13}$$

and

$$\begin{aligned} \mathbb{E}|x(t)|^2 &\leq 2\mathbb{E}|z(t)|^2 + 2\mathbb{E}|D(x(t-\delta(t)))|^2 \\ &\leq 2\frac{H_1}{c_1} e^{\frac{2\alpha_1}{c_1}h} + 2k_0\mathbb{E}\|\xi\|^2 < \infty \end{aligned} \tag{2.14}$$

for all $t \in [0, h]$. Setting $t = h$ in (2.11) and then letting $k \to \infty$, we also get $\alpha_2\mathbb{E}\int_0^h |x(s)|^\gamma ds \leq H_1 + 2\alpha_1\mathbb{E}\int_0^h |z(s)|^2 ds$. Consequently, using (2.13), we obtain

$$\mathbb{E}\int_0^h |x(s)|^\gamma ds \leq \frac{1}{\alpha_2}(H_1 + 2\alpha_1 h\frac{H_1}{c_1} e^{\frac{2\alpha_1}{c_1}h}). \tag{2.15}$$

Let us now proceed to prove $\tau_\infty > 2h$ a.s. Given that we have shown (2.14) and (2.15). For any $k \geq \bar{k}_0$ and $t \in [0, 2h]$, it follows from (2.6) that

$$c_1\mathbb{E}|z(\tau_k \wedge t)|^2 \leq H_2 + 2\alpha_1\mathbb{E}\int_0^{\tau_k \wedge t} |z(s)|^2 ds - \alpha_2\mathbb{E}\int_0^{\tau_k \wedge t} |x(s)|^\gamma ds \tag{2.16}$$

where $H_2 = H_1 + \mathbb{E}\int_h^{2h} \Big(\alpha_1(1 + 2k_0|x(s-\delta(s))|^2) + \alpha_3|x(s-\delta(s))|^\gamma\Big) ds$. Noting that $t \in [0, 2h]$, we get $-\bar{h} \leq t - \delta(t) \leq h$. By Lemma 2.10, (2.14) and (2.15), we have

$$\begin{aligned} H_2 &\leq H_1 + \alpha_1 h + h_0\mathbb{E}\int_{h-\bar{h}}^h \Big(2\alpha_1 k_0|x(s)|^2 + \alpha_3|x(s)|^\gamma\Big) ds \\ &\leq (1 + h_0)H_1 + h_0(2\alpha_1 k_0\mathbb{E}\|\xi\|^2 + \alpha_3\mathbb{E}\|\xi\|^\gamma)(\bar{h} - h) < \infty. \end{aligned}$$

Consequently $\mathbb{E}|z(\tau_k \wedge t)|^2 \leq \frac{H_2}{c_1} + \frac{2\alpha_1}{c_1}\mathbb{E}\int_0^t |z(\tau_k \wedge s)|^2 ds$. Hence, by the Gronwall inequality, we have

$$\mathbb{E}|z(\tau_k \wedge t)|^2 \leq \frac{H_2}{c_1} e^{\frac{4\alpha_1}{c_1}h}, \quad 0 \leq t \leq 2h \tag{2.17}$$

for any $k \geq \bar{k}_0$. In particular, $\mathbb{E}|z(\tau_k \wedge 2h)|^2 \leq \frac{H_2}{c_1} e^{\frac{4\alpha_1}{c_1}h}$, $\forall k \geq \bar{k}_0$. This implies $k^2\mathbb{P}(\tau_k \leq 2h) \leq \frac{H_2}{c_1} e^{\frac{4\alpha_1}{c_1}h}$. Letting $k \to \infty$, we hence obtain that $\mathbb{P}(\tau_\infty \leq 2h) = 0$, namely $\mathbb{P}(\tau_\infty > 2h) = 1$. Letting $k \to \infty$ in (2.17) yields

$$\mathbb{E}|z(t)|^2 \leq \frac{H_2}{c_1} e^{\frac{4\alpha_1}{c_1}h}, \quad 0 \leq t \leq 2h. \tag{2.18}$$

By (2.18), it follows that

$$\begin{aligned}\mathbb{E}|x(t)|^2 &\leq 2\mathbb{E}|z(t)|^2 + 2\mathbb{E}|D(x(t-\delta(t)))|^2 \\ &\leq 2\frac{H_2}{c_1}e^{\frac{4\alpha_1}{c_1}h} + 2k_0 \sup_{-\bar{h}\leq t\leq h}\mathbb{E}|x(t)|^2 < \infty\end{aligned}$$

for any $t \in [0, 2h]$. Setting $t = 2h$ in (2.16) and then letting $k \to \infty$, we also get $\alpha_2\mathbb{E}\int_0^{2h}|x(s)|^\gamma ds \leq H_2 + 2\alpha_1\mathbb{E}\int_0^{2h}|z(s)|^2 ds$. Consequently, using (2.18), we obtain

$$\mathbb{E}\int_0^{2h}|x(s)|^\gamma ds \leq \frac{1}{\alpha_2}(H_2 + 4\alpha_1 h\frac{H_2}{c_1}e^{\frac{4\alpha_1}{c_1}h}).$$

Repeating this procedure, we can show that, for any integer $i \geq 1$, $\tau_\infty > ih$ a.s.,

$$\mathbb{E}|z(t)|^2 \leq \frac{H_i}{c_1}e^{\frac{2^i\alpha_1}{c_1}h}, \quad 0 \leq t \leq ih$$

where

$$\begin{aligned}H_i &= H_{i-1} + \mathbb{E}\int_{(i-1)h}^{ih}\Big(\alpha_1(1 + 2k_0|x(s-\delta(s))|^2) + \alpha_3|x(s-\delta(s))|^\gamma\Big)ds \\ &\leq (1+h_0)H_{i-1} + h_0\mathbb{E}\int_{(i-2)h-(\bar{h}-h)}^{(i-2)h}[2\alpha_1 k_0|x(s-\delta(s))|^2 + \alpha_3|x(s-\delta(s))|^\gamma]ds \\ &< \infty.\end{aligned}$$

We must therefore have $\tau_\infty = \infty$ a.s.. Similarly, for any $t \in [0, ih]$, we can show

$$\begin{aligned}\mathbb{E}|x(t)|^2 &\leq 2\mathbb{E}|z(t)|^2 + 2\mathbb{E}|D(x(t-\delta(t)))|^2 \\ &\leq 2\frac{H_i}{c_1}e^{\frac{2^i\alpha_1}{c_1}h} + 2k_0 \sup_{-\bar{h}\leq t\leq (i-1)h}\mathbb{E}|x(t)|^2 < \infty\end{aligned}$$

and $\mathbb{E}\int_0^{ih}|x(s)|^\gamma ds \leq \frac{1}{\alpha_2}(H_i + 2i\alpha_1 h\frac{H_i}{c_1}e^{\frac{2^i\alpha_1}{c_1}h})$. The proof is therefore complete. □

Remark 2.12 In fact, the proof of Theorem 2.11 is also applicable to the case of bounded delay, namely $\delta(t) \in [h, \tilde{h})$. On the other hand, if the lower bound h is 0, we can obtain the existence and uniqueness of the solution to equation (1.3), but the condition $\alpha_2 > \alpha_3 h_0$ is required.

Theorem 2.13 *Let Assumptions 2.1, 2.5, 2.6 and 2.7 hold with $h = 0$ and $\alpha_2 > \alpha_3 h_0$. Then for any given initial data ξ, there is a unique global solution $x(t)$ to Eq.(1.3) on $t \in [-\bar{h}, \infty)$. Moreover, the solution has the properties that*

$$\sup_{0\leq t\leq T}\mathbb{E}|x(t)|^2 < \infty \quad \textit{and} \quad \mathbb{E}\int_0^T|x(s)|^\gamma ds < \infty \tag{2.19}$$

for all $T > 0$.

Proof. Fix $T > 0$ arbitrarily. For all $t \in [0, T]$ and $k \geq \bar{k}_0$, using Lemma 2.10, we can obtain from (2.10) that

$$\begin{aligned}\mathbb{E}V(z(\tau_k \wedge t), \tau_k \wedge t, r(\tau_k \wedge t)) &\leq \tilde{H}_1 + \alpha_1(1+h_0)\int_0^{\tau_k \wedge t} \mathbb{E}|x(s)|^2 ds \\ &- (\alpha_2 - \alpha_3 h_0)\int_0^{\tau_k \wedge t} \mathbb{E}|x(s)|^\gamma ds \end{aligned} \tag{2.20}$$

where $\tilde{H}_1 = \mathbb{E}V(z(0), 0, r(0)) + \alpha_1 T + h_0 \mathbb{E}\int_{-\bar{h}}^0 (\alpha_1|x(s)|^2 + \alpha_3|x(s)|^\gamma)ds < \infty$. Then, it follows from (2.20) that

$$c_1 \mathbb{E}|z(\tau_k \wedge t)|^2 \leq \tilde{H}_1 + \alpha_1(1+h_0)\int_0^{\tau_k \wedge t} \mathbb{E}|x(s)|^2 ds. \tag{2.21}$$

By Lemma 2.9 and Assumption 2.6, we get

$$\begin{aligned}\mathbb{E}|x(\tau_k \wedge t)|^2 &\leq \frac{\mathbb{E}|z(\tau_k \wedge t)|^2}{1-k_0} + \frac{\mathbb{E}|D(x(\tau_k \wedge t - \delta(\tau_k \wedge t), r(\tau_k \wedge t)))|^2}{k_0} \\ &\leq \frac{\mathbb{E}|z(\tau_k \wedge t)|^2}{1-k_0} + k_0 \mathbb{E}|x(\tau_k \wedge t - \delta(\tau_k \wedge t))|^2.\end{aligned}$$

This implies

$$\begin{aligned}\sup_{0\leq t\leq T} \mathbb{E}|x(\tau_k \wedge t)|^2 &\leq \frac{\sup_{0\leq t\leq T}\mathbb{E}|z(\tau_k \wedge t)|^2}{1-k_0} + k_0 \sup_{0\leq t\leq T} \mathbb{E}|x(\tau_k \wedge t - \delta(\tau_k \wedge t))|^2 \\ &\leq \frac{\tilde{H}_1}{c_1(1-k_0)} + \frac{\alpha_1(1+h_0)}{c_1(1-k_0)}\int_0^T \sup_{0\leq s\leq t} \mathbb{E}|x(\tau_k \wedge s)|^2 dt \\ &+ k_0 \mathbb{E}\|\xi\|^2 + k_0 \sup_{0\leq t\leq T} \mathbb{E}|x(\tau_k \wedge t)|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}\sup_{0\leq t\leq T} \mathbb{E}|x(\tau_k \wedge t)|^2 &\leq \frac{\tilde{H}_1}{c_1(1-k_0)^2} + \frac{k_0}{1-k_0}\mathbb{E}\|\xi\|^2 \\ &+ \frac{\alpha_1(1+h_0)}{c_1(1-k_0)^2}\int_0^T \sup_{0\leq s\leq t} \mathbb{E}|x(\tau_k \wedge s)|^2 dt.\end{aligned}$$

An application of the Gronwall inequality yields

$$\sup_{0\leq t\leq T} \mathbb{E}|x(\tau_k \wedge t)|^2 \leq [\frac{\tilde{H}_1}{c_1(1-k_0)^2} + \frac{k_0}{1-k_0}\mathbb{E}\|\xi\|^2] e^{\frac{\alpha_1(1+h_0)}{c_1(1-k_0)^2}T}. \tag{2.22}$$

Similar to the proof of Theorem 2.11, we can show that $\tau_k \geq T$ a.s. Since $T > 0$ is arbitrary, we must have $\tau_k = \infty$. Letting $k \to \infty$ in (2.22), we obtain that $\sup_{0\leq t\leq T} \mathbb{E}|x(t)|^2 < \infty$. Similarly, we can get $\mathbb{E}\int_0^T |x(s)|^\gamma ds < \infty$. The proof is therefore complete. □

3 Almost sure asymptotic stability

In this section, we shall study the H_∞ stability and almost sure asymptotic stability of HSDSwNT (1.3).

Definition 3.1 *The solution of Eq.(1.3) is said to be H_∞ stable in pth moment ($p > 0$) if*

$$\int_0^\infty \mathbb{E}|x(s)|^p ds < \infty$$

for all $\xi \in \mathcal{L}^2_{\mathcal{F}_0}([-\bar{h}, 0]; R^n)$.

Definition 3.2 *The solution of Eq.(1.3) is said to be almost surely asymptotically stable if*

$$\lim_{t\to\infty} x(t) = 0 \quad a.s.$$

for all $\xi \in \mathcal{L}^2_{\mathcal{F}_0}([-\bar{h}, 0]; R^n)$.

Theorem 3.3 *Let Assumptions 2.1, 2.5, 2.6 and 2.7 hold except that in Assumption 2.7, (2.6) is replaced by*

$$\mathcal{L}V(x, y, t, i) \le -\beta_1|x|^2 + \beta_2|y|^2 - \beta_3|x|^\gamma + \beta_4|y|^\gamma \tag{3.1}$$

for all $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$, where $\beta_1 > \beta_2 h_0 \ge 0$ and $\beta_3 > \beta_4 h_0 \ge 0$. Then for any given initial data ξ, the solution $x(t)$ of Eq.(1.3) has the property

$$\int_0^\infty \mathbb{E}|x(s)|^\gamma ds < \infty. \tag{3.2}$$

Proof. We first observe that (3.1) is stronger than (2.6). So, by Theorems 2.11 and 2.13, for any given initial data ξ, Eq.(1.3) has a unique global solution $x(t)$ on $t \ge 0$. For each integer $k \ge \bar{k}_0$, define the stopping time $\sigma_k = \inf\{t \ge 0 : |z(t)| \ge k\}$. By the Itô formula and Lemma 2.10, it is not difficult to show that

$$c_1\mathbb{E}|z(\sigma_k \wedge t)|^2 \quad \le \quad \tilde{H}_2 - (\beta_3 - \beta_4 h_0)\mathbb{E}\int_0^{\sigma_k \wedge t} |x(s)|^\gamma ds \tag{3.3}$$

where $\tilde{H}_2 = \mathbb{E}V(z(0), 0, r(0)) + h_0\mathbb{E}\int_{-\bar{h}}^0 (\beta_2|x(s)|^2 + \beta_4|x(s)|^\gamma)ds < \infty$. Letting $k \to \infty$, by applying the Fubini theorem, we get $\int_0^t \mathbb{E}|x(s)|^\gamma ds \le \frac{\tilde{H}_2}{\beta_3 - \beta_4 h_0}$. Letting $t \to \infty$ yields the desired assertion (3.2). The proof is therefore complete. □

Theorem 3.4 *Let conditions of Theorem 3.3 hold. Then for any given initial data ξ, the solution $x(t)$ of Eq.(1.3) has the property that*

$$\lim_{t\to\infty} x(t) = 0 \quad a.s. \tag{3.4}$$

Proof. Let us first show

$$\sup_{0\le t<\infty}|z(t)|<\infty \quad a.s. \tag{3.5}$$

If this were false, then there is a $\bar{\Omega}_1=\{\omega\in\Omega:\sup_{0\le t<\infty}|z(t)|=\infty\}$ and ε_1 such that $\mathbb{P}(\bar{\Omega}_1)=\varepsilon_1>0$. Choose a number $M_1>\frac{4\tilde{H}_2}{c_1\varepsilon_1}$. Define the stopping time $\sigma=\inf\{t\ge 0:|z(t)|^2\ge M_1\}$. Then we can find a sufficiently large number T such that $\mathbb{P}(0<\sigma\le T)\ge 0.25\varepsilon_1$. In a similar way as we did in (3.3), we can show $c_1\mathbb{E}|z(\sigma\wedge t)|^2\le\tilde{H}_2$. This implies that

$$\begin{aligned}\frac{\tilde{H}_2}{c_1} &\ge \mathbb{E}(|z(\sigma\wedge t)|^2)\\ &\ge \mathbb{E}(|z(\sigma\wedge t)|^2 I_{\{0<\sigma\le T\}})\\ &\ge M_1P(0<\sigma\le T)\ge 0.25M_1\varepsilon_1.\end{aligned}$$

But this contradicts the fact $M_1>\frac{4\tilde{H}_2}{c_1\varepsilon_1}$. Hence, (3.5) must hold. Now, we will show that

$$\sup_{0\le t<\infty}|x(t)|<\infty \quad a.s. \tag{3.6}$$

For any $T>0$, by Assumption 2.6, we have

$$\begin{aligned}|x(t)| &\le |z(t)|+|D(x(t-\delta(t)),r(t)))|\\ &\le |z(t)|+k_0|x(t-\delta(t))|, \quad t\in[0,T].\end{aligned} \tag{3.7}$$

This implies

$$\begin{aligned}\sup_{0\le t\le T}|x(t)| &\le \sup_{0\le t\le T}|z(t)|+k_0\sup_{0\le t\le T}|x(t-\delta(t))|\\ &\le \sup_{0\le t\le T}|z(t)|+k_0\|\xi\|+k_0\sup_{0\le t\le T}|x(t)|.\end{aligned}$$

Hence, we get $\sup_{0\le t\le T}|x(t)|\le\frac{1}{1-k_0}\Big(\sup_{0\le t\le T}|z(t)|+k_0\|\xi\|\Big)$. Letting $T\to\infty$ and using (3.5), we obtain that (3.6). In what follows, we shall claim that

$$\lim_{t\to\infty}|z(t)|=0 \quad a.s. \tag{3.8}$$

By Lemmas 2.9, 2.10 and Assumption 2.6, we obtain

$$\begin{aligned}\int_0^t\mathbb{E}|z(s)|^\gamma ds &\le \frac{1}{(1-k_0)^{\gamma-1}}\int_0^t\mathbb{E}|x(s)|^\gamma ds+\frac{1}{k_0^{\gamma-1}}\int_0^t\mathbb{E}|D(x(s-\delta(s)),r(s))|^\gamma ds\\ &\le \frac{1}{(1-k_0)^{\gamma-1}}\int_0^t\mathbb{E}|x(s)|^\gamma ds+k_0\int_0^t\mathbb{E}|x(s-\delta(s))|^\gamma ds\\ &\le \frac{1}{(1-k_0)^{\gamma-1}}\int_0^t\mathbb{E}|x(s)|^\gamma ds+k_0h_0\int_{-\bar{h}}^t\mathbb{E}|x(s)|^\gamma ds\\ &\le k_0h_0\int_{-\bar{h}}^0\mathbb{E}|x(s)|^\gamma ds+\Big[\frac{1}{(1-k_0)^{\gamma-1}}+k_0h_0\Big]\int_0^t\mathbb{E}|x(s)|^\gamma ds.\end{aligned}$$

By (3.2) and the Fubini's theorem, it follows that

$$\mathbb{E}\int_0^\infty |z(s)|^\gamma ds < \infty. \tag{3.9}$$

This implies $\int_0^\infty |z(s)|^\gamma ds < \infty \ \ a.s.$ Hence

$$\liminf_{t\to\infty} |z(t)| = 0 \ \ a.s.$$

Now, we turn to proving that

$$\limsup_{t\to\infty} |z(t)| = 0 \ \ a.s \tag{3.10}$$

by contradiction. Suppose (3.10) is false, then there exists $\varepsilon_2 > 0$ such that

$$\mathbb{P}(\bar{\Omega}_2) \geq 3\varepsilon_2, \tag{3.11}$$

where $\bar{\Omega}_2 = \{\omega \in \Omega : \ \limsup_{t\to\infty} |z(t)| > 2\varepsilon_2\}$. Recalling (3.6) as well as the boundedness of the initial data ξ, we can find a positive number d sufficiently large for

$$\mathbb{P}(\bar{\Omega}_3) \geq 1 - \varepsilon_2, \tag{3.12}$$

where $\bar{\Omega}_3 = \{\omega \in \Omega : \ \sup_{0\leq t<\infty} |z(t)| < d\}$. Hence, by (3.11) and (3.12), we have

$$\mathbb{P}(\bar{\Omega}_2 \cap \bar{\Omega}_3) \geq 2\varepsilon_2.$$

Let us now define a sequence of stopping times,

$$\begin{aligned}
&\tau_d := \inf\{t \geq 0 : \ |z(t)| \geq d\},\\
&\rho_1 := \inf\{t \geq 0 : \ |z(t)|^2 \geq 2\varepsilon_2\},\\
&\rho_{2k} := \inf\{t \geq \rho_{2k-1} : \ |z(t)|^2 \leq \varepsilon_2\}, \quad k = 1, 2, \cdots\\
&\rho_{2k+1} := \inf\{t \geq \rho_{2k} : \ |z(t)|^2 \geq 2\varepsilon_2\}, \quad k = 1, 2, \cdots,
\end{aligned}$$

where throughout this paper we set $\inf \emptyset = \infty$. It is easily observed that $\tau_d = \infty$ and $\rho_k < \infty$ if $\omega \in \bar{\Omega}_2 \cap \bar{\Omega}_3$. Noting that $\rho_{2k} < \infty$ whenever $\rho_{2k-1} < \infty$, we can derive from (3.9) that

$$\begin{aligned}
\infty \ &> \ \mathbb{E}\int_0^\infty |z(t)|^\gamma dt\\
&\geq \ \sum_{k=1}^\infty \mathbb{E}\Big(I_{\{\rho_{2k-1}<\infty, \rho_{2k}<\infty, \tau_d=\infty\}} \int_{\rho_{2k-1}}^{\rho_{2k}} |z(t)|^\gamma dt\Big)\\
&\geq \ \varepsilon_2^{\frac{\gamma}{2}} \sum_{k=1}^\infty \mathbb{E}\Big(I_{\{\rho_{2k-1}<\infty, \tau_d=\infty\}}(\rho_{2k} - \rho_{2k-1})\Big).
\end{aligned} \tag{3.13}$$

In the same way as Mao [15] did in the proof of Theorem 3.1, we can obtain that

$$\mathbb{P}\Big(\{\rho_{2k-1} < \infty, \tau_d = \infty\} \cap \Big\{ \sup_{0\leq t\leq T} \Big||z(\rho_{2k-1} + t)|^2 - |z(\rho_{2k-1})|^2\Big| < \varepsilon_2\Big\}\Big) > \varepsilon_2. \tag{3.14}$$

Set $\tilde{\Omega}_k = \Big\{ \sup_{0\le t\le T} \Big||z(\rho_{2k-1}+t)|^2 - |z(\rho_{2k-1})|^2\Big| < \varepsilon_2 \Big\}$. Noting that $\rho_{2k}(\omega) - \rho_{2k-1}(\omega) \ge T$, if $\omega \in \{\rho_{2k-1} < \infty, \tau_d = \infty\} \cap \tilde{\Omega}_k$. Combing (3.13) and (3.14) together, we have

$$\begin{aligned}
\infty &\ge \varepsilon_2^{\frac{\gamma}{2}} \sum_{k=1}^{\infty} \mathbb{E}\Big(I_{\{\rho_{2k-1}<\infty,\tau_d=\infty\}}(\rho_{2k}-\rho_{2k-1})\Big) \\
&\ge \varepsilon_2^{\frac{\gamma}{2}} \sum_{k=1}^{\infty} \mathbb{E}\Big(I_{\{\rho_{2k-1}<\infty,\tau_d=\infty\}\cap\tilde{\Omega}_k}(\rho_{2k}-\rho_{2k-1})\Big) \\
&\ge \varepsilon_2^{\frac{\gamma}{2}} T \sum_{k=1}^{\infty} P(\{\rho_{2k-1}<\infty,\tau_d=\infty\}\cap\tilde{\Omega}_k) \\
&\ge \varepsilon_2^{\frac{\gamma}{2}} T \sum_{k=1}^{\infty} \varepsilon_2 = \infty.
\end{aligned}$$

which is a contradiction. So (3.10) must hold. Finally, let us now show that $\lim_{t\to\infty}|x(t)| = 0$ *a.s.* By (3.7) and letting $t \to \infty$, we obtain that $\limsup_{t\to\infty}|x(t)| \le k_0 \limsup_{t\to\infty}|x(t)|$ *a.s.* This together with (3.6), yields the required assertion (3.4). The proof is therefore complete. □

Remark 3.5 In [15], Mao, Shen and Yuan first studied the almost sure asymptotic stability of the solution for HSDDEs with neutral term, where the coefficients satisfy the local Lipschitz condition and the Khasminskii-type condition. It should be noted that the time delay considered in [15] is a constant delay, which is also a special case of this paper. In this paper, we extend the stability results in [15] to the case of non-differentiable time delay, and the delay can be unbounded by using contradiction and the stopping time, which is different from the existing method by using the semi-martingale convergence theorem in [15].

4 General decay stability

In the previous section, we have discussed the H_∞ stability and almost sure asymptotic stability of the solution to equation (1.3). These two classes if stabilities do not reveal the convergence rate. This section will discuss the stablity with the general decay rate under the conditions of Theorem 3.3. Let us first introduce the following ψ-type function, which will be used as the decay function.

Definition 4.1 *The function $\psi : R \to (0,\infty)$ is said to be ψ-type function if this function satisfies the following conditions:*

(i) *It is continuous and nondecreaing in R and continuously differentiable in R_+.*

(ii) *$\psi(0) = 1$, $\psi(\infty) = \infty$ and $C_\psi = \sup_{t\ge0}|\frac{\psi'(t)}{\psi(t)}| < \infty$.*

(iii) *For any $0 \le s \le t$, $\psi(t) \le \psi(s)\psi(t-s)$.*

Definition 4.2 *The solution of Eq.(1.3) is said to be 2th moment ψ-type stable if there exists a constant $\bar{\gamma}$ such that*

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|x(t)|^2}{\log \psi(t)} < -\bar{\gamma} \quad a.s.$$

Definition 4.3 *The solution of Eq.(1.3) is said to be almost surely ψ-type stable if there exists a constant $\tilde{\gamma}$ such that*

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{\log\psi(t)}<-\tilde{\gamma}\quad a.s.$$

Remark 4.4 Obviously, when $\psi(t)=e^t$ and $\psi(t)=1+t$, this ψ-type stability implies the exponential stability and polynomial stability, respectively.

Theorem 4.5 *Let conditions of Theorem 3.3 hold. Assume that $\delta(t)\le\tilde{h}$, then for any given initial data ξ, the solution $x(t)$ of Eq.(1.3) has the properties that*

$$\limsup_{t\to\infty}\frac{\log\mathbb{E}|x(t)|^2}{\log\psi(t)}<-\eta, \tag{4.1}$$

and

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{\log\psi(t)}<-\frac{\eta}{2},\quad a.s. \tag{4.2}$$

where $\eta\in(0,\varepsilon_1\wedge\varepsilon_2)$, while $\varepsilon_1=[-\frac{\log k_0}{\log\psi(\tilde{h})}]\wedge[\frac{\log\beta_3-\log(h_0\beta_4)}{\log\psi(\tilde{h})}]$ and ε_2 is the unique root to the following equation $\beta_1=2c_2C_\psi x+(2c_2xC_\psi k_0^2+\beta_2)\psi^x(\tilde{h})h_0$.

Proof. For each integer $k\ge\bar{k}_0$, define the stopping time $\sigma_k=\inf\{t\ge0:|z(t)|\ge k\}$. Let $\eta\in(0,\varepsilon_1\wedge\varepsilon_2)$. For any $t\ge0$, by the generalized Itô formula, we obtain that

$$\begin{aligned}
&\mathbb{E}\Big(\psi^\eta(t\wedge\sigma_k)V(z(t\wedge\sigma_k),t\wedge\sigma_k,r(t\wedge\sigma_k))\Big)\\
=\;& V(z(0),0,r(0))+\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)\Big(\eta\frac{\psi'(s)}{\psi(s)}V(z(s),s,r(s))\\
+\;& \mathcal{L}V(x(s),x(s-\delta(s)),s,r(s))\Big)ds.
\end{aligned}$$

By conditions (2.5) and (3.1), we then compute

$$\begin{aligned}
&c_1\mathbb{E}(\psi^\eta(t\wedge\sigma_k)|z(t\wedge\sigma_k)|^2)\\
\le\;& c_2|z(0)|^2+c_2\eta C_\psi\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|z(s)|^2ds-\beta_1\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|x(s)|^2ds\\
+\;& \beta_2\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|x(s-\delta(s))|^2ds-\beta_3\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|x(s)|^\gamma ds\\
+\;& \beta_4\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|x(s-\delta(s))|^\gamma ds.
\end{aligned}\tag{4.3}$$

By the definition of ψ-type function and Lemma 2.10, we have

$$\begin{aligned}
\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|x(s-\delta(s))|^2ds &= \mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s-\delta(s)+\delta(s))|x(s-\delta(s))|^2ds\\
&\le \mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s-\delta(s))\psi^\eta(\delta(s))|x(s-\delta(s))|^2ds\\
&\le \psi^\eta(\tilde{h})\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s-\delta(s))|x(s-\delta(s))|^2ds\\
&\le \psi^\eta(\tilde{h})h_0\mathbb{E}\int_{-\bar{h}}^{t\wedge\sigma_k}\psi^\eta(s)|x(s)|^2ds.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|z(s)|^2ds &\leq 2\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)\Big(|x(s)|^2+k_0^2|x(s-\delta(s))|^2\Big)ds\\ &\leq 2\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|x(s)|^2ds\\ &+ 2k_0^2\psi^\eta(\tilde{h})h_0\mathbb{E}\int_{-\bar{h}}^{t\wedge\sigma_k}\psi^\eta(s)|x(s)|^2ds\end{aligned}$$

and

$$\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|x(s-\delta(s))|^\gamma ds \leq \psi^\eta(\tilde{h})h_0\mathbb{E}\int_{-\bar{h}}^{t\wedge\sigma_k}\psi^\eta(s)|x(s)|^\gamma ds.$$

Substituting these into (4.3) gives

$$\begin{aligned}&c_1\mathbb{E}(\psi^\eta(t\wedge\sigma_k)|z(t\wedge\sigma_k)|^2)\\ \leq\ & \tilde{H}_3-\Big(\beta_1-2c_2\eta C_\psi-(2c_2\eta C_\psi k_0^2+\beta_2)\psi^\eta(\tilde{h})h_0\Big)\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|x(s)|^2ds\\ -\ & (\beta_3-\beta_4\psi^\eta(\tilde{h})h_0)\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^\eta(s)|x(s)|^\gamma ds,\end{aligned}$$

where

$$\begin{aligned}\tilde{H}_3 &= c_2|z(0)|^2+(2c_2\eta C_\psi k_0^2+\beta_2)\psi^\eta(\tilde{h})h_0\int_{-\bar{h}}^0\psi^\eta(s)|x(s)|^2ds\\ &+ \beta_4\psi^\eta(\tilde{h})h_0\int_{-\bar{h}}^0\psi^\eta(s)|x(s)|^\gamma ds<\infty.\end{aligned}$$

On the other hand, for $\eta\leq\varepsilon_1\wedge\varepsilon_2$, we note that

$$\beta_1\geq 2c_2\eta C_\psi+(2c_2\eta C_\psi k_0^2+\beta_2)\psi^\eta(\tilde{h})h_0 \ \text{ and } \ \beta_3\geq\beta_4\psi^\eta(\tilde{h})h_0. \tag{4.4}$$

Then we obtain

$$c_1\mathbb{E}(\psi^\eta(t\wedge\sigma_k)|z(t\wedge\sigma_k)|^2)\leq\tilde{H}_3. \tag{4.5}$$

By Lemma 2.9, it follows that for any $T>0$

$$\begin{aligned}&\sup_{0\leq t\leq T}\mathbb{E}(\psi^\eta(t\wedge\sigma_k)|x(t\wedge\sigma_k)|^2)\\ \leq\ & \frac{\sup_{0\leq t\leq T}\mathbb{E}(\psi^\eta(t\wedge\sigma_k)|z(t\wedge\sigma_k)|^2)}{(1-k_0)}+k_0\sup_{0\leq t\leq T}\mathbb{E}\psi^\eta(t\wedge\sigma_k)|x(t\wedge\sigma_k-\delta(t\wedge\sigma_k))|^2\\ \leq\ & \frac{\tilde{H}_3}{c_1(1-k_0)}+k_0\psi^\eta(\tilde{h})\sup_{0\leq t\leq T}\mathbb{E}\psi^\eta(t\wedge\sigma_k-\delta_{t\wedge\sigma_k})|x(t\wedge\sigma_k-\delta(t\wedge\sigma_k))|^2\\ \leq\ & \frac{\tilde{H}_3}{c_1(1-k_0)}+k_0\psi^\eta(\tilde{h})\sup_{-\bar{h}\leq t\leq T}\mathbb{E}\psi^\eta(t\wedge\sigma_k)|x(t\wedge\sigma_k)|^2.\end{aligned}$$

Note that $\eta \leq \varepsilon_1$, we get $\psi^{\eta}(\tilde{h}) \leq \frac{1}{k_0}$. Hence, we obtain

$$\sup_{0\leq t\leq T} \mathbb{E}(\psi^{\eta}(t\wedge\sigma_k)|x(t\wedge\sigma_k)|^2) \leq \frac{\tilde{H}_3}{c_1(1-k_0)(1-k_0\psi^{\eta}(\tilde{h}))} + \frac{k_0\psi^{\eta}(\tilde{h})}{1-k_0\psi^{\eta}(\tilde{h})} \sup_{-\bar{h}\leq t\leq 0} \mathbb{E}\psi^{\eta}(t)|x(t)|^2.$$

Letting $k \to \infty$, we get that

$$\sup_{0\leq t\leq T} \mathbb{E}(\psi^{\eta}(t)|x(t)|^2) \quad \leq \quad \frac{\tilde{H}_3}{c_1(1-k_0)(1-k_0\psi^{\eta}(\tilde{h}))} + \frac{k_0\psi^{\eta}(\tilde{h})}{1-k_0\psi^{\eta}(\tilde{h})}\mathbb{E}\|\xi\|^2$$

and the assertion (4.1) follows. To show the second assertion, it is sufficient to show

$$\sup_{0<t<\infty} \psi^{\eta}(t)|z(t)|^2 < \infty \quad a.s. \tag{4.6}$$

If this were false, then there is a $\bar{\Omega}_4 = \{\omega \in \Omega : \sup_{0\leq t<\infty} \psi^{\eta}(t)|z(t)|^2 = \infty\}$ and ε_3 such that $\mathbb{P}(\bar{\Omega}_4) = \varepsilon_3 > 0$. Choose a number $M > \frac{5\tilde{H}_3}{2c_1\varepsilon_3}$. Define the stopping time $\tau = \inf\{t \geq 0 : \psi^{\eta}(t)|z(t)|^2 \geq M\}$. Then we can find a sufficiently large number T such that $\mathbb{P}(0 < \tau \leq T) \geq 0.4\varepsilon_3$. Similar to (4.5), we have $c_1\mathbb{E}(\psi^{\eta}(T\wedge\tau)|z(T\wedge\tau)|^2) \leq \tilde{H}_3$. This implies that

$$\begin{aligned}
\frac{\tilde{H}_3}{c_1} &\geq \mathbb{E}(\psi^{\eta}(T\wedge\tau)|z(T\wedge\tau)|^2) \\
&\geq \mathbb{E}(\psi^{\eta}(T\wedge\tau)|z(T\wedge\tau)|^2 I_{\{\tau\leq T\}}) \\
&\geq MP(\tau \leq T) \geq 0.4M\varepsilon_3.
\end{aligned}$$

But this contradicts the fact $M > \frac{5\tilde{H}_3}{2c_1\varepsilon_3}$. Hence, (4.6) must hold. The proof is therefore complete. □

Remark 4.6 By using the semi-martingale convergence theorem, we can also prove that the solution of equation (1.3) is almost surely ψ-type stable. Here, we do not give the details of the proof, please refer to [1].

Remark 4.7 In fact, Theorems 2.11, 2.13, 3.3 and 3.4 mentioned above hold no matter $\delta(t)$ is bounded or not. However, in order to obtain the general decay rate η, we require $\delta(t)$ to be bounded by the constant $\tilde{h}$ in Theorem 4.5, i.e., $\delta(t) \leq \tilde{h}$.

Remark 4.8 In Theorem 4.5, if $\psi(t) = e^t$ and $\psi(t) = 1 + t$, then (4.2) implies that Eq.(1.3) is almost surely exponentially stable and polynomially stable. In other words, we extend these two classes of stability into the general decay stability in this paper. And this will be fully illustrated by Examples 5.1 and 5.2. At the same time, we also note that whether the solution converges to the equilibrium point at the general decay rate depends not only on h_0 of the delay function, but also on the coefficient k_0 of the neutral term.

Remark 4.9 Based on Lemma 2.10 and the stopping time technique, the proof of Theorem 4.5 is completed without the restrictive conditions $\delta(t) = C$ or $\frac{d\delta(t)}{dt} \leq \hat{h} < 1$. In [37, 38], the stability with general decay rate was investigated under these two restrictive conditions. Therefore, the stability results of [37, 38] can be considered as a special case of Theorem 4.5.

Obviously, it is not convenient to check condition (3.1) of Theorem 3.3, since it is not related to coefficients f and g explicitly. Now, we shall impose some conditions on f and g to guarantee Theorem 3.3 and establish a sufficient criteria on the almost sure asymptotic stability and general decay stability in terms of M-matrix. For the convenience of the reader, let us cite some useful results on M-matrices. For more detailed information please see e.g. [1].

Definition 4.10 *A square matrix $A = [a_{ij}]_{N\times N}$ is called a nonsingular M-matrix if A can be expressed in the form $A = sI - B$ with $s > \rho(B)$ while all the elements of B are nonnegative, where I is the identity matrix and $\rho(B)$ the spectral radius of B.*

It is easy to see that a nonsingular M-matrix A has non-positive off-diagonal and positive diagonal entries. There are many conditions which are equivalent to the statement that A is a nonsingular M-matrix and we now cite some of them for the use of this paper.

Lemma 4.11 *If $A \in Z^{N\times N} = \{[a_{ij}]_{N\times N} : a_{ij} \leq 0,\ i \neq j\}$, then the following statements are equivalent:*

(1) A is a nonsingular M-matrix.

(2) A is semi-positive; that is, there exists $x \gg 0$ in R^N such that $Ax \gg 0$.

(3) A^{-1} exists and its elements are all nonnegative.

(4) All the leading principal minors of A are positive; that is

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \text{for every } k = 1, 2, \cdots, N.$$

Let us now give hypothesis in terms of an M-matrix, which will replace condition (3.1). For any $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$, we impose the following condition on the coefficients f and g.

Assumption 4.12 *Let $\gamma > 0$. There exist a real number $\bar{\alpha}_{1i}$ and nonnegative numbers $\bar{\alpha}_{2i}, \bar{\alpha}_{3i}, \bar{\alpha}_{4i}$ such that*

$$x^\top f(x, y, t, i) \leq \bar{\alpha}_{1i}|x|^2 + \bar{\alpha}_{2i}|y|^2 - \bar{\alpha}_{3i}|x|^{\gamma+2} + \bar{\alpha}_{4i}|y|^{\gamma+2}. \tag{4.7}$$

There exist nonnegative numbers $\bar{\beta}_{1i}, \bar{\beta}_{2i}, \bar{\beta}_{3i}, \bar{\beta}_{4i}$ and $\bar{\sigma}_{1i}, \bar{\sigma}_{2i}, \bar{\sigma}_{3i}, \bar{\sigma}_{4i}$ such that

$$|f(x, y, t, i)| \leq \bar{\beta}_{1i}|x| + \bar{\beta}_{2i}|y| + \bar{\beta}_{3i}|x|^{\gamma+1} + \bar{\beta}_{4i}|y|^{\gamma+1}, \tag{4.8}$$

and

$$|g(x, y, t, i)|^2 \leq \bar{\sigma}_{1i}|x|^2 + \bar{\sigma}_{2i}|y|^2 + \bar{\sigma}_{3i}|x|^{\gamma+2} + \bar{\sigma}_{4i}|y|^{\gamma+2}. \tag{4.9}$$

Remark 4.13 In fact, if $\gamma = 0$, then the condition (4.7) will degenerate to the classical one-side linear growth condition, while conditions (4.8) and (4.9) will reduce to the classical linear growth condition.

Assumption 4.14 *Define the $N \times N$ matrix $\tilde{\Gamma} = (|\gamma_{ij}|k_i)_{N\times N}$. Assume that*

$$\mathcal{A} := -\text{diag}(2\bar{\alpha}_{11} + \bar{\beta}_{11}k_1 + \bar{\sigma}_{11}, \cdots, 2\bar{\alpha}_{1N} + \bar{\beta}_{1N}k_N + \bar{\sigma}_{1N}) - \Gamma - \tilde{\Gamma} \tag{4.10}$$

is a nonsingular M-matrix.

Theorem 4.15 *Let Assumptions 2.1, 2.5 2.6 4.12 and 4.14 hold. If*

$$\theta_i(2\bar{\alpha}_{2i} + \bar{\beta}_{1i}k_i + \bar{\beta}_{2i}k_i + \bar{\sigma}_{2i}) + \sum_{j=1}^{N}|\gamma_{ij}|\theta_j k_i^2 + \sum_{j=1,j\neq i}^{N}\gamma_{ij}\theta_j k_i^2 < \frac{1}{h_0}, \quad \forall\ i \in S, \tag{4.11}$$

and

$$\min_{i\in S}\Big(2\bar{\alpha}_{3i} - 2\bar{\beta}_{3i}k_i\frac{\gamma+1}{\gamma+2} - \bar{\sigma}_{3i}\Big)\theta_i > h_0\max_{i\in S}\Big(2\bar{\alpha}_{4i} + 2\bar{\beta}_{3i}k_i\frac{1}{\gamma+2} + \bar{\beta}_{4i}k_i + \bar{\sigma}_{4i}\Big)\theta_i. \tag{4.12}$$

Then for any given initial data ξ, the solution $x(t)$ of Eq.(1.3) has the properties (3.2), (3.4), (4.1) and (4.2).

Proof. By Lemma 4.11 and Assumption 4.14, it follows that

$$\theta = (\theta_1, \cdots, \theta_N)^\top := \mathcal{A}^{-1}\overrightarrow{1} > 0 \tag{4.13}$$

for all $i \in S$, where $\overrightarrow{1} = (1, \cdots, 1)^\top$. Let us define the function $V(z,t,i) = \theta_i|z|^2$. Clearly, V obeys conditions (2.5) with $c_1 = \min_{i\in S}\theta_i$ and $c_2 = \max_{i\in S}\theta_i$. To verify condition (3.1), we compute the operator $\mathcal{L}V$ as follows

$$\begin{aligned}
\mathcal{L}V(x,y,t,i) &= 2\theta_i[x - D(y,i)]^\top f(x,y,t,i) + \theta_i|g(x,y,t,i)|^2 \\
&+ \sum_{j=1}^{N}\gamma_{ij}\theta_j[x - D(y,i)]^\top[x - D(y,i)] \\
&\leq 2\theta_i[x^\top f(x,y,t,i) + |D(y,i)||f(x,y,t,i)|] + \theta_i|g(x,y,t,i)|^2 \\
&+ \sum_{j=1}^{N}\gamma_{ij}\theta_j|x|^2 - 2\sum_{j=1}^{N}\gamma_{ij}\theta_j x^\top D(y,i) + \sum_{j=1}^{N}\gamma_{ij}\theta_j|D(y,i)|^2.
\end{aligned}$$

By Assumption 4.12, we have

$$\begin{aligned}
\mathcal{L}V(x,y,t,i) &\leq \Big(\theta_i(2\bar{\alpha}_{1i} + \bar{\beta}_{1i}k_i + \bar{\sigma}_{1i}) + \sum_{j=1}^{N}\gamma_{ij}\theta_j + \sum_{j=1}^{N}|\gamma_{ij}|\theta_j k_i\Big)|x|^2 \\
&+ \Big(\theta_i(2\bar{\alpha}_{2i} + \bar{\beta}_{1i}k_i + \bar{\beta}_{2i}k_i + \bar{\sigma}_{2i}) + \sum_{j=1}^{N}|\gamma_{ij}|\theta_j k_i + \sum_{j=1,j\neq i}^{N}\gamma_{ij}\theta_j k_i^2\Big)|y|^2 \\
&- (2\bar{\alpha}_{3i} - 2\bar{\beta}_{3i}k_i\frac{\gamma+1}{\gamma+2} - \bar{\sigma}_{3i})\theta_i|x|^{\gamma+2} \\
&+ (2\bar{\alpha}_{4i} + 2\bar{\beta}_{3i}k_i\frac{1}{\gamma+2} + \bar{\beta}_{4i}k_i + \bar{\sigma}_{4i})\theta_i|y|^{\gamma+2}.
\end{aligned}$$

By (4.13), we have $\theta_i(2\bar{\alpha}_{1i} + \bar{\beta}_{1i}k_i + \bar{\sigma}_{1i}) + \sum_{j=1}^{N}\gamma_{ij}\theta_j + \sum_{j=1}^{N}|\gamma_{ij}|\theta_j = -1$. Hence,

$$\mathcal{L}V(x,y,t,i) \leq -\beta_1|x|^2 + \beta_2|y|^2 - \beta_3|x|^\gamma + \beta_4|y|^\gamma, \tag{4.14}$$

where

$$\beta_1 = 1, \beta_2 = \max_{i\in S}\Big[\theta_i(2\bar{\alpha}_{2i} + \bar{\beta}_{1i}k_i + \bar{\beta}_{2i}k_i + \bar{\sigma}_{2i}) + \sum_{j=1}^{N}|\gamma_{ij}|\theta_j k_i^2 + \sum_{j=1,j\neq i}^{N}\gamma_{ij}\theta_j k_i^2\Big],$$
$$\beta_3 = \min_{i\in S}(2\bar{\alpha}_{3i} - 2\bar{\beta}_{3i}k_i\tfrac{\gamma+1}{\gamma+2} - \bar{\sigma}_{3i})\theta_i, \beta_4 = \max_{i\in S}(2\bar{\alpha}_{4i} + 2\bar{\beta}_{3i}k_i\tfrac{1}{\gamma+2} + \bar{\beta}_{4i}k_i + \bar{\sigma}_{4i})\theta_i. \tag{4.15}$$

Recalling (4.11) and (4.12), condition (3.1) is fulfilled. By Theorems 3.3, 3.4 and 4.5, we can conclude that for any given initial data ξ, there is a unique global solution $x(t)$ and the solution of Eq.(1.3) is H_∞ stable, almost surely asymptotically stable and ψ-type stable. The proof is therefore complete. □

Remark 4.16 In fact, Theorem 4.15 is less conservative than Theorem 3 of [37]. It is easy to see that if $\delta(t)$ degenerates to the constant delay τ, then $h_0 = 1$ and the conditions (4.11) and (4.12) are still hold. That is to say, the stability of the following equation

$$d[x(t) - D(x(t-\tau), r(t))] = f(x(t), x(t-\tau), t, r(t))dt + g(x(t), x(t-\tau), t, r(t))dw(t) \tag{4.16}$$

only depends on the coefficients f, g, and is independent of the delay function τ. This is also known as delay independent stability.

Remark 4.17 By conditions (4.11) and (4.12), it can be seen that the stability of equation (1.3) depends on the value of h_0. In other words, this paper mainly studies the delay-dependent stability of stochastic systems. It can be seen from Example 5.1 that different delay functions $\delta(t)$ and $\bar{\delta}(t)$ will affect the stability of neutral stochastic systems (5.1).

5 Two Examples

In this section, we shall give two examples to illustrate the applications of our results.

Example 5.1 Let $w(t)$ is a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}.$$

Of course, $w(t)$ and $r(t)$ are assumed to be independent. Consider the following scalar HSDSwNT

$$d[x(t) - 0.1x(t-\delta(t))] = f(x(t), x(t-\delta(t)), t, r(t))dt + g(x(t), x(t-\delta(t)), t, r(t))dw(t) \tag{5.1}$$

on $t \geq 0$. Here the coefficients f, g are defined by

$$f(x,y,t,1) = 0.25x + 0.1y - 0.5x^3, \quad g(x,y,t,1) = 0.5x + \frac{\sqrt{2}}{4}y,$$
$$f(x,y,t,2) = -1.5x - x^3, \quad g(x,y,t,2) = -0.5y + \frac{\sqrt{2}}{4}x^2$$

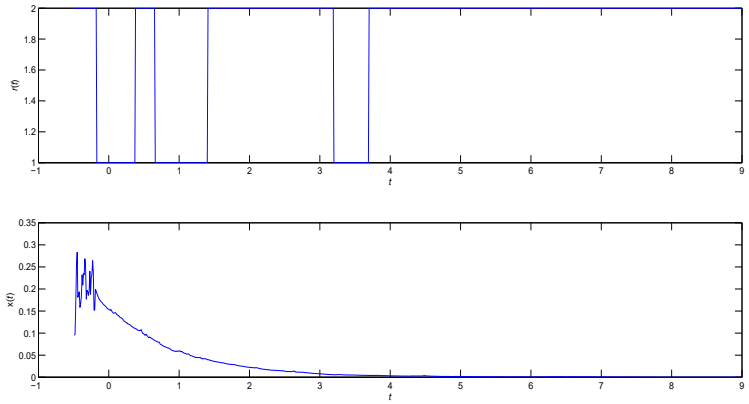

Figure 1: The sample paths of HSDSwNT (5.1) with $\delta(t) = 0.25(1 + |sint|)$.

where $\delta(t) = \theta(1 + |sint|)$, $\theta \in (0, 1)$. It is easy to obtain that $\delta(t)$ satisfies Assumption 2.1 with $h = \theta, \bar{h} = 2\theta$ and $h_0 = 1/(1 - \theta)$. Note that the coefficients f and g satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. Through a straight computation, we have $k_1 = k_2 = 0.1$ and

$$\begin{aligned}
&\bar{\alpha}_{11} = 0.3,\ \bar{\alpha}_{21} = 0.05,\ \bar{\alpha}_{31} = 0.5,\ \bar{\alpha}_{12} = -1.5,\ \ \bar{\alpha}_{32} = 1,\ \bar{\alpha}_{22} = \bar{\alpha}_{41} = \bar{\alpha}_{42} = 0;\\
&\bar{\beta}_{11} = 0.25, \bar{\beta}_{21} = 0.1,\ \bar{\beta}_{31} = 0.5,\ \bar{\beta}_{12} = 1.5,\ \bar{\beta}_{32} = 1,\ \bar{\beta}_{41} = \bar{\beta}_{22} = \bar{\beta}_{42} = 0,\\
&\bar{\sigma}_{11} = 0.5,\ \bar{\sigma}_{21} = 0.25,\ \bar{\sigma}_{22} = 0.5,\ \bar{\sigma}_{32} = 0.25,\ \bar{\sigma}_{31} = \bar{\sigma}_{41} = \bar{\sigma}_{12} = \bar{\sigma}_{42} = 0.
\end{aligned}$$

By (4.10), we get the matrix $\mathcal{A} = \begin{pmatrix} 2.575 & -3.3 \\ -1.1 & 3.75 \end{pmatrix}$ is a non-singular M-matrix. Compute $(\theta_1, \theta_2)^\top = \mathcal{A}^{-1}\vec{1} = (1.1699, 0.6098)^\top$ and $\beta_2 = 0.5221$, $\beta_3 = 0.9757$, $\beta_4 = 0.0305$. Hence, we conclude that the conditions (4.11) and (4.12) hold if $\theta < \min\{1 - \beta_2, 1 - \frac{\beta_4}{\beta_3}\} = 0.4779$. Let us choose $\theta = 0.25$, then by Theorem 4.15, it follows that $\limsup_{t\to\infty} \frac{\log \mathbb{E}|x(t)|^2}{t} \le -\eta$ and $\limsup_{t\to\infty} \frac{\log |x(t)|}{t} \le -\frac{\eta}{2}$ $a.s.$ where $\psi(t) = e^t$ and $\eta \in (0, 0.1113)$. That is to say, the solution of Eq.(5.1) decays at the exponential rate of at least 0.0556. We perform a computer simulation with the time-delay $\delta(t) = 0.25(1 + |sint|)$ for all $t \ge 0$ and the initial data $\xi(t) = 1 + t$ for $t \in [-0.5, 0]$ and $r(0) = 1$. The computer simulation (see Figure 1) supports our theoretical results clearly.

Remark 5.1 If we choose the delay function $\bar{\delta}(t) = 0.75(1 + |sint|)$, then the following equation

$$d[x(t) - 0.1x(t - \bar{\delta}(t))] = f(x(t), x(t - \bar{\delta}(t)), t, r(t))dt + g(x(t), x(t - \bar{\delta}(t)), t, r(t))dw(t) \qquad (5.2)$$

is almost surely exponentially unstable. In fact, it is easy to get $h_0 = 4$ and $\beta = 0.522 > 0.25 = \frac{1}{h_0}$, which means that condition (4.11) does not hold. The computer simulation (see Figure 2) supports our theoretical results clearly.

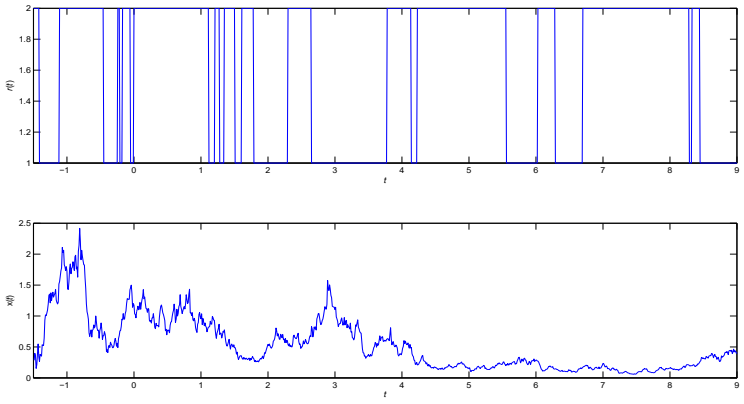

Figure 2: The sample paths of HSDSwNT (5.2) with $\bar{\delta}(t) = 0.75(1 + |sint|)$.

Example 5.2 Let $w(t)$ is a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}.$$

Of course, $w(t)$ and $r(t)$ are assumed to be independent. Consider the following scalar HSDSwNT

$$d[x(t) - D(x(t - \delta(t)), r(t))] = f(x(t), x(t - \delta(t)), t, r(t))dt + g(x(t), x(t - \delta(t)), t, r(t))dw(t) \quad (5.3)$$

on $t \geq 0$. Here the coefficients f, g, D are defined by

$$f(x, y, t, 1) = -2x - 1.5x^3, \quad g(x, y, t, 1) = \rho_1 y^2, \quad D(y, 1) = 0.1y$$
$$f(x, y, t, 2) = x - x^3, \quad g(x, y, t, 2) = \rho_2 y^2, \quad D(y, 2) = 0.05y$$

where ρ_1, ρ_2 are unknown parameters and $\delta(t) = \sum_{k=0}^{\infty} \Big\{0.1I_{[3k,3k+1)}(t) + [0.1 + 0.05(t - 3k - 1)]I_{[3k+1,3k+2)}(t) + [0.15 - 0.05(t - 3k - 2)]I_{[3k+2,3k+3)}(t)\Big\}$. It is easy to obtain that $\delta(t)$ satisfies Assumption 2.1 with $h = 0.1, \bar{h} = 0.15$ and $h_0 = 1.0526$. Note that the coefficients f and g satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. Through a straight computation, we have $k_1 = 0.1, k_2 = 0.05$ and

$$\bar{\alpha}_{11} = -2,\ \bar{\alpha}_{31} = 1.5,\ \bar{\alpha}_{12} = 1,\ \bar{\alpha}_{32} = 1,\ \bar{\alpha}_{21} = \bar{\alpha}_{41} = \bar{\alpha}_{22} = \bar{\alpha}_{42} = 0;$$
$$\bar{\beta}_{11} = 2,\ \bar{\beta}_{31} = 1.5,\ \bar{\beta}_{12} = 1,\ \bar{\beta}_{32} = 1,\ \bar{\beta}_{21} = \bar{\beta}_{41} = \bar{\beta}_{22} = \bar{\beta}_{42} = 0,$$
$$\bar{\sigma}_{41} = 0.04,\ \bar{\sigma}_{42} = 0.25,\ \bar{\sigma}_{1i} = \bar{\sigma}_{2i} = \bar{\sigma}_{3i} = 0, i = 1, 2, 3.$$

By (4.10), we get the matrix $\mathcal{A} = \begin{pmatrix} 4.94 & -1.02 \\ -4.04 & 1.95 \end{pmatrix}$ which is a non-singular M-matrix. By (4.13), we have $\theta_1 = 0.5388$ and $\theta_2 = 1.6291$. Clearly, $\beta_2 = 0.5204 < \frac{1}{h_0}$, while condition (4.12) becomes $\min\{2.775, 1.925\} \geq 1.0526 \times \max\{0.04041 + 0.5388\rho_1^2, 0.04072 + 1.6291\rho_2^2\}$. If $\rho_1 < 1.8218$ and

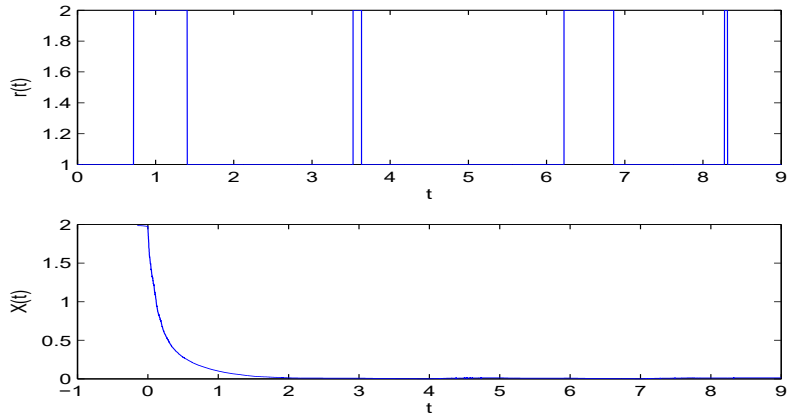

Figure 3: The sample paths of HSDSwNT (5.3) with $\rho_1 = 0.2$ and $\rho_1 = 0.1$.

$\rho_2 < 1.0476$, then by Theorems 3.3 and 3.4, we can conclude that the solution of (5.3) has the properties $\int_0^\infty \mathbb{E}|x(t)|^4 dt < \infty$ and $\lim_{t\to\infty} x(t) = 0$ a.s. On the other hand, if we choose $\rho_1 = 0.2$, $\rho_2 = 0.1$ and by Theorem 4.5, it follows that $\limsup_{t\to\infty} \frac{\log|x(t)|}{\log t} \leq -\frac{\eta}{2}$ $a.s.$ where $\psi(t) = 1+t$ and $\eta \in (0, 0.1342)$. That is to say, the solution of Eq.(5.3) decays at the polynomial rate of at least 0.0671.

Now, let us perform a computer simulation for the scalar HSDSwNT. Assume the initial data $\xi(t) = 1 + \cos t$ for $t \in [-0.15, 0]$ and $r(0) = 1$. In fact, Eq.(5.3) can be regarded as the result of the two equations

$$d[x(t) - 0.1x(t-\delta(t))] = [-2x(t) - 1.5x^3(t)]dt + 0.2x^2(t-\delta(t))dw(t) \tag{5.4}$$

and

$$d[x(t) - 0.05x(t-\delta(t))] = [x(t) - x^3(t)]dt + 0.1x^2(t-\delta(t))dw(t) \tag{5.5}$$

switching among each other according to the movement of the Markov chain $r(t)$. It is obvious that Eq.(5.4) is asymptotically stable, but Eq.(5.5) is unstable. However, we shall see that due to the Markovian switching, the overall system (5.3) will be asymptotically stable. The computer simulation (see Figure 3) supports our results clearly. On the other hand, if we choose $\rho_1 = 2$, $\rho_2 = 1.5$, Theorem 4.5 shows that Eq.(5.3) is not asymptotically stable, for which the condition (4.12) does not hold. The computer simulation (see Figure 4) supports this results clearly. In fact, as shown in Figures 3 and 4, stochastic noise with smaller intensity can play a stabilizing role, but stochastic noise with large intensity will destroy the stability of the stochastic system.

6 Conclusion

In this paper, the existence and uniqueness, the stability with general decay rate of the solution for nonlinear hybrid stochastic systems with neutral term and non-differentiable time delay were

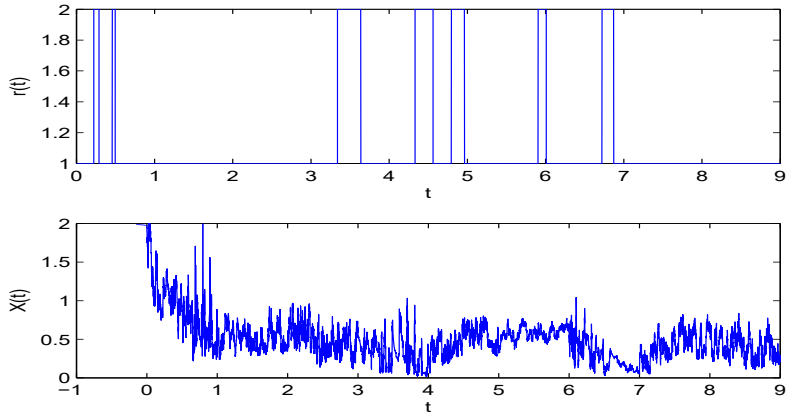

Figure 4: The sample paths of HSDSwNT (5.3) with $\rho_1 = 2$ and $\rho_1 = 1.5$.

considered under the Khasminskii-type condition. By using the Lyapunov function approach and some stochastic analysis techniques, some sufficient conditions were established for the almost sure asymptotic stability and the stability with the general decay rate. New techniques were developed to cope with the time delay which may be non-differentiable. The application of our results was shown by discussing two examples.

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