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Improved delay-dependent stability of superlinear hybrid stochastic systems with general time-varying delays

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ABSTRACT

In the recent paper (Fei et al., 2019), the study of delay-dependent stability of hybrid stochastic differential delay equations (SDDEs) was generalized to superlinear ones (namely, do not satisfy the usual linear growth condition). However, the theory developed there could not be applied to hybrid SDDEs with non-differentiable time delays, or whose drift coefficients miss the key decomposition in (Fei et al., 2019) (see Assumption 1 below). This paper therefore is to deal with these two challenging problems so that the delay-dependent stability criteria derived in (Fei et al., 2019) could be improved. The decomposition scheme is modified in order to include more general hybrid SDDEs. The differentiability assumption on time-varying delays is replaced by a relatively weaker one. Also the Lyapunov functional used in this paper is modulated to adapt to these new changes. Finally, two interesting examples, an application to mosquito model, and design of nonlinear delay feedback control, respectively, are given to demonstrate the effectiveness of our new theory.

1. Introduction

Since 1990's, there have been enormous papers on the analysis of stochastic differential delay equations with Markovian chains (also known as hybrid SDDEs), due to their intensive applications to various real-world problems, such as chemical refining processes [1], mobile manipulators [2], optimal consumption problem [3]. As one of the interesting topics, stability draws many researchers' attention. As the literature in this area is very huge, we only mention [4–10] among others. Two types of stability criteria are often discussed: delay-independent, establishing stability results for any delay value, or delay-dependent, taking into account some limited size of delays. Compared with the first type, delay-dependent stability makes the most of additional information of time delays, and hence it seems less conservative, especially for systems with small delays. We cite [11–18] to the reader for reference.

A general hybrid SDDE is described as

$$dx(t) = f(x(t), x(t-h(t)), t, r(t))dt + g(x(t), x(t-h(t)), t, r(t))dW(t) \tag{1.1}$$

on $t \geq 0$, where $h(t)$ denotes the time delay. Detailed explanation of Eq. (1.1) and other notations will be given in Section 2. Traditionally, the linear growth condition on the drift coefficient f and the diffusion coefficient g is needed for delay-dependent stability (e.g., see [12–16]). In 2017, Fei et al. [17] made an important breakthrough in this area as they got rid of this restriction and brought the study into superlinear systems. Unfortunately, in [17], $f(x, y, t, i)$ was assumed

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to satisfy the global Lipschitz condition in the delay component y, which is relatively restrictive in practice. Later on, to overcome this difficulty, Fei et al. [11] developed generalized results. Although their theory could cover more hybrid SDDEs, there are still two issues that require our further investigation.

1. The results established in work [11] rely closely on the following key assumption.

Assumption 1. Assume that f can be decomposed as $f(x,y,t,i)=f_1(x,y,t,i)+f_2(x,y,t,i)$, where there exists a positive number β such that for any $(x,y,t,i)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times S$,

$$|f_1(x,y,t,i)-f_1(x,x,t,i)|\le\beta|x-y|. \tag{1.2}$$

This assumption is in fact a little harsh since we could not always see this particular structure, for example, in function f, the delay segment y is mixed with the current component x, such as $f(x,y,t,i)=xy-x^3$, or the delay component appears only in the superlinear term, such as $f(x,y,t,i)=x+y^2-x^3$. In these cases, the theory in [11] is not applicable, and a class of hybrid SDDEs might be excluded.

2. The time delay function $h(t)$ is supposed to be a differentiable function and satisfies

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t}\le\bar{h}<1,\qquad\forall t\ge 0 \tag{1.3}$$

for some constant $\bar{h}$. Condition (1.3) also appears widely in the study of delayed systems (e.g., see [8,19,20]), but just being imposed owing to the mathematical need to deal with the time lag. However, many real-world time delays might miss this condition (e.g., [21–26]). For example, in the networked control systems, sawtooth delay appears frequently, such as $h(t)=\tau\sum_{k=0}^{\infty}1_{[k,k+1)}(t)\,(t-k)$ (e.g., see [24,25]). It was also found in [26] that the energy of a vibrating system could decay exponentially with $2T$-periodic switching delay, namely, $h(t)=4T$ in the first half of one period and $6T$ in the latter, where T represents the wave period. These delays are even discontinuous, let alone meeting condition (1.3). Therefore, it seems a little unreasonable to continue imposing this condition.

In summary, these two restrictions make the delay-dependent stability results derived in [11] less applicable in reality. Therefore, this paper is aimed to remove them, and the main contributions can be summarized as follows. (1) We will modify the decomposition scheme of function f in Assumption 1 by changing the Lipschitz coefficient on the second segment of f_1 from a constant β into a polynomial. (2) In theory, such a replacement will bring new mathematical challenge, which results in the Lyapunov functional used in this paper different from that in [11] (more details can be found in Section 3). (3) More general time-varying delay functions will be considered, which meet a weaker assumption (namely Assumption 2, firstly being proposed by [23]) than differentiability assumption (namely condition (1.3)). (4) Compared with [23], the delay function $h(t)$ studied in this paper is no longer needed to be bounded below by a positive number.

A description of the organization of this paper follows. In Section 2, we will present our model of hybrid SDDEs with general time delays and impose some standing hypotheses to guarantee the unique solution of the underlying hybrid SDDE. In Section 3, we will introduce our new scheme of drift coefficient decomposition and the Lyapunov functional used in this paper under some extra conditions for the purpose of stability. After these preparations, we will give a method to determine the upper bound of time delays and our delay-dependent stability results in Section 4, including H_∞ stability, moment asymptotic stability and almost sure asymptotic stability. Ultimately, Sections 5 and 6 are devoted to examples and conclusion, respectively.

2. Model formulation and global solution

In Section 2.1, we will list some basic notations, and then discuss the time delay in detail. The delay function $h(t)$ takes values in $[0,\tau]$ for some positive constant τ, and needs to satisfy Assumption 2 (firstly proposed in [23]), which is obviously weaker than differentiability condition (1.3). In Section 2.2, we will show our existence-and-uniqueness theorem.

2.1. Hybrid SDDEs with general time delays

We first provide the notations to be used widely in this paper. If a and b are both real numbers, then $a\wedge b=\min\{a,b\}$ and $a\vee b=\max\{a,b\}$. Denote by $\mathbb{R}_+$ the collection of all non-negative real numbers. Let $\mathbb{R}^n$ be the n-dimensional vector space over the reals with Euclidean norm $|\cdot|$. For a vector or matrix M, M^T represents its transpose. If M is a matrix, denote its trace norm by $|M|=\sqrt{\mathrm{trace}(M^TM)}$. Let $C([-\tau,0];\mathbb{R}^n)$ represent the family of all continuous functions ϕ from $[-\tau,0]$ to $\mathbb{R}^n$ and designate the norm of its element ϕ by $\|\phi\|=\sup_{-\tau\le\theta\le 0}|\phi(\theta)|$. For a set A, let 1_A be its indicator function, that is, $1_A(a)=1$ if $a\in A$, and 0 otherwise.

We also let $(\Omega,\mathcal{F},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (that is, it is increasing, right-continuous and $\mathcal{F}_0$ contains all P-null sets). Denote by $W(t)=(W_1(t),\ldots,W_m(t))^T$ an m-dimensional

Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with transition rate matrix $Q = (q_{ij})_{N\times N}$ given by

$$P\left(r(t+H) = j | r(t) = i\right) = \begin{cases} 1 + q_{ij}H + o(H), & \text{if } i = j \\ q_{ij}H + o(H), & \text{if } i \neq j \end{cases}$$

as $H \downarrow 0$. Here $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $q_{ii} = -\sum_{j\neq i} q_{ij}$. We assume that the Markov chain $r(t)$ and the Brownian motion $W(t)$ are independent under the probability measure P.

Consider the hybrid SDDE (1.1) with the initial data

$$\left\{x(t)\big| -\tau \leq t \leq 0\right\} = \xi \in C([-\tau, 0]; \mathbb{R}^n), \quad r(0) = r_0 \in S. \tag{2.1}$$

Here, drift coefficient $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$ and diffusion coefficient $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times m}$ are both Borel measurable functions, $h : \mathbb{R}_+ \to [0, \tau]$ is the system delay.

As mentioned before, the mathematical techniques used in many papers to tackle the delay effect, such as [8,19,20], force the authors to impose the differentiability condition on the time delay $h(t)$, which is too restrictive in many real models. Consequently, in this paper, we will consider a more general situation, by imposing the following assumption.

Assumption 2. Suppose that $h(t)$ is a Borel measurable function with the property that

$$h^* := \limsup_{H\to 0^+} \left(\sup_{s\geq -\tau} \frac{\mu(I_{s,H})}{H} \right) < \infty, \tag{2.2}$$

where $\mu(\cdot)$ denotes the Lebesgue measure on the real line and

$$I_{s,H} = \left\{t \in \mathbb{R}_+ \big| t - h(t) \in [s, s+H)\right\}.$$

It should be pointed out that this assumption is not so strong and can be met by many time-varying delay functions in practice. For example, the piecewise constant function $h(t) = T\sum_{k=0}^{\infty} 1_{[(2k+1)T,(2k+2)T)}(t)$ satisfies Assumption 2 with $h^* = 2$, where T is a positive constant. Moreover, if $h(t)$ is a Lipschitz continuous function with Lipschitz coefficient $\hat{h} \in (0, 1)$, then Assumption 2 is satisfied with $h^* = \frac{1}{1-\hat{h}}$. For more details about Assumption 2 and these two examples, we refer the reader to [23]. But different from [23], the delay function $h(t)$ considered in this paper is not needed to be bounded below by a positive constant. Of course we do not want to consider the case where $h(t) = 0$ for all $t \geq 0$ as the SDDE reduces to a stochastic differential equation (SDE).

Next, we need to prepare a useful lemma, which plays a fundamental role when we discuss the properties of the hybrid SDDE (1.1).

Lemma 1. *Let Assumption 2 hold. Let $T > 0$ and $\varphi : [-\tau, T] \to \mathbb{R}_+$ be a continuous function. Then*

$$\int_0^T \varphi(v - h(v))\mathrm{d}v \leq h^* \int_{-\tau}^T \varphi(v)\mathrm{d}v. \tag{2.3}$$

This lemma tells us how to tackle the effect of time delays under our new Assumption 2. We also refer the reader to Lemma 2.2 in [23] for more details. While a little differently, the integral of $\varphi(v)$ in the right-hand side of (2.3) is from $-\tau$ to T, since the delay function $h(t)$ in this paper could reach zero. But one can still use the same way as Lemma 2.2 in [23] was proved to show Lemma 1. So we omit the proof here. Moreover, it should be pointed out that h^* given in Assumption 2 always satisfies that $h^* \geq 1$. In fact, if we let $\varphi(t) \equiv 1$ for all $t \geq -\tau$ in Lemma 1. Then this lemma tells us that $T \leq h^*(T + \tau)$ for any $T > 0$, which implies that $h^* \geq \lim_{T\to\infty} \frac{T}{T+\tau} = 1$.

2.2. Global solution

At first, we do not want the system coefficients to grow very rapidly, so the following polynomial growth condition is required.

Assumption 3. Both coefficients f and g are locally Lipschitz continuous. Also assume that there exist constants $q_1 > 1$ and $K_j \geq 0$, $\hat{K}_j \geq 0$, $(j = 1, 2, 3, 4)$ such that

$$|f(x, y, t, i)| \leq K_1|x| + K_2|y| + K_3|x|^{q_1} + K_4|y|^{q_1} \tag{2.4}$$

and

$$|g(x, y, t, i)|^2 \leq \hat{K}_1|x|^2 + \hat{K}_2|y|^2 + \hat{K}_3|x|^{q_1+1} + \hat{K}_4|y|^{q_1+1} \tag{2.5}$$

for all $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$.

Condition (2.4) will be widely used in the subsequent stability analysis, particularly, in determining the upper bound of τ, which relies on the coefficients K_1, K_2, K_3 and K_4. In order to have a larger τ as possible, we do not take the maximum of these four numbers to control f, that is, $|f(x,y,t,i)| \le (K_1 \vee K_2 \vee K_3 \vee K_4)(|x| + |y| + |x|^{q_1} + |y|^{q_1})$. The same reason applies for condition (2.5).

But Assumption 3 cannot ensure the global solution of hybrid SDDE (1.1). Therefore we impose the following generalized Khasminskii-type assumption.

Assumption 4. Assume that there exist non-negative constants $q \ge 2q_1$, α_1, α_2, $\alpha_{3,i}$, and $\alpha_{4,i}$ $(i \in S)$, where

$$\min_{i\in S}\left\{\alpha_{3,i} - \frac{h^*(q_1+1)+q-2}{q_1+q-1}\alpha_{4,i}\right\} \ge 0, \tag{2.6}$$

such that

$$x^{\mathrm{T}}f(x,y,t,i) + \frac{q-1}{2}|g(x,y,t,i)|^2 \le \alpha_1|x|^2 + \alpha_2|y|^2 - \alpha_{3,i}|x|^{q_1+1} + \alpha_{4,i}|y|^{q_1+1} \tag{2.7}$$

for all $(x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$.

The classical Khasminskii test (see [27]) is a powerful technique for SDEs to have non-explosion solutions without the linear growth condition. More precisely, for an SDE $\mathrm{d}x(t) = f(x(t),t)\mathrm{d}t + g(x(t),t)\mathrm{d}W(t)$, if there is a Lyapunov function $\hat{V}(x,t)$ such that

$$\hat{V}_t(x,t) + \hat{V}_x(x,t)f(x,t) + \frac{1}{2}\mathrm{trace}\left(g^{\mathrm{T}}(x,t)\hat{V}_{xx}(x,t)g(x,t)\right) \le \hat{\alpha}_1\hat{V}(x,t),$$

(i.e., the diffusion operator is bounded by a linear function of $\hat{V}$), then the SDE has a global solution (no explosion at a finite time). For our SDDEs, this test has been generalized to deal with superlinear delay terms (e.g., Assumption 2.2 in [17], Assumption 2.3 in [23]), that is, the condition above is described in a generalized form

$$\begin{aligned}&\hat{V}_t(x,t) + \hat{V}_x(x,t)f(x,y,t,i) + \frac{1}{2}\mathrm{trace}\left(g^{\mathrm{T}}(x,y,t,i)\hat{V}_{xx}(x,t)g(x,y,t,i)\right)\\ \le&\hat{\alpha}_1(\hat{V}(x,t) + \hat{V}(y,t-h(t))) - \hat{\alpha}_2\hat{U}(x,t) + \hat{\alpha}_3\hat{U}(y,t-h(t)),\end{aligned}$$

where in general $\hat{U}$ grows faster than $\hat{V}$. The significant generalization here is that the diffusion operator is no longer bounded by a linear function of $\hat{V}$. For example, in Assumption 4, we take the special form $\hat{V}(x,t) = |x|^q$ and $\hat{U}(x,t) = |x|^{q_1+q-1}$ (see Theorem 1 and its proof). The latter is often used to eliminate the delay effect of superlinear term $|y|^{q_1+1}$, which is given in Assumption 3. However, condition (2.6) is a little stronger than Assumption 2.3 in [23] since we require $\alpha_{3,i} - \frac{h^*(q_1+1)+q-2}{q_1+q-1}\alpha_{4,i} \ge 0$. But this is needed to deal with the difficulty arising from the time delay, which is bounded below by zero rather than a positive constant. It is because this stronger condition that the proof of the existence of a global solution becomes easier.

Theorem 1. *Let Assumptions 2, 3 and 4 hold. Then for any given initial data (2.1), there is a unique global solution $x(t)$ of hybrid SDDE (1.1) on $t \in [0,\infty)$ with the property that*

$$\sup_{0\le v\le t} E|x(v)|^q < \infty, \quad \forall t \ge 0. \tag{2.8}$$

Proof. Fix the initial data $\xi \in C([-\tau,0];\mathbb{R}^n)$ and $r_0 \in S$. Since the system coefficients are locally Lipschitz continuous, by Theorem 7.12 in [5], there is a unique maximal local solution $x(t)$ on $t \in [0,\sigma_e)$, where σ_e is the explosion time. Let $k_0 > 0$ be sufficiently large for $k_0 \ge \|\xi\|$. For each integer $k \ge k_0$, define the stopping time

$$\sigma_k = \inf\left\{t \in [0,\sigma_e)\big||x(t)| \ge k\right\}.$$

Clearly, σ_k is increasing as $k \to \infty$. Set $\sigma_\infty = \lim_{k\to\infty}\sigma_k$, whence $\sigma_\infty \le \sigma_e$ a.s. If we can show that $\sigma_\infty = \infty$ a.s., then $\sigma_e = \infty$ a.s., and the solution $x(t)$ is the global solution.

Now, for any $k \ge k_0$ and $t \ge 0$, we derive from the Itô formula and condition (2.7) that

$$\begin{aligned}&E|x(t\wedge\sigma_k)|^q - |\xi(0)|^q\\ \le&E\int_0^{t\wedge\sigma_k} q|x(s)|^{q-2}\left(x^{\mathrm{T}}(s)f(x(s),x(s-h(s)),s,r(s)) + \frac{q-1}{2}|g(x(s),x(s-h(s)),s,r(s))|^2\right)\mathrm{d}s\\ \le&E\int_0^{t\wedge\sigma_k} q|x(s)|^{q-2}\left(\alpha_1|x(s)|^2 + \alpha_2|x(s-h(s))|^2 - \alpha_{3,r(s)}|x(s)|^{q_1+1} + \alpha_{4,r(s)}|x(s-h(s))|^{q_1+1}\right)\mathrm{d}s.\end{aligned}$$

By the Young inequality, we further obtain that

$$\begin{aligned} E|x(t\wedge\sigma_k)|^q-|\xi(0)|^q \le &(q\alpha_1+(q-2)\alpha_2)E\int_0^{t\wedge\sigma_k}|x(s)|^q\mathrm{d}s+2\alpha_2E\int_0^{t\wedge\sigma_k}|x(s-h(s))|^q\mathrm{d}s\\ &-E\int_0^{t\wedge\sigma_k}\left(q\alpha_{3,r(s)}-\frac{q(q-2)\alpha_{4,r(s)}}{q_1+q-1}\right)|x(s)|^{q_1+q-1}\mathrm{d}s\\ &+E\int_0^{t\wedge\sigma_k}\frac{q(q_1+1)\alpha_{4,r(s)}}{q_1+q-1}|x(s-h(s))|^{q_1+q-1}\mathrm{d}s. \end{aligned} \tag{2.9}$$

Making use of Lemma 1, we see that

$$\int_0^{t\wedge\sigma_k}|x(s-h(s))|^q\mathrm{d}s\le h^*\int_{-\tau}^{t\wedge\sigma_k}|x(s)|^q\mathrm{d}s$$

and

$$\int_0^{t\wedge\sigma_k}|x(s-h(s))|^{q_1+q-1}\mathrm{d}s\le h^*\int_{-\tau}^{t\wedge\sigma_k}|x(s)|^{q_1+q-1}\mathrm{d}s.$$

Substituting these into (2.9) yields that

$$\begin{aligned} E|x(t\wedge\sigma_k)|^q\le &C_1+(q\alpha_1+(q-2+2h^*)\alpha_2)E\int_0^{t\wedge\sigma_k}|x(s)|^q\mathrm{d}s\\ &-E\int_0^{t\wedge\sigma_k}q\left(\alpha_{3,r(s)}-\frac{h^*(q_1+1)+q-2}{q_1+q-1}\alpha_{4,r(s)}\right)|x(s)|^{q_1+q-1}\mathrm{d}s\\ \le &C_1+(q\alpha_1+(q-2+2h^*)\alpha_2)E\int_0^{t\wedge\sigma_k}|x(s)|^q\mathrm{d}s\\ &-q\min_{i\in S}\left\{\alpha_{3,i}-\frac{h^*(q_1+1)+q-2}{q_1+q-1}\alpha_{4,i}\right\}E\int_0^{t\wedge\sigma_k}|x(s)|^{q_1+q-1}\mathrm{d}s, \end{aligned} \tag{2.10}$$

where

$$C_1=|\xi(0)|^q+2h^*\alpha_2\tau\|\xi\|^q+\frac{h^*q(q_1+1)\alpha_4}{q_1+q-1}\tau\|\xi\|^{q_1+q-1}.$$

In particular, by (2.6),

$$\begin{aligned} E|x(t\wedge\sigma_k)|^q\le &C_1+(q\alpha_1+(q-2+2h^*)\alpha_2)\int_0^t E|x(s\wedge\sigma_k)|^q\mathrm{d}s\\ \le &C_1+(q\alpha_1+(q-2+2h^*)\alpha_2)\int_0^t\sup_{0\le v\le s}E|x(v\wedge\sigma_k)|^q\mathrm{d}s. \end{aligned}$$

Since the right-hand-side term is increasing in t, we must have

$$\sup_{0\le v\le t}E|x(v\wedge\sigma_k)|^q\le C_1+(q\alpha_1+(q-2+2h^*)\alpha_2)\int_0^t\sup_{0\le v\le s}E|x(v\wedge\sigma_k)|^q\mathrm{d}s.$$

Applying the Gronwall inequality, we have

$$\sup_{0\le v\le t}E|x(v\wedge\sigma_k)|^q\le C_1e^{(q\alpha_1+(q-2+2h^*)\alpha_2)t}. \tag{2.11}$$

This implies that

$$k^qP(\sigma_k\le t)\le E|x(t\wedge\sigma_k)|^q\le\sup_{0\le v\le t}E|x(v\wedge\sigma_k)|^q<\infty.$$

We can hence let $k\to\infty$ in the inequality above to obtain that $P(\sigma_\infty\le t)=0$, namely, $P(\sigma_\infty>t)=1$. Since $t\ge 0$ is arbitrary, we must have that $P(\sigma_\infty=\infty)=1$ as required. Letting $k\to\infty$ in (2.11) gives the assertion (2.8) immediately. The proof is therefore complete. □

Theorem 1 implies that for any $t\ge 0$, the solution $x(t)$ is in L^q, while both $f(x(t),x(t-h(t)),t,r(t))$ and $g(x(t),x(t-h(t)),t,r(t))$ are in L^2. These properties are significant when we discuss the stability of hybrid SDDE (1.1).

3. Conditions for stability

In Section 2, we have shown that there admits a global solution of hybrid SDDE (1.1) under our standing Assumptions 2, 3 and 4. But this is not enough to derive the delay-dependent stability. For this purpose, we will impose some additional conditions on the system in this section. At the meantime, we will talk about the new decomposition of the drift coefficient in Section 3.1, and the Lyapunov functional in Section 3.2.

3.1. New decomposition scheme

The key assumption in [11] is that f could be decomposed into two parts, one of which is globally Lipschitz continuous in delay component. But as we mentioned before, this is a little restrictive. Therefore, in this paper, we will modify this decomposition, which is stated as the following assumption.

Assumption 5. Assume that the drift coefficient f can be decomposed as

$$f(x,y,t,i)=f_1(x,y,t,i)+f_2(x,y,t,i) \tag{3.1}$$

and, moreover, there exist three constants $r \in \left[0, \frac{q_1-1}{2}\right]$, $\beta_1 \geq 0$, $\beta_2 \geq 0$ and $\beta_3 \geq 0$, where β_1, β_2, β_3 cannot be zero simultaneously, such that

$$|f_1(x,y,t,i)-f_1(x,x,t,i)| \leq \left(\beta_1+\beta_2|x|^r+\beta_3|y|^r\right)|x-y| \tag{3.2}$$

for all $(x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$.

Remark 1. Here, in line with condition (2.4), we mention that the numbers r, β_1, β_2 and β_3 also play a big part in the determination of the upper bound of τ. As a result, we do not express the right-hand side of condition (3.2) in the form of $(\beta_1 \vee \beta_2 \vee \beta_3)\left(1+|x|^r+|y|^r\right)|x-y|$.

It should also be underlined that our new decomposition scheme of f indeed enables us to include a wider class of hybrid SDDEs. Recalling the examples in the introduction part, if $f(x,y,t,i)=xy-x^3$, then $f_1(x,y,t,i)=xy$ with $r=1$, $\beta_1=0$, $\beta_2=1$, $\beta_3=0$ and $f_2(x,y,t,i)=-x^3$; if $f(x,y,t,i)=x+y^2-x^3$, then $f_1(x,y,t,i)=y^2$ with $r=1$, $\beta_1=0$, $\beta_2=1$, $\beta_3=1$ and $f_2(x,y,t,i)=x-x^3$. Moreover, when $\beta_2=\beta_3=0$ or $r=0$, condition (3.2) becomes the familiar condition (1.2) studied in [11].

Finally, we strengthen that the decomposition plan for many examples is always not fixed and there might exist other possible schemes. But unfortunately, we currently could not provide a standard to determine which one is the best. In practice, for convenience, we always tend to put the delay terms whose order are not larger than $\frac{q_1+1}{2}$ in f_1, and others including non-delay terms and higher-order delay terms in f_2. This is because f_1 represents the delay-dependent property and we want it as simple as possible.

The key idea in this paper is to make use of the decomposition of $f(x,y,t,i)$ defined in Assumption 5. We then rewrite $f(x,y,t,i)$ as

$$f(x,y,t,i)=(f_1(x,y,t,i)-f_1(x,x,t,i))+(f_1(x,x,t,i)+f_2(x,y,t,i)). \tag{3.3}$$

Thus the drift coefficient can be analyzed by two parts. The second part, $f_1(x,x,t,i)+f_2(x,y,t,i)$, will be dealt with in Section 3.2. Now, let us focus on the first part, $f_1(x,y,t,i)-f_1(x,x,t,i)$. Recalling condition (3.2), we derive that

$$\begin{aligned}
&|f_1(x,y,t,i)-f_1(x,x,t,i)| \\
\leq&\beta_1|x-y|+\beta_2\sqrt{|x|^{2r}(|x|+|y|)}\sqrt{|x-y|}+\beta_3\sqrt{|y|^{2r}(|x|+|y|)}\sqrt{|x-y|} \\
\leq&\beta_1|x-y|+\frac{\beta_2}{2\varepsilon_1}|x-y|+\frac{\beta_2\varepsilon_1}{2}|x|^{2r}(|x|+|y|)+\frac{\beta_3}{2\varepsilon_1}|x-y|+\frac{\beta_3\varepsilon_1}{2}|y|^{2r}(|x|+|y|) \\
\leq&\left(\beta_1+\frac{\beta_2+\beta_3}{2\varepsilon_1}\right)|x-y|+\frac{\beta_2(4r+1)+\beta_3}{4r+2}\varepsilon_1|x|^{2r+1}+\frac{\beta_2+\beta_3(4r+1)}{4r+2}\varepsilon_1|y|^{2r+1},
\end{aligned} \tag{3.4}$$

where $\varepsilon_1>0$ is a parameter to be determined later. This then forces us to estimate the difference between $x(t)$ and $x(t-h(t))$, which is the following lemma.

Lemma 2. *Let all the conditions in Theorem 1 hold. Then for any $t \geq 2\tau$, we have*

$$\begin{aligned}
E|x(t)-x(t-h(t))|^2 \leq&(H_1\tau+H_2)\int_{t-2\tau}^t E|x(v)|^2\mathrm{d}v+(H_3\tau+H_4\tau)\int_{t-2\tau}^t E|x(v)|^{2q_1}\mathrm{d}v \\
&+(H_5\tau+H_6)\int_{t-2\tau}^t E|x(v)|^{q_1+1}\mathrm{d}v,
\end{aligned} \tag{3.5}$$

where $H_1=4\left(K_1^2+K_2^2h^*\right)$, $H_2=2(\hat{K}_1+\hat{K}_2h^*)$, $H_3=4K_3^2$, $H_4=4K_4^2h^*$, $H_6=2(\hat{K}_3+\hat{K}_4h^*)$,

$$H_5=4\left(\left(K_1K_3+\frac{K_1K_4+K_2K_3q_1}{q_1+1}\right)+\left(K_2K_4+\frac{K_2K_3+K_1K_4q_1}{q_1+1}\right)h^*\right).$$

Proof. It is easy to see from hybrid SDDE (1.1) that

$$x(t)-x(t-h(t))=\int_{t-h(t)}^t f(x(v),x(v-h(v)),v,r(v))\mathrm{d}v+\int_{t-h(t)}^t g(x(v),x(v-h(v)),v,r(v))\mathrm{d}W(v).$$

Using the elementary inequality $|a+b|^2 \le 2(|a|^2+|b|^2)$ and the Hölder inequality, we can show that

$$|x(t)-x(t-h(t))|^2 \le 2\tau \int_{t-h(t)}^{t} |f(x(v),x(v-h(v)),v,r(v))|^2 \mathrm{d}v + 2\left|\int_{t-h(t)}^{t} |g(x(v),x(v-h(v)),v,r(v))|^2 \mathrm{d}W(v)\right|^2.$$

Recalling the discussions below Theorem 1, we can take expectations on both sides and apply the Doob martingale inequality to obtain

$$E|x(t)-x(t-h(t))|^2 \le 2E\int_{t-h(t)}^{t}\left(\tau|f(x(v),x(v-h(v)),v,r(v))|^2 + |g(x(v),x(v-h(v)),v,r(v))|^2\right)\mathrm{d}v. \tag{3.6}$$

For any $(x,y,t,i)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times S$, by condition (2.4) and the Young inequality, compute

$$\begin{aligned}|f(x,y,t,i)|^2 \le& 2K_1^2|x|^2 + 2K_2^2|y|^2 + 2K_3^2|x|^{2q_1} + 2K_4^2|y|^{2q_1} + 2\left(K_1K_3 + \frac{K_1K_4}{q_1+1} + \frac{K_2K_3q_1}{q_1+1}\right)|x|^{q_1+1}\\ &+ 2\left(K_2K_4 + \frac{K_2K_3}{q_1+1} + \frac{K_1K_4q_1}{q_1+1}\right)|y|^{q_1+1}.\end{aligned}$$

This, combining with condition (2.5) then yields that

$$\begin{aligned}&2\left(\tau|f(x,y,t,i)|^2 + |g(x,y,t,i)|^2\right)\\ \le&\left(4K_1^2\tau + 2\hat{K}_1\right)|x|^2 + \left(4K_2^2\tau+2\hat{K}_2\right)|y|^2 + 4K_3^2\tau|x|^{2q_1} + 4K_4^2\tau|y|^{2q_1}\\ &+\left(4\left(K_1K_3+\frac{K_1K_4+K_2K_3q_1}{q_1+1}\right)\tau + 2\hat{K}_3\right)|x|^{q_1+1} + \left(4\left(K_2K_4+\frac{K_2K_3+K_1K_4q_1}{q_1+1}\right)\tau+2\hat{K}_4\right)|y|^{q_1+1}.\end{aligned}$$

Substituting this into (3.6) gives that

$$\begin{aligned}&E|x(t)-x(t-h(t))|^2\\ \le&\left(4K_1^2\tau+2\hat{K}_1\right)\int_{t-\tau}^{t}E|x(v)|^2\mathrm{d}v + \left(4K_2^2\tau+2\hat{K}_2\right)\int_{t-\tau}^{t}E|x(v-h(v))|^2\mathrm{d}v\\ &+4K_3^2\tau\int_{t-\tau}^{t}E|x(v)|^{2q_1}\mathrm{d}v + 4K_4^2\tau\int_{t-\tau}^{t}E|x(v-h(v))|^{2q_1}\mathrm{d}v\\ &+\left(4\left(K_1K_3+\frac{K_1K_4+K_2K_3q_1}{q_1+1}\right)\tau+2\hat{K}_3\right)\int_{t-\tau}^{t}E|x(v)|^{q_1+1}\mathrm{d}v\\ &+\left(4\left(K_2K_4+\frac{K_2K_3+K_1K_4q_1}{q_1+1}\right)\tau+2\hat{K}_4\right)\int_{t-\tau}^{t}E|x(v-h(v))|^{q_1+1}\mathrm{d}v\\ \le&(H_1\tau+H_2)\int_{t-2\tau}^{t}E|x(v)|^2\mathrm{d}v + (H_3\tau+H_4\tau)\int_{t-2\tau}^{t}E|x(v)|^{2q_1}\mathrm{d}v + (H_5\tau+H_6)\int_{t-2\tau}^{t}E|x(v)|^{q_1+1}\mathrm{d}v,\end{aligned}$$

where Lemma 1 has been used. This ends the proof. □

Before closing this subsection, we make some comments about the advantages and challenges of delay-dependent stability.

Remark 2. In the study delay-independent stability, we often use the delay-free information to suppress the impact of time delays. Thus the delay size is of no use, but the non-delay term is always strengthened. For example, to obtain a delay-independent stability criterion for the underlying hybrid SDDE in this paper, the following assumption

$$x^{\mathrm{T}}f(x,y,t,i) + \frac{q_1}{2}|g(x,y,t,i)|^2 \le -\nu_1|x|^2 + \nu_2|y|^2 - \nu_3|x|^{q_1+1} + \nu_4|y|^{q_1+1},$$

where $\nu_1 > \nu_2 h^*$ and $\nu_3 > \frac{h^*(q_1+1)+q_1-1}{2q_1}\nu_4$, might be needed. Certainty, such a requirement is sometimes a little strong. To ease this restriction, in the research of delay-dependent stability, we use another way to cope with the effect of time delays. More precisely, we estimate $x(t)-x(t-h(t))$, the difference between current-time state and past-time state, and hope it could be small enough if the time lag is sufficiently small. Consequently, the conditions on the non-delay term could be relaxed.

Remark 3. However, the estimation of $x(t)-x(t-h(t))$ is challenging. We are always forced to impose some extra conditions. In [17], $f(x,y,t,i)$ was required to be globally Lipschitz continuous in y. In [11], $f(x,y,t,i)$ could be decomposed into two parts, one of which should satisfy global Lipschitz condition in the delay segment. While, in this paper, the global Lipschitz condition is replaced by condition (3.2). But here we still need $r \le \frac{q_1-1}{2}$ to achieve the stability purpose.

Moreover, due to the mathematical skills, we can currently just estimate $x(t)-x(t-h(t))$ in the sense of mean square. In fact, instead of (3.4), $|f_1(x,y,t,i)-f_1(x,x,t,i)|$ can be computed in another way,

$$\begin{aligned}|f_1(x,y,t,i)-f_1(x,x,t,i)| &\le \left(\beta_1+(\beta_2+\beta_3R)|x|^r+\beta_3R|x-y|^r\right)|x-y| \\ &\le \beta_1|x-y|+\frac{r(\beta_2+\beta_3R)}{r+1}|x|^{r+1}+\left(\frac{\beta_2+\beta_3R}{r+1}+\beta_3R\right)|x-y|^{r+1}\end{aligned}$$

where $R=2^{(r-1)\vee 0}$. In this situation, the delay part in $f_1(x,y,t,i)$ could be handled completely by estimating $x-y$. Our conditions on the system coefficients could hence be less conservative (e.g., $\frac{q_1}{2}|g(x,y,t,i)|^2$ could be replaced by $\frac{2r+1}{2}|g(x,y,t,i)|^2$ in Assumption 6). Nevertheless, we are not able do with $E|x(t)-x(t-h(t))|^{2r+2}$. As a result, in this paper, we set free parameter ε_1 and use the integral transform (Lemma 1) to reduce the influence of time delays. More details can be found in Lemma 3 below.

Anyway, how to overcome these difficulties deserves our further investigation.

3.2. Lyapunov functional

In the previous subsection, we have worked on the part $f_1(x,y,t,i)-f_1(x,y,t,i)$ in decomposition (3.3). Now let us pay attention to $f_1(x,x,t,i)+f_2(x,y,t,i)$. But we have known little about this term. This forces us to give the following assumption.

Assumption 6. For each $i\in S$, assume that there exist real constants a_i, $\bar{a}_i$, positive constants c_i, $\bar{c}_i$, and non-negative constants b_i, $\bar{b}_i$, d_i, $\bar{d}_i$ such that

$$x^{\mathrm{T}}(f_1(x,x,t,i)+f_2(x,y,t,i))+\frac{1}{2}|g(x,y,t,i)|^2\le a_i|x|^2+b_i|y|^2-c_i|x|^{q_1+1}+d_i|y|^{q_1+1} \tag{3.7}$$

and

$$x^{\mathrm{T}}(f_1(x,x,t,i)+f_2(x,y,t,i))+\frac{q_1}{2}|g(x,y,t,i)|^2\le \bar{a}_i|x|^2+\bar{b}_i|y|^2-\bar{c}_i|x|^{q_1+1}+\bar{d}_i|y|^{q_1+1} \tag{3.8}$$

for all $(x,y,t,i)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times S$, while both

$$A=-2\mathrm{diag}(a_1,a_2,\ldots,a_N)-Q,\quad \bar{A}=-(q_1+1)\mathrm{diag}(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_N)-Q \tag{3.9}$$

are non-singular M-matrices.

We set $(\eta_1,\ldots,\eta_N)^{\mathrm{T}}:=A^{-1}(1,\ldots,1)^{\mathrm{T}}$, $(\bar{\eta}_1,\ldots,\bar{\eta}_N)^{\mathrm{T}}:=\bar{A}^{-1}(1,\ldots,1)^{\mathrm{T}}$. As A and $\bar{A}$ are non-singular M-matrices, all η_i and $\bar{\eta}_i$ are positive. Denote by $C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+\times S;\mathbb{R}_+)$ the family of all continuous non-negative functions, which are continuously twice differentiable in x and once in t for each $i\in S$. Define a function $U(x,t,i)\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+\times S;\mathbb{R}_+)$ by

$$U(x,t,i)=\eta_i|x|^2+\bar{\eta}_i|x|^{q_1+1},\quad (x,t,i)\in\mathbb{R}^n\times\mathbb{R}_+\times S \tag{3.10}$$

while define a function $LU:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times S\to\mathbb{R}$ by

$$\begin{aligned}LU(x,y,t,i)=&2\eta_i\left(x^{\mathrm{T}}(f_1(x,x,t,i)+f_2(x,y,t,i))+\frac{1}{2}|g(x,y,t,i)|^2\right)+\sum_{j=1}^N q_{ij}\eta_j|x|^2\\ &+(q_1+1)\bar{\eta}_i|x|^{q_1-1}\left(x^{\mathrm{T}}(f_1(x,x,t,i)+f_2(x,y,t,i))+\frac{q_1}{2}|g(x,y,t,i)|^2\right)+\sum_{j=1}^N q_{ij}\bar{\eta}_j|x|^{q_1+1}.\end{aligned}$$

Making use of Assumption 6 and the Young inequality, we observe that

$$\begin{aligned}LU(x,y,t,i)\le&-|x|^2+2b_i\eta_i|y|^2-\left((q_1+1)\bar{c}_i\bar{\eta}_i-\frac{q_1^2-1}{2q_1}\bar{d}_i\bar{\eta}_i\right)|x|^{2q_1}+\frac{(q_1+1)^2}{2q_1}\bar{d}_i\bar{\eta}_i|y|^{2q_1}\\ &-(2c_i\eta_i+1-(q_1-1)\bar{b}_i\bar{\eta}_i)|x|^{q_1+1}+(2d_i\eta_i+2\bar{b}_i\bar{\eta}_i)|y|^{q_1+1}.\end{aligned} \tag{3.11}$$

From this observation, to cope with the effect of time lag, Assumption 6 is not enough to ensure the stability and should be strengthened.

Assumption 7. Let Assumption 6 hold. Additionally, assume the following three numbers B_j ($j = 1, 2, 3$) are positive

$$B_1 = 1 - h^*\varpi_1, \quad B_2 = \varpi_2 - \frac{q_1 - 1 + (q_1+1)h^*}{2q_1}\varpi_3, \quad B_3 = \varpi_4 - h^*\varpi_5 + 1 - \left(\frac{q_1-1}{2} + h^*\right)\varpi_6, \tag{3.12}$$

in which

$$\begin{aligned} \varpi_1 &= 2\max_{i\in S} b_i\eta_i, & \varpi_2 &= (q_1+1)\min_{i\in S}\bar{c}_i\bar{\eta}_i, & \varpi_3 &= (q_1+1)\max_{i\in S}\bar{d}_i\bar{\eta}_i, \\ \varpi_4 &= 2\min_{i\in S} c_i\eta_i, & \varpi_5 &= 2\max_{i\in S} d_i\eta_i, & \varpi_6 &= 2\max_{i\in S}\bar{b}_i\bar{\eta}_i. \end{aligned} \tag{3.13}$$

The main method to investigate the delay-dependent stability in this paper is the technique of Lyapunov functionals. For this aim, we define a segment $\hat{x}_t = \{x(t+s)| - 2\tau \le s \le 0\}$ for $t \ge 0$, where we set $x(s) = \xi(-\tau)$ for $-2\tau \le s < \tau$. The Lyapunov functional used will be of the form

$$V(\hat{x}_t, t, r(t)) = U(x(t), t, r(t)) + \int_{-2\tau}^{0}\int_{t+s}^{t}\left(\theta_1|x(v)|^2 + \theta_2|x(v)|^{2q_1} + \theta_3|x(v)|^{q_1+1}\right)\mathrm{d}v\mathrm{d}s, \tag{3.14}$$

for $t \ge 0$, where θ_j ($j = 1, 2, 3$) are positive constants to be determined later.

By the generalized Itô formula and the fundamental theory of calculus, we can show that $V(\hat{x}_t, t, r(t))$ is in fact an Itô process on $t \ge 0$ with its Itô differential

$$\mathrm{d}V(\hat{x}_t, t, r(t)) = \mathcal{L}V(\hat{x}_t, t, r(t))\mathrm{d}t + U_x(x(t), t, r(t))g(x(t), x(t-h(t)), t, r(t))\mathrm{d}W(t) + \mathrm{d}M(t), \tag{3.15}$$

where

$$\begin{aligned} \mathcal{L}V(\hat{x}_t, t, r(t)) =& LU(x(t), x(t-h(t)), t, r(t)) \\ &+ U_x(x(t), t, r(t))(f_1(x(t), x(t-h(t)), t, r(t)) - f_1(x(t), x(t), t, r(t))) \\ &+ 2\theta_1\tau|x(t)|^2 + 2\theta_2\tau|x(t)|^{2q_1} + 2\theta_3\tau|x(t)|^{q_1+1} \\ &- \theta_1\int_{t-2\tau}^{t}|x(v)|^2\mathrm{d}v - \theta_2\int_{t-2\tau}^{t}|x(v)|^{2q_1}\mathrm{d}v - \theta_3\int_{t-2\tau}^{t}|x(v)|^{q_1+1}\mathrm{d}v \end{aligned} \tag{3.16}$$

and $M(t)$ is a continuous martingale vanishing at $t = 0$. The explicit form of $M(t)$ is of no use in this paper so we omit it here, but it can be found in Theorem 1.45 in [5].

Remark 4. From (3.16), we can see that the last three terms, the integrals of $|x(v)|^2$, $|x(v)|^{q_1+1}$ and $|x(v)|^{2q_1}$, are the same as those terms in the right-hand-side of (3.5). This is the reason why we establish the Lyapunov functional in the form of (3.14). It also makes the construction of this kind of Lyapunov functionals more flexible according to different needs.

4. Delay-dependent stability

In Section 3, we have presented our new decomposition scheme of $f(x, y, t.i)$ and dealt with its decomposing parts, $f_1(x, y, t, i) - f_1(x, x, t, i)$ and $f_1(x, x, t, i) + f_2(x, y, t, i)$, respectively. But before starting our stability criteria, for the convenience of the reader, we give the following lemma, which will be used to estimate $\mathcal{L}V(\hat{x}_t, t, r(t))$ later.

Lemma 3. *Set three positive free parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$, where ε_1 has been already given in (3.4). Then under Assumption 5, for any $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$, we have*

$$\begin{aligned} &U_x(x, t, i)(f_1(x, y, t, i) - f_1(x, x, t, i)) \\ \le& (\eta_M\varepsilon_2 + J_1\varepsilon_1)|x|^2 + J_2\varepsilon_1|y|^2 + \left(\frac{(q_1+1)\bar{\eta}_M}{2}\varepsilon_3 + J_3\varepsilon_1\right)|x|^{2q_1} + J_4\varepsilon_1|y|^{2q_1} \\ &+ (J_1+J_3)\varepsilon_1|x|^{q_1+1} + (J_2+J_4)\varepsilon_1|y|^{q_1+1} + \left(\beta_1 + \frac{\beta_2+\beta_3}{2\varepsilon_1}\right)^2\left(\frac{\eta_M}{\varepsilon_2} + \frac{q_1+1}{2}\frac{\bar{\eta}_M}{\varepsilon_3}\right)|x-y|^2, \end{aligned} \tag{4.1}$$

where $\eta_M = \max_{i\in S}\eta_i$, $\bar{\eta}_M = \max_{i\in S}\bar{\eta}_i$, and

$$\begin{aligned} J_1 =& \frac{\beta_2(4r+1)+\beta_3}{2r+1}\eta_M + \frac{\beta_2+\beta_3(4r+1)}{(2r+1)(2r+2)}\eta_M, \\ J_2 =& \frac{\beta_2+\beta_3(4r+1)}{2r+2}\eta_M, \\ J_3 =& (q_1+1)\frac{\beta_2(4r+1)+\beta_3}{4r+2}\bar{\eta}_M + \frac{(q_1+1)q_1}{q_1+2r+1}\frac{\beta_2+\beta_3(4r+1)}{4r+2}\bar{\eta}_M, \\ J_4 =& \frac{(q_1+1)(2r+1)}{q_1+2r+1}\frac{\beta_2+\beta_3(4r+1)}{4r+2}\bar{\eta}_M. \end{aligned}$$

Proof. Recalling the estimation of $f_1(x,y,t,i)-f_1(x,x,t,i)$ in (3.4), we have for any $(x,y,t,i)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times S$

$$\begin{aligned} &U_x(x,t,i)(f_1(x,y,t,i)-f_1(x,x,t,i))\\ &\le \left(2\eta_M|x|+(q_1+1)\bar{\eta}_M|x|^{q_1}\right)\left(\left(\beta_1+\frac{\beta_2+\beta_3}{2\varepsilon_1}\right)|x-y|+\frac{\beta_2(4r+1)+\beta_3}{4r+2}\varepsilon_1|x|^{2r+1}\right.\\ &\quad\left.+\frac{\beta_2+\beta_3(4r+1)}{4r+2}\varepsilon_1|y|^{2r+1}\right). \end{aligned} \tag{4.2}$$

By the elementary inequality and the Young inequality, compute

$$\begin{aligned} &2\eta_M|x|\left(\beta_1+\frac{\beta_2+\beta_3}{2\varepsilon_1}\right)|x-y|\le \eta_M\varepsilon_2|x|^2+\left(\beta_1+\frac{\beta_2+\beta_3}{2\varepsilon_1}\right)^2\frac{\eta_M}{\varepsilon_2}|x-y|^2,\\ &(q_1+1)\bar{\eta}_M|x|^{q_1}\left(\beta_1+\frac{\beta_2+\beta_3}{2\varepsilon_1}\right)|x-y|\le\frac{q_1+1}{2}\bar{\eta}_M\varepsilon_3|x|^{2q_1}+\frac{q_1+1}{2}\left(\beta_1+\frac{\beta_2+\beta_3}{2\varepsilon_1}\right)^2\frac{\bar{\eta}_M}{\varepsilon_3}|x-y|^2,\\ &2\eta_M|x|\frac{\beta_2+\beta_3(4r+1)}{4r+2}\varepsilon_1|y|^{2r+1}\le\frac{\beta_2+\beta_3(4r+1)}{(2r+1)(2r+2)}\eta_M\varepsilon_1|x|^{2r+2}+\frac{\beta_2+\beta_3(4r+1)}{2r+2}\eta_M\varepsilon_1|y|^{2r+2},\\ &(q_1+1)\bar{\eta}_M|x|^{q_1}\frac{\beta_2+\beta_3(4r+1)}{4r+2}\varepsilon_1|y|^{2r+1}\le\frac{(q_1+1)q_1}{q_1+2r+1}\frac{\beta_2+\beta_3(4r+1)}{4r+2}\bar{\eta}_M\varepsilon_1|x|^{q_1+2r+1}\\ &\qquad+\frac{(q_1+1)(2r+1)}{q_1+2r+1}\frac{\beta_2+\beta_3(4r+1)}{4r+2}\bar{\eta}_M\varepsilon_1|y|^{q_1+2r+1}. \end{aligned}$$

Substituting these into (4.2) and rearranging terms gives that

$$\begin{aligned} &U_x(x,t,i)(f_1(x,y,t,i)-f_1(x,x,t,i))\\ &\le\eta_M\varepsilon_2|x|^2+J_1\varepsilon_1|x|^{2r+2}+J_2\varepsilon_1|y|^{2r+2}+J_3\varepsilon_1|x|^{q_1+2r+1}+J_4\varepsilon_1|y|^{q_1+2r+1}+\frac{(q_1+1)\bar{\eta}_M}{2}\varepsilon_3|x|^{2q_1}\\ &\quad+\left(\beta_1+\frac{\beta_2+\beta_3}{2\varepsilon_1}\right)^2\left(\frac{\eta_M}{\varepsilon_2}+\frac{q_1+1}{2}\frac{\bar{\eta}_M}{\varepsilon_3}\right)|x-y|^2, \end{aligned}$$

where J_1, J_2, J_3, J_4 have been given before. Noting that $|x|^{2r+2}\le|x|^2+|x|^{q_1+1}$ and $|x|^{q_1+2r+1}\le|x|^{q_1+1}+|x|^{2q_1}$, since $0\le r\le\frac{q_1-1}{2}$, which is required in Assumption 5, we further derive that

$$\begin{aligned} &U_x(x,t,i)(f_1(x,y,t,i)-f_1(x,x,t,i))\\ &\le(\eta_M\varepsilon_2+J_1\varepsilon_1)|x|^2+J_2\varepsilon_1|y|^2+\left(\frac{(q_1+1)\bar{\eta}_M}{2}\varepsilon_3+J_3\varepsilon_1\right)|x|^{2q_1}+J_4\varepsilon_1|y|^{2q_1}\\ &\quad+(J_1+J_3)\varepsilon_1|x|^{q_1+1}+(J_2+J_4)\varepsilon_1|y|^{q_1+1}+\left(\beta_1+\frac{\beta_2+\beta_3}{2\varepsilon_1}\right)^2\left(\frac{\eta_M}{\varepsilon_2}+\frac{q_1+1}{2}\frac{\bar{\eta}_M}{\varepsilon_3}\right)|x-y|^2. \end{aligned}$$

This completes the proof. □

From the proof of Lemma 3, we find that parameter ε_1 is used to eliminate the influence of $|x|^{2r+1}$ and $|y|^{2r+1}$, and parameters ε_2, ε_3 are used to cope with $|U_x(x,t,i)|$. By making use of Assumption 7 and selecting an appropriate τ, we are able to let $\mathcal{L}V(\hat{x}_t,t,r(t))$ become negative. Base on this, we can now present our stability results. The first one is H_∞ stability.

Theorem 2. *Let all the conditions in Theorem 1 and Assumptions 5, 7 hold. Then there is a positive number τ^* such that for any initial data (2.1), the solution of hybrid SDDE (1.1) has the property that*

$$\int_0^\infty E|x(t)|^{2q_1}\mathrm{d}t<\infty \tag{4.3}$$

and

$$\sup_{0\le t<\infty} E|x(t)|^{q_1+1}<\infty \tag{4.4}$$

as long as $\tau<\tau^*$.

Before giving the proof, to make this theorem can be implemented in practice, we make some comments on how to determine the value of τ^*.

Remark 5. We define a domain Λ on $\mathbb{R}^3_+$ by

$$\Lambda = \Big\{(\varepsilon_1, \varepsilon_2, \varepsilon_3)\Big| \varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0, \quad \eta_M\varepsilon_2 + (J_1 + J_2h^*)\varepsilon_1 < B_1, \\ \frac{q_1+1}{2}\bar{\eta}_M\varepsilon_3 + (J_3 + J_4h^*)\varepsilon_1 < B_2, \quad (J_1 + J_2h^* + J_3 + J_4h^*)\varepsilon_1 < B_3\Big\},$$

and four functions on Λ by

$$\begin{aligned}
\varphi_1(\varepsilon_1, \varepsilon_2, \varepsilon_3) =& B_1 - \eta_M\varepsilon_2 - (J_1 + J_2h^*)\varepsilon_1,\\
\varphi_2(\varepsilon_1, \varepsilon_2, \varepsilon_3) =& B_2 - \frac{q_1+1}{2}\bar{\eta}_M\varepsilon_3 - (J_3 + J_4h^*)\varepsilon_1,\\
\varphi_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) =& B_3 - (J_1 + J_2h^* + J_3 + J_4h^*)\varepsilon_1,\\
\varphi_4(\varepsilon_1, \varepsilon_2, \varepsilon_3) =& \left(\beta_1 + \frac{\beta_2+\beta_3}{2\varepsilon_1}\right)^2\left(\frac{\eta_M}{\varepsilon_2} + \frac{q_1+1}{2}\frac{\bar{\eta}_M}{\varepsilon_3}\right).
\end{aligned}$$

Then τ^* is given by

$$\tau^* = \sup_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in\Lambda} \varphi(\varepsilon_1, \varepsilon_2, \varepsilon_3),$$

where

$$\varphi(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \frac{-H_2 + \sqrt{H_2^2 + 4H_1\frac{\varphi_1(\varepsilon_1,\varepsilon_2,\varepsilon_3)}{2\varphi_4(\varepsilon_1,\varepsilon_2,\varepsilon_3)}}}{2H_1} \wedge \sqrt{\frac{1}{H_3 + H_4}\frac{\varphi_2(\varepsilon_1, \varepsilon_2, \varepsilon_3)}{2\varphi_4(\varepsilon_1, \varepsilon_2, \varepsilon_3)}} \wedge \frac{-H_6 + \sqrt{H_6^2 + 4H_5\frac{\varphi_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)}{2\varphi_4(\varepsilon_1,\varepsilon_2,\varepsilon_3)}}}{2H_5}.$$

Since we require that β_1, β_2 and β_3 cannot be zero at the same time in Assumption 5, $\varphi_4(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is always positive, which implies that we must have $\tau^* < \infty$. With a little effort, we find that when $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ approaches the boundary of Λ, the continuous function $\varphi(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ tends to zero. As a result, there exists $(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3) \in \Lambda$ such that

$$\tau^* = \max_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in\Lambda} \varphi(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \varphi(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3). \tag{4.5}$$

From now on, the free parameters ε_1, ε_2, ε_3 are fixed as $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$, $\hat{\varepsilon}_3$, respectively. Meanwhile, we denote by $\hat{\varphi}_1 = \varphi_1(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3)$, $\hat{\varphi}_2 = \varphi_2(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3)$, $\hat{\varphi}_3 = \varphi_3(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3)$, $\hat{\varphi}_4 = \varphi_4(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3)$ for convenience. Next, we show that Theorem 2 is true.

Proof. Fix the initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$ and $r_0 \in S$ arbitrarily. We obtain from (3.15) that

$$V(\hat{x}_t, t, r(t)) = V(\hat{x}_0, 0, r(0)) + \int_0^t \mathcal{L}V(\hat{x}_s, s, r(s))\mathrm{d}s + \int_0^t U_x(x(s), s, r(s))g(x(s), x(s-h(s)), s, r(s))\mathrm{d}W(s) + M(t).$$

Then we can use the same stopping time $\{\sigma_k\}_{k\ge k_0}$ defined in the proof of Theorem 1 to obtain that

$$EV(\hat{x}_{t\wedge\sigma_k}, t\wedge\sigma_k, r(t\wedge\sigma_k)) = V(\hat{x}_0, 0, r(0)) + E\int_0^{t\wedge\sigma_k} \mathcal{L}V(\hat{x}_s, s, r(s))\mathrm{d}s.$$

For each $k \ge k_0$,

$$\left|\int_0^{t\wedge\sigma_k} \mathcal{L}V(\hat{x}_s, s, r(s))\mathrm{d}s\right| \le \int_0^{t\wedge\sigma_k} \left|\mathcal{L}V(\hat{x}_s, s, r(s))\right| \mathrm{d}s \le \int_0^t \left|\mathcal{L}V(\hat{x}_s, s, r(s))\right| \mathrm{d}s.$$

Recalling the definition of $\mathcal{L}V(\hat{x}_t, t, r(t))$ in (3.16) and using condition (3.2), the Hölder inequality, the Young inequality, we have

$$\begin{aligned}
\left|\mathcal{L}V(\hat{x}_t, t, r(t))\right| \le& |LU(x(t), x(t-h(t)), t, r(t))|\\
&+ |U_x(x(t), t, r(t))\,\|\, f_1(x(t), x(t-h(t)), t, r(t)) - f_1(x(t), x(t), t, r(t))|\\
&+ 2\theta_1\tau|x(t)|^2 + 2\theta_2\tau|x(t)|^{2q_1} + 2\theta_3\tau|x(t)|^{q_1+1}\\
&+ \theta_1\left|\int_{t-2\tau}^t |x(v)|^2\mathrm{d}v\right| + \theta_2\left|\int_{t-2\tau}^t |x(v)|^{2q_1}\mathrm{d}v\right| + \theta_3\left|\int_{t-2\tau}^t |x(v)|^{q_1+1}\mathrm{d}v\right|\\
\le& C\left(1 + |x(t)|^{2q_1} + |x(t-h(t))|^{2q_1} + \int_{t-2\tau}^t \left(|x(v)|^2 + |x(v)|^{2q_1}\right)\mathrm{d}v\right),
\end{aligned}$$

where C is a positive number independent from t. This then yields that

$$
\begin{aligned}
&E\int_0^t \left|\mathcal{L}V(\hat{x}_s,s,r(s))\right|\mathrm{d}s=\int_0^t E\left|\mathcal{L}V(\hat{x}_s,s,r(s))\right|\mathrm{d}s\\
&\le\int_0^t C\left(1+E|x(s)|^{2q_1}+E|x(s-h(s))|^{2q_1}+\int_{s-2\tau}^{s}\left(E|x(v)|^2+E|x(v)|^{2q_1}\right)\mathrm{d}v\right)\mathrm{d}s<\infty,
\end{aligned}
$$

where we have used (2.8) and the Fubini theorem. Letting $k\to\infty$ and using the Fatou lemma, the dominated convergence theorem gives that

$$
\begin{aligned}
EV(\hat{x}_t,t,r(t))=&E\left(\liminf_{k\to\infty}V(\hat{x}_{t\wedge\sigma_k},t\wedge\sigma_k,r(t\wedge\sigma_k))\right)\le\liminf_{k\to\infty}EV(\hat{x}_{t\wedge\sigma_k},t\wedge\sigma_k,r(t\wedge\sigma_k))\\
=&V(\hat{x}_0,0,r(0))+\liminf_{k\to\infty}E\int_0^{t\wedge\sigma_k}\mathcal{L}V(\hat{x}_s,s,r(s))\mathrm{d}s\\
=&V(\hat{x}_0,0,r(0))+E\int_0^t\mathcal{L}V(\hat{x}_s,s,r(s))\mathrm{d}s.
\end{aligned}\tag{4.6}
$$

By (3.11) and Lemma 3, it is easy to show from (3.16) that

$$
\begin{aligned}
\mathcal{L}V(\hat{x}_s,s,r(s))\le&-\left(1-\eta_M\hat{\varepsilon}_2-J_1\hat{\varepsilon}_1-2\theta_1\tau\right)|x(s)|^2+(\varpi_1+J_2\hat{\varepsilon}_1)|x(s-h(s))|^2\\
&-\left(\varpi_2-\frac{q_1-1}{2q_1}\varpi_3-\frac{q_1+1}{2}\bar{\eta}_M\hat{\varepsilon}_3-J_3\hat{\varepsilon}_1-2\theta_2\tau\right)|x(s)|^{2q_1}\\
&+\left(\frac{q_1+1}{2q_1}\varpi_3+J_4\hat{\varepsilon}_1\right)|x(s-h(s))|^{2q_1}\\
&-\left(\varpi_4+1-\frac{q_1-1}{2}\varpi_6-(J_1+J_3)\hat{\varepsilon}_1-2\theta_3\tau\right)|x(s)|^{q_1+1}\\
&+\left(\varpi_5+\varpi_6+(J_2+J_4)\hat{\varepsilon}_1\right)|x(s-h(s))|^{q_1+1}\\
&+\left(\beta_1+\frac{\beta_2+\beta_3}{2\hat{\varepsilon}_1}\right)^2\left(\frac{\eta_M}{\hat{\varepsilon}_2}+\frac{q_1+1}{2}\frac{\bar{\eta}_M}{\hat{\varepsilon}_3}\right)|x(s)-x(s-h(s))|^2\\
&-\theta_1\int_{s-2\tau}^{s}|x(v)|^2\mathrm{d}v-\theta_2\int_{s-2\tau}^{s}|x(v)|^{2q_1}\mathrm{d}v-\theta_3\int_{s-2\tau}^{s}|x(v)|^{q_1+1}\mathrm{d}v.
\end{aligned}
$$

Substituting this into (4.6), then using Lemma 1, and recalling the definition of constants B_1, B_2, B_3 in (3.12), and $\hat{\varphi}_1$, $\hat{\varphi}_2$, $\hat{\varphi}_3$, $\hat{\varphi}_4$ in Remark 5, we obtain that

$$
\begin{aligned}
EV(\hat{x}_t,t,r(t))\le&C_2-(\hat{\varphi}_1-2\theta_1\tau)E\int_0^t|x(s)|^2\mathrm{d}s-(\hat{\varphi}_2-2\theta_2\tau)E\int_0^t|x(s)|^{2q_1}\mathrm{d}s\\
&-(\hat{\varphi}_3-2\theta_3\tau)E\int_0^t|x(s)|^{q_1+1}\mathrm{d}s+\hat{\varphi}_4E\int_0^t|x(s)-x(s-h(s))|^2\mathrm{d}s\\
&-\theta_1E\int_0^t\int_{s-2\tau}^{s}|x(v)|^2\mathrm{d}v\mathrm{d}s-\theta_2E\int_0^t\int_{s-2\tau}^{s}|x(v)|^{2q_1}\mathrm{d}v\mathrm{d}s-\theta_3E\int_0^t\int_{s-2\tau}^{s}|x(v)|^{q_1+1}\mathrm{d}v\mathrm{d}s,
\end{aligned}\tag{4.7}
$$

where

$$
C_2=V(\hat{x}_0,0,r(0))+(\varpi_1+J_2\hat{\varepsilon}_1)h^*\tau\|\xi\|^2+\left(\frac{q_1+1}{2q_1}\varpi_3+J_4\hat{\varepsilon}_1\right)h^*\tau\|\xi\|^{2q_1}+(\varpi_5+\varpi_6+(J_2+J_4)\hat{\varepsilon}_1)h^*\tau\|\xi\|^{q_1+1}.
$$

For $t\in[0,2\tau]$, we clearly have

$$
\int_0^t E|x(s)-x(s-h(s))|^2\mathrm{d}s\le\int_0^{2\tau}2\left(E|x(s)|^2+E|x(s-h(s))|^2\right)\mathrm{d}s\le\int_0^{2\tau}4\sup_{-\tau\le v\le2\tau}E|x(v)|^2\mathrm{d}s\le8\tau\sup_{-\tau\le v\le2\tau}E|x(v)|^2.
$$

For $t>2\tau$, recall the estimation in Lemma 2, that is

$$
\begin{aligned}
\int_0^t E|x(s)-x(s-h(s))|^2\mathrm{d}s\le&(H_1\tau+H_2)\int_0^t\int_{s-2\tau}^{s}E|x(v)|^2\mathrm{d}v\mathrm{d}s+(H_3\tau+H_4\tau)\int_0^t\int_{s-2\tau}^{s}E|x(v)|^{2q_1}\mathrm{d}v\mathrm{d}s\\
&+(H_5\tau+H_6)\int_0^t\int_{s-2\tau}^{s}E|x(v)|^{q_1+1}\mathrm{d}v\mathrm{d}s.
\end{aligned}
$$

Then for any $t \geq 0$, we obtain that

$$\begin{aligned}
&\hat{\varphi}_4 \int_0^t E|x(s) - x(s-h(s))|^2 \mathrm{d}s \\
\leq & C_3 + (H_1\tau + H_2)\hat{\varphi}_4 \int_0^t \int_{s-2\tau}^s E|x(v)|^2 \mathrm{d}v\mathrm{d}s + (H_3\tau + H_4\tau)\hat{\varphi}_4 \int_0^t \int_{s-2\tau}^s E|x(v)|^{2q_1} \mathrm{d}v\mathrm{d}s \\
& + (H_5\tau + H_6)\hat{\varphi}_4 \int_0^t \int_{s-2\tau}^s E|x(v)|^{q_1+1} \mathrm{d}v\mathrm{d}s,
\end{aligned} \tag{4.8}$$

where $C_3 = 8\tau\hat{\varphi}_4 \sup_{-\tau \leq v \leq 2\tau} E|x(v)|^2$. Putting (4.8) into (4.7) and letting

$$\theta_1 = (H_1\tau + H_2)\hat{\varphi}_4, \quad \theta_2 = (H_3\tau + H_4\tau)\hat{\varphi}_4, \quad \theta_3 = (H_5\tau + H_6)\hat{\varphi}_4,$$

gives that

$$\begin{aligned}
EV(\hat{x}_t, t, r(t)) \leq & C_2 + C_3 - \left(\hat{\varphi}_1 - 2\left(H_1\tau^2 + H_2\tau\right)\hat{\varphi}_4\right) \int_0^t E|x(s)|^2 \mathrm{d}s - \left(\hat{\varphi}_2 - 2\left(H_3\tau^2 + H_4\tau^2\right)\hat{\varphi}_4\right) \int_0^t E|x(s)|^{2q_1} \mathrm{d}s \\
& - \left(\hat{\varphi}_3 - 2\left(H_5\tau^2 + H_6\tau\right)\hat{\varphi}_4\right) \int_0^t E|x(s)|^{q_1+1} \mathrm{d}s.
\end{aligned}$$

Recalling the discussion in Remark 5 and using the properties of quadratic functions, when $\tau < \tau^*$, we have

$$H_1\tau^2 + H_2\tau < \frac{\hat{\varphi}_1}{2\hat{\varphi}_4}, \quad H_3\tau^2 + H_4\tau^2 < \frac{\hat{\varphi}_2}{2\hat{\varphi}_4}, \quad H_5\tau^2 + H_6\tau < \frac{\hat{\varphi}_3}{2\hat{\varphi}_4}.$$

This immediately yields that

$$\bar{\eta}_M E|x(t)|^{q_1+1} \leq C_2 + C_3$$

and

$$C_4 \int_0^t E|x(s)|^{2q_1} \mathrm{d}s \leq C_2 + C_3,$$

where $C_4 = \hat{\varphi}_2 - 2\left(H_3\tau^2 + H_4\tau\right)\hat{\varphi}_4$ is a positive constant. Finally, letting $t \to \infty$ implies the required assertions (4.3) and (4.4). □

In Theorem 2, we have known that $x(t)$ is bounded in L^{q_1+1}. Then it is not difficult to show that $E|x(t)|^2$ is uniformly continuous in t. Combining with the fact that $\int_0^\infty E|x(t)|^{2q_1} dt < \infty$, we can conclude that hybrid SDDE (1.1) is also moment asymptotically stable.

Theorem 3. *Let all the conditions in Theorem 2 hold. Then the solution of hybrid SDDE (1.1) satisfies that*

$$\lim_{t\to\infty} E|x(t)|^{\bar{q}} = 0 \tag{4.9}$$

for any $\bar{q} \in [2, q_1 + 1)$ and any initial data (2.1) provided $\tau < \tau^*$.

We can use the same analysis as in the proof of Theorem 3.6 in [17] to show this theorem so we omit it. Next, making use of the idea of stochastic LaSalle theorem developed in [28], we can show the almost sure asymptotic stability of hybrid SDDE (1.1) in the following theorem.

Theorem 4. *Under the same conditions in Theorem 2, for any initial data (2.1) and $\tau < \tau^*$, hybrid SDDE (1.1) obeys that*

$$\lim_{t\to\infty} x(t) = 0 \quad a.s. \tag{4.10}$$

Proof. Fix the initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$ and $r_0 \in S$ arbitrarily. From Theorem 2 and the Fubini theorem, we know that

$$E\int_0^\infty |x(t)|^{2q_1} \mathrm{d}t < \infty,$$

which yields that

$$\liminf_{t\to\infty} |x(t)| = 0 \quad a.s.$$

If the required assertion (4.10) is false, we can find a sufficiently small number $\varepsilon > 0$ such that

$$P(|x(t)|^{q_1+1} \geq 2\varepsilon) \geq 4\varepsilon.$$

We can use the same stopping time σ_k defined in the proof of Theorem 1, then (4.7) can be rewritten as

$$\begin{aligned} EV(\hat{x}_{t\wedge\sigma_k}, t\wedge\sigma_k, r(t\wedge\sigma_k)) \le & C_2 - (\hat{\varphi}_1 - 2\theta_1\tau)E\int_0^{t\wedge\sigma_k}|x(s)|^2\mathrm{d}s - (\hat{\varphi}_2 - 2\theta_2\tau)E\int_0^{t\wedge\sigma_k}|x(s)|^{2q_1}\mathrm{d}s \\ & - (\hat{\varphi}_3 - 2\theta_3\tau)E\int_0^{t\wedge\sigma_k}|x(s)|^{q_1+1}\mathrm{d}s + \hat{\varphi}_4 E\int_0^{t\wedge\sigma_k}|x(s)-x(s-h(s))|^2\mathrm{d}s \\ & - \theta_1 E\int_0^{t\wedge\sigma_k}\int_{s-2\tau}^{s}|x(v)|^2\mathrm{d}v\mathrm{d}s - \theta_2 E\int_0^{t\wedge\sigma_k}\int_{s-2\tau}^{s}|x(v)|^{2q_1}\mathrm{d}v\mathrm{d}s \\ & - \theta_3 E\int_0^{t\wedge\sigma_k}\int_{s-2\tau}^{s}|x(v)|^{q_1+1}\mathrm{d}v\mathrm{d}s. \end{aligned}$$

Since $k^{q_1+1}P(\sigma_k \le t) \le E|x(t\wedge\sigma_k)|^{q_1+1} \le EV(\hat{x}_{t\wedge\sigma_k}, t\wedge\sigma_k, r(t\wedge\sigma_k))$, we can let $k\to\infty$ and use the $\theta_1, \theta_2, \theta_3$ defined in the proof of Theorem 2 to obtain

$$\begin{aligned} \limsup_{k\to\infty} k^{2r+2}P(\sigma_k \le t) \le & C_2 - (\hat{\varphi}_1 - 2\theta_1\tau)E\int_0^{t}|x(s)|^2\mathrm{d}s - (\hat{\varphi}_2 - 2\theta_2\tau)E\int_0^{t}|x(s)|^{2q_1}\mathrm{d}s \\ & - (\hat{\varphi}_3 - 2\theta_3\tau)E\int_0^{t}|x(s)|^{q_1+1}\mathrm{d}s + \hat{\varphi}_4 E\int_0^{t}|x(s)-x(s-h(s))|^2\mathrm{d}s \\ & - \theta_1 E\int_0^{t}\int_{s-2\tau}^{s}|x(v)|^2\mathrm{d}v\mathrm{d}s - \theta_2 E\int_0^{t}\int_{s-2\tau}^{s}|x(v)|^{2q_1}\mathrm{d}v\mathrm{d}s - \theta_3 E\int_0^{t}\int_{s-2\tau}^{s}|x(v)|^{q_1+1}\mathrm{d}v\mathrm{d}s \\ \le & C_2 + C_3. \end{aligned}$$

There exists a positive integer k_1 such that $k^{q_1+1}P(\sigma_k \le t) \le C_2 + C_3 + 1$ whenever $k \ge k_1$. Since this holds for any $t \ge 0$, we have $k^{q_1+1}P(\sigma_k < \infty) \le C_2 + C_3 + 1$. Then in line with the discussion of Theorem 2 in [11], the required assertion (4.10) must hold. This ends the proof. □

5. Examples

We give two examples here to illustrate the effectiveness of our theory. The first one is an application to mosquito model. The second one is to design nonlinear delay feedback control, whose Lipschitz coefficient is not a constant. To avoid complicated calculations, we let $W(t)$ be a scalar Brownian motion, $r(t)$ be a continuous Markov chain taking values in $S = \{1, 2\}$ with transition rate matrix $Q = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}$.

5.1. Application to mosquito model

Consider the following scalar non-autonomous hybrid SDDE

$$\mathrm{d}x(t) = \left(\hat{b}_{r(t)}(t)e^{-0.1|x(t-h(t))|}x(t-h(t)) - \hat{a}_{r(t)}(t)x(t) - \hat{c}_{r(t)}(t)x^3(t)\right)\mathrm{d}t + \hat{d}_{r(t)}(t)x(t-h(t))\mathrm{d}W(t) \tag{5.1}$$

on $t \ge 0$ with delay function $h(t)$ and

$$\begin{aligned} &\hat{a}_1(t) = 0.6(1+\sin^2(t)), && \hat{b}_1(t) = -(1+\cos^2(t)), && \hat{c}_1(t) = 0.4(2+\sin(t)), && \hat{d}_1(t) = 0.4\cos(t), \\ &\hat{a}_2(t) = 0.2\left(1+\frac{1}{1+t}\right), && \hat{b}_2(t) = -\frac{0.8}{1+t}, && \hat{c}_2(t) = 0.3\left(1+\frac{1}{\sqrt{1+t}}\right), && \hat{d}_2(t) = \frac{0.2}{\sqrt{1+t}}. \end{aligned}$$

This equation can be used to model the behaviors of adult female mosquitoes (see, e.g. [29]).

Let us first pay attention to two special cases, namely, $h(t) \equiv 0$ and $h(t) \equiv 3$. The simulation results with initial data $\xi(t) = 1 + 0.1\sin(t)$ for $t \in [-3, 0]$ and $r_0 = 1$ are shown in Fig. 1 (the second subfigure and the third subfigure, respectively). The former one indicates that hybrid SDDE (5.1) is asymptotically stable with $h(t) \equiv 0$, while the latter one shows us the corresponding SDDE is unstable with $h(t) \equiv 3$. As a consequence, hybrid SDDE (5.1) could become stable when the time delay becomes smaller and smaller. And our theory is aimed to give a bound for the delay.

Let us consider the delay function $h : \mathbb{R}_+ \to [0, \tau]$ meeting Assumption 2 and assume that $h^* = 2$. Letting $f(x, y, t, i) = \hat{b}_i(t)e^{-0.1|y|}y - \hat{a}_i(t)x - \hat{c}_i(t)x^3$ and $g(x, y, t, i) = \hat{d}_i(t)y$, It is easy to show that Assumption 3 is satisfied with $q_1 = 3$, $K_1 = 0.6$, $K_2 = 2$, $K_3 = 1.2$, $K_4 = 0$, $\hat{K}_1 = 0$, $\hat{K}_2 = 0.16$, $\hat{K}_3 = 0$, $\hat{K}_4 = 0$. Then, we observe that

$$\begin{aligned} xf(x, y, t, 1) + \frac{q-1}{2}|g(x, y, t, 1)|^2 \le & -0.6|x|^2 + |xy| - 0.4|x|^4 + \frac{q-1}{2}0.16y^2, \\ xf(x, y, t, 2) + \frac{q-1}{2}|g(x, y, t, 2)|^2 \le & -0.2|x|^2 + 0.8|xy| - 0.3|x|^4 + \frac{q-1}{2}0.04y^2. \end{aligned}$$

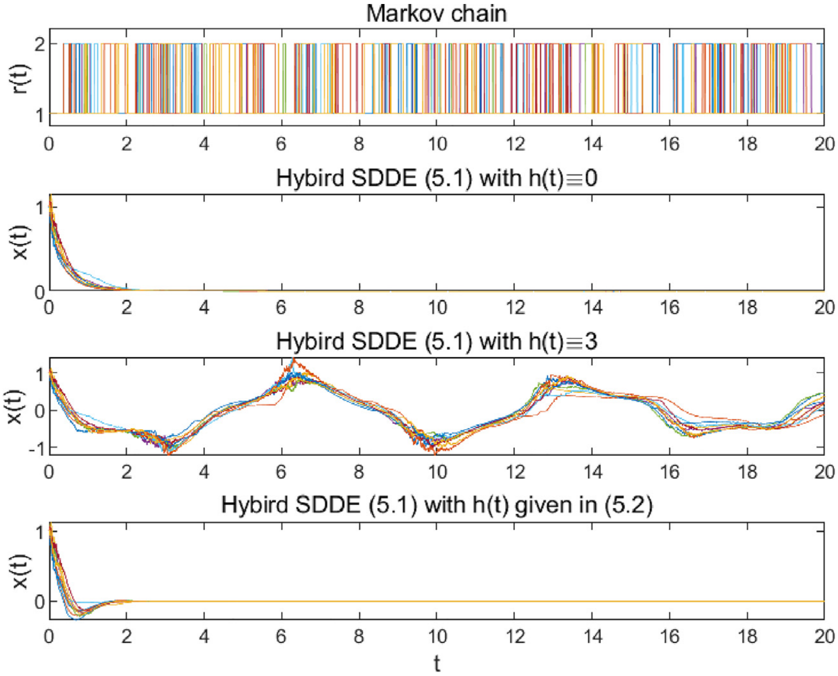

Fig. 1. Ten sample paths of Markov chain, hybrid SDDE (5.1) with $h(t) \equiv 0$, with $h(t) \equiv 3$, with $h(t)$ given in (5.2), using the truncated Euler–Maruyama method with time step size 10^{-4}.

Hence, we can choose $q = 6$, which is no less than $2q_1$, to let Assumption 4 hold with $\alpha_1 = 0.2$, $\alpha_2 = 0.9$, $\alpha_{3,1} = 0.4$, $\alpha_{3,2} = 0.3$, $\alpha_{4,1} = 0$, $\alpha_{4,2} = 0$. Additionally, we see that $\alpha_{3,1} - \frac{h^*(q_1+1)+q-2}{q_1+q-1}\alpha_{4,1} = 0.4$ and $\alpha_{3,2} - \frac{h^*(q_1+1)+q-2}{q_1+q-1}\alpha_{4,2} = 0.3$. Up to mow, all the conditions in Theorem 1 have been checked. Next, we can decompose f as Eq. (3.1) with

$$f_1(x, y, t, i) = \hat{b}_i(t)e^{-0.1|y|}y, \quad f_2(x, y, t, i) = -\hat{a}_i(t)x - \hat{c}_i(t)x^3.$$

Since for any $(x, y, t, i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times S$,

$$|f_1(x, y, t, i) - f_1(x, x, t, i)| \le |\hat{b}_i(t)| \left(e^{-0.1|x|}|x-y| + |y| \left|e^{-|x|} - e^{-|y|}\right|\right) \le |\hat{b}_i(t)| \left(e^{-0.1|x|} + 0.1|y|\right) |x-y|,$$

we see that Assumption 5 is satisfied with $\beta_1 = 2$, $\beta_2 = 0$, $\beta_3 = 0.2$, and $r = 1$. To verify Assumption 7, compute

$$x(f_1(x, x, t, i) + f_2(x, y, t, i)) \le \begin{cases} -0.6|x|^2 - |x|^2e^{-|x|} - 0.4|x|^4, & i = 1, \\ -0.2|x|^2 - 0.8|x|^2e^{-|x|} - 0.3|x|^4, & i = 2. \end{cases}$$

As a result, we obtain

$$\begin{array}{llllllll} a_1 = -0.6, & a_2 = -0.2, & b_1 = 0.08, & b_2 = 0.02, & c_1 = 0.4, & c_2 = 0.3, & d_1 = 0, & d_2 = 0, \\ \bar{a}_1 = -0.6, & \bar{a}_2 = -0.2, & \bar{b}_1 = 0.16, & \bar{b}_2 = 0.04, & \bar{c}_1 = 0.4, & \bar{c}_2 = 0.3, & \bar{d}_1 = 0, & \bar{d}_2 = 0, \end{array}$$

and

$$A = \begin{pmatrix} 2.2 & -1 \\ -6 & 6.4 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 3.4 & -1 \\ -6 & 6.8 \end{pmatrix}.$$

We hence show that $(\eta_1, \eta_2) = (0.9158, 1.0149)$ and $(\bar{\eta}_1, \bar{\eta}_2) = (0.4556, 0.5491)$, which shows that $A, \bar{A}$ are non-singular M-matrices. We also have $B_1 = 0.7069$, $B_2 = 0.6589$, $B_3 = 1.1715$. Consequently, all the conditions in Assumption 7 are fulfilled. Until now, we have check all the conditions in Theorem 2.

By Theorems 3 and 4, we conclude that hybrid SDDE (5.1) is asymptotically stable in both $L^{\bar{q}}$ ($\bar{q} \in [2, 4)$) and almost sure sense if $\tau < \tau^*$. Using the method introduced in Remark 5, we derive that $\tau^* = 0.005$. Finally, we choose

$$h(t) = 0.001 \sum_{k=0}^{\infty} 1_{[0.001(2k+1), 0.001(2k+2))}(t). \tag{5.2}$$

One can easily verify that this delay function satisfies Assumption 2 with $h^* = 2$ and $\tau = 0.001$. The computer simulation is given in Fig. 1 (the bottom subfigure). The simulation supports our results clearly.

5.2. Nonlinear delay feedback control

Consider the modified stochastic van der Pol–Duffing oscillator studied in [30] described by

$$\begin{cases} dx_1(t) = \Big(-(1+\lambda_{r(t)})x_1(t) + \hat{B}_{r(t)}(x_2(t)-x_1(t))^3 + \lambda_{r(t)}x_2(t) - \hat{A}_{r(t)}x_1^3(t)\Big)\,dt + \delta_{r(t)}x_1^2(t)dW(t) \\ dx_2(t) = \Big(\lambda_{r(t)}x_1(t) - \rho_{r(t)}x_3(t) - \hat{B}_{r(t)}(x_2(t)-x_1(t))^3 - (\lambda_{r(t)}+1)x_2(t) - \hat{C}_{r(t)}x_2^3(t)\Big)\,dt + \delta_{r(t)}x_2^2(t)dW(t) \\ dx_3(t) = \Big(x_2(t) + \rho_{r(t)}x_3(t) - \hat{D}_{r(t)}x_3^3(t)\Big)\,dt + \delta_{r(t)}x_3^2(t)dW(t) \end{cases}$$

on $t \ge 0$, where $\lambda_1 = 0.5$, $\lambda_2 = 0.3$, $\rho_1 = 0.2$, $\rho_2 = 0.1$, $\hat{A}_1 = 1$, $\hat{A}_2 = 0.8$, $\hat{B}_1 = 0.2$, $\hat{B}_2 = 0.4$, $\hat{C}_1 = 0.8$, $\hat{C}_2 = 1$, $\hat{D}_1 = 0.8$, $\hat{D}_2 = 1.2$, $\delta_1 = 0.3$, $\delta_2 = 0.2$. Letting $x(t) = (x_1(t), x_2(t), x_3(t))^T$, we can rewrite the oscillator system as

$$dx(t) = F(x(t), r(t))dt + G(x(t), r(t))dW(t), \tag{5.3}$$

where $G(x, i) = \delta_i \left(x_1^2, x_2^2, x_3^2\right)^T$ and

$$F(x,i) = \begin{pmatrix} -(1+\lambda_i)x_1 + \hat{B}_i(x_2-x_1)^3 + \lambda_i x_2 - \hat{A}_i x_1^3 \\ \lambda_i x_1 - \rho_i x_3 - \hat{B}_i(x_2-x_1)^3 - (\lambda_i+1)x_2 - \hat{C}_i x_2^3 \\ x_2 + \rho_i x_3 - \hat{D}_i x_3^3 \end{pmatrix}.$$

Through computer simulation (see Fig. 2 (the middle one)), we see that Eq. (5.3) is unstable. Then we want to design a delay feedback control to stabilize equation (5.3). It should be pointed out that delay feedback controls in the most results (see, e.g. [23,31,32]) are globally Lipschitz continuous, as a result of which the linear ones are usually used. But due to the oscillations of environment such as wind and electricity, we sometimes need to implement the following nonlinear controller

$$u(x,i) = -\kappa_i \text{diag}\left(\sqrt{1+0.02\cos^2(x)}\right)x \tag{5.4}$$

with $\kappa_1 = 0.8$, $\kappa_2 = 0.5$ and time lag $h(t)$ meeting Assumption 2 with $h^* = 1.25$. Obviously, our control function $u(x, i)$ does not meet the global Lipschitz condition. But with further analysis, we derive that for any $x, \tilde{x} \in \mathbb{R}^3$

$$\begin{aligned} |u(x,i) - u(\tilde{x},i)|^2 =& \kappa_i^2 \sum_{j=1}^3 \left(\sqrt{1+0.02\cos^2(x_j)}x_j - \sqrt{1+0.02\cos^2(\tilde{x}_j)}\tilde{x}_j\right)^2 \\ \le& 2\kappa_i^2 \sum_{j=1}^3 \left(\left(1+0.02\cos^2(x_j)\right)(x_j-\tilde{x}_j)^2 + \left(\sqrt{1+0.02\cos^2(x_j)} - \sqrt{1+0.02\cos^2(\tilde{x}_j)}\right)^2 \tilde{x}_j^2\right) \\ \le& 2\kappa_i^2 \left(1.02 + 0.0001|\tilde{x}|^2\right)|x-\tilde{x}|^2, \end{aligned}$$

where we have used the differential mean value theorem. Thus we can apply the theory developed in this paper to the delay-state-feedback controlled problem

$$dx(t) = \big(F(x(t), r(t)) + u(x(t-h(t)), r(t))\big)dt + G(x(t), r(t))dW(t). \tag{5.5}$$

It is easy to check that Assumption 3 holds with $K_1 = 3.6056$, $K_2 = 0.808$, $K_3 = 3.7736$, $K_4 = 0$, $\hat{K}_1 = 0$, $\hat{K}_2 = 0$, $\hat{K}_3 = 3.7736$, $\hat{K}_4 = 0$, $q_1 = 3$. Then, compute

$$x^T F(x,i) \le -x_2^2 - \rho_i x_2 x_3 + \rho_i x_3^2 - \hat{A}_i x_1^4 - \hat{C}_i x_2^4 - \hat{D}_1 x_3^4 \le \left(\frac{\rho_i^2}{4} + \rho_i\right)|x|^2 - \frac{1}{3}(\hat{A}_i \wedge \hat{C}_i \wedge \hat{D}_i)|x|^4$$

and

$$x^T u(\tilde{x}, i) \le \sqrt{1.02}\kappa_i \sum_{j=1}^3 \left(\frac{1}{2}x_j^2 + \frac{1}{2}\tilde{x}_j^2\right) \le \frac{\sqrt{1.02}}{2}\kappa_i(|x|^2 + |\tilde{x}|^2).$$

We can choose $q = 6 \ge 2q_1$ to let Assumption 4 hold with $\alpha_1 = 0.614$, $\alpha_2 = 0.404$, $\alpha_{3,1} = 0.0417$, $\alpha_{3,2} = 0.1667$, $\alpha_{4,1} = 0$, $\alpha_{4,2} = 0$. Until mow, all the conditions in Theorem 1 have been verified. Next, the drift coefficient can be decomposed as $f_1(x, \tilde{x}, t, i) = u(\tilde{x}, i)$, $f_2(x, \tilde{x}, t, i) = F(x, i)$. Hence Assumption 5 is fulfilled with $\beta_1 = 1.1426$, $\beta_2 = 0$, $\beta_3 = 0.0113$, and $r = 1$. Compute

$$x^T F(x,i) + u(x,i) + \frac{1}{2}|G(x,i)|^2 \le \left(\frac{\rho_i^2}{4} + \rho_i - \sqrt{1.02}\kappa_i\right)|x|^2 - \left(\frac{1}{3}(\hat{A}_i \wedge \hat{C}_i \wedge \hat{D}_i) - \frac{1}{2}\delta_i^2\right)|x|^4$$

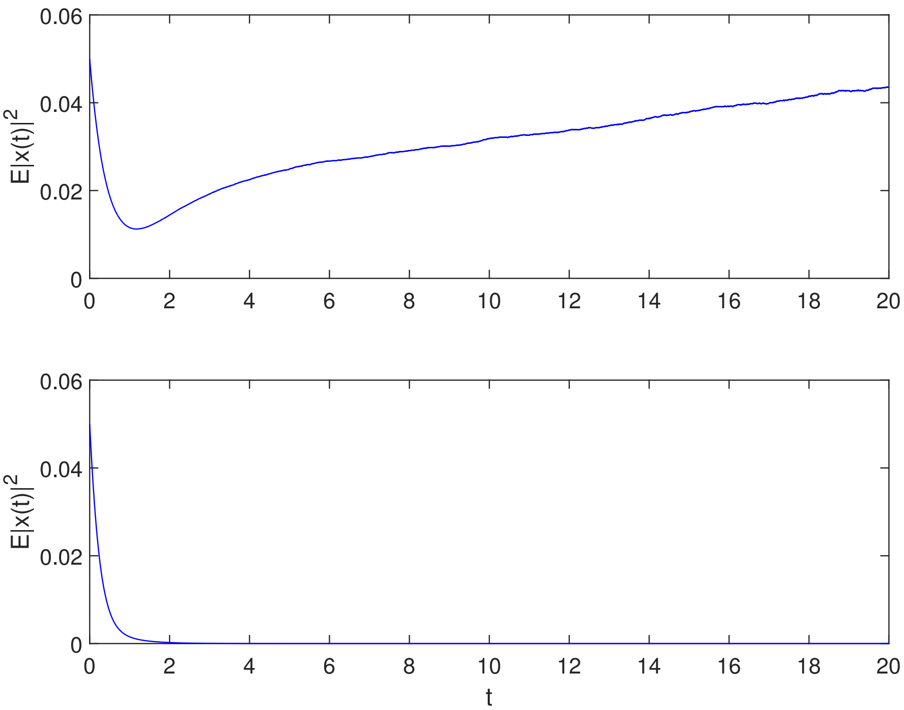

Fig. 2. Computer simulations of $E|x(t)|^2$ of oscillator system (5.3) (top), controlled oscillator system (5.5) (bottom) using the truncated Euler–Maruyama method with step size 10^{-4} and sample size 200 as well as the fixed initial data for $\xi(t) = (\xi_1(t), \dots, \xi_{Ne}(t))^{\mathrm{T}}$, where $\xi(t) = (0.1 + 0.1\cos(t), 0.1 + 0.1\sin(t), 0)^{\mathrm{T}}$ for $t \in [-0.1, 0]$ and $r_0 = 1$ for all 200 samples.

and

$$x^{\mathrm{T}}F(x,i) + u(x,i) + \frac{q_1}{2}|G(x,i)|^2 \le \left(\frac{\rho_i^2}{4} + \rho_i - \sqrt{1.02}\kappa_i\right)|x|^2 - \left(\frac{1}{3}(\hat{A}_i \wedge \hat{C}_i \wedge \hat{D}_i) - \frac{q_1}{2}\delta_i^2\right)|x|^4.$$

Then we derive that $a_1 = -0.598$, $a_2 = -0.4025$, $\bar{a}_1 = -0.598$, $\bar{a}_2 = -0.4025$, $c_1 = 0.2217$, $c_2 = 0.2467$, $\bar{c}_1 = 0.1317$, $\bar{c}_2 = 0.2067$, $b_1 = b_2 = \bar{b}_1 = \bar{b}_2 = d_1 = d_2 = \bar{d}_1 = \bar{d}_2 = 0$ and $A = \begin{pmatrix} 2.1959 & -1 \\ -6 & 6.805 \end{pmatrix}$, $\bar{A} = \begin{pmatrix} 3.3918 & -1 \\ -6 & 7.6099 \end{pmatrix}$. We then see that $(\eta_1, \eta_2) = (0.8727, 0.9164)$ and $(\bar{\eta}_1, \bar{\eta}_2) = (0.4346, 0.4741)$, which shows that A, $\bar{A}$ are non-singular M-matrices. Consequently, Assumption 7 is satisfied with $B_1 = 1$, $B_2 = 0.2289$, $B_3 = 1.3869$. Recalling the discussions in Remark 5, we obtain that $\tau^* = 0.009$. By Theorems 3 and 4, we conclude that controlled Eq. (5.5) is $\bar{q}$th moment asymptotically stable ($\bar{q} \in [2, 4)$) and almost surely asymptotically stable if $\tau < 0.009$. Ultimately, we select

$$h(t) = \sum_{k=0}^{\infty} \left(0.2(t-k)1_{[k,k+0.5)}(t) + (0.2 - 0.2(t-k))1_{[k+0.5,k+1)}(t)\right). \tag{5.6}$$

Delay function (5.6) meets Assumption 2 with $h^* = 1.25$ and $\tau = 0.001$. The simulation in Fig. 2 (the bottom one) shows our results clearly.

6. Conclusion

In this paper, compared with [11], the generalized delay-dependent stability criteria of superlinear hybrid SDDEs have been established with two restrictions lifted, in the sense of H_∞ stability, moment asymptotic stability, almost sure asymptotic stability. The major contributions of our new work could be concluded as follows. (1) The drift coefficient of the underlying system is decomposed into two parts, in one of which the Lipschitz coefficient of the delay component is a polynomial rather than a constant required in the aforementioned work [11]. The results in this paper hence have much wider applications. (2) Time-varying delay function is not necessary to be differentiable (see [8,11,19,20]), or limited by a strictly positive lower bound (see [23]) anymore. Then more general time delays in practice can be covered. (3) The technique of constructing Lyapunov functionals is modified, which could be a reference to other work when using this kind of Lyapunov functionals to study superlinear SDDEs. (4) By setting the free parameters, we can reduce the influence of time delays and let the Lyapunov operators become negative easily.

CRediT authorship contribution statement

Henglei Xu: Investigation, Writing – original draft, Software. **Xuerong Mao:** Supervision, Conceptualization, Methodology, Reviewing and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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