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A PRIORI AND *A POSTERIORI* ERROR ANALYSIS FOR SEMILINEAR PROBLEMS IN LIQUID CRYSTALS

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Abstract. In this paper, we develop a unified framework for the *a priori* and *a posteriori* error control of different lowest-order finite element methods for approximating the regular solutions of systems of partial differential equations under a set of hypotheses. The systems involve cubic nonlinearities in lower order terms, non-homogeneous Dirichlet boundary conditions, and the results are established under minimal regularity assumptions on the exact solution. The key contributions include (i) results for existence and local uniqueness of the discrete solutions using Newton–Kantorovich theorem, (ii) *a priori* error estimates in the energy norm, and (iii) *a posteriori* error estimates that steer the adaptive refinement process. The results are applied to conforming, Nitsche, discontinuous Galerkin, and weakly over penalized symmetric interior penalty schemes for variational models of ferronematics and nematic liquid crystals. The theoretical estimates are corroborated by substantive numerical results.

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1. Introduction

In this paper, we establish a unified framework for the *a priori* and *a posteriori* error analysis for coupled systems of second-order semilinear elliptic partial differential equations (PDEs) with nonlinearities in lower-order terms and non-homogeneous Dirichlet boundary conditions under minimal regularity assumption on the exact solution. We approximate the regular solutions u to $N(u) = 0$, such that u is subject to non-homogeneous Dirichlet boundary condition. Here $N : X \to V^*$ is a nonlinear differentiable operator with cubic nonlinearities in lower-order terms, the Hilbert space X is the continuous trial space, and V is the test space with dual V^*. The framework of the *a priori* error analysis is based on the Newton–Kantorovich theorem, applied to the discrete differentiable function $N_h : X_h \to V_h^*$, associated with N. Here X_h and V_h are finite dimensional trial and test spaces, respectively, associated with a triangulation $\mathcal{T}$ of a bounded two-dimensional domain with a Lipschitz continuous boundary.

Keywords and phrases. Conforming FEM, Nitsche's method, discontinuous Galerkin and WOPSIP methods, *a priori* and *a posteriori* error analysis, non-linear elliptic PDEs, non-homogeneous Dirichlet boundary conditions, nematic liquid crystals, ferronematics.

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Under a set of hypotheses that are sufficient to establish the discrete stability of a well-posed linear problem, and to verify the assumptions in Kantorovich theorem, this article unifies the *a priori* error analysis for lowest-order finite element methods for semi-linear problems with cubic nonlinearity. The *a posteriori* error analysis is established under an additional hypothesis and is based on the local Lipschitz continuity of the Frechét derivative of an extended nonlinear operator $\widehat{N} : \widehat{X} \to V^*$ around the exact solution, and an appropriate decomposition of the error itself to deal with the non-homogeneous Dirichlet boundary data approximation. Here the extended Hilbert space $\widehat{X}$ contains both the continuous and discrete trial spaces. The framework applies to a wide class of variational models in materials science, with free energies that involve a Dirichlet or isotropic elastic energy density and a lower-order nonlinear, non-convex potential. Two good examples of such variational models are the Landau–de Gennes model for liquid crystals [26] and the ferronematic model studied in [13].

The framework in this article covers four popular discrete schemes studied in literature; namely, the conforming, Nitsche, discontinuous Galerkin (dG), and weakly over penalized symmetric interior penalty (WOPSIP) schemes for the approximation of the regular solutions in ferronematic model [13] and a Landau–de Gennes (LDG) model for nematic liquid crystals (NLCs), with special reference to the stable solutions that model the physically observable configurations in experiments. The Nitsche's method [15, 28] imposes non-homogeneous Dirichlet boundary conditions weakly rather than incorporating them into the finite element space (as done in the standard conforming FEM). The dG schemes also impose Dirichlet boundary conditions weakly in the discrete formulation and has other attractive features. The basis functions of the finite element space associated to dG schemes are discontinuous, hence adds flexibility in global assembly, and are parallelizable; they are elementwise conservative, they allow hanging nodes in mesh generation, and different order of polynomials on bordering elements without continuity enforcement, and help in handling complicated geometries. In the last few decades, dG schemes have received a significant amount of attention due to their applications in a wide range of PDEs, see [10] and the references therein. The WOPSIP method [3, 4, 29] is a symmetric variant of the dG scheme well-studied in the literature. This method is intrinsically parallel, does not require the tuning of the penalty parameters, satisfies optimal order error estimates in both energy and L^2-norms and has less computational complexity. Though the over-penalization in the WOPSIP method increases the condition number of the resulting discrete system, this can be offset by a simple block diagonal preconditioner [3]. This article identifies a set of hypotheses that are sufficient to prove optimal order *a priori* and reliable and efficient *a posteriori* error estimates for semilinear problems with nonlinearities in lower order terms and applies to all the aforementioned methods.

An abstract framework for the error control of the conforming and nonconforming discretization of a class of fourth-order semilinear elliptic problems with quadratic nonlinearities and homogeneous Dirichlet boundary conditions is developed in [7]. The *a priori* analysis is extended to other lowest-order quadratic schemes and rough right-hand sides $F \in V^*$ in [8]. The new framework in this article builds on these results; accounts for non-homogeneous Dirichlet boundary conditions, higher order nonlinearities in lower order terms, and *a posteriori* error analysis. In [23], we analyse the dG approximation of the Euler–Lagrange equations of a reduced two-dimensional LDG energy, which are a coupled system of second-order nonlinear elliptic PDEs with cubic nonlinearities with non-homogeneous Dirichlet boundary conditions, and discuss a parameter dependent *a priori* error analysis for the regular solutions that have H^2-regularity. The error analysis in [24] focusses on solutions of the reduced LdG model with reduced regularity for Nitsche's method and dGFEM, and employs *medius analysis* [12], which combines the ideas of *a priori* and *a posteriori* error analyses.

In this paper, *medius analysis* is circumvented employing the properties of the enriching operator developed in [20] and building on the techniques for lowest-order nonstandard FEMs in the context of the biharmonic equation [6]. Though the recipe for treating semilinear PDEs and exact solutions with minimal regularity is well studied in literature; the systems with cubic nonlinearities and non-homogeneous boundary conditions considered in this article introduce significant changes in the hypotheses and the further development of the unified framework. This leads to novelty in the error analysis, and also in the verification of hypotheses for the important applications considered in this article. To state some specific challenges, an elegant representation of the nonlinear operator for the ferronematic case [25] helps in identifying the hypotheses **(B1)** and **(B2)** that are crucial for a systematic analysis in this article. The nonlinear part involve fifteen terms and contain quadratic as well as cubic terms, and this explains the significance of the vector form representation of the nonlinear operator in the analysis. Note that Carstensen *et al.* [7,8] deals with quadratic nonlinearity for which the boundedness in **(B1)**, **(B2)** are simpler to verify (and hence are not stated as explicit assumptions). Moreover, the quadratic nonlinearity implies that the second-order term in the expansion of Taylor series for the nonlinear parts vanish. For the applications considered in this article, owing to cubic nonlinearity, the second-order term in Taylor expansion does not vanish; and the non-zero higher order terms in the Taylor expansion are controlled by local Lipschitz continuity property of the Frechét derivative of the nonlinear operator. The general framework of the *a posteriori* error control in this paper is based on the technique in [24] and adapts the methodology in [7,33] for semilinear systems with cubic nonlinearity and non-homogeneous boundary conditions.

To illustrate the applicability of the analysis in this paper, we show how the framework can be applied to the ferronematic free energy functional [13], that involves solving a *a system of four coupled PDEs* with cubic nonlinearity and non-homogeneous Dirichlet boundary conditions. Ferronematics [27] are exciting composite materials with long range orientational order and magnetic ordering, and are of interest in optics, telecommunications, microfluidics, and pharmacology. In [25], the authors discuss the asymptotic analysis of global ferronematic energy minimizers in a rescaled elastic constant limit, and perform finite element analysis for the approximation of H^2-regular solutions in conforming FEM and Nitsche's frameworks. The general framework of this paper covers the analysis of the ferronematics solutions with milder regularity in $\mathbf{H}^{1+\alpha}(\Omega) := (H^{1+\alpha}(\Omega))^4$, $0 < \alpha \leq 1$, and also works for conforming, Nitsche, dG and WOPSIP schemes, both recovering and substantially extending the work in [25]. More precisely, we work on bounded two-dimensional domains with Lipschitz boundaries and the non-homogeneous Dirichlet boundary conditions in $\mathbf{H}^{\frac{1}{2}+\alpha}(\partial\Omega) := (H^{\frac{1}{2}+\alpha}(\partial\Omega))^4, 0 < \alpha \leq 1$. The second example focuses on the LdG (Landau–de Gennes) model for nematic liquid crystals, which are complex fluids with long-range orientational order. The papers [23,24] focus on the reduced LdG model in two dimensions, with two degrees of freedom. The framework in this paper can be used to derive error estimates for this reduced LdG model, which involves a system of two nonlinear, elliptic coupled partial differential equations with non-homogeneous Dirichlet boundary conditions, and cubic nonlinearities in the two degrees of freedom. This is a substantial improvement that follows from merely checking the hypotheses of the framework, and applies to conforming, Nitsche, dG and WOPSIP schemes simultaneously, and solutions with reduced regularity. Other possible applications include variational models for micromagnetics, elasticity, elastomers and these are not discussed further in this paper.

The principal mathematical results can be summarised as follows. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Let $\mathcal{T}$ denote a shape regular triangulation of $\overline{\Omega}$ into triangles, and $\mathbb{T}(\delta)$ denote the non-empty subset of all such triangulations $\mathcal{T}$ with maximal mesh size smaller than or equal to a positive constant δ. The maximal mesh-size in a triangulation $\mathcal{T}$ is denoted by $h := \max h_{\mathcal{T}}$, where $h_{\mathcal{T}} \in P_0(\mathcal{T})$ with $h_{\mathcal{T}}|_T := h_T := \mathrm{diam}(T) \approx |T|^{\frac{1}{2}}$ for all $T \in \mathcal{T}$. For each triangulation $\mathcal{T}$, the associated finite dimensional trial space is X_h, and the test space is V_h. The main contributions are given below.

- (**Existence and uniqueness of solution).** There exists a positive constant δ such that, for all $\mathcal{T} \in \mathbb{T}(\delta)$, there exists a locally unique discrete solution $u_h \in X_h$ to $N_h(u_h) = 0$ near the regular root u, where $N_h : X_h \to V_h^*$ is the discrete differentiable function associated to N.
- (***A priori* error estimate).** For all $\mathcal{T} \in \mathbb{T}(\delta)$, $\|u - u_h\|_{\widehat{X}} \lesssim h^\alpha, 0 < \alpha \leq 1$, where the norm $\|\cdot\|_{\widehat{X}}$ is associated to an extended Hilbert space $\widehat{X}$ such that $X + X_h \subseteq \widehat{X}$. Here α is the index of elliptic regularity.
- (**Reliable and efficient *a posteriori* error estimate).** There exist $\epsilon > 0$ such that any approximation $\eta_h \in X_h$ with $\|u - \eta_h\|_{\widehat{X}} \leq \epsilon$ satisfies

$$\|u - \eta_h\|_{\widehat{X}} \lesssim \left\|\widehat{N}(\eta_h)\right\|_{V^*} + \min_{\eta \in X(g)} \|\eta_h - \eta\|_{\widehat{X}} + \text{data app}(\cdot) \lesssim \big(\|u - \eta_h\|_{\widehat{X}} + \text{data app}(g)\big).$$

Here the nonlinear function $\widehat{N}:\widehat{X}\to V^*$ extends the function N to $\widehat{X}$, and the set $X(g):=V+g$ for some $g\in X$. Here "data app$(\cdot)$" is a seminorm on X, and in applications it represents the approximation error for the non-homogeneous boundary data in the triangulation $\mathcal{T}$.
- (**Applications).** Application of the results to conforming, dG, Nitsche and WOPSIP schemes for ferronematic and nematic liquid crystal models are discussed.
- (**Numerical results).** Numerical results that corroborate theoretical estimates for uniform mesh-refinements, and show improved empirical convergence rates in adaptive mesh-refinements are presented.

The paper is organized as follows. Section 2 focuses on the preliminary set up and statement of proof of the discrete stability of a well-posed linear problem under a set of hypotheses. Under an additional set of hypotheses we establish the existence and uniqueness of local discrete solution in Section 3. *A posteriori* error control follows in Section 4 with an hypothesis. Sections 5 and 6 apply the analysis to ferronematic systems for conforming, Nitsche, and dG schemes. We discuss the abstract theorems for WOPSIP scheme, applied to the ferronematics model in Section 7. Section 8 contains several numerical experiments that illustrate theoretical results for both uniform and adaptive refinements. The reduced LDG model for NLCs is further discussed in Section 9. Section 10 concludes with some brief perspectives. Appendix contains a proof of the abstract discrete inf-sup condition, a table that summarises the constant dependencies for various examples, proofs of error estimates for Nitsche's method, and the comprehensive numerical results.

2. Abstract discrete inf-sup condition

This section deals with the sufficient conditions for the stability of a well-posed linear problem. The semilinear boundary value problem, its discretization, and the preliminary set up are introduced first.

Let $\widehat{X}$ be a real Hilbert space, and let X and X_h be two complete linear subspaces of $\widehat{X}$. Suppose V (resp. V_h) be a complete linear subspace of X (resp. X_h). Recall that X and V (resp. X_h and V_h) are the continuous (resp. discrete) test and trial spaces. The norm in $\widehat{X}$ (resp. X and X_h) is denoted by $\|\cdot\|_{\widehat{X}}$ (resp. $\|\cdot\|_X$ and $\|\cdot\|_{X_h}$) such that $(\|\cdot\|_{\widehat{X}})|_{X_h}=\|\cdot\|_{X_h}$ and $(\|\cdot\|_{\widehat{X}})|_V\le\|\cdot\|_X$. Let there exist $C_1>0$ such that $\|\cdot\|_X\le C_1\|\cdot\|_{\widehat{X}}$ in X.

Define $N:X\to V^*$ by $N(x):=\mathscr{A}x+\mathscr{B}(x)$ with a leading operator $\mathscr{A}\in L(X;V^*)$ associated to the bilinear form $A:X\times X\to\mathbb{R}$ such that $\langle\mathscr{A}x,\varphi\rangle=A(x,\varphi)$; and a nonlinear function $\mathscr{B}:\widehat{X}\to\widehat{X}^*$ such that $\langle\mathscr{B}(x),\varphi\rangle:=B(x,\varphi)$. Here $B:\widehat{X}\times\widehat{X}\to\mathbb{R}$ corresponds to the nonlinear part of the system of PDEs. We approximate the regular solution $u\in X(g)$ to the continuous nonlinear system

$$\langle N(u),\varphi\rangle=\langle\mathscr{A}u,\varphi\rangle+\langle\mathscr{B}(u),\varphi\rangle=A(u,\varphi)+B(u,\varphi)=0\qquad\text{for all }\varphi\in V.\tag{2.1}$$

A solution of (2.1) is called regular if and only if N is differentiable at u and the Frechét derivative $DN(u)$ is an isomorphism of V onto V^*. For $x_h\in X_h$, define $N_h(x_h):=\mathscr{A}_hx_h+\mathscr{B}(x_h)-F_h$, where $\mathscr{A}_hx_h:=A_h(x_h,\cdot)$ is associated to the discrete bilinear form $A_h:X_h\times X_h\to\mathbb{R}$ and $F_h\in V_h^*$ stems from the non-homogeneous Dirichlet boundary condition. The discrete problem that corresponds to (2.1) seeks $u_h\in X_h$ such that for all $\varphi_h\in V_h$,

$$\langle N_h(u_h),\varphi_h\rangle:=\langle\mathscr{A}_hu_h,\varphi_h\rangle+\langle\mathscr{B}(u_h),\varphi_h\rangle-F_h(\varphi_h)=A_h(u_h,\varphi_h)+B(u_h,\varphi_h)-F_h(\varphi_h)=0.\tag{2.2}$$

Let $B_L:\widehat{X}\times\widehat{X}\to\mathbb{R}$ denote the bilinear form associated to the operator $\mathscr{B}_L\in L(X;V^*)$ obtained by a linearisation of $\mathscr{B}$ around the exact solution u of (2.1), that is $B_L:=D\mathscr{B}(u)$, such that $\mathscr{B}_Lx:=B_L(x,\cdot)$ for all $x\in X$. Let there exist a constant $C_2>0$ such that for all $\widehat{x},\widehat{y}\in\widehat{X}$, $|B_L(\widehat{x},\widehat{y})|\le C_2\|\widehat{x}\|_{\widehat{X}}\|\widehat{y}\|_{\widehat{X}}$. Suppose that the linear operators $\mathscr{A}$ and $\mathscr{A}+\mathscr{B}_L$ in $L(V;V^*)$ associated with the bilinear form A and the Frechét derivative $DN(u;\cdot,\cdot):=A(\cdot,\cdot)+B_L(\cdot,\cdot)$ of N at the regular solution u, are invertible. That is,

$$0<\alpha_0:=\inf_{\substack{x\in V\\ \|x\|_X=1}}\ \sup_{\substack{y\in V\\ \|y\|_X=1}}A(x,y)\ \text{ and }0<\beta:=\inf_{\substack{x\in V\\ \|x\|_X=1}}\ \sup_{\substack{y\in V\\ \|y\|_X=1}}DN(u;x,y).\tag{2.3}$$

Abbreviate the bound $\|\mathscr{A}\| := \|\mathscr{A}\|_{L(X;V^*)}$ and $\|\mathscr{A}^{-1}\| := \|\mathscr{A}^{-1}\|_{L(V^*;V)}$. Also, let the discrete bilinear form $A_h(\cdot,\cdot)$ (that corresponds to $A(\cdot,\cdot)$) satisfy the inf-sup condition,

$$0 < \alpha_h = \inf_{\substack{x_h \in V_h \\ \|x_h\|_{X_h}=1}} \sup_{\substack{y_h \in V_h \\ \|y_h\|_{X_h}=1}} A_h(x_h, y_h). \tag{2.4}$$

Four parameters in **(A1)**–**(A4)** and a smallness condition on them are identified next to establish the discrete inf-sup condition for the discrete bilinear form in Theorem 2.1.

Suppose that there exist two linear operators $I_h \in L(V;V_h)$, $Q \in L(V_h;V)$, and parameters $\delta_1, \delta_2, \Lambda_1, C_A \geq 0$ such that **(A1)**–**(A4)** below hold.

(A1) $\delta_1 := \sup_{\substack{\theta_h \in V_h \\ \|\theta_h\|_{X_h}=1}} \left\|(1-I_h)\mathscr{A}^{-1}(B_L(\theta_h,\cdot)|_V)\right\|_{\widehat{X}}$ is sufficiently small,

(A2) $\delta_2 := \sup_{\substack{\theta_h \in V_h \\ \|\theta_h\|_{X_h}=1}} \sup_{\substack{\varphi_h \in V_h \\ \|\varphi_h\|_{X_h}=1}} B_L(\theta_h, (1-Q)\varphi_h)$ is sufficiently small,

(A3) $\|(1-Q)\varphi_h\|_{\widehat{X}} \leq \Lambda_1 \operatorname{dist}_{\|\cdot\|_{\widehat{X}}}(\varphi_h, V)$ for all $\varphi_h \in V_h$

(A4) $A_h(\xi_h,\varphi_h) - A(Q\xi_h, Q\varphi_h) \leq C_A \|(1-Q)\xi_h\|_{\widehat{X}} \|\varphi_h\|_{X_h}$ for all $\xi_h, \varphi_h \in V_h$.

Remark 2.1 (**(A1)**–**(A4)** in applications)**.** Given $\theta_h \in V_h$, and the solution ξ of the linear problem $A(\xi,\varphi) = B_L(\theta_h,\varphi)$ for all $\varphi \in V$, **(A1)** corresponds to an interpolation error. The boundedness of the bilinear forms B_L, A_h, A and a choice of an appropriate enrichment operator Q [20] establish the smallness of the term $B_L(\theta_h,(1-Q)\varphi_h)$ in **(A2)**, and the bounds **(A3)** and **(A4)** in applications.

Remark 2.2 (Comparison with [7, 8])**.** The article [7] discusses only conforming and nonconforming methods whereas [8] extends [7] to include other lowest-order P_2 methods. Hence we compare the hypotheses of this article with [8]. The hypotheses to establish the discrete inf-sup condition is exactly comparable to the corresponding hypotheses in [8] as the discrete inf-sup condition is associated to linear PDEs with homogeneous boundary conditions. More precisely, **(A1)**, **(A2)**, **(A3)**, and **(A4)** are provided in **(H2)**, **(H3)**, (2.4), and **(H1)**, respectively.

For $\theta_h, \varphi_h \in V_h$, let the Frechét derivative $DN_h(u)$ at u of N_h be defined by

$$DN_h(u;\theta_h,\varphi_h) := A_h(\theta_h,\varphi_h) + B_L(\theta_h,\varphi_h). \tag{2.5}$$

Let $\beta_0 := \alpha_h\widehat{\beta} - ((\alpha_h + C_A\Lambda_1 + C_1^2(\Lambda_1+1)^2\|\mathscr{A}\|)\delta_1 + \delta_2)$ with

$$\widehat{\beta} := \frac{\beta}{\Lambda_1\beta + C_1\|\mathscr{A}\|(1+\Lambda_1(1+C_2\|\mathscr{A}^{-1}\|))} > 0. \tag{2.6}$$

Theorem 2.1 (Discrete inf-sup condition)**.** *Let u be a regular solution to* (2.1)*. The assumptions* (2.3)*,* (2.4)*, and* **(A1)**–**(A4)** *imply the discrete inf-sup condition*

$$\beta_0 \leq \beta_h = \inf_{\substack{\theta_h \in V_h \\ \|\theta_h\|_{X_h}=1}} \sup_{\substack{\varphi_h \in V_h \\ \|\varphi_h\|_{X_h}=1}} DN_h(u;\theta_h,\varphi_h). \tag{2.7}$$

Requiring that the exact solution of (2.1) lies in an affine subspace $X(g)$ of X leads to a different norm equivalence relation of $\|\cdot\|_{\widehat{X}}$ and $\|\cdot\|_X$ on V, and hence the proof of Theorem 2.1 involves a modification of the approach in Theorem 2.1 of [8], and is presented in Appendix A.

3. Existence and uniqueness of the discrete solution

This section presents existence and uniqueness of the discrete solution of the nonlinear system (2.2). We identify four parameters in **(A5)**–**(A8)**, and two boundedness properties in **(B1)** and **(B2)** associated to the nonlinear part B of (2.1) and its perturbation to establish the well-posedness of the discrete problem.

Assume the existence of the linear operator $P \in L(X_h; X)$ and the parameters $\delta_3, \delta_4, \Lambda_2, \widetilde{C}_A \geq 0$ such that

- **(A5)** $\exists\, z_h \in X_h$ with $\delta_3 := \|u - z_h\|_{\widehat{X}}$ sufficiently small,
- **(A6)** $\delta_4 := \sup_{\substack{\varphi_h \in V_h \\ \|\varphi_h\|_{X_h}=1}} B(u, (1-Q)\varphi_h)$ is sufficiently small,
- **(A7)** $\|(1-P)x_h\|_{\widehat{X}} \leq \Lambda_2 \operatorname{dist}_{\|\cdot\|_{\widehat{X}}}(x_h, X) \quad$ for all $x_h \in X_h$,
- **(A8)** $A_h(x_h, \varphi_h) - A(Px_h, Q\varphi_h) - F_h(\varphi_h) \leq \widetilde{C}_A \|x_h - x\|_{\widehat{X}} \|\varphi_h\|_{X_h} \quad \forall x \in X(g),\ x_h \in X_h,\ \varphi_h \in V_h$.

Remark 3.1 (Comparison with [8])**.** The hypotheses **(A5)** and **(A7)** in this paper are exactly comparable to **(H4)** and (2.3), respectively, in [8]. **(A8)** is introduced in this paper to handle the fact that the exact solution of (2.1) lies in an affine subspace $X(g)$ of X, and in applications, the non-homogeneous boundary conditions. Both the articles [7,8] exclusively focus on trilinear nonlinearity, whereas with the assumptions **(A6), (B1), (B2)**, we generalize the results to higher order nonlinearity, for example, cubic nonlinearity in ferronematics.

Let $\widetilde{B}_L : \widehat{X} \times \widehat{X} \to \mathbb{R}$ be a perturbed bilinear form obtained by the linearisation of $\mathscr{B}$ around $z_h \in X_h$. For any $\theta_h, \varphi_h \in V_h$ with $\|\theta_h\|_{X_h} = 1$ and $\|\varphi_h\|_{X_h} = 1$, let there exist constants $C_B, \widetilde{C}_B > 0$ such that

- **(B1)** $D\mathscr{B}(u; \theta_h, \varphi_h) - D\mathscr{B}(z_h; \theta_h, \varphi_h) = B_L(\theta_h, \varphi_h) - \widetilde{B}_L(\theta_h, \varphi_h) \leq C_B \|u - z_h\|_{\widehat{X}}$,
- **(B2)** $B(u, \varphi_h) - B(z_h, \varphi_h) \leq \widetilde{C}_B \|u - z_h\|_{\widehat{X}}$.

The non-negative parameters $\delta_1, \cdots, \delta_4$, and $C_B, \widetilde{C}_B$ depends on the fixed regular solution u to $N(u) = 0$, and this dependence is suppressed in the notation for simplicity.

Remark 3.2 (**(A5)**–**(A8), (B1),** and **(B2)** in applications)**.** **(A5)** assumes the existence of $z_h \in X_h$ sufficiently close to the exact solution u of (2.1), and in applications, z_h is chosen as the interpolation of u. **(A6)** holds by boundedness of B and properties of Q. The operator $P : X_h \to X$ is introduced to treat the non-homogeneous Dirichlet boundary condition for the exact solution and **(A7)** states its approximation properties. **(A8)** is a modification of **(A4)** for non-homogeneous boundary conditions in applications. For the case of homogeneous boundary condition, the analysis follows by choosing $P = Q$ and **(A8)** is **(A4)** for this case. **(B1)** controls the error between the linearised form B_L and its perturbation. The starting point in Newton iteration is a perturbation of the exact solution (denoted as z_h above), and **(B2)** controls perturbation of the nonlinear part $B(\cdot, \cdot)$.

The existence and local uniqueness of the discrete solution is established in Theorem 3.2 using Kantorovich theorem, and proof utilizes the discrete inf-sup condition in Theorem 2.1.

Theorem 3.1 (Kantorovich [16,34])**.** *Let Z, Y be Banach spaces, and $L(Y, Z)$ denotes the Banach space of bounded linear operators of Y into Z. Suppose that the mapping $\mathscr{N} : \mathrm{D} \subset Z \to Y$ is Fréchet differentiable on an open convex set D, and the derivative $D\mathscr{N}(\cdot)$ is Lipschitz continuous on D with Lipschitz constant L. For a fixed starting point $x^0 \in \mathrm{D}$, the inverse $D\mathscr{N}(x^0)^{-1}$ exists as a continuous operator on Z. The real numbers a and b are chosen such that*

$$\left\|D\mathscr{N}(x^0)^{-1}\right\|_{L(Y;Z)} \leq a \quad \text{and} \quad \left\|D\mathscr{N}(x^0)^{-1}\mathscr{N}(x^0)\right\|_Z \leq b \tag{3.1}$$

and $h^* := abL \leq \frac{1}{2}$. Suppose, the first approximation $x^1 := x^0 - D\mathscr{N}(x^0)^{-1}\mathscr{N}(x^0)$ has a property that the closed ball $\overline{B}_Z(x^1, r) := \{x \in Z \mid \|x - x^1\|_Z \leq r\}$ lies within the domain of definition D, where $r = \frac{1-\sqrt{1-2h^*}}{aL} - b$. Then the following are true.

(1) *Existence and uniqueness. There exists a solution,* $x \in \overline{B}_Z(x^1, r)$ *to* $\mathscr{N}(x) = 0$*, and the solution is unique on* $\overline{B}_Z(x^0, r^*) \cap \mathrm{D}$*, that is, on a suitable neighborhood of the initial point,* x^0*, with* $r^* = \frac{1+\sqrt{1-2h^*}}{aL}$.

(2) *Convergence of Newton's method. The Newton's scheme with initial iterate,* x^0*, leads to a sequence,* $x^n := x^{n-1} - D\mathscr{N}(x^{n-1})^{-1}\mathscr{N}(x^{n-1})$*, in* $\overline{B}_Z(x^0, r^*)$*, which converges to,* x*, with error bound* $\|x^n - x\|_Z \leq \frac{(1-(1-2h^*)^{\frac{1}{2}})^{2^n}}{2^n aL}, \quad n = 0, 1 \ldots.$

Theorem 3.2 (Existence, uniqueness, and Newton iterates)**.** *Let* $u \in X(g)$ *be a regular solution to* $N(u) = 0$. *Suppose that* (2.3), (2.4), (**A1**)–(**A8**)*, and* (**B1**) and (**B2**) *hold, and let*

$$\beta_1 := \alpha_h\widehat{\beta} - \Big(\Big(\alpha_h + C_A\Lambda_1 + C_1^2(\Lambda_1 + 1)^2\|\mathscr{A}\|\Big)\delta_1 + \delta_2 + C_B\delta_3\Big) > 0, \quad \text{and} \tag{3.2}$$

$$b := \beta_1^{-1}\Big(\Big(C_1^2\|\mathscr{A}\|(1 + \Lambda_1)(1 + \Lambda_2) + \widetilde{C}_A + \widetilde{C}_B\Big)\delta_3 + \delta_4\Big) \geq 0. \tag{3.3}$$

For z_h *in* **(A5)***, let the Frechét derivative of* $\mathscr{B} : X_h \to V_h^*$ *be Lipschitz continuous on* $\overline{B}_{X_h}(z_h, 2b)$ *with Lipschitz constant* L*. Then* (i) *there exists a solution* $u_h \in X_h$ *to* $N_h(u_h) = 0$ *with* $\|u - u_h\|_{\widehat{X}} \leq \rho$*, where* $\rho := \delta_3 + b + r$,

$$r := \Big(1 - \sqrt{1 - 2h^*}\Big)/m - b \geq 0, \quad \text{and} \quad r^* := \Big(1 + \sqrt{1 - 2h^*}\Big)/m > 0, \tag{3.4}$$

with $m := \beta_1^{-1}L > 0$ *and* $h^* := bm \geq 0$*, and* (ii) *if* $\rho m \leq \frac{1}{2}$*, then the solution* u_h *to* $N_h(u_h) = 0$ *is unique in* $B_{\widehat{X}}(u, \rho)$, (iii) *the Newton iterates* x_h^n *converges to* u_h *with* $\|x_h^n - u_h\|_{X_h} \leq \frac{(1-(1-2h^*)^{\frac{1}{2}})^{2^n}}{2^n\beta_1^{-1}L}, \quad n = 0, 1, \ldots$

Proof. The proof utilizes the Kantorovich theorem stated in Theorem 3.1. The assumptions of Theorem 3.1 are verified in Steps 1–4 and the conclusions are established.

Step 1 (Settings)**.** Let $\mathscr{N} := N_h$, $Z = X_h, Y := V_h^*$, $\mathrm{D} := B_{X_h}(z_h, 2b)$, and $x^0 := z_h$.

Step 2 (Lipschitz continuity of DN_h)**.** The definition of DN_h with the cancellation of the linear A_h terms yields

$$DN_h(\eta_h; \theta_h, \varphi_h) - DN_h(\chi_h; \theta_h, \varphi_h) = D\mathscr{B}(\eta_h; \theta_h, \varphi_h) - D\mathscr{B}(\chi_h; \theta_h, \varphi_h) \qquad \text{for all } \eta_h, \chi_h \in \mathrm{D}.$$

Since the Frechét derivative of $\mathscr{B} : X_h \to V_h^*$ is Lipschitz continuous on D with Lipschitz constant L, the above displayed identity shows that the Frechét derivative DN_h is also Lipschitz continuous. Also, the smallness assumptions on the parameters δ_3 and δ_4 implies $2bL < \beta_1$.

Step 3 (Verification of (3.1))**.** For $z_h \in X_h$ in **(A5)**, define the perturbed bilinear form

$$DN_h(z_h; \theta_h, \varphi_h) := A_h(\theta_h, \varphi_h) + \widetilde{B}_L(\theta_h, \varphi_h) \qquad \text{for } \theta_h, \varphi_h \in V_h.$$

The definition of DN_h yields $DN_h(u; \theta_h, \varphi_h) = DN_h(z_h; \theta_h, \varphi_h) + (B_L(\theta_h, \varphi_h) - \widetilde{B}_L(\theta_h, \varphi_h))$. Theorem 2.1, **(B1)**, $\delta_3 = \|u - z_h\|_{\widehat{X}}$ from **(A5)**, and the smallness assumption on δ_3 lead to

$$0 < \beta_1 = \beta_0 - C_B\delta_3 \leq \inf_{\substack{\theta_h \in V_h \\ \|\theta_h\|_{X_h} = 1}} \sup_{\substack{\varphi_h \in V_h \\ \|\varphi_h\|_{X_h} = 1}} DN_h(z_h; \theta_h, \varphi_h). \tag{3.5}$$

This yields $\|DN_h(z_h)^{-1}\|_{L(V_h^*; V_h)} \leq \beta_1^{-1} =: a$. The definition of a and the inequality $2bL < \beta_1$ from Step 2 imply that $h^* = abL = \beta_1^{-1}bL < \frac{1}{2}$. Furthermore,

$$\big\|DN_h(z_h)^{-1}N_h(z_h)\big\|_{X_h} \leq \Big\|DN_h(z_h)^{-1}\Big\|_{L\left(V_h^*; V_h\right)}\|N_h(z_h)\|_{V_h^*} \leq \beta_1^{-1}\|N_h(z_h)\|_{V_h^*}. \tag{3.6}$$

Next we estimate $\|N_h(z_h)\|_{V_h^*}$. Given $\varphi_h \in V_h$ with $\|\varphi_h\|_{X_h} = 1$, the definition of $N_h(\cdot)$ in (2.2), an addition and subtraction of $A(Pz_h, Q\varphi_h)$, and the continuous nonlinear system $\langle N(u), Q\varphi_h\rangle = 0$ in (2.1) yield

$$\begin{aligned} N_h(z_h;\varphi_h) &= A_h(z_h,\varphi_h) + B(z_h,\varphi_h) - F_h(\varphi_h) \\ &= A(Pz_h - u, Q\varphi_h) + (A_h(z_h,\varphi_h) - A(Pz_h, Q\varphi_h) - F_h(\varphi_h)) + (B(z_h,\varphi_h) - B(u, Q\varphi_h)). \end{aligned} \tag{3.7}$$

The boundedness of $A(\cdot,\cdot)$, a triangle inequality, $\|\cdot\|_X \leq C_1\|\cdot\|_{\widehat{X}}$, **(A3)**, and **(A7)** lead to

$$A(Pz_h - u, Q\varphi_h) \leq \|\mathscr{A}\|\|Pz_h - u\|_X\|Q\varphi_h\|_X \leq C_1^2\|\mathscr{A}\|(1+\Lambda_1)(1+\Lambda_2)\|u - z_h\|_{\widehat{X}}.$$

Utilize **(A8)** to estimate the second term on the right hand side of (3.7) as

$$A_h(z_h,\varphi_h) - A(Pz_h, Q\varphi_h) - F_h(\varphi_h) \leq \widetilde{C}_A\|u - z_h\|_{\widehat{X}}.$$

Employ **(B2)**, **(A5)** and **(A6)** to establish

$$B(z_h,\varphi_h) - B(u, Q\varphi_h) = (B(z_h,\varphi_h) - B(u,\varphi_h)) + B(u, \varphi_h - Q\varphi_h) \leq \widetilde{C}_B\delta_3 + \delta_4.$$

A substitution of the aforementioned three estimates in (3.7) and a use of **(A5)** establish

$$\|N_h(z_h)\|_{V_h^*} \leq \Big(C_1^2\|\mathscr{A}\|(1+\Lambda_1)(1+\Lambda_2) + \widetilde{C}_A + \widetilde{C}_B\Big)\delta_3 + \delta_4.$$

This, equation (3.6), and the definition of b in (3.3) yield

$$\big\|DN_h(z_h)^{-1}N_h(z_h)\big\|_{X_h} \leq \beta_1^{-1}\|N_h(z_h)\|_{V_h^*} \leq b. \tag{3.8}$$

The possible case $b = 0$ implies $\delta_3 = \delta_4 = 0$, and $u = z_h$ in **(A5)**, and $N_h(z_h) = 0$ from (2.2). Therefore, $u = z_h$ is the discrete solution u_h. In this particular situation, the Newton Scheme is a constant sequence $z_h = x_h^0 = x_h^1 = x_h^2 = \ldots$, and converges to $\lim_{n\to\infty} z_h = u_h$. Theorem 3.1 applies with $r = 0, r^* = 0$ and $\rho = 0$.

Step 4 (Ball condition)**.** Recall that the Newton's scheme with initial iterate, $x_h^0 := z_h$, leads to a sequence, $x_h^n := x_h^{n-1} - DN_h(x_h^{n-1})^{-1}N_h(x_h^{n-1})$, $n = 0, 1, \ldots$ The definition $x_h^1 := z_h - DN_h(z_h)^{-1}N_h(z_h)$ and (3.8) imply

$$\big\|x_h^1 - z_h\big\|_{X_h} \leq \big\|DN_h(z_h)^{-1}N_h(z_h)\big\|_{X_h} \leq b.$$

For $y_h \in \overline{B}_{X_h}(x_h^1, r)$, this plus a triangle inequality yields

$$\|y_h - z_h\|_{X_h} \leq \big\|y_h - x_h^1\big\|_{X_h} + \big\|x_h^1 - z_h\big\|_{X_h} \leq r + b. \tag{3.9}$$

The definitions of r, h^*, m and $2bL < \beta_1$ show

$$r + b = \frac{1 - \sqrt{1 - 2\beta_1^{-1}bL}}{\beta_1^{-1}L} = \frac{2\beta_1^{-1}bL}{\beta_1^{-1}L\big(1 + \sqrt{1 - 2\beta_1^{-1}bL}\big)} < 2b. \tag{3.10}$$

This applied to (3.9) implies that the closed ball $\overline{B}_{X_h}(x_h^1, r)$ lies within the domain of definition $\mathrm{D} = B_{X_h}(z_h, 2b)$.

Step 5 (Conclusions)**.** (i) Theorem 3.1 applies to the Frechét differentiable function $N_h : \mathrm{D} \subset X_h \to V_h^*$ and ensures the existence of the discrete solution $u_h \in \overline{B}_{X_h}(x_h^1, r)$, and the uniqueness in $\overline{B}_{X_h}(z_h, r^*) \cap \mathrm{D}$. A triangle inequality, **(A5)**, the definition of x_h^1, and $u_h \in \overline{B}_{X_h}(x_h^1, r)$ lead to

$$\|u - u_h\|_{\widehat{X}} \leq \|u - z_h\|_{\widehat{X}} + \big\|z_h - x_h^1\big\|_{\widehat{X}} + \big\|x_h^1 - u_h\big\|_{\widehat{X}} \leq \delta_3 + b + r =: \rho.$$

(ii) The proof of the uniqueness of u_h in $B_{\widehat{X}}(u,\rho)$ follows verbatim from Theorem 4.1(iii) of [8] and is skipped.
(iii) The convergence of the Newton iterates directly follow with $a = \beta_1^{-1}$. This concludes the proof. □

Remark 3.3 (*A priori* error control)**.** In all the applications considered in this paper, $\rho \lesssim h^\alpha$, $0 < \alpha \le 1$, in the displayed inequality above thus establishing the *a priori* error control $\|u - u_h\|_{\widehat{X}} \lesssim h^\alpha$. Here α is the index of elliptic regularity.

Remark 3.4 (Uniqueness of u_h in $B_{\widehat{X}}(u,\rho)$)**.** The assumption $\rho m \le \frac{1}{2}$ for the uniqueness of u_h in $B_{\widehat{X}}(u,\rho)$ is justified for sufficiently small values of the discretization parameter, h, since $m \lesssim 1$, $\rho \lesssim h^\alpha$, $0 < \alpha \le 1$.

4. *A POSTERIORI* ERROR CONTROL

Suppose that there exists a bounded extension $\widehat{A} : \widehat{X} \times \widehat{X} \to \mathbb{R}$ of A with $\|\widehat{A}\|_{\widehat{X}\times\widehat{X}} := \sup_{\substack{\widehat{x}\in\widehat{X} \\ \|\widehat{x}\|_{\widehat{X}}=1}} \sup_{\substack{\widehat{y}\in\widehat{X} \\ \|\widehat{y}\|_{\widehat{X}}=1}} \widehat{A}(\widehat{x},\widehat{y})$ such that $\widehat{A}|_{X\times X} = A$. Define $\widehat{N} : \widehat{X} \to V^*$ by $\langle \widehat{N}(\widehat{\eta}), \varphi\rangle := \widehat{A}(\widehat{\eta},\varphi) + B(\widehat{\eta},\varphi)$ for all $\widehat{\eta} \in \widehat{X}, \varphi \in V$. Assume that the Frechét derivative of $\mathscr{B} : \widehat{X} \to V^*$ is locally Lipschitz continuous at $u \in \widehat{X}$, that is, there is a $R_0 > 0$ such that in $B_{\widehat{X}}(u, R_0)$,

$$\gamma := \sup_{\widehat{\eta}\in B_{\widehat{X}}(u,R_0)} \frac{\|D\mathscr{B}(\widehat{\eta}) - D\mathscr{B}(u)\|_{L(\widehat{X},V^*)}}{\|\widehat{\eta} - u\|_{\widehat{X}}} < \infty, \tag{4.1}$$

where $\|D\mathscr{B}(\widehat{\eta}) - D\mathscr{B}(u)\|_{L(\widehat{X},V^*)} := \sup_{\theta\in\widehat{X}\setminus\{0\}} \sup_{\varphi\in V\setminus\{0\}} \frac{D\mathscr{B}(\widehat{\eta};\theta,\varphi) - D\mathscr{B}(u;\theta,\varphi)}{\|\theta\|_{\widehat{X}}\|\varphi\|_X}$. Recall that data app$(\cdot)$ is a seminorm on X and $X(g) = V + g$ for some $g \in X$. In addition to (4.1), assume that there exist an operator $\mathscr{G} : X_h \to X(g)$ and a parameter $\Lambda_3 < \infty$ such that

$$\textbf{(AP)}\ \ \|\mathscr{G}\eta_h - \eta_h\|_{\widehat{X}} \le \Lambda_3\Big(\mathrm{dist}_{\|\cdot\|_{\widehat{X}}}(\eta_h, X(g)) + \text{data app}(g)\Big) \qquad \text{for all } \eta_h \in X_h.$$

Theorem 4.1. *Given a regular solution $u \in X(g)$ to $N(u) = 0$, let the Frechét derivative of $\mathscr{B} : \widehat{X} \to V^*$ be locally Lipschitz continuous at $u \in \widehat{X}$. Assume the existence of Λ_3 in* **(AP)***. Then there exists a constant $0 < R < \min(\frac{1}{2}\beta\gamma^{-1}, R_0)$ such that for all $\eta_h \in B_{\widehat{X}}(u,R) \cap X_h$,*

$$\text{(i)}\ \ \|u - \eta_h\|_{\widehat{X}} \le 2\beta^{-1}\Big\|\widehat{N}(\eta_h)\Big\|_{V^*} + 2\Lambda_3\left(1 + \beta^{-1}\Big\|D\widehat{N}(u)\Big\|_{L(\widehat{X},V^*)}\right)\|\mathscr{G}\eta_h - \eta_h\|_{\widehat{X}},$$

$$\text{(ii)}\ \ \Big\|\widehat{N}(\eta_h)\Big\|_{V^*} \le \left(\frac{1}{2}\beta + \Big\|D\widehat{N}(u)\Big\|_{L(\widehat{X},V^*)}\right)\|u - \eta_h\|_{\widehat{X}}.$$

Proof. (i) Let $u \in X(g)$ be a regular solution to $N(u) = 0$. The definition of $D\widehat{N}(x;\cdot,\cdot) := \widehat{A}(\cdot,\cdot) + D\mathscr{B}(\widehat{x};\cdot,\cdot)$ for $x \in \widehat{X}$, and the Lipschitz continuity of the Frechét derivative, $D\mathscr{B} : \widehat{X} \to V^*$ at $u \in \widehat{X}$, imply

$$\sup_{\widehat{\eta}\in B_{\widehat{X}}(u,R_0)} \frac{\Big\|D\widehat{N}(\widehat{\eta}) - D\widehat{N}(u)\Big\|_{L(\widehat{X},V^*)}}{\|\widehat{\eta} - u\|_{\widehat{X}}} = \sup_{\widehat{\eta}\in B_{\widehat{X}}(u,R_0)} \frac{\|D\mathscr{B}(\widehat{\eta}) - D\mathscr{B}(u)\|_{L(\widehat{X},V^*)}}{\|\widehat{\eta} - u\|_{\widehat{X}}} = \gamma < \infty. \tag{4.2}$$

Let $\eta_h \in B_{\widehat{X}}(u, R_0)$. A Taylor expansion of $\widehat{N}$ around u, and $N(u) = 0$ in (2.1) lead to

$$0 = N(u;\varphi) = \widehat{N}(\eta_h;\varphi) + \int_0^1 D\widehat{N}(u + t(\eta_h - u); u - \eta_h, \varphi)\,\mathrm{d}t.$$

Introduce $\pm D\widehat{N}(u; u-\eta_h, \varphi)$ in the above displayed expression and re-arrange the terms as

$$D\widehat{N}(u; u-\eta_h, \varphi) = -\widehat{N}(\eta_h; \varphi) - \int_0^1 \Big(D\widehat{N}(u + t(\eta_h - u); u-\eta_h, \varphi) - D\widehat{N}(u; u-\eta_h, \varphi)\Big)\,\mathrm{d}t. \tag{4.3}$$

Rewrite $u-\eta_h$ as $(u-\mathscr{G}\eta_h) + (\mathscr{G}\eta_h - \eta_h)$ in the left-hand side of the above term and use linearity of $D\widehat{N}(u;\cdot,\cdot)$, to obtain

$$\begin{aligned} D\widehat{N}(u; u-\mathscr{G}\eta_h, \varphi) &= -\widehat{N}(\eta_h; \varphi) - D\widehat{N}(u; \mathscr{G}\eta_h - \eta_h, \varphi) \\ &\quad - \int_0^1 \Big(D\widehat{N}(u + t(\eta_h - u); u-\eta_h, \varphi) - D\widehat{N}(u; u-\eta_h, \varphi)\Big)\,\mathrm{d}t \\ &\leq \left(\left\|\widehat{N}(\eta_h)\right\|_{V^*} + \left\|D\widehat{N}(u)\right\|_{L(\widehat{X}, V^*)} \|\mathscr{G}\eta_h - \eta_h\|_{\widehat{X}} + \gamma\|u-\eta_h\|_{\widehat{X}}^2 \right)\|\varphi\|_X \end{aligned} \tag{4.4}$$

with (4.2) in the last step. Since $\mathscr{G}\eta_h \in X(g)$, $u - \mathscr{G}\eta_h \in V$. For $\tau > 0$ small enough, the continuous inf-sup condition (2.3) implies that there exists $\varphi \in V$ with $\|\varphi\|_X = 1$ such that

$$(\beta - \tau)\|u - \mathscr{G}\eta_h\|_X \leq DN(u; u - \mathscr{G}\eta_h, \varphi).$$

A triangle inequality, norm equivalence $(\|\cdot\|_{\widehat{X}})|_V \equiv \|\cdot\|_X$, and the last displayed inequality yield

$$(\beta-\tau)\|u-\eta_h\|_{\widehat{X}} \leq (\beta-\tau)\big(\|u-\mathscr{G}\eta_h\|_X + \|\mathscr{G}\eta_h - \eta_h\|_{\widehat{X}}\big) \leq DN(u; u-\mathscr{G}\eta_h, \varphi) + (\beta-\tau)\|\mathscr{G}\eta_h - \eta_h\|_{\widehat{X}}.$$

For $\tau \to 0$, equation (4.4) leads to

$$\beta\|u-\eta_h\|_{\widehat{X}} \leq \left\|\widehat{N}(\eta_h)\right\|_{V^*} + \left(\beta + \left\|D\widehat{N}(u)\right\|_{L(\widehat{X}, V^*)}\right)\|\mathscr{G}\eta_h - \eta_h\|_{\widehat{X}} + \gamma\|u-\eta_h\|_{\widehat{X}}^2.$$

Re-arrange the terms to obtain

$$\big(\beta - \gamma\|u-\eta_h\|_{\widehat{X}}\big)\|u-\eta_h\|_{\widehat{X}} \leq \left\|\widehat{N}(\eta_h)\right\|_{V^*} + \left(\beta + \left\|D\widehat{N}(u)\right\|_{L(\widehat{X}, V^*)}\right)\|\mathscr{G}\eta_h - \eta_h\|_{\widehat{X}}.$$

A choice of $R > 0$ such that $\|u-\eta_h\|_{\widehat{X}} \leq R < \min(\frac{1}{2}\beta\gamma^{-1}, R_0)$ leads to

$$\beta\|u-\eta_h\|_{\widehat{X}} \leq 2\left\|\widehat{N}(\eta_h)\right\|_{V^*} + \left(2\beta + 2\left\|D\widehat{N}(u)\right\|_{L(\widehat{X}, V^*)}\right)\|\mathscr{G}\eta_h - \eta_h\|_{\widehat{X}}.$$

This completes the proof of (i).

(ii) The identity (4.2), (4.3), and $\|u-\eta_h\|_{\widehat{X}} < \frac{1}{2}\beta\gamma^{-1}$ yield

$$\left\|\widehat{N}(\eta_h)\right\|_{V^*} \leq \gamma\|u-\eta_h\|_{\widehat{X}}^2 + \left\|D\widehat{N}(u)\right\|_{L(\widehat{X}, V^*)}\|u-\eta_h\|_{\widehat{X}} \leq \left(1/2\beta + \left\|D\widehat{N}(u)\right\|_{L(\widehat{X}, V^*)}\right)\|u-\eta_h\|_{\widehat{X}}.$$

This concludes the proof. □

5. Application to ferronematic system

In this section, we discuss the preliminaries needed to apply the frameworks of *a priori* and *a posteriori* error estimates, discussed in Sections 2–4, to ferronematic system [13, 25].

5.1. General notations

Standard notations on Sobolev spaces and their norms are employed throughout in the sequel. The standard semi-norm and norm on $H^s(\Omega)$ (resp. $W^{s,p}(\Omega)$) for s,p positive real numbers, are denoted by $|\cdot|_s$ and $\|\cdot\|_s$ (resp. $|\cdot|_{s,p}$ and $\|\cdot\|_{s,p}$). The $L^2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$. The notation $\mathbf{H}^s(\Omega)$ (resp. $\mathbf{L}^p(\Omega)$) is used to denote the product space $(H^s(\Omega))^4$ (resp. $(L^p(\Omega))^4$. The norms $\|\!|\cdot|\!\|_s$ (resp. $\|\!|\cdot|\!\|_{s,p}$) in the Sobolev spaces $\mathbf{H}^s(\Omega)$ (resp. $\mathbf{W}^{s,p}(\Omega)$) are defined by $\|\!|\Phi|\!\|_s = (\sum_{i=1}^4 \|\varphi_i\|_s^2)^{\frac{1}{2}}$ for all $\Phi = (\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in \mathbf{H}^s(\Omega)$ (resp. $\|\!|\Phi|\!\|_{s,p} = (\sum_{i=1}^4 \|\varphi_i\|_{s,p}^2)^{\frac{1}{2}}$ for all $\Phi = (\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in \mathbf{W}^{s,p}(\Omega)$). The norm on $\mathbf{L}^2(\Omega)$ space is defined by $\|\!|\Phi|\!\|_0 = (\sum_{i=1}^4 \|\varphi_i\|_0^2)^{\frac{1}{2}}$ for all $\Phi = (\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in \mathbf{L}^2(\Omega)$. Let $\mathbf{C}^0(\partial\Omega) := (\mathrm{C}^0(\partial\Omega))^4$, $V := H_0^1(\Omega)$, $\mathbf{V} := \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^4$, $X := H^1(\Omega)$, and $\mathbf{X} := \mathbf{H}^1(\Omega)$.

Recall that $\mathbb{T}$ is a set of shape regular triangulation of $\overline{\Omega}$ into triangles and $h = \max h_{\mathcal{T}}$ is the discretization parameter associated with the triangulation $\mathcal{T} \in \mathbb{T}$. Let $\mathcal{E}(\Omega)$ (resp. $\mathcal{E}(\partial\Omega)$) denote the interior (resp. boundary) edges of $\mathcal{T}$ and $\mathcal{E} := \mathcal{E}(\Omega) \cup \mathcal{E}(\partial\Omega)$. The length of an edge E is denoted by h_E. Define $H^1(\mathcal{T}) := \{\varphi \in L^2(\Omega) : \varphi|_T \in H^1(T)$ for all $T \in \mathcal{T}\}$, $P_1(\mathcal{T}) := \{\varphi \in L^2(\Omega)|\ \varphi|_T \in P_1(T)$ for all $T \in \mathcal{T}\}$, and set $\mathbf{H}^1(\mathcal{T}) := (H^1(\mathcal{T}))^4$, $\mathbf{P}_1(\mathcal{T}) := (P_1(\mathcal{T}))^4$. Define $\mathbf{H}^1(E) := (H^1(E))^4$ for $E \in \mathcal{E}$. Let $\|\cdot\|_{\mathrm{pw}} := |\cdot|_{H^1(\mathcal{T})} := \|\nabla_{\mathrm{pw}}\cdot\|_0$ associated with the piecewise gradient ∇_{pw}, and $\|\!|\cdot|\!\|_{\mathrm{pw}}^2 := \sum_{i=1}^4 \|\cdot\|_{\mathrm{pw}}^2$. The jump $[\varphi]_E$ and average $\{\varphi\}_E$ of piecewise H^1 function φ, across an interior edge E shared by the triangles T^+ and T^-, are defined as $[\varphi]_E := \varphi|_{T^+} - \varphi|_{T^-}$ and $\{\varphi\}_E := \frac{1}{2}(\varphi|_{T^+} + \varphi|_{T^-})$, respectively, and for an boundary edge E of the triangle T, are defined by $[\varphi]_E := \varphi|_T$ and $\{\varphi\}_E := \varphi|_T$, respectively. For $\Phi := (\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in \mathbf{H}^1(\mathcal{T})$, and the unit outward normal ν_E to an edge $E \cap \partial T$, define $\nabla\Phi\nu_E|_T := (\nabla\varphi_1|_T\cdot\nu_E, \nabla\varphi_2|_T\cdot\nu_E, \nabla\varphi_3|_T\cdot\nu_E, \nabla\varphi_4|_T\cdot\nu_E)$, $\|\!|\Phi|\!\|_{0,E}^2 := \sum_{i=1}^4 \|\varphi_i\|_{0,E}^2$ and $\|\!|\Phi|\!\|_{0,T}^2 := \sum_{i=1}^4 \|\varphi_i\|_{0,T}^2$, where for $\varphi \in H^1(\mathcal{T})$, $\|\varphi\|_{0,E}^2 := \int_E \varphi^2\,\mathrm{d}s$ and $\|\varphi\|_{0,T}^2 := \int_T \varphi^2\,\mathrm{d}x$. The inequality $a \lesssim b$ abbreviates $a \le Cb$ with the constant $C > 0$ independent of the discretization parameter h, and depends on the constants from trace inequalities, Sobolev embedding results, enrichment operator estimates, interpolation estimates, penalty parameters in different FEMs, the constants in the continuous problem, and the exact solution of the continuous problem. The exact dependence of various constants in the applications are outlined in Table B.1.

5.2. Background

The free energy of a dilute suspension of magnetic nanoparticles in a nematic liquid crystal filled two dimensional (2D) square well [1] is an example of a ferronematic system. It is described by two macroscopic order parameters (i) LDG $\mathbf{Q}$-tensor order parameter, and (ii) magnetization vector, $M := (M_1, M_2)$. In 2D, $\mathbf{Q} := s(2\mathbf{n}\otimes\mathbf{n} - I)$, where $\mathbf{Q} \in \mathbf{S}_0 := \{\mathbf{Q} = (Q_{ij})_{1\le i,j\le 2} \in \mathbb{R}^{2\times 2} : \mathbf{Q} = \mathbf{Q}^T, \mathrm{tr}\mathbf{Q} = 0\}$ and I is the 2×2 identity matrix. Here the nematic director $\mathbf{n}$ is an eigenvector of $\mathbf{Q}$ with largest positive eigenvalue and represents the preferred direction of nematic molecular alignment, and $s \in \mathbb{R}$ is a scalar order parameter that measures the degree of orientational ordering. Let Ω be a 2D domain of unit characteristic length L. For Q_{11} and Q_{12}, two independent components of $\mathbf{Q}$, the total re-scaled and dimensionless free energy [13] is given by

$$\mathcal{E}(\mathbf{Q},\mathbf{M}) := \int_\Omega \frac{1}{2}\Big(|\nabla Q_{11}|^2 + |\nabla Q_{12}|^2 + |\nabla M_1|^2 + |\nabla M_2|^2\Big)\,\mathrm{d}x + \frac{1}{\ell}\int_\Omega f_B(\mathbf{Q},\mathbf{M})\,\mathrm{d}x, \tag{5.1}$$

where f_B is the quartic bulk energy density given by

$$f_B(\mathbf{Q},\mathbf{M}) := \frac{1}{4}\big(Q_{11}^2 + Q_{12}^2 - 1\big)^2 + \frac{1}{4}\big(M_1^2 + M_2^2 - 1\big)^2 - \frac{c}{2}\big(Q_{11}\big(M_1^2 - M_2^2\big) + 2Q_{12}M_1M_2\big).$$

The parameter $\ell > 0$ depends on nematic elastic constant, magnetic stiffness constant, temperature, and L. c is a nemato-magnetic coupling parameter. Set the admissible set $\mathcal{X} := \{\mathbf{w} \in \mathbf{X} : \mathbf{w} = \mathbf{g}$ on $\partial\Omega\}$, and $\tilde{Q} := Q_{11}^2 + Q_{12}^2 - 1$, $\tilde{M} := M_1^2 + M_2^2 - 1$. The local or global minimizers of (5.1) are weak solutions, $\Psi :=$

$(Q_{11}, Q_{12}, M_1, M_2) \in \mathcal{X}$, of the associated Euler–Lagrange equations:

$$\begin{aligned} \Delta Q_{11} - \ell^{-1}\Big(\widetilde{Q} Q_{11} - \frac{c}{2}(M_1^2 - M_2^2)\Big) &= 0,\\ \Delta Q_{12} - \ell^{-1}\Big(\widetilde{Q} Q_{12} - cM_1M_2\Big) &= 0,\\ \Delta M_1 - \ell^{-1}\Big(\widetilde{M} M_1 - c(Q_{11}M_1 + Q_{12}M_2)\Big) &= 0,\\ \Delta M_2 - \ell^{-1}\Big(\widetilde{M} M_2 - c(Q_{12}M_1 - Q_{11}M_2)\Big) &= 0. \end{aligned} \tag{5.2}$$

5.3. Weak formulation

The weak formulation of (5.2) seeks $\Psi \in \mathcal{X}$ such that for all $\Phi \in \mathbf{V}$,

$$N(\Psi;\Phi) := A(\Psi,\Phi) + B(\Psi,\Phi) = 0, \tag{5.3}$$

where for $\Xi := (\xi_1, \xi_2, \xi_3, \xi_4)$, $\boldsymbol{\eta} := (\eta_1, \eta_2, \eta_3, \eta_4)$, $\Theta := (\theta_1, \theta_2, \theta_3, \theta_4)$ and $\Phi := (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbf{X}$,

$$A(\Theta,\Phi) := \sum_{i=1}^{4} \int_\Omega \nabla\theta_i \cdot \nabla\varphi_i \,\mathrm{d}x, B(\Psi,\Phi) := B_1(\Psi,\Phi) + B_2(\Psi,\Psi,\Phi) + B_3(\Psi,\Psi,\Psi,\Phi). \tag{5.4}$$

Here

$$\begin{aligned} B_1(\Theta,\Phi) &:= -\sum_{i=1}^{4} \frac{1}{\ell} \int_\Omega \theta_i\varphi_i \,\mathrm{d}x,\\ B_2(\boldsymbol{\eta},\Theta,\Phi) &:= \frac{c}{2\ell}\Big(\int_\Omega (\eta_4\theta_4 - \eta_3\theta_3)\varphi_1 \,\mathrm{d}x - \int_\Omega (\eta_3\theta_4 + \eta_4\theta_3)\varphi_2 \,\mathrm{d}x - \int_\Omega (\eta_1\theta_3 + \eta_3\theta_1)\varphi_3 \,\mathrm{d}x\\ &\quad - \int_\Omega (\eta_2\theta_4 + \eta_4\theta_2)\varphi_3 \,\mathrm{d}x - \int_\Omega (\eta_2\theta_3 + \eta_3\theta_2)\varphi_4 \,\mathrm{d}x + \int_\Omega (\eta_1\theta_4 + \eta_4\theta_1)\varphi_4 \,\mathrm{d}x\Big). \end{aligned}$$

For $\bar{\xi}_{ij} = (\xi_i, \xi_j), \bar{\eta}_{ij} = (\eta_i, \eta_j)$, $\bar{\theta}_{ij} = (\theta_i, \theta_j)$, $\bar{\varphi}_{ij} = (\varphi_i, \varphi_j) \in (H^1(\Omega))^2$ with $(i,j) = (1,2)$ or $(3,4)$,

$$\begin{aligned} B_3(\Xi,\boldsymbol{\eta},\Theta,\Phi) &:= \frac{1}{3\ell}\int_\Omega \Big(\big(\bar{\xi}_{12}\cdot\bar{\eta}_{12}\big)\big(\bar{\theta}_{12}\cdot\bar{\varphi}_{12}\big) + \big(\bar{\xi}_{12}\cdot\bar{\theta}_{12}\big)\big(\bar{\eta}_{12}\cdot\bar{\varphi}_{12}\big) + \big(\bar{\eta}_{12}\cdot\bar{\theta}_{12}\big)\big(\bar{\xi}_{12}\cdot\bar{\varphi}_{12}\big)\Big) \,\mathrm{d}x\\ &\quad + \frac{1}{3\ell}\int_\Omega \Big(\big(\bar{\xi}_{34}\cdot\bar{\eta}_{34}\big)\big(\bar{\theta}_{34}\cdot\bar{\varphi}_{34}\big) + \big(\bar{\xi}_{34}\cdot\bar{\theta}_{34}\big)\big(\bar{\eta}_{34}\cdot\bar{\varphi}_{34}\big) + \big(\bar{\eta}_{34}\cdot\bar{\theta}_{34}\big)\big(\bar{\xi}_{34}\cdot\bar{\varphi}_{34}\big)\Big) \,\mathrm{d}x. \end{aligned}$$

Thanks to the compact representation of the nonlinear system in the vector form displayed in (5.3), the results in the abstract framework presented in the previous sections apply here for $X := \mathbf{X}$, $V := \mathbf{V}$, $X(g) := \mathcal{X}$, and the nonlinear function $N : \mathbf{X} \to \mathbf{V}^*$ defined in (5.3).

For all $\Theta, \Phi \in \mathbf{V}$, the Frechét derivative of N at the regular solution Ψ is defined as

$$\langle DN(\Psi)\Theta, \Phi\rangle := A(\Theta,\Phi) + \langle D\mathscr{B}(\Psi)\Theta, \Phi\rangle \tag{5.5}$$

with $\langle D\mathscr{B}(\Psi)\Theta, \Phi\rangle := B_1(\Theta,\Phi) + 2B_2(\Psi,\Theta,\Phi) + 3B_3(\Psi,\Psi,\Theta,\Phi)$. The solution Ψ of (5.3) is regular implies that the inf-sup condition stated below holds

$$0 < \beta := \inf_{\substack{\Theta\in\mathbf{V}\\ \|\!|\Theta|\!\|_1 = 1}} \sup_{\substack{\Phi\in\mathbf{V}\\ \|\!|\Phi|\!\|_1 = 1}} \langle DN(\Psi)\Theta, \Phi\rangle. \tag{5.6}$$

Regularity results

Now we discuss the regularity results for the solutions of (5.3). It follows that any weak solution

$$\Psi \in \mathbf{H}^2(\Omega) \text{ if } \Omega \text{ is convex and } \mathbf{g} \in \mathbf{H}^{\frac{3}{2}}(\Omega), \quad \text{and } \Psi \in \mathbf{H}^{1+\alpha}(\Omega) \text{ if } \Omega \text{ is nonconvex and } \mathbf{g} \in \mathbf{H}^{\frac{1}{2}+\alpha}(\partial\Omega),$$

where $\alpha := \frac{\pi}{\omega} - \epsilon_1$ and ω is the maximum of the interior angles at the reentrant corners and ϵ_1 is any positive number. The proof is skipped as it is an adaptation of the technique utilized in [23] for bounded convex polygonal domains in $\mathbb{R}^2$, and employs Sobolev embedding results, a bootstrapping argument [11], and standard elliptic regularity result ([3], Sect. 2).

In this paper, we discuss the discrete approximations of the regular solution Ψ of (5.3) such that $\Psi \in \boldsymbol{\mathcal{X}} \cap \mathbf{H}^{1+\alpha}(\Omega)$, with $0 < \alpha \leq 1$.

5.4. Auxiliary results

This section presents some auxiliary results that are used extensively in the rest of the article. The first part discusses interpolation, smoothing operators; the second part establishes bounds that are helpful in verifying hypotheses in applications, and third part describes discrete norms and their properties.

Interpolation and smoothing operators

Lemma 5.1 (Interpolation estimates [2])**.** *For any $\zeta \in H^{1+\alpha}(\Omega)$ with $\alpha \in (0,1]$, and $T \in \mathcal{T}$, there exist $I_{\mathrm{C}}\zeta \in P_1(\mathcal{T}) \cap H^1(\Omega)$ such that $\|\zeta - I_{\mathrm{C}}\zeta\|_{H^l(T)} \lesssim h_T^{1+\alpha-l}\|\zeta\|_{H^{1+\alpha}(T)}$ for $l = 0, 1$.*

Next we describe the estimates for the Scott–Zhang interpolation operator [31] and a lifting operator [19] important for the *a posteriori* error analysis.

Lemma 5.2 (Scott–Zhang interpolation [31])**.** *For $l, m \in \mathbb{N}_0$ with $1 \leq l < \infty$, the Scott–Zhang interpolation operator $I_{\mathrm{SZ}} : H_0^l(\Omega) \to V_{\mathrm{C}} := P_1(\mathcal{T}) \cap H_0^1(\Omega)$ satisfies the stability and approximation properties stated below:*

(i) *for all $0 \leq m \leq 1$, $\|I_{\mathrm{SZ}}\zeta\|_{m,\Omega} \leq C_{\mathrm{SZ}}\|\zeta\|_{l,\Omega}$ for all $\zeta \in H_0^l(\Omega)$,*
(ii) *provided $l \leq 2$, for all $0 \leq m \leq l$, $\|\zeta - I_{\mathrm{SZ}}\zeta\|_{m,T} \leq C_{\mathrm{SZ}} h_T^{l-m}|\zeta|_{l,\omega_T}$ for all $\zeta \in H_0^l(\omega_T), T \in \mathcal{T}$. Here the constant $C_{\mathrm{SZ}} > 0$ is independent of h, and ω_T is the set of all triangles in $\mathcal{T}$ that share at least one vertex with T.*

Define the lifting operator [19] $\mathscr{G} : \mathbf{P}_1(\mathcal{T}) \to \mathcal{X}$ such that $\boldsymbol{\eta}_{\mathbf{g}} := \mathscr{G}\boldsymbol{\eta}_h$ satisfies

$$\int_\Omega \nabla\boldsymbol{\eta}_{\mathbf{g}} \cdot \nabla\Phi \,\mathrm{d}x = \sum_{T\in\mathcal{T}} \int_T \nabla\boldsymbol{\eta}_h \cdot \nabla\Phi \,\mathrm{d}x \qquad \text{for all } \Phi \in \mathbf{V}$$

subject to the Dirichlet boundary condition $\boldsymbol{\eta}_{\mathbf{g}}|_{\partial\Omega} = \mathbf{g} \in \mathbf{C}^0(\partial\Omega)$ with $\mathbf{g}|_E \in \mathbf{H}^1(E)$ for all $E \in \mathcal{E}(\partial\Omega)$.

Let $\tilde{\mathbf{g}}$ be the continuous piecewise linear Lagrange interpolant of $\mathbf{g}$ on $\mathcal{E}(\partial\Omega)$. The approximation error for the non-homogeneous Dirichlet boundary data $\mathbf{g}$ is defined by

$$\big(\text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega))\big)^2 := \sum_{E\in\mathcal{E}(\partial\Omega)} h_E |||\partial_{\mathcal{E}}(\mathbf{g} - \tilde{\mathbf{g}})|||_{0,E}^2, \tag{5.7}$$

where $\partial_{\mathcal{E}}(\mathbf{g} - \tilde{\mathbf{g}})$ is the piecewise first order derivative of $\mathbf{g} - \tilde{\mathbf{g}}$ along $\partial\Omega$. Then the following results hold.

Lemma 5.3 (Estimates for an auxiliary problem ([19], Lems. 4.4, 4.7))**.** *There exists a constant $C_{\mathscr{G}} > 0$, depending only on the shape regularity of $\mathcal{T}$ such that*

$$\sum_{T\in\mathcal{T}} |||\nabla(\boldsymbol{\eta}_{\mathbf{g}} - \boldsymbol{\eta}_h)|||_{0,T}^2 \leq C_{\mathscr{G}} \left(\sum_{E\in\mathcal{E}(\Omega)} h_E^{-1} |||[\boldsymbol{\eta}_h]_E|||_{0,E}^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-1} |||\boldsymbol{\eta}_h - \mathbf{g}|||_{0,E}^2 + (\text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega)))^2 \right).$$

Remark 5.1. Note that the non-homogeneous Dirichlet boundary condition $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ is sufficient to define the lifting operator $\mathscr{G}$. Further regularity that $\mathbf{g} \in \mathbf{C}^0(\partial\Omega)$ with $\mathbf{g}|_E \in \mathbf{H}^1(E)$ for all $E \in \mathcal{E}(\partial\Omega)$, is crucial to obtain the data approximation bound in (5.7), but not necessary condition for the definition of $\mathscr{G}$.

For $\Phi := (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbf{H}^1(\mathcal{T})$, introduce a discrete norm defined by $|||\Phi|||_h^2 := \sum_{i=1}^4 \|\varphi_i\|_h^2$, where for $\varphi \in H^1(\mathcal{T})$,

$$\|\varphi\|_h^2 := \sum_{T\in\mathcal{T}} \|\nabla\varphi\|_{0,T}^2 + \sum_{E\in\mathcal{E}} h_E^{-1}\|[\varphi]_E\|_{0,E}^2. \tag{5.8}$$

The next lemma introduces two enrichment (smoothing) operators J_1 and J_2, with J_1 helpful for the analysis with non-homogeneous boundary conditions.

Lemma 5.4 (Enrichment operators)**.**

(a) *The linear map* $J_1 : P_1(\mathcal{T}) \to X_{\mathrm{C}} := \{\varphi_h \in C^0(\overline{\Omega}) |\ \varphi_h|_T \in P_1(T)$ *for all* $T \in \mathcal{T}\}$ *defined by* $J_1\varphi_h(p) = |\mathcal{T}_p|^{-1}\sum_{T\in\mathcal{T}_p} \varphi_h|_T(p)$ *for a node* p *on* $\overline{\Omega}$, *where* $\mathcal{T}_p$ *is the set of all triangles sharing the node* p, *and* $|\mathcal{T}_p|$ *is the cardinality of* $\mathcal{T}_p$, *satisfies*

$$\sum_{m=0}^{1} \left|h_{\mathcal{T}}^{m-1}(\varphi_h - J_1\varphi_h)\right|_{H^m(\mathcal{T})}^2 \lesssim \sum_{E\in\mathcal{E}(\Omega)} h_E^{-1}\|[\varphi_h]_E\|_{0,E}^2.$$

(b) *The linear map* $J_2 : P_1(\mathcal{T}) \to H_0^1(\Omega)$ *defined by* $J_2\varphi_h := \widetilde{J}_1\varphi_h + \sum_{E\in\mathcal{E}(\Omega)} J_{2,E}(\{\varphi_h\}_E - \widetilde{J}_1\varphi_h)b_E$ *where* $J_{2,E} : L^2(E) \to P_0(E)$ *with* $\int_E J_{2,E}\varphi_h b_E \,\mathrm{d}s = \int_E \varphi_h \,\mathrm{d}s$ *for the quadratic bubble function* b_E *supported on the two elements containing* E, *and*

$$\widetilde{J}_1\varphi_h(p) = \begin{cases} |\mathcal{T}_p|^{-1}\sum_{T\in\mathcal{T}_p} \varphi_h|_T(p) & \text{for a node } p \text{ on } \Omega, \\ 0 & \text{for a node } p \text{ on } \partial\Omega, \end{cases}$$

satisfies

$$\text{(i)} \int_E J_2\varphi_h \,\mathrm{d}s = \int_E \{\varphi_h\} \,\mathrm{d}s \quad \text{for all } E \in \mathcal{E}(\Omega),\ \text{(ii)} \sum_{m=0}^{1} \left|h_{\mathcal{T}}^{m-1}(\varphi_h - J_2\varphi_h)\right|_{H^m(\mathcal{T})}^2 \lesssim \sum_{E\in\mathcal{E}} h_E^{-1}\|[\varphi_h]_E\|_{0,E}^2.$$

Any $\varphi_h \in P_1(\mathcal{T})$ *satisfies*

(c) $\sum_{m=0}^{1} \left|h_{\mathcal{T}}^{m-1}(\varphi_h - J_1\varphi_h)\right|_{H^m(\mathcal{T})} + \|\varphi_h - J_1\varphi_h\|_h \lesssim \min_{\eta\in X} \|\eta - \varphi_h\|_h$ *and*

(d) $\sum_{m=0}^{1} \left|h_{\mathcal{T}}^{m-1}(\varphi_h - J_2\varphi_h)\right|_{H^m(\mathcal{T})} + \|\varphi_h - J_2\varphi_h\|_h \lesssim \min_{\varphi\in V} \|\varphi - \varphi_h\|_h$.

For the proofs of (a) and (b), we refer to Section 2 in [17], Section 3 in [20]. The proofs of (c) and (d) are given in Appendix C.

Boundedness and coercivity

This section discusses the bounds for the continuous and discrete forms that are useful in the applications.

Lemma 5.5 (Boundedness and coercivity [25])**.**

(i) *For all* Θ, $\Phi \in \mathbf{X}$, *and* $\Xi \in \mathbf{V}$, *there exists a constant* $\alpha_0 > 0$ *such that*

$$A(\Theta, \Phi) \le |||\Theta|||_1 |||\Phi|||_1 \quad \text{and} \quad \alpha_0 |||\Xi|||_1^2 \le A(\Xi, \Xi).$$

(ii) *For all* $\Xi, \boldsymbol{\eta}, \Theta, \Phi \in \mathbf{X}$,

$$B_1(\Theta, \Phi) \leq \|\Theta\|_0 \|\Phi\|_0,\ B_2(\boldsymbol{\eta}, \Theta, \Phi) \lesssim \|\boldsymbol{\eta}\|_1 \|\Theta\|_1 \|\Phi\|_1, \quad \text{and } B_3(\Xi, \boldsymbol{\eta}, \Theta, \Phi) \lesssim \|\Xi\|_1 \|\boldsymbol{\eta}\|_1 \|\Theta\|_1 \|\Phi\|_1.$$

(iii) *For* $\Xi, \boldsymbol{\eta} \in \mathbf{H}^{1+\alpha}(\Omega), \Theta,\ \Phi \in \mathbf{X}$,

$$B_2(\boldsymbol{\eta}, \Theta, \Phi) \lesssim \|\boldsymbol{\eta}\|_{1+\alpha} \|\Theta\|_0 \|\Phi\|_0 \quad \text{and } B_3(\Xi, \boldsymbol{\eta}, \Theta, \Phi) \lesssim \|\Xi\|_{1+\alpha} \|\boldsymbol{\eta}\|_{1+\alpha} \|\Theta\|_0 \|\Phi\|_0.$$

(iv) *For* $\Xi, \boldsymbol{\eta}, \boldsymbol{\chi} \in \mathbf{H}^{1+\alpha}(\Omega)$ *and* $\Phi \in \mathbf{H}^1(\mathcal{T})$,

$$\begin{aligned} B_1(\boldsymbol{\chi}, \Phi) &\lesssim \|\boldsymbol{\chi}\|_{1+\alpha} \|\Phi\|_0, B_2(\boldsymbol{\eta}, \boldsymbol{\chi}, \Phi) \lesssim \|\boldsymbol{\eta}\|_{1+\alpha} \|\boldsymbol{\chi}\|_{1+\alpha} \|\Phi\|_0, \\ B_3(\Xi, \boldsymbol{\eta}, \boldsymbol{\chi}, \Phi) &\lesssim \|\Xi\|_{1+\alpha} \|\boldsymbol{\eta}\|_{1+\alpha} \|\boldsymbol{\chi}\|_{1+\alpha} \|\Phi\|_0, \end{aligned}$$

where the constants in $\lesssim$ *depend on* ℓ, c, *and Sobolev embedding results.*

The proof is a consequence of the Sobolev embedding results $H^1(\Omega) \hookrightarrow L^p(\Omega)$, $p = 3, 4$, and $H^{1+\alpha}(\Omega) \hookrightarrow L^\infty(\Omega)$, $0 < \alpha \leq 1$, for $\Omega \subset \mathbb{R}^2$ and Hölder's inequality, and is omitted.

Lemma 5.6 (Poincaré-type inequality ([21], Rem. 2.3))**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with Lipschitz boundary* $\partial\Omega$. *For* $\varphi \in H^1(\mathcal{T})$, *there exists a constant* $C_{\mathrm{P}} > 0$ *independent of* h *and* φ *such that for* $1 \leq r < \infty$, $\|\varphi\|_{L^r(\Omega)} \leq C_{\mathrm{P}} \|\varphi\|_h$.

Lemma 5.7 (Trace inequalities ([10], Lems. 1.46, 1.49))**.** (i) *For* $\varphi \in H^1(T), T \in \mathcal{T},\ \|\varphi\|_{0,\partial T}^2 \lesssim h_T^{-1} \|\varphi\|_{0,T}^2 + \|\varphi\|_{0,T} \|\nabla \varphi\|_{0,T}$. (ii) *For* $\varphi \in P_1(T)$ *and* $E \subset \partial T,\ h_E \|\nabla \varphi\|_{0,E}^2 \lesssim \|\nabla \varphi\|_{0,T}^2$.

Recall the definitions of B and $\langle D\mathscr{B}(\Psi)\cdot, \cdot\rangle$ from (5.4) and (5.5), respectively. Note that the bilinear form $B_L(\cdot,\cdot)$ (resp. $\widetilde{B}_L(\cdot,\cdot)$) in Sections 2 and 3 is denoted by $\langle D\mathscr{B}(\Psi)\cdot, \cdot\rangle$ (resp. $\langle D\mathscr{B}(z_h)\cdot, \cdot\rangle$) in applications. The bounds in the next two lemmas are instrumental to verify the hypotheses for dG, Nitsche, and WOPSIP schemes.

Lemma 5.8 (Intermediate bounds)**.**

(i) *Any* $\Psi \in \mathbf{H}^{1+\alpha}(\Omega)$ *and* $\Theta, \Phi \in \mathbf{P}_1(\mathcal{T})$ *satisfy*

$$\begin{aligned} \langle D\mathscr{B}(\Psi)\Theta, (1 - J_2)\Phi\rangle &\lesssim h\Big(1 + \|\Psi\|_{1+\alpha} + \|\Psi\|_{1+\alpha}^2\Big) \|\Theta\|_h \|\Phi\|_h, \\ B(\Psi, (1 - J_2)\Phi) &\lesssim h\Big(\|\Psi\|_{1+\alpha} + \|\Psi\|_{1+\alpha}^2 + \|\Psi\|_{1+\alpha}^3\Big) \|\Phi\|_h. \end{aligned}$$

(ii) *Any* $\Psi \in \mathbf{H}^1(\Omega)$ *and* $z_h, \Theta, \Phi \in \mathbf{P}_1(\mathcal{T})$ *satisfy*

$$\begin{aligned} \langle D\mathscr{B}(\Psi)\Theta, \Phi\rangle - \langle D\mathscr{B}(z_h)\Theta, \Phi\rangle &\lesssim (1 + \|\Psi\|_1 + \|z_h\|_h) \|\Psi - z_h\|_h \|\Theta\|_h \|\Phi\|_h, \\ B(\Psi, \Phi) - B(z_h, \Phi) &\lesssim \Big(1 + \|\Psi\|_1 + \|z_h\|_h + \|\Psi\|_1^2 + \|z_h\|_h^2\Big) \|\Psi - z_h\|_h \|\Phi\|_h. \end{aligned}$$

(iii) *For* $\Psi, \Phi \in \mathbf{X}$ *and* $\boldsymbol{\eta}, \Theta \in \mathbf{X} + \mathbf{P}_1(\mathcal{T})$,

$$\langle D\mathscr{B}(\boldsymbol{\eta})\Theta, \Phi\rangle - \langle D\mathscr{B}(\Psi)\Theta, \Phi\rangle \lesssim \|\boldsymbol{\eta} - \Psi\|_h (1 + \|\boldsymbol{\eta} - \Psi\|_h + 2\|\Psi\|_1) \|\Theta\|_h \|\Phi\|_1.$$

(iv) *For* $\boldsymbol{\eta}, \boldsymbol{\chi}, \Theta, \Phi \in \mathbf{H}^1(\mathcal{T})$,

$$\begin{aligned} B_1(\Theta, \Phi) &\lesssim \|\Theta\|_h \|\Phi\|_h, B_2(\boldsymbol{\eta}, \Theta, \Phi) \lesssim \|\boldsymbol{\eta}\|_h \|\Theta\|_h \|\Phi\|_h, \\ B_3(\boldsymbol{\chi}, \boldsymbol{\eta}, \Theta, \Phi) &\lesssim \|\boldsymbol{\chi}\|_h \|\boldsymbol{\eta}\|_h \|\Theta\|_h \|\Phi\|_h, \\ \langle D\mathscr{B}(\boldsymbol{\eta})\Theta, \Phi\rangle - \langle D\mathscr{B}(\boldsymbol{\chi})\Theta, \Phi\rangle &\lesssim \|\boldsymbol{\eta} - \boldsymbol{\chi}\|_h (1 + \|\boldsymbol{\eta}\|_h + \|\boldsymbol{\chi}\|_h) \|\Theta\|_h \|\Phi\|_h. \end{aligned}$$

Proof of (i). The definition of $\langle D\mathscr{B}(\Psi)\cdot,\cdot\rangle$ (resp. $B(\cdot,\cdot)$) and Lemmas 5.5(ii), 5.5(iii) (resp. Lem. 5.5(iv)) imply

$$\begin{aligned}\langle D\mathscr{B}(\Psi)\Theta,(1-J_2)\Phi\rangle &= B_1(\Theta,(1-J_2)\Phi)+B_2(\Psi,\Theta,(1-J_2)\Phi)+B_3(\Psi,\Psi,\Theta,(1-J_2)\Phi)\\ &\lesssim \Big(1+|||\Psi|||_{1+\alpha}+|||\Psi|||_{1+\alpha}^2\Big)|||\Theta|||_0|||(1-J_2)\Phi|||_0\\ \Big(\text{resp. } B(\Psi,(1-J_2)\Phi) &\lesssim \Big(|||\Psi|||_{1+\alpha}+|||\Psi|||_{1+\alpha}^2+|||\Psi|||_{1+\alpha}^3\Big)|||(1-J_2)\Phi|||_0\Big).\end{aligned}$$

This plus Lemma 5.4 at the first step and then Lemma 5.6 concludes the proof. □

Proof of (ii). Recall the perturbed bilinear form $\langle D\mathscr{B}(z_h)\cdot,\cdot\rangle$ defined by

$$\langle D\mathscr{B}(z_h)\Theta,\Phi\rangle := B_1(\Theta,\Phi)+2B_2(z_h,\Theta,\Phi)+3B_3(z_h,z_h,\Theta,\Phi).$$

The definitions of $\langle D\mathscr{B}(\Psi)\cdot,\cdot\rangle$, $\langle D\mathscr{B}(z_h)\cdot,\cdot\rangle$, the linearity of B_2 (resp. B_3) in first variable (resp. first and second variables), and a re-grouping of terms imply

$$\langle D\mathscr{B}(\Psi)\Theta,\Phi\rangle-\langle D\mathscr{B}(z_h)\Theta,\Phi\rangle = 2B_2(\Psi-z_h,\Theta,\Phi)+3B_3(\Psi,\Psi-z_h,\Theta,\Phi)+3B_3(\Psi-z_h,z_h,\Theta,\Phi). \tag{5.9}$$

Utilize the Cauchy–Schwarz inequality first, and then the Sobolev embedding result $H^1(\Omega)\hookrightarrow L^4(\Omega)$, Lemma 5.6 to obtain

$$\begin{aligned}\langle D\mathscr{B}(\Psi)\Theta,\Phi\rangle-\langle D\mathscr{B}(z_h)\Theta,\Phi\rangle &\lesssim |||\Psi-z_h|||_{0,3}|||\Theta|||_{0,3}|||\Phi|||_{0,3}+\Big(|||\Psi|||_{0,4}+|||z_h|||_{0,4}\Big)|||\Psi-z_h|||_{0,4}|||\Theta|||_{0,4}|||\Phi|||_{0,4}\\ &\lesssim (1+|||\Psi|||_1+|||z_h|||_h)|||\Psi-z_h|||_h|||\Theta|||_h|||\Phi|||_h.\end{aligned}$$

The definition of B, the linearity of B_i, $i=1,2,3$, in each variable, and a re-grouping of terms reveal

$$\begin{aligned}B(\Psi,\Phi)-B(z_h,\Phi) &= B_1(\Psi-z_h,\Phi)+B_2(\Psi-z_h,\Psi,\Phi)+B_2(z_h,\Psi-z_h,\Phi)\\ &\quad+B_3(\Psi-z_h,\Psi,\Psi,\Phi)+B_3(z_h,\Psi-z_h,\Psi,\Phi)+B_3(z_h,z_h,\Psi-z_h,\Phi).\end{aligned} \tag{5.10}$$

Now follow the approach to estimate (5.9) and obtain the second inequality in (ii). □

Proof of (iii). For $\boldsymbol{\eta},\Theta\in\mathbf{X}+\mathbf{P}_1(\mathcal{T}),\Phi\in\mathbf{X}$, recall that the Frechét derivative of $\mathscr{B}$ at $\boldsymbol{\eta}$ given by

$$\langle D\mathscr{B}(\boldsymbol{\eta})\Theta,\Phi\rangle := B_1(\Theta,\Phi)+2B_2(\boldsymbol{\eta},\Theta,\Phi)+3B_3(\boldsymbol{\eta},\boldsymbol{\eta},\Theta,\Phi).$$

This, a re-grouping of the terms and the Cauchy–Schwarz inequality lead to

$$\begin{aligned}\langle D\mathscr{B}(\boldsymbol{\eta})\Theta,\Phi\rangle-\langle D\mathscr{B}(\Psi)\Theta,\Phi\rangle &= 2B_2(\boldsymbol{\eta}-\Psi,\Theta,\Phi)+3B_3(\boldsymbol{\eta}-\Psi,\boldsymbol{\eta},\Theta,\Phi)+3B_3(\Psi,\boldsymbol{\eta}-\Psi,\Theta,\Phi)\\ &\lesssim |||\boldsymbol{\eta}-\Psi|||_{0,3}|||\Theta|||_{0,3}|||\Phi|||_{0,3}+|||\boldsymbol{\eta}-\Psi|||_{0,4}\Big(|||\boldsymbol{\eta}|||_{0,4}+|||\Psi|||_{0,4}\Big)|||\Theta|||_{0,4}|||\Phi|||_{0,4}.\end{aligned} \tag{5.11}$$

The above inequality, Sobolev embedding results $H^1(\Omega)\hookrightarrow L^p(\Omega)$, $p=3,4$, and Lemma 5.6 lead to the desired inequality in (iii). □

Proof of (iv). For $\boldsymbol{\eta},\boldsymbol{\chi},\Theta,\Phi\in\mathbf{H}^1(\mathcal{T})$, the definitions of $B_i, i=1,2,3$, Sobolev embedding results $H^1(T)\hookrightarrow L^p(T), 1\le p<\infty$, the Cauchy–Schwarz inequality and Lemma 5.6 leads to the boundedness results in the first part of (iv). The definition of $D\mathscr{B}(\cdot)$, a re-grouping of terms, and the bounds in the first part yield

$$\begin{aligned}\langle D\mathscr{B}(\boldsymbol{\eta})\Theta,\Phi\rangle-\langle D\mathscr{B}(\boldsymbol{\chi})\Theta,\Phi\rangle &= 2B_2(\boldsymbol{\eta}-\boldsymbol{\chi},\Theta,\Phi)+3B_3(\boldsymbol{\eta}-\boldsymbol{\chi},\boldsymbol{\eta},\Theta,\Phi)+3B_3(\boldsymbol{\chi},\boldsymbol{\eta}-\boldsymbol{\chi},\Theta,\Phi)\\ &\lesssim |||\boldsymbol{\eta}-\boldsymbol{\chi}|||_h(1+|||\boldsymbol{\eta}|||_h+|||\boldsymbol{\chi}|||_h)|||\Theta|||_h|||\Phi|||_h.\end{aligned} \tag{5.12}$$

□

Remark 5.2. The estimates in Lemma 5.8(ii) with different choices of z_h for different schemes and Lemma 5.9 stated in the next subsection lead to the estimates in **(B1)** and **(B2)** for each scheme, and are described later.

Discrete norms and properties

This section defines the discrete norms employed in the applications and establishes norm equivalence results crucial to prove several boundedness properties in energy norms for the discrete schemes discussed in this article. Let the orthogonal projection operator $\Pi_E^0 : L^2(E) \to P_0(E)$ be defined by $\Pi_E^0\varphi = h_E^{-1}\int_E \varphi\, \mathrm{d}s$. Define the energy norms

$$\begin{aligned}
&\text{(a) } \|\varphi\|_{\mathrm{dG}}^2 := \sum_{T\in\mathcal{T}} \|\nabla\varphi\|_{0,T}^2 + \sum_{E\in\mathcal{E}} \sigma_{\mathrm{dG}} h_E^{-1} \|[\varphi]_E\|_{0,E}^2 && \text{for } \varphi \in H^1(\mathcal{T}),\\
&\text{(b) } \|\varphi\|_{\mathrm{N}}^2 := \|\nabla\varphi\|_0^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} \sigma h_E^{-1} \|\varphi\|_{0,E}^2 && \text{for } \varphi \in H^1(\Omega),\\
&\text{(c) } \|\varphi\|_{\mathrm{P}}^2 := \sum_{T\in\mathcal{T}} \|\nabla\varphi\|_{0,T}^2 + \sum_{E\in\mathcal{E}} \sigma_{\mathrm{P}} h_E^{-2} \big(\Pi_E^0[\varphi]_E\big)^2 && \text{for } \varphi \in H^1(\mathcal{T}).
\end{aligned}$$

Here σ_{dG}, σ, and σ_{P} are the positive penalty parameters that appear in dG, Nitsche, and WOPSIP schemes defined in Sections 6 and 7.

Lemma 5.9 (Norm equivalence). *The following norm equivalence relations hold:*

(i) $\|\varphi\|_h \approx \|\varphi\|_{\mathrm{dG}}$, $\|\varphi\|_{L^r(\Omega)} \lesssim \|\varphi\|_{\mathrm{dG}}$ for $\varphi \in H^1(\mathcal{T})$, (ii) $\|\varphi\|_h \approx \|\varphi\|_{\mathrm{N}}$ and $\|\varphi\|_{L^r(\Omega)} \lesssim \|\varphi\|_{\mathrm{N}}$ for $\varphi \in H^1(\Omega)$, and (iii) $\|\varphi\|_h \lesssim \|\varphi\|_{\mathrm{P}}$ and $\|\varphi\|_{L^r(\Omega)} \lesssim \|\varphi\|_{\mathrm{P}}$ for $\varphi \in H^1(\mathcal{T})$.

Proof of (i). The proof is a direct application of the definitions of $\|\cdot\|_h$ and $\|\cdot\|_{\mathrm{dG}}$ norms, and Lemma 5.6. □

Proof of (ii). For $\varphi \in H^1(\Omega)$, $[\varphi]_E = 0$ for all $\in \mathcal{E}(\Omega)$. This plus the definitions of $\|\cdot\|_h$ and $\|\cdot\|_{\mathrm{N}}$ norms, and Lemma 5.6 leads to the inequalities in (ii). □

Proof of (iii). An application of the definition of $\|\cdot\|_h$ and $\|\cdot\|_{\mathrm{P}}$ norms, and the estimate ([3], Lem. 3.1) stated below

$$\sum_{E\in\mathcal{E}} h_E^{-1}\|[\varphi]\|_{0,E}^2 \lesssim \sum_{T\in\mathcal{T}} \|\nabla\varphi\|_{0,T}^2 + \sum_{E\in\mathcal{E}} h_E^{-1}\big\|\Pi_E^0[\varphi]\big\|_{0,E}^2 \lesssim \|\varphi\|_{\mathrm{P}}^2$$

leads to the first inequality. Lemma 5.6 and $\|\varphi\|_h \lesssim \|\varphi\|_{\mathrm{P}}$ imply the second inequality in (iii). □

6. Finite element methods for ferronematics

In this section we discuss the error estimates for conforming, dG, Nitsche and WOPSIP schemes for the ferronematics system. For $\Phi_{\mathrm{dG}} = (\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in \mathbf{P}_1(\mathcal{T})$, define the product norms

$$|||\Phi_{\mathrm{dG}}|||_{\mathrm{dG}}^2 = \sum_{i=1}^4 \|\varphi_i\|_{\mathrm{dG}}^2, \; |||\Phi|||_{\mathrm{N}}^2 := \sum_{i=1}^4 \|\varphi_i\|_{\mathrm{N}}^2, \quad \text{and } |||\Phi_{\mathrm{P}}|||_{\mathrm{P}}^2 := \sum_{i=1}^4 \|\varphi_i\|_{\mathrm{P}}^2.$$

Set $X_{\mathrm{C}} := P_1(\mathcal{T}) \cap H^1(\Omega)$, $V_{\mathrm{C}} := X_{\mathrm{C}} \cap H_0^1(\Omega)$, and $\mathbf{X} := (H^1(\Omega))^4$. Table 1 presents a summary of the relevant spaces and operators for these schemes, applied to the ferronematic system.

Note that the nonhomogeneous Dirichlet boundary condition is enforced *via* the initial guess in the Newton's scheme for conforming FEM, differently from the other methods Nitsche, dG, and WOPSIP, where it is enforced *via* a loading term.

TABLE 1. Overview of discrete spaces and operators for the ferronematic system.

	Notations								
Methods	X_h	$V_h \subseteq X_h$	$\widehat{X}$	$\|\cdot\|_{\widehat{X}}$	A_h	F_h	I_h	P	Q
Conforming FEM	$\mathbf{X}_{\rm C} := (X_{\rm C})^4$	$\mathbf{V}_{\rm C} := (V_{\rm C})^4$	$\mathbf{X}$	$\vert\vert\vert \cdot \vert\vert\vert_1$	A in (5.4)	0	$I_{\rm C}$	id	id
dGFEM	$X_{\rm dG} := P_1(\mathcal{T})$ $\mathbf{X}_{\rm dG} := (X_{\rm dG})^4$	$\mathbf{X}_{\rm dG}$	$\mathbf{X} + \mathbf{X}_{\rm dG}$	$\vert\vert\vert \cdot \vert\vert\vert_{\rm dG}$	$A_{\rm dG}$ in (6.8)	$F_{\rm dG}$ in (6.8)	$I_{\rm C}$	J_1	J_2
Nitsche's method	$\mathbf{X}_{\rm C}$	$\mathbf{X}_{\rm C}$	$\mathbf{X}$	$\vert\vert\vert \cdot \vert\vert\vert_{\rm N}$	$A_{\rm N}$ in (6.15)	$F_{\rm N}$ in (6.15)	$I_{\rm C}$	id	J_2
WOPSIP	$\mathbf{X}_{\rm P} := (X_{\rm dG})^4$	$\mathbf{X}_{\rm P}$	$\mathbf{X} + \mathbf{X}_{\rm P}$	$\vert\vert\vert \cdot \vert\vert\vert_{\rm P}$	$A_{\rm P}$ in (7.2)	$F_{\rm P}$ in (7.3)	$I_{\rm CR}$ in Lemma 7.1	J_1	J_2

6.1. Conforming finite element method

The discrete problem that corresponds to (5.3) seeks $\Psi_{\rm C} \in \mathbf{X}_{\rm C}$ with $\Psi_{\rm C} = \mathbf{g}_{\rm C}$ on $\partial\Omega$,

$$N(\Psi_{\rm C}; \Phi_{\rm C}) := A(\Psi_{\rm C}, \Phi_{\rm C}) + B(\Psi_{\rm C}, \Phi_{\rm C}) = 0 \qquad \text{for all } \Phi_{\rm C} \in \mathbf{V}_{\rm C}. \tag{6.1}$$

Here $\mathbf{g}_{\rm C}$ is the Lagrange P_1 interpolation of $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}+\alpha}(\partial\Omega)$ with $0 < \alpha \leq 1$.

Lemma 6.1 (Linearised systems)**.** *Let Ψ be a regular solution of (5.3) with $\Psi \in \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \leq 1$. For a given $\Theta_{\rm C} \in \mathbf{V}_{\rm C}$ with $|||\Theta_{\rm C}|||_1 = 1$, there exists a unique solution $\zeta \in \mathbf{V} \cap \mathbf{H}^{1+\alpha}(\Omega)$ to $A(\zeta, \Phi) = \langle D\mathscr{B}(\Psi)\Theta_{\rm C}, \Phi\rangle$ for all $\Phi \in \mathbf{V}$ with $|||\zeta|||_{1+\alpha} \lesssim 1$.*

Proof. Recall $\langle D\mathscr{B}(\Psi)\Theta, \Phi\rangle := B_1(\Theta, \Phi) + 2B_2(\Psi, \Theta, \Phi) + 3B_3(\Psi, \Psi, \Theta, \Phi)$ from (5.5). Since $\Psi \in \mathbf{H}^{1+\alpha}(\Omega)$ and $\Theta_{\rm C} \in \mathbf{V}_{\rm C}$, Lemmas 5.5(ii) and 5.5(iii) shows

$$\langle D\mathscr{B}(\Psi)\Theta_{\rm C}, \Phi\rangle \lesssim \left(1 + |||\Psi|||_{1+\alpha} + |||\Psi|||_{1+\alpha}^2\right)|||\Phi|||_0 \qquad \text{for all } \Phi \in \mathbf{V}.$$

This and elliptic regularity yield the existence of a unique solution $\zeta \in \mathbf{V} \cap \mathbf{H}^{1+\alpha}(\Omega)$ to the linear system $A(\zeta, \Phi) = \langle D\mathscr{B}(\Psi)\Theta_{\rm C}, \Phi\rangle$ for all $\Phi \in \mathbf{V}$, such that $|||\zeta|||_{1+\alpha} \lesssim (1 + |||\Psi|||_{1+\alpha} + |||\Psi|||_{1+\alpha}^2)$. ☐

Remark 6.1 (Linearised systems for all schemes)**.** For a given $\theta_h \in V_h$ with $\|\theta_h\|_{X_h} = 1$, arguments exactly as Lemma 6.1 establish the existence of a unique $\zeta := \mathscr{A}^{-1}(\langle D\mathscr{B}(\Psi)\theta_h, \cdot\rangle|_V) \in \mathbf{V} \cap \mathbf{H}^{1+\alpha}(\Omega)$ for the dG, Nitsche, and WOPSIP schemes.

Theorem 6.1 (Discrete inf-sup condition)**.** *There exists a positive constant δ such that, any $\mathcal{T} \in \mathbb{T}(\delta)$ and a regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \leq 1$ of (5.3) satisfy (2.7).*

Proof. The proof applies Theorem 2.1; thus it is enough to verify **(A1)–(A4)**. Lemmas 6.1 and 5.1 imply $|||\zeta - I_{\rm C}\zeta|||_1 \lesssim h^\alpha |||\zeta|||_{1+\alpha} \lesssim h^\alpha$. This leads to $\delta_1 \lesssim h^\alpha$ in **(A1)**. Since $Q = id$ for the conforming scheme, the parameters $\delta_2 = 0$, $\Lambda_1 = 0$ and $C_A = 0$ satisfy **(A2), (A3)**, and **(A4)**, respectively. Therefore, for a sufficiently small choice of the maximal mesh-size h, the discrete inf-sup condition in Theorem 2.1 is verified. This concludes the proof. ☐

Theorem 6.2 (*A priori* error estimate)**.** *There exists $\delta > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \leq 1$ of (5.3) and the locally unique discrete solution $\Psi_{\rm C} \in \mathbf{X}_{\rm C}$ to (6.1) such that $\Psi_{\rm C} = \mathbf{g}_{\rm C}$ on $\partial\Omega$ satisfy $|||\Psi - \Psi_{\rm C}|||_1 \lesssim h^\alpha$.*

Proof. Since **(A1)**–**(A4)** hold, it is enough to verify **(A5)**–**(A8)**, **(B1)**, **(B2)**, and the Lipschitz continuity of the Frechét derivative of $\mathscr{B} : \mathbf{X}_{\rm C} \to \mathbf{V}_{\rm C}^*$ on $\overline{B}_{\mathbf{X}_{\rm C}}(I_{\rm C}\Psi, 2b)$ to apply Theorem 3.1. This is done in three steps below.

Step 1: Verification of **(A5)**–**(A8).** A choice of $z_h := I_{\rm C}\Psi$ and an application of Lemma 5.1 leads to **(A5)** with $\delta_3 \lesssim h^\alpha$. Since $P = Q = id$ and $F_h = 0$ (see Tab. 1) for the conforming case, $\delta_4 = 0$ in **(A6)**, $\Lambda_2 = 0$ in **(A7)** and $\widetilde{C}_A = 0$ in **(A8)**.

Step 2: Verification of **(B1)**, **(B2).** Let $\theta_h := \Theta_{\rm C}, \varphi_h := \Phi_{\rm C} \in \mathbf{V}_{\rm C}$ with $|||\Theta_{\rm C}|||_1 = |||\Phi_{\rm C}|||_1 = 1$. Follow the re-grouping of terms as (5.9) (resp. (5.10)) with $z_h := I_{\rm C}\Psi$ and use Lemmas 5.5(ii) and 5.1 to obtain

$$\langle D\mathscr{B}(\Psi)\Theta_{\rm C}, \Phi_{\rm C}\rangle - \langle D\mathscr{B}(I_{\rm C}\Psi)\Theta_{\rm C}, \Phi_{\rm C}\rangle \lesssim (1 + |||\Psi|||_1 + |||I_{\rm C}\Psi|||_1)|||\Psi - I_{\rm C}\Psi|||_1 \leq C_B|||\Psi - I_{\rm C}\Psi|||_1.$$

$$\text{(resp. } B(\Psi, \Phi_{\rm C}) - B(I_{\rm C}\Psi, \Phi_{\rm C})) \lesssim \Big(1 + |||\Psi|||_1 + |||I_{\rm C}\Psi|||_1 + |||\Psi|||_1^2 + |||I_{\rm C}\Psi|||_1^2\Big)|||\Psi - I_{\rm C}\Psi|||_1|||\Phi_{\rm C}|||_1$$
$$\leq \widetilde{C}_B|||\Psi - I_{\rm C}\Psi|||_1|||\Phi_{\rm C}|||_1.$$

For the dependency of constants $C_B, \widetilde{C}_B$ on various parameters, see Table B.1.

Step 3: Verification of Lipschitz continuity of $D\mathscr{B}(\cdot) \in L(\mathbf{X}_{\rm C}, \mathbf{V}_{\rm C}^*)$ in $\overline{B}_{\mathbf{X}_{\rm C}}(I_{\rm C}\Psi, 2b)$**.** For $\boldsymbol{\eta}_{\rm C}, \boldsymbol{\chi}_{\rm C} \in \overline{B}_{\mathbf{X}_{\rm C}}(I_{\rm C}\Psi, 2b)$, $\Theta_{\rm C} \in \mathbf{X}_{\rm C}$, $\Phi_{\rm C} \in \mathbf{V}_{\rm C}$, the combination of terms as done in (5.12) and Lemma 5.5(ii), triangle inequalities that lead to

$$|||\boldsymbol{\eta}_{\rm C}|||_1 \leq |||\boldsymbol{\eta}_{\rm C} - I_{\rm C}\Psi|||_1 + |||I_{\rm C}\Psi|||_1 \leq 2b + |||I_{\rm C}\Psi|||_1, |||\boldsymbol{\chi}_{\rm C}|||_1 \leq 2b + |||I_{\rm C}\Psi|||_1, \tag{6.2}$$

and Lemma 5.1 yield

$$\langle D\mathscr{B}(\boldsymbol{\eta}_{\rm C})\Theta_{\rm C}, \Phi_{\rm C}\rangle - \langle D\mathscr{B}(\boldsymbol{\chi}_{\rm C})\Theta_{\rm C}, \Phi_{\rm C}\rangle \lesssim |||\boldsymbol{\eta}_{\rm C} - \boldsymbol{\chi}_{\rm C}|||_1(1 + |||\boldsymbol{\eta}_{\rm C}|||_1 + |||\boldsymbol{\chi}_{\rm C}|||_1)|||\Theta_{\rm C}|||_1|||\Phi_{\rm C}|||_1$$
$$\leq L|||\boldsymbol{\eta}_{\rm C} - \boldsymbol{\chi}_{\rm C}|||_1|||\Theta_{\rm C}|||_1|||\Phi_{\rm C}|||_1.$$

Therefore, Theorem 3.2 applies and yields the existence and local uniqueness of the discrete solution $\Psi_{\rm C}$ to (6.1) with $|||\Psi - \Psi_{\rm C}|||_1 \leq \rho := \delta_3 + b + r$. Thus $\delta_3 \lesssim h^\alpha$, $\delta_4 = 0$, and (3.2) imply $b := \beta_1^{-1}\big((C_I^2\|\mathscr{A}\|(1 + \Lambda_2) + \widetilde{C}_A + \widetilde{C}_B)\delta_3 + \delta_4\big) \lesssim h^\alpha$. This and $b + r < 2b$ from (3.10) imply that $|||\Psi - \Psi_{\rm C}|||_1 \lesssim h^\alpha$.

Step 4: Boundary condition for the discrete solution $\Psi_{\rm C}$**.** Recall the initial iterate $\Psi_{\rm C}^0 := I_{\rm C}\Psi$, in the Newton's scheme, $\Psi_{\rm C}^n := \Psi_{\rm C}^{n-1} - DN(\Psi_{\rm C}^{n-1})^{-1}N(\Psi_{\rm C}^{n-1})$. The discrete inf-sup condition in (3.5) implies that $DN(I_{\rm C}\Psi)|_{\mathbf{V}_{\rm C}}$ is invertible. Therefore, $DN(\Psi_{\rm C}^0)^{-1}N(\Psi_{\rm C}^0) \in \mathbf{V}_{\rm C}$. This implies $\Psi_{\rm C}^1 := \Psi_{\rm C}^0 - DN(\Psi_{\rm C}^0)^{-1}N(\Psi_{\rm C}^0) = \mathbf{g}_{\rm C}$ on $\partial\Omega$. Subsequently, the sequence $\Psi_{\rm C}^n|_{\partial\Omega} = \mathbf{g}_{\rm C}$ and hence the sequential limit, $\Psi_{\rm C}$, which is indeed the solution of discrete problem (6.1), satisfies the boundary condition $\mathbf{g}_{\rm C}$ on $\partial\Omega$. For any function $\mathbf{w} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{w} = \mathbf{g}_{\rm C}$ on $\partial\Omega$, the sequence $\Psi_{\rm C}^n - \mathbf{w} \to \Psi_{\rm C} - \mathbf{w}$ in $\mathbf{H}_0^1(\Omega)$. Then by Mazur's theorem ([11], p. 723) $\Psi_{\rm C} - \mathbf{w} \in \mathbf{H}_0^1(\Omega)$. Consequently the trace of $\Psi_{\rm C}$ on $\partial\Omega$ is $\mathbf{g}_{\rm C}$. This concludes the proof. ◻

For the solution $u_h := (u_{1,h}, u_{2,h}, u_{3,h}, u_{4,h}) \in X_h$ of the discrete nonlinear problem $N_h(u_h) = 0$ in (2.2),

$$\mathscr{B}(u_h) := \frac{1}{\ell}\bigg(u_{1,h} + \frac{c}{2}(u_{3,h}^2 - u_{4,h}^2) - (u_{1,h}^2 + u_{2,h}^2)u_{1,h},\;\; u_{2,h} + cu_{3,h}u_{4,h} - (u_{1,h}^2 + u_{2,h}^2)u_{2,h},$$
$$u_{3,h} + c(u_{1,h}u_{3,h} + u_{2,h}u_{4,h}) - (u_{3,h}^2 + u_{4,h}^2)u_{3,h},\;\; u_{4,h} + c(u_{2,h}u_{3,h} - u_{1,h}u_{4,h}) - (u_{3,h}^2 + u_{4,h}^2)u_{4,h}\bigg). \tag{6.3}$$

Given the unique discrete solution to (6.1) with respect to the triangulation $\mathcal{T}$, define the error estimators

$$\vartheta_T^2 := h_T^2|||\mathscr{B}(\Psi_{\rm C})|||_{0,T}^2 \quad \text{for all } T \in \mathcal{T}, \quad \big(\vartheta_E^i\big)^2 := h_E|||[\nabla\Psi_{\rm C}\nu_E]_E|||_{0,E}^2 \quad \text{for all } E \in \mathcal{E}(\Omega), \tag{6.4}$$

$$\text{and } \vartheta_{\mathrm{C}}^2 := \sum_{T\in\mathcal{T}} \vartheta_T^2 + \sum_{E\in\mathcal{E}(\Omega)} (\vartheta_E^i)^2. \tag{6.5}$$

Theorem 6.3 (*A posteriori* error estimate)**.** *Let Ψ be a regular solution of* (5.3) *with $\mathbf{g} \in \mathbf{C}^0(\partial\Omega)$ and $\mathbf{g}|_E \in \mathbf{H}^1(E)$ for all $E \in \mathcal{E}(\partial\Omega)$. There exist $\delta, R, C_{\mathrm{rel}}$, and $C_{\mathrm{eff}} > 0$ such that, any $\mathcal{T} \in \mathbb{T}(\delta)$ and the unique solution Ψ_{C} to* (6.1) *with $|||\Psi - \Psi_{\mathrm{C}}|||_1 < R$ satisfy*

$$C_{\mathrm{eff}}^{-1}\vartheta_{\mathrm{C}} \le |||\Psi - \Psi_{\mathrm{C}}|||_1 \le C_{\mathrm{rel}}(\vartheta_{\mathrm{C}} + \text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega))).$$

*Here recall that the approximation error "*data app$(\mathbf{g}, \mathcal{E}(\partial\Omega))$*" for the non-homogeneous Dirichlet boundary data* $\mathbf{g}$ *in the triangulation $\mathcal{T}$ is defined in* (5.7)*.*

Proof. We verify the conditions of Theorem 4.1. Set $\widehat{A} := A$ and $\widehat{N} := N$.

Verification of local Lipschitz continuity. The assumption of local Lipschitz continuity of the Frechét derivative of $\mathscr{B} : \mathbf{X} \to \mathbf{V}^*$ at $\Psi \in \mathbf{X}$ involves the ball $B_{\mathbf{X}}(\Psi, R_0) := \{\boldsymbol{\eta} \in \mathbf{X} |\ |||\Psi - \boldsymbol{\eta}|||_1 < R_0\}$ for some $R_0 > 0$. For $\boldsymbol{\eta} \in B_{\mathbf{X}}(\Psi, R_0)$ and $\Theta, \Phi \in \mathbf{X}$, bound similar to (5.11), Sobolev embedding results $H^1(\Omega) \hookrightarrow L^p(\Omega)$, $p = 3, 4$, for $\Omega \subset \mathbb{R}^2$, and $|||\boldsymbol{\eta}|||_{0,4} \le |||\boldsymbol{\eta} - \Psi|||_{0,4} + |||\Psi|||_{0,4}$ lead to

$$|||D\mathscr{B}(\boldsymbol{\eta}) - D\mathscr{B}(\Psi)|||_{L(\mathbf{X},V^*)} \lesssim |||\boldsymbol{\eta} - \Psi|||_1(1 + R_0 + 2|||\Psi|||_1),$$

where the constant in "$\lesssim$" depends on the parameters ℓ, c and C_S. This verifies (4.1) with $\gamma := \gamma_{\mathrm{C}} \lesssim (1 + R_0 + 2|||\Psi|||_1)$.

Verification of **(AP)**. Choose $\boldsymbol{\eta}_h := \Psi_{\mathrm{C}} \in \mathbf{X}_{\mathrm{C}}$ in Lemma 5.3. Then the function $\mathscr{G} : \mathbf{P}_1(\mathcal{T}) \to \mathcal{X}$ at Ψ_{C} satisfies

$$\int_\Omega \nabla(\mathscr{G}\Psi_{\mathrm{C}}) \cdot \nabla\Phi \,\mathrm{d}x = \sum_{T\in\mathcal{T}} \int_T \nabla\Psi_{\mathrm{C}} \cdot \nabla\Phi \,\mathrm{d}x \qquad \text{for all } \Phi \in \mathbf{V}$$

subject to the Dirichlet boundary condition $(\mathscr{G}\Psi_{\mathrm{C}})|_{\partial\Omega} = \mathbf{g}$. The definition of $|||\cdot|||_1$, and Lemmas 5.6, (5.8), Lemma 5.3 with $[\mathscr{G}\Psi_{\mathrm{C}} - \Psi_{\mathrm{C}}]_E = 0$ for all $E \in \mathcal{E}(\Omega)$, and $\mathscr{G}\Psi_{\mathrm{C}} - \Psi_{\mathrm{C}} = \mathbf{g} - \mathbf{g}_{\mathrm{C}}$ on $\partial\Omega$ reveal

$$\begin{aligned}
|||\mathscr{G}\Psi_{\mathrm{C}} - \Psi_{\mathrm{C}}|||_1^2 &= |||\nabla(\mathscr{G}\Psi_{\mathrm{C}} - \Psi_{\mathrm{C}})|||_0^2 + |||\mathscr{G}\Psi_{\mathrm{C}} - \Psi_{\mathrm{C}}|||_0^2 \\
&\le (1 + C_{\mathrm{P}}^2)|||\nabla(\mathscr{G}\Psi_{\mathrm{C}} - \Psi_{\mathrm{C}})|||_0^2 + C_{\mathrm{P}}^2 \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-1}|||\mathscr{G}\Psi_{\mathrm{C}} - \Psi_{\mathrm{C}}|||_{0,E}^2 \\
&\lesssim \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-1}|||\mathbf{g} - \mathbf{g}_{\mathrm{C}}|||_{0,E}^2 + (\text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega)))^2,
\end{aligned}$$

where the constants in "$\lesssim$" depends on $C_{\mathscr{G}}$ and C_{P}. Note that $\mathbf{g} - \mathbf{g}_{\mathrm{C}} = 0$ at the two end points of each edge $E \in \mathcal{E}(\partial\Omega)$. Therefore Poincaré inequality [18] on each edge implies $|||\mathbf{g} - \mathbf{g}_{\mathrm{C}}|||_{0,E} \lesssim h_E|||\partial_{\mathcal{E}}(\mathbf{g} - \mathbf{g}_{\mathrm{C}})|||_{0,E}$. This applied to the above inequality proves **(AP)** with Λ_3 depending on $C_{\mathscr{G}}$ and C_{P}.

Conclusions of Theorem 4.1. Suppose that δ satisfies Theorem 6.2, and if necessary, is chosen smaller such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the unique discrete solution $\Psi_{\mathrm{C}} \in \mathbf{X}_{\mathrm{C}}$ to (6.1) satisfies $|||\Psi - \Psi_{\mathrm{C}}|||_1 < R < \min(R_0, \frac{\beta}{2\gamma_{\mathrm{C}}})$. Thus Theorem 4.1(i) lead to

$$|||\Psi - \Psi_{\mathrm{C}}|||_1 \lesssim |||N(\Psi_{\mathrm{C}})|||_{\mathbf{V}^*} + \Big(1 + |||DN(\Psi)|||_{L(\mathbf{X},\mathbf{V}^*)}\Big) \text{ data app}(\mathbf{g}, \mathcal{E}(\partial\Omega)). \tag{6.6}$$

Since $\mathbf{V}$ is a Hilbert space, there exists a $\Phi \in \mathbf{V}$ with $|||\Phi|||_1 = 1$ such that

$$|||N(\Psi_{\mathrm{C}})|||_{\mathbf{V}^*} = N(\Psi_{\mathrm{C}}; \Phi) = N(\Psi_{\mathrm{C}}; \Phi - I_{\mathrm{SZ}}\Phi) + N(\Psi_{\mathrm{C}}; I_{\mathrm{SZ}}\Phi),$$

where $I_{\mathrm{SZ}} : \mathbf{V} \to \mathbf{V}_{\mathrm{C}}$ is the Scott–Zhang interpolation in Lemma 5.2. Note that for $\Phi_{\mathrm{C}} := I_{\mathrm{SZ}}\Phi$, equation (6.1) implies $N(\Psi_{\mathrm{C}}; I_{\mathrm{SZ}}\Phi) = 0$. Next we estimate the term $N(\Psi_{\mathrm{C}}; \Phi - I_{\mathrm{SZ}}\Phi) = A(\Psi_{\mathrm{C}}, \Phi - I_{\mathrm{SZ}}\Phi) + B(\Psi_{\mathrm{C}}, \Phi - I_{\mathrm{SZ}}\Phi)$. Apply integration by parts elementwise for $A(\Psi_{\mathrm{C}}, \Phi - I_{\mathrm{SZ}}\Phi)$, utilize $\Delta(\Psi_{\mathrm{C}}|_T) = 0$ for all $T \in \mathcal{T}$ and $[\Phi - I_{\mathrm{SZ}}\Phi]_E = 0$ for all $E \in \mathcal{E}$ to obtain $A(\Psi_{\mathrm{C}}, \Phi - I_{\mathrm{SZ}}\Phi) = \sum_{E\in\mathcal{E}(\Omega)} \langle [\nabla\Psi_{\mathrm{C}}\nu_E]_E, \Phi - I_{\mathrm{SZ}}\Phi\rangle_E$. Recall the definition $B(\Psi_{\mathrm{C}}, \Phi - I_{\mathrm{SZ}}\Phi) = \langle \mathscr{B}(\Psi_{\mathrm{C}}), \Phi - I_{\mathrm{SZ}}\Phi\rangle$ and the estimators in (6.4), $\vartheta_T^2 := h_T^2 |||\mathscr{B}(\Psi_{\mathrm{C}})|||_{0,T}^2$ for all $T \in \mathcal{T}$, $(\vartheta_E^i)^2 := h_E |||[\nabla\Psi_{\mathrm{C}}\nu_E]_E|||_{0,E}^2$ for all $E \in \mathcal{E}(\Omega)$ with $\mathscr{B}(\Psi_{\mathrm{C}})$ as defined in (6.3). This plus the Cauchy–Schwarz inequality and (6.4) applied to the above displayed identity leads to

$$N(\Psi_{\mathrm{C}}; \Phi - I_{\mathrm{SZ}}\Phi) \lesssim \left(\sum_{T\in\mathcal{T}} \vartheta_T^2 + \sum_{E\in\mathcal{E}(\Omega)} (\vartheta_E^i)^2\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}} h_T^{-2} |||\Phi - I_{\mathrm{SZ}}\Phi|||_{0,T}^2 + \sum_{E\in\mathcal{E}(\Omega)} h_E^{-1} |||\Phi - I_{\mathrm{SZ}}\Phi|||_{0,E}^2\right)^{\frac{1}{2}}.$$

This, Lemma 5.2 with $|||\Phi|||_1 = 1$, and a combination of the last two displayed estimates lead to

$$|||N(\Psi_{\mathrm{C}})|||_{\mathbf{V}^*} \lesssim \left(\sum_{T\in\mathcal{T}} \vartheta_T^2 + \sum_{E\in\mathcal{E}(\Omega)} \left(\vartheta_E^i\right)^2\right)^{\frac{1}{2}}.$$

The above inequality applied to (6.6) concludes the proof of the reliability estimate of Theorem 6.3. The proof of the efficiency is a straightforward application of standard local efficiency estimates [33]. □

6.2. Discontinuous Galerkin finite element methods

The dG formulation associated to (5.3) seeks $\Psi_{\mathrm{dG}} \in \mathbf{X}_{\mathrm{dG}}$ such that for all $\Phi_{\mathrm{dG}} \in \mathbf{X}_{\mathrm{dG}}$,

$$A_{\mathrm{dG}}(\Psi_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) + B(\Psi_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) - F_{\mathrm{dG}}(\Phi_{\mathrm{dG}}) = 0, \tag{6.7}$$

where for $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ and $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbf{X}_{\mathrm{dG}}$, and the penalty parameter $\sigma_{\mathrm{dG}} > 0$,

$$A_{\mathrm{dG}}(\Theta, \Phi) := A_{\mathrm{pw}}(\Theta, \Phi) + \boldsymbol{\mathcal{J}}(\Theta, \Phi) + \boldsymbol{\mathcal{J}}_{\sigma_{\mathrm{dG}}}(\Theta, \Phi), \quad \text{and } F_{\mathrm{dG}}(\Phi) := F_{\mathrm{dG}}^{\mathcal{J}}(\Phi) + F_{\mathrm{dG}}^{\mathcal{J}_{\sigma_{\mathrm{dG}}}}(\Phi). \tag{6.8}$$

Here for $\theta, \varphi \in X_{\mathrm{dG}}$, $-1 \le \lambda \le 1$,

$$\begin{aligned}
A_{\mathrm{pw}}(\Theta, \Phi) &:= \sum_{i=1}^{4} a_{\mathrm{pw}}(\theta_i, \varphi_i) \text{ with } a_{\mathrm{pw}}(\theta, \varphi) := \sum_{T\in\mathcal{T}} \int_T \nabla\theta \cdot \nabla\varphi \,\mathrm{d}x,\\
\boldsymbol{\mathcal{J}}(\Theta, \Phi) &:= \sum_{i=1}^{4} \mathcal{J}(\theta_i, \varphi_i) \text{ with } \mathcal{J}(\theta, \varphi) := -\sum_{E\in\mathcal{E}} \langle \{\nabla_{\mathrm{pw}}\theta \cdot \nu_E\}_E, [\varphi]_E\rangle_E - \lambda \sum_{E\in\mathcal{E}} \langle \{\nabla_{\mathrm{pw}}\varphi \cdot \nu_E\}_E, [\theta]_E\rangle_E,\\
\boldsymbol{\mathcal{J}}_{\sigma_{\mathrm{dG}}}(\Theta, \Phi) &:= \sum_{i=1}^{4} \mathcal{J}_{\sigma_{\mathrm{dG}}}(\theta_i, \varphi_i) \text{ with } \mathcal{J}_{\sigma_{\mathrm{dG}}}(\theta, \varphi) := \sum_{E\in\mathcal{E}} \frac{\sigma_{\mathrm{dG}}}{h_E} \langle [\theta]_E, [\varphi]_E\rangle_E,\\
F_{\mathrm{dG}}^{\mathcal{J}}(\Phi) &:= \sum_{i=1}^{4} F_{\mathrm{dG}}^{i,\mathcal{J}}(\varphi_i) \text{ with } F_{\mathrm{dG}}^{i,\mathcal{J}}(\varphi) := -\lambda \sum_{E\in\mathcal{E}(\partial\Omega)} \langle g_i, \nabla_{\mathrm{pw}}\varphi \cdot \nu_E\rangle_E \qquad \text{for } 1 \le i \le 4,\\
F_{\mathrm{dG}}^{\mathcal{J}_{\sigma_{\mathrm{dG}}}}(\Phi) &:= \sum_{i=1}^{4} F_{\mathrm{dG}}^{i,\mathcal{J}_{\sigma_{\mathrm{dG}}}}(\varphi_i) \text{ with } F_{\mathrm{dG}}^{i,\mathcal{J}_{\sigma_{\mathrm{dG}}}}(\varphi) := \sum_{E\in\mathcal{E}(\partial\Omega)} \frac{\sigma_{\mathrm{dG}}}{h_E} \langle g_i, \varphi\rangle_E.
\end{aligned}$$

We refer to Table 1 for the discrete spaces, norms and the choices of operators for the dGFEM.

For $\Phi_{\rm dG} \in \mathbf{X}_{\rm dG}$, the coercivity $\alpha_{\rm dG}|||\Phi_{\rm dG}|||_{\rm dG}^2 \leq A_{\rm dG}(\Phi_{\rm dG}, \Phi_{\rm dG})$ follows from Section 3.4 of [30], for a sufficiently large value of $\sigma_{\rm dG}$ and hence (2.4) holds. Lemmas 5.8(iv) and 5.9(iii) lead to the boundedness results stated next.

$$\text{For } \boldsymbol{\chi}, \boldsymbol{\eta}, \Theta, \Phi \in \mathbf{H}^1(\mathcal{T}),\ B_1(\Theta, \Phi) \lesssim |||\Theta|||_{\rm dG}|||\Phi|||_{\rm dG}, B_2(\boldsymbol{\eta}, \Theta, \Phi) \lesssim |||\boldsymbol{\eta}|||_{\rm dG}|||\Theta|||_{\rm dG}|||\Phi|||_{\rm dG},$$
$$B_3(\boldsymbol{\chi}, \boldsymbol{\eta}, \Theta, \Phi) \lesssim |||\boldsymbol{\chi}|||_{\rm dG}|||\boldsymbol{\eta}|||_{\rm dG}|||\Theta|||_{\rm dG}|||\Phi|||_{\rm dG}.$$

Theorem 6.4 (Discrete inf-sup condition)**.** *There exist $\delta > 0$ and a sufficiently large $\sigma_{\rm dG} > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$ and a regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \leq 1$ of (5.3), (2.7) holds for $V_h = \mathbf{X}_{\rm dG}$ and $\langle DN_h(\Psi)\Theta_{\rm dG}, \Phi_{\rm dG}\rangle := A_{\rm dG}(\Theta_{\rm dG}, \Phi_{\rm dG}) + B_L(\Theta_{\rm dG}, \Phi_{\rm dG})$.*

Proof. We verify **(A1)**–**(A4)** to apply Theorem 2.1.

Step 1: Verification of **(A1)**–**(A3).** Let $\Theta_{\rm dG}, \Phi_{\rm dG} \in \mathbf{X}_{\rm dG}$ with $|||\Theta_{\rm dG}|||_{\rm dG} = 1 = |||\Phi_{\rm dG}|||_{\rm dG}$. Remark 6.1 and Lemmas 5.7(ii), 5.1 lead to $|||\zeta - I_{\rm C}\zeta|||_{\rm dG} \lesssim h^\alpha |||\zeta|||_{1+\alpha} \lesssim h^\alpha$, and establish **(A1)** with $\delta_1 \lesssim h^\alpha$. Lemmas 5.8(i) and 5.9(i) with $|||\Theta_{\rm dG}|||_{\rm dG} = 1 = |||\Phi_{\rm dG}|||_{\rm dG}$ imply $\delta_2 \lesssim h$ in **(A2)**. Lemmas 5.4 and 5.9 establish **(A3)**.

Step 2: Verification of **(A4).** Let $\xi_h := \Theta_{\rm dG}$, $\varphi_h := \Phi_{\rm dG} \in \mathbf{X}_{\rm dG}$. The definition of $A_{\rm dG}(\cdot,\cdot)$, an algebraic manipulation and a re-arrangement of terms yield

$$\begin{aligned} A_{\rm dG}(\Theta_{\rm dG}, \Phi_{\rm dG}) - A(J_2\Theta_{\rm dG}, J_2\Phi_{\rm dG}) &= (A_{\rm pw}(\Theta_{\rm dG}, \Phi_{\rm dG} - J_2\Phi_{\rm dG}) + \boldsymbol{\mathcal{J}}(\Theta_{\rm dG}, \Phi_{\rm dG})) \\ &\quad + A_{\rm pw}(\Theta_{\rm dG} - J_2\Theta_{\rm dG}, J_2\Phi_{\rm dG}) + \boldsymbol{\mathcal{J}}_{\sigma_{\rm dG}}(\Theta_{\rm dG}, \Phi_{\rm dG}) \\ &=: T_1 + T_2 + T_3. \end{aligned} \tag{6.9}$$

For $\Theta_{\rm dG} := (\theta_1, \theta_2, \theta_3, \theta_4), \Phi_{\rm dG} := (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbf{X}_{\rm dG}, 1 \leq i \leq 4$, an integration by parts and (6.8) for the components of T_1 yield

$$\begin{aligned} a_{\rm pw}(\theta_i, \varphi_i - J_2\varphi_i) + \mathcal{J}(\theta_i, \varphi_i) &= -\sum_{T\in\mathcal{T}} \int_T \Delta\theta_i(\varphi_i - J_2\varphi_i)\,{\rm d}x + \sum_{E\in\mathcal{E}} \langle\{\nabla_{\rm pw}\theta_i \cdot \nu_E\}_E, [\varphi_i - J_2\varphi_i]_E\rangle_E \\ &+ \sum_{E\in\mathcal{E}(\Omega)} \langle[\nabla_{\rm pw}\theta_i \cdot \nu_E]_E, \{\varphi_i - J_2\varphi_i\}_E\rangle_E - \sum_{E\in\mathcal{E}} \langle\{\nabla_{\rm pw}\theta_i \cdot \nu_E\}_E, [\varphi_i]_E\rangle_E - \lambda \sum_{E\in\mathcal{E}} \langle\{\nabla_{\rm pw}\varphi_i \cdot \nu_E\}_E, [\theta_i]_E\rangle_E. \end{aligned}$$

Since $\theta_i \in P_1(\mathcal{T})$, $\Delta\theta_i = 0$ on $T \in \mathcal{T}$ in the first term of the above expression. This plus Lemma 5.4(b) applied to the third term, $[J_2\varphi_i]_E = [J_2\theta_i]_E = 0$ for all $E \in \mathcal{E}$, and a cancellation of terms lead to

$$a_{\rm pw}(\theta_i, \varphi_i - J_2\varphi_i) + \mathcal{J}(\theta_i, \varphi_i) = \lambda \sum_{E\in\mathcal{E}} \langle\{\nabla_{\rm pw}\varphi_i \cdot \nu_E\}_E, [J_2\theta_i - \theta_i]_E\rangle_E.$$

Combine this with all the components, and apply the Cauchy–Schwarz inequality and Lemma 5.7(ii) to arrive at

$$T_1 \lesssim |||\Theta_{\rm dG} - J_2\Theta_{\rm dG}|||_{\rm dG}|||\Phi_{\rm dG}|||_{\rm dG}.$$

The Cauchy–Schwarz inequality, Lemmas 5.4(d) and 5.9(i) lead to

$$T_2 := A_{\rm pw}(\Theta_{\rm dG} - J_2\Theta_{\rm dG}, J_2\Phi_{\rm dG}) \lesssim |||\Theta_{\rm dG} - J_2\Theta_{\rm dG}|||_{\rm dG}|||\Phi_{\rm dG}|||_{\rm dG}.$$

The definition of $\mathcal{J}_{\sigma_{\rm dG}}(\cdot,\cdot)$ in (6.8) and $[J_2\theta_i]_E = 0$ for all $E \in \mathcal{E}$, $1 \leq i \leq 4$ yield $\mathcal{J}_{\sigma_{\rm dG}}(\theta_i, \varphi_i) = \sum_{E\in\mathcal{E}} \frac{\sigma_{\rm dG}}{h_E}\langle[\theta_i - J_2\theta_i]_E, [\varphi_i]_E\rangle_E$. A combination of all the components of $\boldsymbol{\mathcal{J}}_{\sigma_{\rm dG}}$, and the Cauchy–Schwarz inequality imply

$$T_3 \lesssim |||\Theta_{\rm dG} - J_2\Theta_{\rm dG}|||_{\rm dG}|||\Phi_{\rm dG}|||_{\rm dG}.$$

The above displayed estimates for T_1, T_2, T_3 in (6.9) establishes **(A4)**.

Consequently, Theorem 2.1 verifies the discrete inf-sup condition with $1 \lesssim \beta_h$ for a sufficiently small choice of the maximal mesh-size h, and this completes the proof. $\square$

Theorem 6.5 (*A priori* error estimate)**.** *There exist $\delta > 0$ and a sufficiently large $\sigma_{\mathrm{dG}} > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \leq 1$ of (5.3) and the locally unique discrete solution $\Psi_{\mathrm{dG}} \in \mathbf{X}_{\mathrm{dG}}$ to (6.7) satisfy $\|\Psi - \Psi_{\mathrm{dG}}\|_{\mathrm{dG}} \lesssim h^\alpha$.*

Proof. We verify **(A5)**–**(A8)**, **(B1)**, **(B2)**, and the Lipschitz continuity of the Frechét derivative of $\mathscr{B} : \mathbf{X}_{\mathrm{dG}} \to \mathbf{X}_{\mathrm{dG}}^*$ on $\overline{B}_{\mathbf{X}_{\mathrm{dG}}}(I_{\mathrm{C}}\Psi, 2b)$ in next two steps to apply Theorem 3.1.

Step 1: Verification of **(A5)**–**(A8).** Choose $z_h := I_{\mathrm{C}}\Psi$ in **(A5)**. Lemmas 5.1 and 5.7(i) reveal $\|\Psi - I_{\mathrm{C}}\Psi\|_{\mathrm{dG}} \lesssim h^\alpha \|\Psi\|_{1+\alpha}$, and hence $\delta_3 \lesssim h^\alpha$. For $\Phi_{\mathrm{dG}} \in \mathbf{X}_{\mathrm{dG}}$ with $\|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}} = 1$, Lemmas 5.8(i) and 5.9(i) verify $\delta_4 \lesssim h$ in **(A6)**. Lemmas 5.4(d), and 5.9(i) imply **(A7)**. For **(A8)**, set $x_h := \boldsymbol{\eta}_{\mathrm{dG}}, \varphi_h := \Phi_{\mathrm{dG}} \in \mathbf{X}_{\mathrm{dG}}$. Use the definition of $A_{\mathrm{dG}}(\cdot,\cdot), F_{\mathrm{dG}}(\cdot)$, add and subtract $A_{\mathrm{pw}}(\boldsymbol{\eta}_{\mathrm{dG}}, J_2\Phi_{\mathrm{dG}})$, and then re-arrange the terms to treat the non-homogeneous Dirichlet boundary conditions as

$$\begin{aligned} A_{\mathrm{dG}}(\boldsymbol{\eta}_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) - A(J_1\boldsymbol{\eta}_{\mathrm{dG}}, J_2\Phi_{\mathrm{dG}}) - F_{\mathrm{dG}}(\Phi_{\mathrm{dG}}) &= \big(A_{\mathrm{pw}}(\boldsymbol{\eta}_{\mathrm{dG}}, \Phi_{\mathrm{dG}} - J_2\Phi_{\mathrm{dG}}) + \boldsymbol{\mathcal{J}}(\boldsymbol{\eta}_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) - F^{\mathcal{J}}_{\mathrm{dG}}(\Phi_{\mathrm{dG}})\big) \\ &+ A_{\mathrm{pw}}(\boldsymbol{\eta}_{\mathrm{dG}} - J_1\boldsymbol{\eta}_{\mathrm{dG}}, J_2\Phi_{\mathrm{dG}}) + \Big(\boldsymbol{\mathcal{J}}_{\sigma_{\mathrm{dG}}}(\boldsymbol{\eta}_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) - F^{\mathcal{J}_{\sigma_{\mathrm{dG}}}}_{\mathrm{dG}}(\Phi_{\mathrm{dG}})\Big) := T_4 + T_5 + T_6. \end{aligned}$$

For $\boldsymbol{\eta}_{\mathrm{dG}} := (\eta_1, \eta_2, \eta_3, \eta_4)$, $\Phi_{\mathrm{dG}} := (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbf{X}_{\mathrm{dG}}$, an integration by parts for the term $a_{\mathrm{pw}}(\cdot,\cdot)$, and then the steps similar to estimate T_4 in Theorem 6.4 and the definition of $F^{\mathcal{J}}_{\mathrm{dG}}(\cdot)$ in (6.8) yield

$$a_{\mathrm{pw}}(\eta_i, \varphi_i - J_2\varphi_i) + \mathcal{J}(\eta_i, \varphi_i) - F^{\mathcal{J}}_{\mathrm{dG}}(\varphi_i) = -\lambda \sum_{E\in\mathcal{E}} \langle \{\nabla_{\mathrm{pw}}\varphi_i \cdot \nu_E\}_E, [\eta_i]_E\rangle_E + \lambda \sum_{E\in\mathcal{E}(\partial\Omega)} \langle \nabla_{\mathrm{pw}}\varphi_i \cdot \nu_E, g_i\rangle_E.$$

with $1 \leq i \leq 4$. Therefore, for all $\boldsymbol{\chi} \in \mathbf{X}$ and $\boldsymbol{\chi} = \mathbf{g}$ on $\partial\Omega$, this leads to

$$T_4 \lesssim \|\boldsymbol{\chi} - \boldsymbol{\eta}_{\mathrm{dG}}\|_{\mathrm{dG}} \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}.$$

The estimate $T_5 := A_{\mathrm{pw}}(\boldsymbol{\eta}_{\mathrm{dG}} - J_1\boldsymbol{\eta}_{\mathrm{dG}}, J_2\Phi_{\mathrm{dG}}) \lesssim \|\boldsymbol{\eta}_{\mathrm{dG}} - \boldsymbol{\chi}\|_{\mathrm{dG}} \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}$ follows from Lemmas 5.4(c), 5.4(d) and 5.9(i). An analogous treatment as that of T_3 of Theorem 6.4, and a use of $[\boldsymbol{\chi}]_E = 0$ for all $E \in \mathcal{E}(\Omega)$, $[\boldsymbol{\chi}]_E = \mathbf{g}$ for all $E \in \mathcal{E}(\partial\Omega)$, in addition lead to

$$T_6 := \boldsymbol{\mathcal{J}}_{\sigma_{\mathrm{dG}}}(\boldsymbol{\eta}_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) - F^{\mathcal{J}_\sigma}_{\mathrm{dG}}(\Phi_{\mathrm{dG}}) \leq \|\boldsymbol{\eta}_{\mathrm{dG}} - \boldsymbol{\chi}\|_{\mathrm{dG}} \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}.$$

A combination of the estimates of T_4, T_5, T_6 completes the proof of **(A8)**.

Step 2: Verification of **(B1)**, **(B2).** For $\theta_h := \Theta_{\mathrm{dG}}, \varphi_h := \Phi_{\mathrm{dG}} \in \mathbf{X}_{\mathrm{dG}}$ with $\|\Theta_{\mathrm{dG}}\|_{\mathrm{dG}} = 1 = \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}$, Lemma 5.8(ii) with $z_h = I_{\mathrm{C}}\Psi$, Lemmas 5.9(i), and 5.1 lead to the desired estimates in **(B1)** and **(B2)**.

Step 3: Verification of Lipschitz continuity of $D\mathscr{B}(\cdot) \in L(\mathbf{X}_{\mathrm{dG}}, \mathbf{X}_{\mathrm{dG}}^*)$ in $\overline{B}_{\mathbf{X}_{\mathrm{dG}}}(I_{\mathrm{C}}\Psi, 2b)$**.** Let $\boldsymbol{\eta}_{\mathrm{dG}}, \boldsymbol{\chi}_{\mathrm{dG}} \in \overline{B}_{\mathbf{X}_{\mathrm{dG}}}(I_{\mathrm{C}}\Psi, 2b)$, and $\Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}} \in \mathbf{X}_{\mathrm{dG}}$. For this case, Lemmas 5.8(iv) and 5.9(i), the triangle inequalities analogous to (6.2) in the energy norm, and Lemma 5.1 lead to

$$\langle D\mathscr{B}(\boldsymbol{\eta}_{\mathrm{dG}})\Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}\rangle - \langle D\mathscr{B}(\boldsymbol{\chi}_{\mathrm{dG}})\Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}\rangle \leq L\|\boldsymbol{\eta}_{\mathrm{dG}} - \boldsymbol{\chi}_{\mathrm{dG}}\|_{\mathrm{dG}} \|\Theta_{\mathrm{dG}}\|_{\mathrm{dG}} \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}.$$

Therefore, Theorem 3.2 verifies the existence and local uniqueness of the discrete solution Ψ_{dG} to (6.7) with $\|\Psi - \Psi_{\mathrm{dG}}\|_{\mathrm{dG}} \leq \rho := \delta_3 + b + r$. The estimates $\delta_3 \lesssim h^\alpha$, $\delta_4 \lesssim h$, and (3.2) imply $b := \beta_1^{-1}((C_1^2\|\mathscr{A}\|(1 + \Lambda_2) + \widetilde{C}_A + \widetilde{C}_B)\delta_3 + \delta_4) \lesssim h^\alpha$. This and $b + r < 2b$ implies $\|\Psi - \Psi_{\mathrm{dG}}\|_{\mathrm{dG}} \lesssim h^\alpha$, and concludes the proof. $\square$

Remark 6.2 (Quasi best approximation result for $H^2(\Omega)$ regularity)**.** There exist $\delta > 0$ and a sufficiently large $\sigma_{\rm dG} > 0$ such that, any $\mathcal{T} \in \mathbb{T}(\delta)$, the regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^2(\Omega)$, and the discrete solution $\Psi_{\rm dG} \in \mathbf{X}_{\rm dG}$ to (6.7) satisfy

$$|||\Psi - \Psi_{\rm dG}|||_{\rm dG} \lesssim \min_{\Theta_{\rm dG} \in \mathbf{X}_{\rm dG}} |||\Psi - \Theta_{\rm dG}|||_{\rm dG},$$

where the constant in "$\lesssim$" is independent of h.

The proof of this quasi best approximation result follows the methodology utilized in Theorem 3.3 of [23], and entail modifications to deal with the quadratic non-linear terms that appear in the Euler–Lagrange PDE (5.2) for ferronematic system. The idea can be extended for the Nitsche's method. For conforming FEM, a result of this form has been established up to a data approximation term (see [25], Thm. 3.5). Note that, Maity *et al.* [23] deal with the non-linear PDEs associated to the LDG model for NLCs that does not involve any quadratic non-linear terms in the formulation.

Recall $\mathscr{B}(\cdot)$ from (6.3). Given the unique discrete solution to (6.7) with respect to the triangulation $\mathcal{T}$, the error estimators read

$$\vartheta_T^2 := h_T^2 |||\mathscr{B}(\Psi_{\rm dG})|||_{0,T}^2, \;\; \left(\vartheta_E^i\right)^2 := h_E |||[\nabla \Psi_{\rm dG} \nu_E]_E|||_{0,E}^2 + h_E^{-1} |||[\Psi_{\rm dG}]_E|||_{0,E}^2 \quad \text{for all } E \in \mathcal{E}(\Omega),$$
$$\left(\vartheta_E^\partial\right)^2 := h_E^{-1} |||\Psi_{\rm dG} - \mathbf{g}|||_{0,E}^2 \quad \text{for all } E \in \mathcal{E}(\partial\Omega), \;\; \text{and } \vartheta_{\rm dG}^2 := \sum_{T \in \mathcal{T}} \vartheta_T^2 + \sum_{E \in \mathcal{E}(\Omega)} \left(\vartheta_E^i\right)^2 + \sum_{E \in \mathcal{E}(\partial\Omega)} \left(\vartheta_E^\partial\right)^2. \tag{6.10}$$

Theorem 6.6 (*A posteriori* error estimate)**.** *Let Ψ be a regular solution of* (5.3) *and $\mathbf{g} \in \mathbf{C}^0(\partial\Omega)$ with $\mathbf{g}|_E \in \mathbf{H}^1(E)$ for all $E \in \mathcal{E}(\partial\Omega)$. There exist $\delta, R, C_{\rm rel}$, and $C_{\rm eff} > 0$ such that, any $\mathcal{T} \in \mathbb{T}(\delta)$ and the unique solution $\Psi_{\rm dG}$ to* (6.7) *with $|||\Psi - \Psi_{\rm dG}|||_{\rm dG} < R$ satisfy*

$$C_{\rm eff}^{-1} \vartheta_{\rm dG} \le |||\Psi - \Psi_{\rm dG}|||_{\rm dG} \le C_{\rm rel}(\vartheta_{\rm dG} + \text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega))).$$

*Here the approximation error "*data app$(\mathbf{g}, \mathcal{E}(\partial\Omega))$*" for the non-homogeneous Dirichlet boundary data $\mathbf{g}$ in the triangulation $\mathcal{T}$ is defined in* (5.7).

Proof. We apply Theorem 4.1, and hence it is enough to verify the conditions of this theorem. Set

$$\widehat{A} := A_{\rm pw}, \; \widehat{N}(\cdot;\cdot) := A_{\rm pw}(\cdot,\cdot) + B(\cdot,\cdot).$$

Verification of local Lipschitz continuity. For $\boldsymbol{\eta} \in B_{\widehat{X}}(\Psi, R_0)$ with $R_0 > 0$, $\Theta \in \widehat{X}$ and $\Phi \in \mathbf{V}$, Lemmas 5.8(iii) and 5.9(i) lead to

$$|||D\mathscr{B}(\boldsymbol{\eta}) - D\mathscr{B}(\Psi)|||_{L(\widehat{X}, \mathbf{V}^*)} \lesssim |||\boldsymbol{\eta} - \Psi|||_{\rm dG}(1 + R_0 + |||\Psi|||_1),$$

where the constant in "$\lesssim$" depends on $\ell, c, C_S, C_{\rm P}$. Therefore, $\gamma := \gamma_{\rm dG} \lesssim (1 + R_0 + |||\Psi|||_1)$ proves (4.1).
Verification of **(AP)**. Apply Lemma 5.3 with $\boldsymbol{\eta}_h := \Psi_{\rm dG}$, the discrete solution of (6.7), and set $\Psi_{\mathbf{g}} := \mathscr{G}\Psi_{\rm dG}$, to arrive at

$$|||\mathscr{G}\Psi_{\rm dG} - \Psi_{\rm dG}|||_{\rm dG}^2 \lesssim \sum_{E \in \mathcal{E}(\Omega)} h_E^{-1} |||[\Psi_{\rm dG}]_E|||_{0,E}^2 + \sum_{E \in \mathcal{E}(\partial\Omega)} h_E^{-1} |||\Psi_{\rm dG} - \mathbf{g}|||_{0,E}^2 + (\text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega)))^2,$$

where the constants in "$\lesssim$" depends on $C_{\mathscr{G}}$ and $\sigma_{\rm dG}$.

Conclusions of Theorem 4.1. Suppose that δ satisfy Theorem 6.5, and if necessary, are chosen smaller such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the unique discrete solution $\Psi_{\rm dG} \in \mathbf{X}_{\rm dG}$ to (6.7) satisfies $|||\Psi - \Psi_{\rm dG}|||_{\rm dG} < R < \min(R_0, \frac{\beta}{2\gamma_{\rm dG}})$. This applied to Theorem 4.1(i) establishes

$$|||\Psi - \Psi_{\rm dG}|||_{\rm dG} \lesssim \left\| \widehat{N}(\Psi_{\rm dG}) \right\|_{\mathbf{V}^*} + \left(1 + |||D\widehat{N}(\Psi)|||_{L(\widehat{X}, \mathbf{V}^*)}\right)$$

$$\times \left(\sum_{E\in\mathcal{E}(\Omega)} \left(\vartheta_E^i\right)^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} \left(\vartheta_E^\partial\right)^2 + (\text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega)))^2 \right)^{\frac{1}{2}}.$$

Since $\mathbf{V}$ is a Hilbert space, there exists a $\Phi \in \mathbf{V}$ with $\|\!|\Phi|\!\|_1 = 1$ such that

$$\left\|\!\left| \widehat{N}(\Psi_{\mathrm{dG}}) \right|\!\right\|_{\mathbf{V}^*} = \widehat{N}(\Psi_{\mathrm{dG}}; \Phi) = \widehat{N}(\Psi_{\mathrm{dG}}; \Phi - I_{\mathrm{SZ}}\Phi) + \widehat{N}(\Psi_{\mathrm{dG}}; I_{\mathrm{SZ}}\Phi). \tag{6.11}$$

The fact that $I_{\mathrm{SZ}}\Phi = 0$ on $\partial\Omega$, and the jump and average terms of $A_{\mathrm{dG}}(\cdot,\cdot)$ in the expansion of $N_{\mathrm{dG}}(\Psi_{\mathrm{dG}}; I_{\mathrm{SZ}}\Phi)$ lead to

$$\widehat{N}(\Psi_{\mathrm{dG}}; I_{\mathrm{SZ}}\Phi) = \lambda \sum_{E\in\mathcal{E}(\Omega)} \langle [\Psi_{\mathrm{dG}}]_E, \{\nabla(I_{\mathrm{SZ}}\Phi)\nu_E\}_E\rangle_E + \lambda \sum_{E\in\mathcal{E}(\partial\Omega)} \langle \Psi_{\mathrm{dG}} - \mathbf{g}, \nabla(I_{\mathrm{SZ}}\Phi)\nu_E\rangle_E. \tag{6.12}$$

We follow the approach of applying integration by parts for $A_{\mathrm{pw}}(\Psi_{\mathrm{dG}}, \Phi - I_{\mathrm{SZ}}\Phi)$, and cancellation of terms as in Step 3 of Theorem 6.3 to attain

$$\begin{aligned} \widehat{N}(\Psi_{\mathrm{dG}}; \Phi - I_{\mathrm{SZ}}\Phi) &= A_{\mathrm{pw}}(\Psi_{\mathrm{dG}}, \Phi - I_{\mathrm{SZ}}\Phi) + B(\Psi_{\mathrm{dG}}, \Phi - I_{\mathrm{SZ}}\Phi) \\ &= \sum_{T\in\mathcal{T}} \int_T \boldsymbol{\eta}_T \cdot (\Phi - I_{\mathrm{SZ}}\Phi)\,\mathrm{d}x + \sum_{E\in\mathcal{E}(\Omega)} \langle \boldsymbol{\eta}_E, \Phi - I_{\mathrm{SZ}}\Phi\rangle_E, \end{aligned} \tag{6.13}$$

with $\boldsymbol{\eta}_T := (\mathscr{B}(\Psi_{\mathrm{dG}}))|_T$ in $T \in \mathcal{T}$, and $\boldsymbol{\eta}_E := [\nabla\Psi_{\mathrm{dG}}\nu_E]_E$, where E is an edge of T. The Cauchy–Schwarz inequality, Lemmas 5.7(ii) and 5.2 plus $\|\!|\Phi|\!\|_1 = 1$ applied to (6.12), (6.13) lead to the estimate of $\|\!|\widehat{N}(\Psi_{\mathrm{dG}})|\!\|_{\mathbf{V}^*}$ in (6.11). The bound

$$\sum_{E\in\mathcal{E}(\Omega)} h_E^{-1} \|\!|[\Psi_{\mathrm{dG}}]_E|\!\|_{0,E}^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-1} \|\!|\Psi_{\mathrm{dG}} - \mathbf{g}|\!\|_{0,E}^2 \le \|\!|\Psi - \Psi_{\mathrm{dG}}|\!\|_{\mathrm{dG}}^2$$

and standard local efficiency estimates conclude the proof of the efficiency estimate. □

6.3. Nitsche's method

The Nitsche's formulation that corresponds to (5.3) seeks $\Psi_{\mathrm{N}} \in \mathbf{X}_{\mathrm{C}}$ such that for all $\Phi_{\mathrm{N}} \in \mathbf{X}_{\mathrm{C}}$,

$$A_{\mathrm{N}}(\Psi_{\mathrm{N}}, \Phi_{\mathrm{N}}) + B(\Psi_{\mathrm{N}}, \Phi_{\mathrm{N}}) - F_{\mathrm{N}}(\Phi_{\mathrm{N}}) = 0, \tag{6.14}$$

where for $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$, $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbf{X}_{\mathrm{C}}$,

$$A_{\mathrm{N}}(\Theta, \Phi) := A(\Theta, \Phi) + \boldsymbol{\mathcal{J}}(\Theta, \Phi) + \boldsymbol{\mathcal{J}}_\sigma(\Theta, \Phi), \quad \text{and } F_{\mathrm{N}}(\Phi) := F_{\mathrm{N}}^{\mathcal{J}}(\Phi) + F_{\mathrm{N}}^{\mathcal{J}_\sigma}(\Phi). \tag{6.15}$$

Here for $\theta, \varphi \in X_{\mathrm{C}}$, and the unit outward normal ν associated to $\partial\Omega$,

$$\boldsymbol{\mathcal{J}}(\Theta, \Phi) := \sum_{i=1}^4 \mathcal{J}(\theta_i, \varphi_i) \text{ with } \mathcal{J}(\theta, \varphi) := -\langle \nabla\theta\cdot\nu, \varphi\rangle_{\partial\Omega} - \langle \theta, \nabla\varphi\cdot\nu\rangle_{\partial\Omega},$$

$$\boldsymbol{\mathcal{J}}_\sigma(\Theta, \Phi) := \sum_{i=1}^4 \mathcal{J}_\sigma(\theta_i, \varphi_i) \text{ with } \mathcal{J}_\sigma(\theta, \varphi) := \sum_{E\in\mathcal{E}(\partial\Omega)} \frac{\sigma}{h_E}\langle \theta, \varphi\rangle_E$$

$$\text{For } 1 \le i \le 4,\ F_{\mathrm{N}}^{\mathcal{J}}(\Phi) := \sum_{i=1}^4 F_{\mathrm{N}}^{i,\mathcal{J}}(\varphi_i) \text{ with } F_{\mathrm{N}}^{i,\mathcal{J}}(\varphi) := -\langle g_i, \nabla\varphi\cdot\nu\rangle_{\partial\Omega},$$

$$F_{\rm N}^{\mathcal{J}_\sigma}(\Phi) := \sum_{i=1}^{4} F_{\rm N}^{i,\mathcal{J}_\sigma}(\varphi_i) \text{ with } F_{\rm N}^{i,\mathcal{J}_\sigma}(\varphi) := \sum_{E\in\mathcal{E}(\partial\Omega)} \frac{\sigma}{h_E}\langle g_i, \varphi\rangle_E.$$

The discrete space is equipped with the mesh-dependent product norm $\|\!|\cdot|\!\|_{\rm N}$. The details of the discrete spaces, norms and the choices of operators are presented in Table 1. For a sufficiently large choice of σ, the inf-sup condition in (2.4) follows from Section 3.4 of [30]. Lemmas 5.8(iv) and 5.9(ii) establish the following boundedness results in discrete norm for $\Xi, \boldsymbol{\eta}, \Theta, \Phi \in \mathbf{X}$,

$$B_1(\Theta,\Phi) \lesssim \|\!|\Theta|\!\|_{\rm N}\|\!|\Phi|\!\|_{\rm N}, B_2(\boldsymbol{\eta},\Theta,\Phi) \lesssim \|\!|\boldsymbol{\eta}|\!\|_{\rm N}\|\!|\Theta|\!\|_{\rm N}\|\!|\Phi|\!\|_{\rm N}, B_3(\Xi,\boldsymbol{\eta},\Theta,\Phi) \lesssim \|\!|\Xi|\!\|_{\rm N}\|\!|\boldsymbol{\eta}|\!\|_{\rm N}\|\!|\Theta|\!\|_{\rm N}\|\!|\Phi|\!\|_{\rm N}.$$

Next we state the results on the discrete inf-sup condition, *a priori* and *a posteriori* error estimates for Nitsche's method. The proofs apply Theorems 2.1, 3.2, and 4.1, respectively. The verification of the hypotheses to apply the aforementioned theorems follows analogous to Theorems 6.4, 6.5, and 6.6, respectively. Though the analysis for Nitsche's method is simpler due to the conforming discrete space, some of the techniques in the proof for the dG scheme needs to be blended as the treatment of the boundary conditions are similar to that of the dG scheme. (See Tab. 1 for details of spaces and operators.) The proof of Theorem 6.7 below is presented in Appendix D for brevity.

Recall $\mathscr{B}(\cdot)$ from (6.3). Given the unique discrete solution (6.14) with respect to the triangulation $\mathcal{T}$, the error estimators read

$$\vartheta_T^2 := h_T^2\|\!|\mathscr{B}(\Psi_{\rm N})|\!\|_{0,T}^2, \;\; (\vartheta_E^i)^2 := h_E\|\!|[\nabla\Psi_{\rm N}\nu_E]_E|\!\|_{0,E}^2 \quad \text{for all } E\in\mathcal{E}(\Omega), \tag{6.16}$$
$$\text{and } \left(\vartheta_E^\partial\right)^2 := h_E^{-1}\|\!|\Psi_{\rm N} - \mathbf{g}|\!\|_{0,E}^2 \quad \text{for all } E\in\mathcal{E}(\partial\Omega), \;\; \text{and } \vartheta_{\rm N}^2 := \sum_{T\in\mathcal{T}}\vartheta_T^2 + \sum_{E\in\mathcal{E}(\Omega)}\left(\vartheta_E^i\right)^2 + \sum_{E\in\mathcal{E}(\partial\Omega)}\left(\vartheta_E^\partial\right)^2.$$

Theorem 6.7 (Stability and error control)**.**

(a) *(Discrete inf-sup condition). There exist $\delta > 0$ and a sufficiently large $\sigma > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$ and a regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \le 1$ of (5.3), (2.7) holds for $V_h = \mathbf{X}_{\rm C}$ and $\langle DN_h(\Psi)\Theta_{\rm N}, \Phi_{\rm N}\rangle := A_{\rm N}(\Theta_{\rm N}, \Phi_{\rm N}) + B_L(\Theta_{\rm N}, \Phi_{\rm N})$.*

(b) *(A priori error estimate). There exist $\delta > 0$ and a sufficiently large $\sigma > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \le 1$ of (5.3) and the locally unique discrete solution $\Psi_{\rm N} \in \mathbf{X}_{\rm C}$ to (6.14) satisfy $\|\!|\Psi - \Psi_{\rm N}|\!\|_{\rm N} \lesssim h^\alpha$.*

(c) *(A posteriori error estimate). Let Ψ be a regular solution of (5.3) with $\mathbf{g} \in \mathbf{C}^0(\partial\Omega)$ and $\mathbf{g}|_E \in \mathbf{H}^1(E)$ for all $E \in \mathcal{E}(\partial\Omega)$. There exist $\delta, R, C_{\rm rel}$, and $C_{\rm eff} > 0$ such that, any $\mathcal{T} \in \mathbb{T}(\delta)$ and the unique solution $\Psi_{\rm N}$ to (6.14) with $\|\!|\Psi - \Psi_{\rm N}|\!\|_{\rm N} < R$ satisfy*

$$C_{\rm eff}^{-1}\vartheta_{\rm N} \le \|\!|\Psi - \Psi_{\rm N}|\!\|_{\rm N} \le C_{\rm rel}(\vartheta_{\rm N} + \text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega))).$$

Note that the approximation error "data app$(\mathbf{g}, \mathcal{E}(\partial\Omega))$" for the non-homogeneous Dirichlet boundary data $\mathbf{g}$ in the triangulation $\mathcal{T}$ is defined in (5.7).

7. Weakly over penalized symmetric interior penalty method

The mesh dependent norm $\|\cdot\|_{\rm P}$ (see Lem. 5.9) for the WOPSIP scheme is not equivalent to $\|\cdot\|_h$, and **(A4)**, **(A7)**, **(A8)** do not hold for this case. However, the proofs of the discrete inf-sup condition and the *a priori* error control hold with minor modifications in the corresponding proofs of the abstract framework, and are established in Theorems 7.1 and 7.2.

See Table 1 for the discrete spaces and the choices of operators. The discrete space $\mathbf{X}_{\rm P} := (X_{\rm P})^4$ with $X_{\rm P} := P_1(\mathcal{T})$ for the WOPSIP method is equipped with $\|\!|\cdot|\!\|_{\rm P}$. The discrete formulation seeks $\Psi_{\rm P} \in \mathbf{X}_{\rm P}$ such that for all $\Phi_{\rm P} \in \mathbf{X}_{\rm P}$,

$$A_{\rm P}(\Psi_{\rm P}, \Phi_{\rm P}) + B_{\rm N}(\Psi_{\rm P}, \Phi_{\rm P}) - F_{\rm P}(\Phi_{\rm P}) = 0, \tag{7.1}$$

where for $\Theta_{\rm P} := (\theta_1, \theta_2, \theta_3, \theta_4), \Phi_{\rm P} := (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbf{X}_{\rm P}$, $1 \leq i \leq 4$,

$$A_{\rm P}(\Theta_{\rm P}, \Phi_{\rm P}) := A_{\rm pw}(\Theta_{\rm P}, \Phi_{\rm P}) + \boldsymbol{\mathcal{J}}_{\sigma_{\rm P}}(\Theta_{\rm P}, \Phi_{\rm P}) \text{ with } \boldsymbol{\mathcal{J}}_{\sigma_{\rm P}}(\Theta_{\rm P}, \Phi_{\rm P}) := \sum_{i=1}^{4} \mathcal{J}_{\sigma_{\rm P}}(\theta_i, \varphi_i), \text{ and for } \theta, \varphi \in X_{\rm P}, \quad (7.2)$$

$$\mathcal{J}_{\sigma_{\rm P}}(\theta, \varphi) := \sum_{E\in\mathcal{E}} \frac{\sigma_{\rm P}}{h_E^2} \big(\Pi_E^0[\theta]_E\big)\big(\Pi_E^0[\varphi]_E\big),\; F_{\rm P}(\Phi_{\rm P}) := \sum_{i=1}^{4} F_{\rm P}^{i,\mathcal{J}_{\sigma_{\rm P}}}(\varphi_i) \text{ with}$$
$$F_{\rm P}^{i,\mathcal{J}_{\sigma_{\rm P}}}(\varphi) := \sum_{E\in\mathcal{E}(\partial\Omega)} \frac{\sigma_{\rm P}}{h_E^2} \big(\Pi_E^0 g_i\big)\big(\Pi_E^0 \varphi\big). \quad (7.3)$$

The coercivity $\|\!|\Phi_{\rm P}|\!\|_{\rm P}^2 = A_{\rm P}(\Phi_{\rm P}, \Phi_{\rm P})$ for all $\Phi_{\rm P} \in \mathbf{X}_{\rm P}$ follows from the definition of $\|\!| \cdot |\!\|_{\rm P}$. For $\boldsymbol{\chi}, \boldsymbol{\eta}, \Theta, \Phi \in \mathbf{H}^1(\mathcal{T})$, Lemmas 5.8(iv) and 5.9(iii) lead to

$$B_1(\Theta, \Phi) \lesssim \|\!|\Theta|\!\|_{\rm P} \|\!|\Phi|\!\|_{\rm P}, B_2(\boldsymbol{\eta}, \Theta, \Phi) \lesssim \|\!|\boldsymbol{\eta}|\!\|_{\rm P}\|\!|\Theta|\!\|_{\rm P}\|\!|\Phi|\!\|_{\rm P}, B_3(\boldsymbol{\chi}, \boldsymbol{\eta}, \Theta, \Phi) \lesssim \|\!|\boldsymbol{\chi}|\!\|_{\rm P}\|\!|\boldsymbol{\eta}|\!\|_{\rm P}\|\!|\Theta|\!\|_{\rm P}\|\!|\Phi|\!\|_{\rm P}.$$

The *a priori* error estimate, for this case, utilizes the Crouzeix–Raviart interpolation operator and its properties are stated below.

Lemma 7.1 (Crouzeix–Raviart interpolation [9])**.** *The CR interpolation operator $I_{\rm CR} : X \to P_1(\mathcal{T})$ is defined by $I_{\rm CR}\xi(\mathrm{mid}(E)) = h_E^{-1} \int_E \xi \,\mathrm{d}s$ for all $E \in \mathcal{E}$. For all $\xi \in H^{1+\alpha}(\Omega)$ with $0 < \alpha \leq 1$, it holds that*

$$\|I_{\rm CR}\xi - \xi\|_0 + h|I_{\rm CR}\xi - \xi|_{H^1(\mathcal{T})} \lesssim h^{1+\alpha}\|\xi\|_{1+\alpha}.$$

Remark 7.1. For $\xi \in H^{1+\alpha}(\Omega)$ with $0 < \alpha \leq 1$, $[\xi]_E = 0$, $\int_E [I_{\rm CR}\xi]_E \,\mathrm{d}s = h_E[I_{\rm CR}\xi]_E(\mathrm{mid}E) = 0$ for all $E \in \mathcal{E}(\Omega)$, and the identity $\int_E I_{\rm CR}\xi \,\mathrm{d}s = \int_E \xi \,\mathrm{d}s$ for all $E \in \mathcal{E}(\partial\Omega)$ imply $\|\!|\xi - I_{\rm CR}\xi|\!\|_{\rm P} = \|\!|\xi - I_{\rm CR}\xi|\!\|_{\rm pw} \lesssim h^\alpha|\xi|_{H^{1+\alpha}(\Omega)}$.

Theorem 7.1 (Discrete inf-sup condition)**.** *There exist $\delta > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$ and a regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \leq 1$ of (5.3), (2.7) holds for $V_h = \mathbf{X}_{\rm P}$ and $\langle DN_h(\Psi)\Theta_{\rm P}, \Phi_{\rm P}\rangle := A_{\rm P}(\Theta_{\rm P}, \Phi_{\rm P}) + B_L(\Theta_{\rm P}, \Phi_{\rm P})$.*

Proof. Let $\Theta_{\rm P}, \Phi_{\rm P} \in \mathbf{X}_{\rm P}$ with $\|\!|\Theta_{\rm P}|\!\|_{\rm P} = \|\!|\Phi_{\rm P}|\!\|_{\rm P} = 1$. Remarks 6.1 and 7.1 lead to $\|\!|\zeta - I_{\rm CR}\zeta|\!\|_{\rm P} \lesssim h^\alpha$, and this implies $\delta_1 \lesssim h^\alpha$ in **(A1)**. Lemmas 5.8(i) and 5.9(iii) lead to $\delta_2 \lesssim h$ in **(A2)**. For $\Phi \in \mathbf{V}$, $[J_2\Phi_{\rm P}]_E = 0 = [\Phi]_E$ for all $E \in \mathcal{E}$, the definition of $\|\!| \cdot |\!\|_{\rm P}$ at the first step, and then Lemmas 5.4(d), 5.9(iii) prove **(A3)** as

$$\|\!|(1 - J_2)\Phi_{\rm P}|\!\|_{\rm P}^2 = |(1 - J_2)\Phi_{\rm P}|_{H^1(\mathcal{T})}^2 + \sum_{E\in\mathcal{E}} h_E^{-2}\big(\Pi_E^0[\Phi_{\rm P} - \Phi]_E\big)^2 \lesssim \|\!|\Phi_{\rm P} - \Phi|\!\|_{\rm P}^2.$$

For $\Theta_{\rm P} = (\theta_1, \theta_2, \theta_3, \theta_4), \Phi_{\rm P} = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbf{X}_{\rm P}$, (7.2), and an algebraic manipulation lead to

$$A_{\rm P}(\Theta_{\rm P}, \Phi_{\rm P}) - A(J_2\Theta_{\rm P}, J_2\Phi_{\rm P}) = A_{\rm pw}(\Theta_{\rm P}, \Phi_{\rm P} - J_2\Phi_{\rm P}) + \boldsymbol{\mathcal{J}}_{\sigma_{\rm P}}(\Theta_{\rm P}, \Phi_{\rm P}) + A_{\rm pw}(\Theta_{\rm P} - J_2\Theta_{\rm P}, J_2\Phi_{\rm P}).$$

Apply integration by parts, and then utilize $\Delta(\Theta_{\rm P}|_T) = 0$ on each $T \in \mathcal{T}$ to obtain

$$T_1 := a_{\rm pw}(\theta_i, \varphi_i - J_2\varphi_i) = \sum_{E\in\mathcal{E}} \langle\{\nabla_{\rm pw}\theta_i \cdot \nu_E\}_E, [\varphi_i - J_2\varphi_i]_E\rangle_E + \sum_{E\in\mathcal{E}(\Omega)} \langle[\nabla_{\rm pw}\theta_i \cdot \nu_E]_E, \{\varphi_i - J_2\varphi_i\}_E\rangle_E.$$

Employ $[J_2\varphi_i]_E = 0$ for all $E \in \mathcal{E}$, definition of Π_E^0, the Cauchy–Schwarz inequality, and Lemmas 5.7(ii), 5.4(b) to obtain

$$T_1 \leq \left(\sum_{E\in\mathcal{E}} h_E \left\|\!\left|\{\nabla_{\mathrm{pw}}\theta_i \cdot \nu_E\}_E\right\|\!\right|_{0,E}^2\right)^{\frac{1}{2}} \left(\sum_{E\in\mathcal{E}} h_E^{-1} \left\|\!\left|\Pi_E^0[\varphi_i]_E\right\|\!\right|_{0,E}^2\right)^{\frac{1}{2}} \lesssim h\|\theta_i\|_{\mathrm{P}}\|\varphi_i\|_{\mathrm{P}}.$$

A combination of all the components lead to $A_{\mathrm{pw}}(\Theta_{\mathrm{P}}, \Phi_{\mathrm{P}} - J_2\Phi_{\mathrm{P}}) \lesssim h\|\!|\Theta_{\mathrm{P}}\|\!|_{\mathrm{P}}\|\!|\Phi_{\mathrm{P}}\|\!|_{\mathrm{P}}$. The estimates of the other two terms in the expansion of $A_{\mathrm{P}}(\Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}) - A(J_2\Theta_{\mathrm{P}}, J_2\Phi_{\mathrm{P}})$ follow the steps in Theorem 6.4, and hence we skip the details. In this case, we obtain

$$A_{\mathrm{P}}(\Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}) - A(J_2\Theta_{\mathrm{P}}, J_2\Phi_{\mathrm{P}}) \lesssim (\|\!|\Theta_{\mathrm{P}} - J_2\Theta_{\mathrm{P}}\|\!|_{\mathrm{P}} + h\|\!|\Theta_{\mathrm{P}}\|\!|_{\mathrm{P}})\|\!|\Phi_{\mathrm{P}}\|\!|_{\mathrm{P}}. \tag{7.4}$$

Observe that the estimate of **(A4)** above for WOPSIP method contains an extra term, which can be controlled by the smallness of the term, and hence allow us to apply the abstract Theorem 2.1. This concludes the proof. ☐

Theorem 7.2 (*A priori* error estimate)**.** *There exists $\delta > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \leq 1$ of (5.3) and the locally unique discrete solution $\Psi_{\mathrm{P}} \in \mathbf{X}_{\mathrm{P}}$ to (7.1) satisfy $\|\!|\Psi - \Psi_{\mathrm{P}}\|\!|_{\mathrm{P}} \lesssim h^\alpha$.*

Remark 7.2. The proof of Theorem 7.2 follows the proof of abstract Theorem 3.2 with some modifications in **(A7)** and **(A8)**. For WOPSIP method, we establish the smallness of $A(Pz_h - u, Q\varphi_h)$ term in (3.7) of Theorem 3.2 instead of verifying **(A7)**. Besides, for **(A8)**, we obtain

$$T := A_{\mathrm{P}}(\boldsymbol{\eta}_{\mathrm{P}}, \Phi_{\mathrm{P}}) - A(J_1\boldsymbol{\eta}_{\mathrm{P}}, J_2\Phi_{\mathrm{P}}) - F_{\mathrm{P}}(\Phi_{\mathrm{P}}) \lesssim (\|\!|\boldsymbol{\eta}_{\mathrm{P}} - \boldsymbol{\chi}\|\!|_{\mathrm{P}} + h\|\!|\boldsymbol{\eta}_{\mathrm{P}}\|\!|_{\mathrm{P}})\|\!|\Phi_{\mathrm{P}}\|\!|_{\mathrm{P}},\ \boldsymbol{\eta}_{\mathrm{P}}, \Phi_{\mathrm{P}} \in \mathbf{X}_{\mathrm{P}}, \boldsymbol{\chi} \in \mathcal{X}.$$

The additional term in the bound of T for WOPSIP is small, and this ensures that the proof of Theorem 7.2 follows the steps of Theorem 3.2 with minor modifications.

Proof of Theorem 7.2. Remark 7.1 applied for $\xi := \Psi$ leads to the choice of $\delta_3 \lesssim h^\alpha$ in **(A5)**. For $\Phi_{\mathrm{P}} \in \mathbf{X}_{\mathrm{P}}$ with $\|\!|\Phi_{\mathrm{P}}\|\!|_{\mathrm{P}} = 1$, Lemmas 5.8(i) and 5.9(iii) lead to $\delta_4 \lesssim h$ in **(A6)**. The Cauchy–Schwarz inequality plus a triangle inequality $\|\!|J_1(I_{\mathrm{CR}}\Psi) - \Psi\|\!|_{\mathrm{pw}} \leq \|\!|(J_1 - 1)I_{\mathrm{CR}}\Psi\|\!|_{\mathrm{pw}} + \|\!|I_{\mathrm{CR}}\Psi - \Psi\|\!|_{\mathrm{pw}}$, Lemmas 5.4(c), 5.4(d), 5.9(iii) and 7.1 imply

$$A(J_1(I_{\mathrm{CR}}\Psi) - \Psi, J_2\Phi_{\mathrm{P}}) \leq \Big(\|\!|(J_1 - 1)I_{\mathrm{CR}}\Psi\|\!|_{\mathrm{pw}} + \|\!|I_{\mathrm{CR}}\Psi - \Psi\|\!|_{\mathrm{pw}}\Big)\|\!|J_2\Phi_{\mathrm{P}}\|\!|_{\mathrm{pw}} \lesssim h^\alpha \|\!|\Phi_{\mathrm{P}}\|\!|_{\mathrm{P}}.$$

The definitions of $A_{\mathrm{P}}(\cdot,\cdot)$, $F_{\mathrm{P}}(\cdot)$, an addition and subtraction of $A_{\mathrm{pw}}(\boldsymbol{\eta}_{\mathrm{P}}, J_2\Phi_{\mathrm{P}})$, and a re-grouping of terms yield

$$\begin{aligned} A_{\mathrm{P}}(\boldsymbol{\eta}_{\mathrm{P}}, \Phi_{\mathrm{P}}) - A(J_1\boldsymbol{\eta}_{\mathrm{P}}, J_2\Phi_{\mathrm{P}}) - F_{\mathrm{P}}(\Phi_{\mathrm{P}}) &= A_{\mathrm{pw}}(\boldsymbol{\eta}_{\mathrm{P}}, \Phi_{\mathrm{P}} - J_2\Phi_{\mathrm{P}}) + A_{\mathrm{pw}}(\boldsymbol{\eta}_{\mathrm{P}} - J_1\boldsymbol{\eta}_{\mathrm{P}}, J_2\Phi_{\mathrm{P}}) \\ &\quad + (\boldsymbol{\mathcal{J}}_{\sigma_{\mathrm{P}}}(\boldsymbol{\eta}_{\mathrm{P}}, \Phi_{\mathrm{P}}) - F_{\mathrm{P}}(\Phi_{\mathrm{P}})). \end{aligned}$$

We follow the steps as for the estimation of T_1 in Theorem 7.1 to obtain $A_{\mathrm{pw}}(\boldsymbol{\eta}_{\mathrm{P}}, \Phi_{\mathrm{P}} - J_2\Phi_{\mathrm{P}}) \lesssim h\|\!|\boldsymbol{\eta}_{\mathrm{P}}\|\!|_{\mathrm{P}}\|\!|\Phi_{\mathrm{P}}\|\!|_{\mathrm{P}}$. Lemmas 5.4(c), 5.4(d) and 5.9(iii) with $\boldsymbol{\chi} \in \mathbf{X}$ and $\boldsymbol{\chi} = \mathbf{g}$ on $\partial\Omega$ lead to $A_{\mathrm{pw}}(\boldsymbol{\eta}_{\mathrm{P}} - J_1\boldsymbol{\eta}_{\mathrm{P}}, J_2\Phi_{\mathrm{P}}) \lesssim \|\!|\boldsymbol{\eta}_{\mathrm{P}} - \boldsymbol{\chi}\|\!|_{\mathrm{P}}\|\!|\Phi_{\mathrm{P}}\|\!|_{\mathrm{P}}$. For the estimate of the term $\boldsymbol{\mathcal{J}}_{\sigma_{\mathrm{P}}}(\boldsymbol{\eta}_{\mathrm{P}}, \Phi_{\mathrm{P}}) - F_{\mathrm{P}}(\Phi_{\mathrm{P}})$, proceed with the steps utilized for T_6 of Theorem 6.5.

The proofs of **(B1)**, **(B2)**, and the local Lipschitz continuity of $D\mathscr{B}(\cdot)$ on $\overline{B}_{\mathbf{X}_{\mathrm{P}}}(I_{\mathrm{CR}}\Psi, 2b)$ utilize Lemma 5.8(ii), (iv) with $z_h := I_{\mathrm{CR}}\Psi$, and then apply Lemma 5.9(iii) and Remark 7.1. The rest of the proof adopts the techniques utilized in Theorem 6.5, and is skipped here. ☐

Recall the definition of $\mathscr{B}(\cdot)$ from (6.3). Given the unique discrete solution (7.1) with respect to the triangulation $\mathcal{T}$, the error estimators read

$$\vartheta_T^2 := h_T^2 \|\mathscr{B}(\Psi_{\mathrm{P}})\|_{0,T}^2, \ \left(\vartheta_E^{\partial}\right)^2 := h_E^{-2}\left|\Pi_E^0(\Psi_{\mathrm{P}} - \mathbf{g})\right|^2 + h_E^{-1}\|\Psi_{\mathrm{P}} - \mathbf{g}\|_{0,E}^2 \quad \text{for all } E \in \mathcal{E}(\partial\Omega),$$
$$\left(\vartheta_E^{i}\right)^2 := h_E \|[\nabla \Psi_{\mathrm{P}} \nu_E]_E\|_{0,E}^2 + h_E^{-2}\left|\Pi_E^0[\Psi_{\mathrm{P}}]_E\right|^2 + h_E^{-1}\|[\Psi_{\mathrm{P}}]_E\|_{0,E}^2 \quad \text{for all } E \in \mathcal{E}(\Omega),$$
$$\text{and } \vartheta_{\mathrm{P}}^2 := \sum_{T\in\mathcal{T}} \vartheta_T^2 + \sum_{E\in\mathcal{E}(\Omega)} \left(\vartheta_E^{i}\right)^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} \left(\vartheta_E^{\partial}\right)^2. \tag{7.5}$$

Theorem 7.3 (*A posteriori* error estimate)**.** *Let* Ψ *be a regular solution of* (5.3) *with* $\mathbf{g} \in \mathbf{C}^0(\partial\Omega)$ *and* $\mathbf{g}|_E \in \mathbf{H}^1(E)$ *for all* $E \in \mathcal{E}(\partial\Omega)$*. There exist* $\delta, R, C_{\mathrm{rel}}$*, and* $C_{\mathrm{eff}} > 0$ *such that, any* $\mathcal{T} \in \mathbb{T}(\delta)$ *and the unique solution* Ψ_{P} *to* (7.1) *with* $\|\Psi - \Psi_{\mathrm{P}}\|_{\mathrm{P}} < R$ *satisfy*

$$C_{\mathrm{eff}}^{-1} \vartheta_{\mathrm{P}} \le \|\Psi - \Psi_{\mathrm{P}}\|_{\mathrm{P}} \le C_{\mathrm{rel}}(\vartheta_{\mathrm{P}} + \text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega))).$$

*Here the approximation error "*data app$(\mathbf{g}, \mathcal{E}(\partial\Omega))$*" for the non-homogeneous Dirichlet boundary data* $\mathbf{g}$ *in the triangulation* $\mathcal{T}$ *is defined in* (5.7)*.*

Proof. For $\boldsymbol{\eta} \in B_{\widehat{X}}(\Psi, R_0)$ with $R_0 > 0$, Lemmas 5.8(iii) and 5.9(iii) lead to $\|D\mathscr{B}(\boldsymbol{\eta}) - D\mathscr{B}(\Psi)\|_{L(\widehat{X},\mathbf{V}^*)} \lesssim \|\boldsymbol{\eta} - \Psi\|_{\mathrm{P}}(1 + R_0 + \|\Psi\|_1)$, where the constant in "$\lesssim$" depends on $\ell, c, C_S, C_{\mathrm{P}}$ and the constant from Lemma 5.9(iii). This proves the locally Lipschitz continuity of $D\mathscr{B}(\cdot)$ at $\Psi \in \widehat{X}$ for the choice of $\gamma := \gamma_{\mathrm{P}} \lesssim (1 + R_0 + \|\Psi\|_1)$ in (4.1). Next we focus on **(AP)**. Apply Lemma 5.3 with $\boldsymbol{\eta}_h := \Psi_{\mathrm{P}}$, for the discrete solution of (7.1), and set $\Psi_{\mathbf{g}} := \mathscr{G}\Psi_{\mathrm{P}}$, to arrive at

$$\|\mathscr{G}\Psi_{\mathrm{P}} - \Psi_{\mathrm{P}}\|_{\mathrm{pw}}^2 \lesssim \sum_{E\in\mathcal{E}(\Omega)} h_E^{-1}\|[\Psi_{\mathrm{P}}]_E\|_{0,E}^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-1}\|\Psi_{\mathrm{P}} - \mathbf{g}\|_{0,E}^2 + (\text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega)))^2,$$
$$\sum_{E\in\mathcal{E}} h_E^{-2}\Pi_E^0([\mathscr{G}\Psi_{\mathrm{P}} - \Psi_{\mathrm{P}}]_E)^2 \lesssim \sum_{E\in\mathcal{E}(\Omega)} h_E^{-2}\left|\Pi_E^0([\Psi_{\mathrm{P}}]_E)\right|^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-2}\left|\Pi_E^0(\mathbf{g} - \Psi_{\mathrm{P}})\right|^2.$$

Therefore, we obtain $\|\mathscr{G}\Psi_{\mathrm{P}} - \Psi_{\mathrm{P}}\|_{\mathrm{P}}^2 \lesssim \sum_{E\in\mathcal{E}(\Omega)}(\vartheta_E^i)^2 + \sum_{E\in\mathcal{E}(\partial\Omega)}(\vartheta_E^{\partial})^2 + (\text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega)))^2$, where the constants in "$\lesssim$" depend on $C_{\mathscr{G}}$ and σ_{P}. Suppose that δ satisfies Theorem 7.2, and if necessary, is chosen smaller such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the unique discrete solution $\Psi_{\mathrm{P}} \in \mathbf{X}_{\mathrm{P}}$ to (7.1) satisfies $\|\Psi - \Psi_{\mathrm{P}}\|_{\mathrm{P}} < R \le \min(R_0, \frac{\beta}{2\gamma_{\mathrm{P}}})$. This applied to the abstract residual in Theorem 4.1(i) reveals

$$\|\Psi - \Psi_{\mathrm{P}}\|_{\mathrm{P}} \lesssim \left\|\widehat{N}(\Psi_{\mathrm{P}})\right\|_{\mathbf{V}^*} + \left(1 + \|D\widehat{N}(\Psi)\|_{L(\widehat{X},\mathbf{V}^*)}\right)$$
$$\times \left(\sum_{E\in\mathcal{E}(\Omega)} \left(\vartheta_E^i\right)^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} \left(\vartheta_E^{\partial}\right)^2 + (\text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega)))^2\right)^{\frac{1}{2}}.$$

Since $\mathbf{V}$ is a Hilbert space, there exists a $\Phi \in \mathbf{V}$ with $\|\Phi\|_1 = 1$ such that

$$\left\|\widehat{N}(\Psi_{\mathrm{P}})\right\|_{\mathbf{V}^*} = \widehat{N}(\Psi_{\mathrm{P}}; \Phi) = \widehat{N}(\Psi_{\mathrm{P}}; \Phi - I_{\mathrm{SZ}}\Phi) + \widehat{N}(\Psi_{\mathrm{P}}; I_{\mathrm{SZ}}\Phi).$$

Since $[I_{\mathrm{SZ}}\Phi]_E = 0$ for all $E \in \mathcal{E}$, $\boldsymbol{\mathcal{J}}_{\sigma_{\mathrm{P}}}(\Psi_{\mathrm{P}}, I_{\mathrm{SZ}}\Phi) = 0$, $F_{\mathrm{P}}(I_{\mathrm{SZ}}\Phi) = 0$. Therefore, $\widehat{N}(\Psi_{\mathrm{P}}; I_{\mathrm{SZ}}\Phi) = 0$. To estimate the term $\widehat{N}(\Psi_{\mathrm{P}}; \Phi - I_{\mathrm{SZ}}\Phi)$, follow the technique employed for the first term on the right hand side of (6.11) with the local terms $\boldsymbol{\eta}_T := (\mathscr{B}(\Psi_{\mathrm{P}}))|_T$ in $T \in \mathcal{T}$, and $\boldsymbol{\eta}_E := [\nabla\Psi_{\mathrm{P}}\nu_E]_E$ for $E \in \mathcal{E}(\Omega)$.

Lemma 5.9(iii), $[\Psi]_E = 0$ for all $E \in \mathcal{E}(\Omega)$ and $\Psi = \mathbf{g}$ on $\partial\Omega$ imply

$$\sum_{E\in\mathcal{E}(\Omega)} \left(h_E^{-1}\|[\Psi_{\mathrm{P}}]_E\|_{0,E}^2 + h_E^{-2}\left|\Pi_E^0[\Psi_{\mathrm{P}}]_E\right|^2\right) + \sum_{E\in\mathcal{E}(\partial\Omega)} \left(h_E^{-1}\|\Psi_{\mathrm{P}} - \mathbf{g}\|_{0,E}^2 + h_E^{-2}\left|\Pi_E^0(\Psi_{\mathrm{P}} - \mathbf{g})\right|^2\right) \lesssim \|\Psi - \Psi_{\mathrm{P}}\|_{\mathrm{P}}^2.$$

The above bound and standard local efficiency estimates conclude the proof of the efficiency bound. □

TABLE 2. Tangential boundary conditions for solution components Q_{11}, Q_{12}, M_1, M_2.

Solution	$x = 0$	$x = 1$	$y = 0$	$y = 1$
Q_{11}	-1	-1	1	1
Q_{12}	0	0	0	0
M_1	0	0	-1	1
M_2	1	-1	0	0

8. Numerical experiments

This section reports on two numerical examples for the problem (5.3) that validate the theoretical estimates.

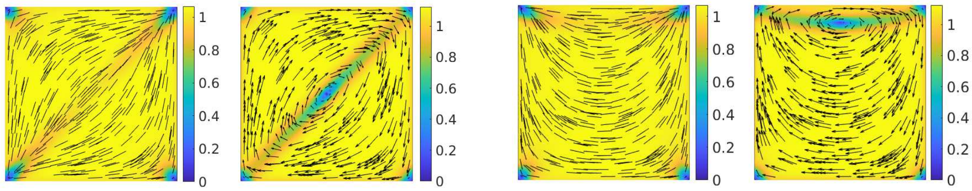
8.1. Benchmark example for ferronematic system on square domain

Consider the benchmark example of dilute suspensions of magnetic nanoparticles in a nematic host within the square domain $\Omega := [0,1] \times [0,1]$. The Dirichlet boundary condition $\mathbf{g}$ is a Lipschitz continuous function constructed using trapezoidal shape function and is compatible with the tangent boundary conditions (see Tab. 2) for the nematic director ($\mathbf{n}$) and the magnetization vector ($\mathbf{M}$) (see [25]). Tangent boundary conditions means the liquid crystal molecules in contact with the well surfaces are constrained to be in the plane of the surfaces, and are commonly used for confined NLC systems both experimentally and theoretically. Consequently, the nematic director/ magnetic field has to be tangent to the square edges. This leads to natural mismatch at the four vertices of square domain and are described in Table 2. The discontinuities at the vertices are circumvented by defining the Dirichlet boundary condition $\mathbf{g}$ using trapizoidal shape function ($T_d(\cdot)$) following [13, 25] with the parameter $d = 3\sqrt{\ell}$, where we choose $0 < \ell \ll 1$ sufficiently small so that the qualitative solution profiles are preserved. The boundary condition is given by

$$\mathbf{g} = \begin{cases} (T_d(x), 0, -T_d(x), 0) & \text{on } y = 0, \\ (T_d(x), 0, T_d(x), 0) & \text{on } y = 1, \\ (-T_d(y), 0, 0, T_d(y)) & \text{on } x = 0, \\ (-T_d(y), 0, 0, -T_d(y)) & \text{on } x = 1, \end{cases} \quad \text{and} \quad T_d(t) = \begin{cases} t/d, & 0 \le t \le d, \\ 1, & d \le t \le 1-d, \\ (1-t)/d, & 1-d \le t \le 1, \end{cases}$$

and the parameter "d" is referred as the size of mismatch region. Note that this benchmark example has been studied extensively for conforming FEM in [13, 25]. The discrete solution landscapes, numerical errors and convergence rates in energy and $\mathbf{L}^2$ norms for various values of the parameters ℓ, c are discussed therein. In this paper, we present the results on computational errors and convergence rates for the Nitsche and dG schemes.

Recall the diagonal and rotated solution landscapes for the uncoupled system ($c = 0$) from [25]. The nematic directors align along the diagonals of the square domain leading to D1, D2 diagonal solutions, whereas for the stable rotated solutions, the in-plane director $\mathbf{n}$ rotates by π radians between a pair of parallel square edges leading to R1, R2, R3, R4 rotated solutions. The director is the leading eigenvector of the $\mathbf{Q}$-tensor with the largest positive eigenvalue, and describes the preferred in-plane alignment of the nematic molecules. The associated ferronematic solutions for positive coupling ($c > 0$) are denoted by $(\mathbf{Q}_s, \mathbf{M}_s)$, where the suffix s refers to diagonal or rotated states for the nematic directors. The negative coupling enhances the multistability of the solutions and leads to two distinct magnetic profiles for each diagonal or rotated nematic stable state. In this case the ferronematic solutions are denoted by $(\mathbf{Q}_s, \mathbf{M}_s^i)$, $i = 1, 2$. In the nematic (resp. magnetic) profiles, the black lines (resp. arrows) represent the nematic director $\mathbf{n} = (\cos\theta, \sin\theta)$ (resp. magnetic field $\mathbf{M}$), where θ is the director angle in the plane, and the color bar represents the value of the scalar order parameter $s = \sqrt{Q_{11}^2 + Q_{12}^2}$ (resp. $|\mathbf{M}| = \sqrt{M_1^2 + M_2^2}$). Figure 1 (resp. Fig. 2) displays the discrete solutions $\Psi_{\mathrm{N}} := (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$ and $\Psi_{\mathrm{N}} := (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ (resp. $\Psi_{\mathrm{N}} := (\mathbf{Q}_{D1}, \mathbf{M}_{D1}^1)$, $\Psi_{\mathrm{N}} := (\mathbf{Q}_{D1}, \mathbf{M}_{D1}^2)$ and $\Psi_{\mathrm{N}} := (\mathbf{Q}_{R4}, \mathbf{M}_{R4}^1)$, $\Psi_{\mathrm{N}} := (\mathbf{Q}_{R4}, \mathbf{M}_{R4}^1)$), respectively for the parameter values $\ell = 0.001$, $c = 0.25$ (resp. $c = -0.25$). The convergence rates obtained in



(a) (b)

FIGURE 1. Discrete solution profiles $\Psi_{\mathrm{N}} := (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$ and $\Psi_{\mathrm{N}} := (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ for $\ell = 0.001$, $c = 0.25$. (a) $\mathbf{Q}_{D1}$ and $\mathbf{M}_{D1}$ profile. (b) $\mathbf{Q}_{R4}$ and $\mathbf{M}_{R4}$ profile.

energy and $\mathbf{L}^2$ norms are $O(h)$ and $O(h^2)$, respectively, for both positive ($c > 0$) and negative coupling ($c < 0$). Numerical errors and order of convergences in energy and $\mathbf{L}^2$ norms for the discrete solutions in the Nitsche and dG schemes are presented in Tables E.1 and E.2. The penalty parameter values have been chosen as 10 for numerical computations in both the schemes. Similar numerical results and convergence histories hold for the stable nematic states: D2, R1, R2 and R3.

Adaptive mesh-refinement

This ferronematic example is studied with adaptive mesh refinement for conforming, Nitsche, and dG schemes in this section. We briefly describe the adaptive algorithm.

Given an initial triangulation $\mathcal{T}_0$, run the steps **SOLVE**, **ESTIMATE**, **MARK** and **REFINE** described below successively for different levels $l = 0, 1, 2, \dots$

SOLVE. Compute the solution $\Psi_l := \Psi_{\mathrm{C}}$ (resp. $\Psi_l := \Psi_{\mathrm{dG}}$ and $\Psi_l := \Psi_{\mathrm{N}}$) of the discrete problem (6.1) (resp. (6.7) and (6.14)) for the triangulation $\mathcal{T}_l$.

ESTIMATE. Calculate the error indicator $\Xi_{T,l} := \left(\vartheta_T^2 + \sum_{E \in \partial T \cap \mathcal{E}(\Omega)} (\vartheta_E^i)^2 + \sum_{E \in \partial T \cap \mathcal{E}(\partial\Omega)} (\vartheta_E^{\partial})^2\right)^{\frac{1}{2}}$ for each element $T \in \mathcal{T}_l$. Recall the volume and edge estimators for conforming (resp. dG schemes and Nitsche's method) given by (6.5) (resp. (6.10) and (6.16)).

MARK. For the next level of refinement, choose the elements $T \in \mathcal{T}_l$ using Dörfler marking [33] such that $0.3 \sum_{T \in \mathcal{T}_l} \Xi_{T,l}^2 \leq \sum_{T \in \tilde{\mathcal{T}}} \Xi_{T,l}^2$ and collect those elements to construct a subset $\tilde{\mathcal{T}} \subset \mathcal{T}_l$.

REFINE. Compute the closure of $\tilde{\mathcal{T}}$ and use newest vertex bisection [32] refinement strategy to construct the new triangulation $\mathcal{T}_{l+1}$.

Figures 3a, 3b and 3c, 3d (resp. Figs. 4a, 4b and 4c, 4d) plot the adaptive mesh refinements near the defect points (four corners) and the domain wall (in magnetic profile) for the discrete solutions $\Psi_{\mathrm{C}} := (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$ and $\Psi_{\mathrm{C}} := (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ (resp. $\Psi_{\mathrm{C}} := (\mathbf{Q}_{D1}, \mathbf{M}_{D1}^1)$ and $\Psi_{\mathrm{C}} := (\mathbf{Q}_{R4}, \mathbf{M}_{R4}^1)$), respectively, for the conforming scheme in (6.1). A defect or domain wall is a region of low order (small values of $|\mathbf{Q}|$ or $|\mathbf{M}|$), and defects are a consequence of the tangent boundary conditions, and the topologically non-trivial boundary conditions for $\mathbf{M}$. The domain walls regularise any jumps or discontinuities in the eigenvectors of $\mathbf{Q}$ and/or the directions of $\mathbf{M}$, so that the solution has finite Dirichlet energy. For $n \in \mathbb{N}$, the adaptive refinements at the n-th level of triangulation are denoted by $\mathcal{T}_n$. The convergence history of the estimator ϑ_{C} defined in (6.5) is presented in Tables E.3 and E.4 for the discrete solutions displayed in Figures 1 and 2. Moreover, to bring out a comparison between both uniform and adaptive refinements, we present the numerical results for the estimators ϑ_{C} (resp. ϑ_{N}, ϑ_{dG}, and ϑ_{P} defined in (6.16), (6.10), and (7.5)) for both uniform refinements and some selected level of adaptive refinements with respect to the number of degrees of freedom in Table E.5 (resp. Tabs. E.6, E.7, and E.8) for those discrete solutions. The notation N in the tables represent the number of mesh points in a triangulation.

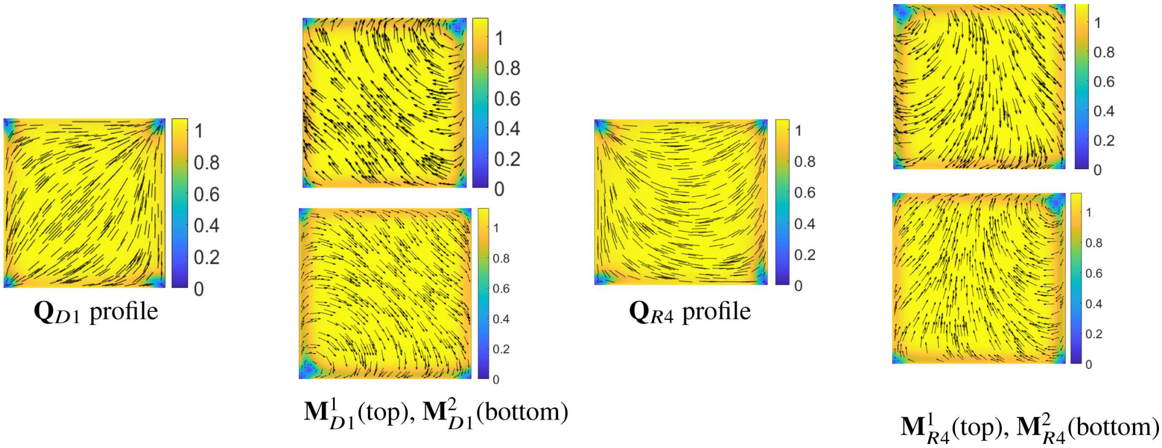


FIGURE 2. Discrete solution profiles $\Psi_{\mathrm{N}} := (\mathbf{Q}_{D1}, \mathbf{M}^i_{D1})$, and $\Psi_{\mathrm{N}} := (\mathbf{Q}_{R4}, \mathbf{M}^i_{R4}), i = 1, 2$, for $\ell = 0.001$, $c = -0.25$.

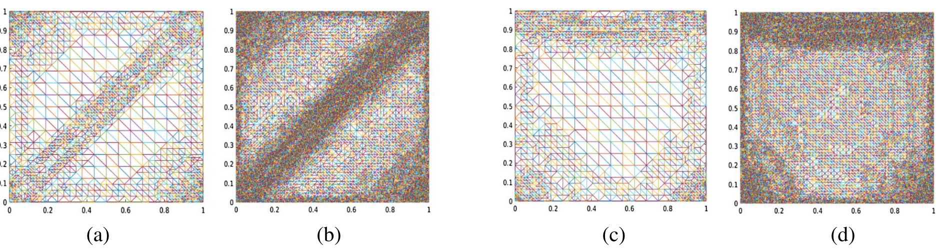


FIGURE 3. Adaptive mesh refinements in triangulations $\mathcal{T}_7$ and $\mathcal{T}_{13}$ for the discrete solution profiles $\Psi_{\mathrm{C}} := (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$ and $\Psi_{\mathrm{C}} := (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ for $\ell = 0.001$, $c = 0.25$. (a) $\mathcal{T}_7$. (b) $\mathcal{T}_{13}$. (c) $\mathcal{T}_7$. (d) $\mathcal{T}_{13}$.

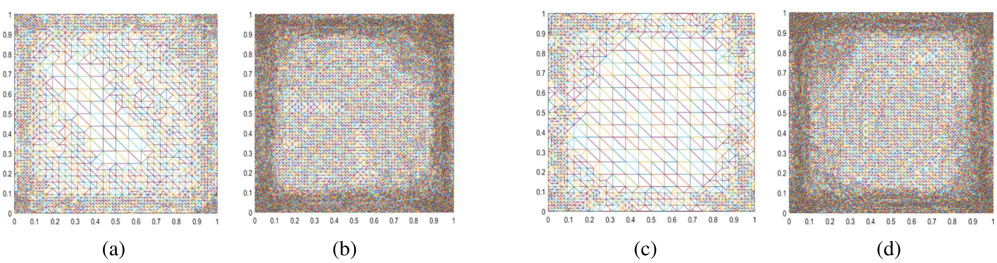


FIGURE 4. Adaptive mesh refinements in triangulations $\mathcal{T}_7$, and $\mathcal{T}_{13}$ for the discrete solution profiles $\Psi_{\mathrm{C}} := (\mathbf{Q}_{D1}, \mathbf{M}^1_{D1})$ and $\Psi_{\mathrm{C}} := (\mathbf{Q}_{R4}, \mathbf{M}^1_{R4})$ for $\ell = 0.001$, $c = -0.25$. (a) $\mathcal{T}_7$. (b) $\mathcal{T}_{13}$. (c) $\mathcal{T}_7$. (d) $\mathcal{T}_{13}$.

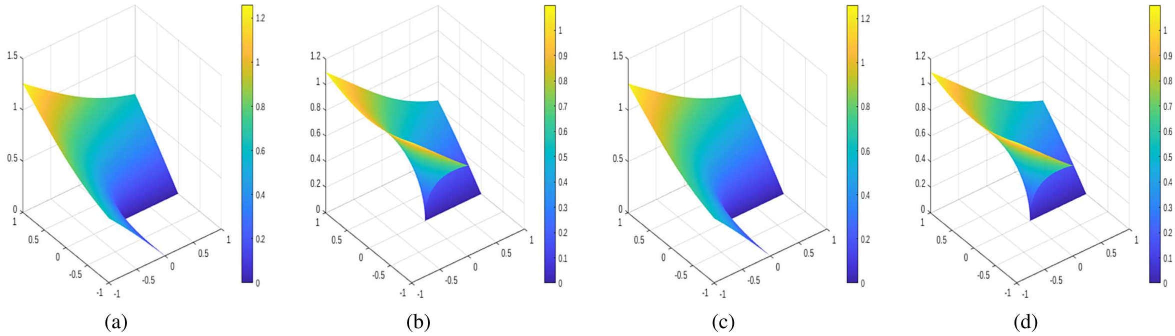


FIGURE 5. Solution profiles for conforming FEM for the parameter values $\ell = 1, c = 0.25$. (a) Q_{11}. (b) Q_{12}. (c) M_1. (d) M_2.

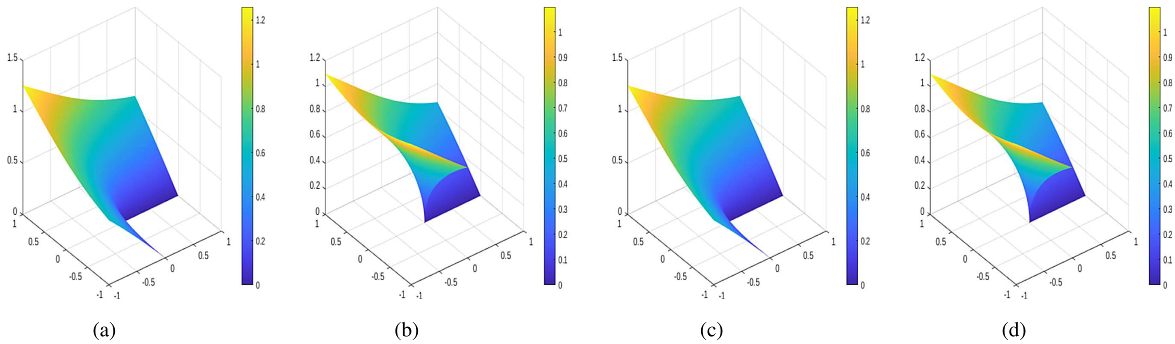


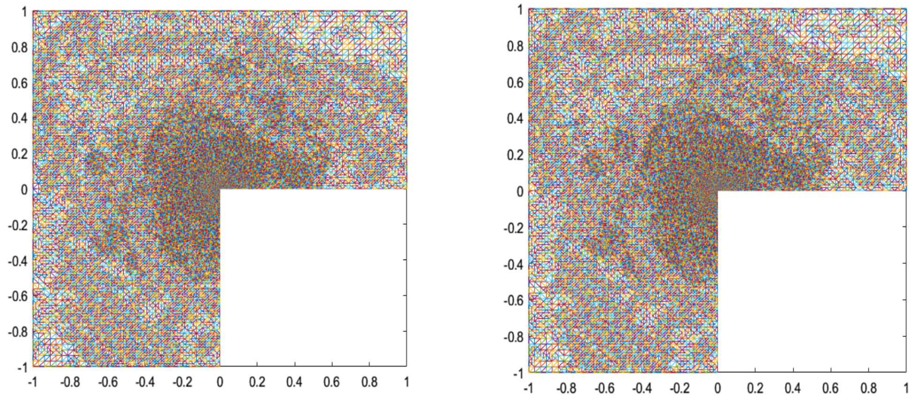
FIGURE 6. Solution profiles for conforming FEM for the parameter values $\ell = 1, c = -0.25$. (a) Q_{11}. (b) Q_{12}. (c) M_1. (d) M_2.

The number of degrees of freedom in each triangulation is $4 \times N$ for conforming FEM and Nitsche's method, and $4 \times (3 \times \text{number of triangles in } \mathcal{T})$ for dG and WOPSIP schemes.

Next we focus on two more numerical examples to illustrate the practical performances of the error indicators in adaptive mesh refinement for conforming, Nitsche, dG, and WOPSIP schemes.

8.2. Example on a L-shaped domain with zero load

Consider the PDE (5.3) in a L-shaped domain, $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$ with zero load function and the Dirichlet boundary condition $\mathbf{g} = (g_1, g_2, g_3, g_4)$ given by $g_1 = r^{2/3}\sin(2\theta/3)$, $g_2 = r^{1/2}\sin(\theta/2)$, $g_3 = r^{2/3}\sin(2\theta/3)$, $g_4 = r^{1/2}\sin(\theta/2)$ on $\partial\Omega$. Figures 5 and 6 display the solution profiles for the parameter values $\ell = 1$, $c = \pm 0.25$ for conforming FEM. Tables E.9–E.11 present the numerical errors and convergence rates obtained in energy and $\mathbf{L}^2$ norms for the approximation of discrete solutions for conforming, Nitsche and dG schemes. The convergence rates obtained in energy and $\mathbf{L}^2$ norms are $O(h^{1/2})$ and $O(h)$, respectively. The convergence history of the estimators for both uniform and adaptive refinements are presented in Tables E.12, E.13, E.14, E.15 for conforming, Nitsche, dG, and WOPSIP schemes, respectively. The convergence rate of the estimator is sub-optimal in uniform refinements. The empirical improved convergence rate 0.5 of the estimator is observed in adaptive refinements.



(a) (b)

FIGURE 7. Adaptive mesh refinements at 19th level for conforming FEM for the parameter values (a) $\ell = 1, c = 0.25$, (b) $\ell = 1, c = -0.25$.

8.3. Example on a L-shaped domain

Consider the PDE (5.3) in a L-shaped domain, $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$ with Dirichlet boundary condition, and calculate the load function $\mathbf{f} := N(\Psi)$ for the manufactured solution $\Psi := (u_1, u_2, u_3, u_4)$ with $u_1 = r^{2/3}\sin(2\theta/3)$, $u_2 = r^{1/2}\sin(\theta/2)$, $u_3 = r^{2/3}\sin(2\theta/3)$, $u_4 = r^{1/2}\sin(\theta/2)$. In this case, the volume estimators are modified as $\vartheta_T^2 := h_T^2 \|\mathbf{f} - \mathscr{B}(\Psi_{\rm C})\|_{0,T}^2$, $\vartheta_T^2 := h_T^2 \|\mathbf{f} - \mathscr{B}(\Psi_{\rm N})\|_{0,T}^2$, $\vartheta_T^2 := h_T^2 \|\mathbf{f} - \mathscr{B}(\Psi_{\rm dG})\|_{0,T}^2$, and $\vartheta_T^2 := h_T^2 \|\mathbf{f} - \mathscr{B}(\Psi_{\rm P})\|_{0,T}^2$ for conforming, Nitsche, dG, and WOPSIP schemes, respectively, to account the non-zero load function $\mathbf{f}$. Figure 7 displays the adaptive mesh-refinement near the singularity at origin for $\ell = 1$ and both positive and negative coupling $c = \pm 0.25$ for conforming FEM. The theoretically expected convergence rate 0.25 is obtained in energy norm for uniform refinement for parameter values $\ell = 1, c = \pm 0.25$, and is calculated with respect to the number of degrees of freedom. This convergence rate is sub-optimal due to the corner singularity. The adaptive mesh-refinement leads to the empirical optimal convergence rate 0.5. The numerical results for conforming, Nitsche, dG and WOPSIP schemes are displayed in Appendix E.5 (see Tab. E.16 for the conforming, Tab. E.17 for the Nitsche, and Tab. E.18 for the dG, and Tab. E.19 for the WOPSIP schemes).

Remark 8.1 (Comparison among the different discretizations)**.** In dG and WOPSIP discrete schemes, the basis functions are discontinuous, and consequently leads to high number of degrees of freedom in computations. However, dG and WOPSIP schemes add flexibility in global assembly, are parallelizable, and also helps in handling complicated geometries. The over-penalization in the WOPSIP discretization increases the condition number of the resulting discrete system. An appropriate preconditioner and parallelization are more suitable for this scheme. In terms of the theoretical order of convergences, all the schemes (conforming, Nitsche, dG, and WOPSIP) are comparable. The nonconforming schemes are also attractive and will be studied in another work with details of adaptive convergence.

9. Application to LDG model for nematic liquid crystals

In this section, the framework in Sections 2–4 is applied to the reduced LDG model for nematic liquid crystals discussed in [23, 24]. In the absence of surface energy and external field, the dimensionless reduced LDG free energy [22] is given by

$$\mathcal{E}(\Psi) = \int_\Omega \left(|\nabla \Psi|^2 + \epsilon^{-2} \left(|\Psi|^2 - 1 \right)^2 \right) \mathrm{d}x,$$

where $\Psi = \mathbf{g}$ on $\partial\Omega$. Here $\Psi := (Q_{11}, Q_{12})$ denotes the components of the Q tensor, $\mathbf{g} : \partial\Omega \to \mathbb{R}^2$ is the given non-homogeneous Dirichlet boundary condition, and ϵ is a small positive parameter that depends on the elastic constant, bulk energy parameters and the size of the domain. Recall that we study the model problem in a polygonal bounded domain Ω with Lipschitz boundary $\partial\Omega$. The corresponding Euler–Lagrange equation for a critical point Ψ of the reduced LDG free energy is [22]

$$-\Delta \Psi = 2\epsilon^{-2} \left(1 - |\Psi|^2 \right) \Psi \quad \text{ in } \Omega \text{ with } \Psi = \mathbf{g} \quad \text{ on } \partial\Omega.$$

The analysis for this model follows immediately for all the discrete schemes discussed in this article by setting $c = 0$ and appropriately modifying the spaces, the bilinear form and the nonlinear terms in the ferronematic model. The modifications are indicated below.

In the abstract theory, choose $X := \mathbf{X} = (H^1(\Omega))^2, V := \mathbf{V} = (H^1_0(\Omega))^2$, and $X(g) := \mathcal{X} := \{\mathbf{w} \in \mathbf{X} | \ \mathbf{w} = \mathbf{g} \text{ on } \partial\Omega\}$. The weak formulation of the energy minimizing PDE for the reduced LDG model for NLCs seeks $\Psi \in \mathcal{X}$ such that

$$N(\Psi; \Phi) := A(\Psi, \Phi) + B(\Psi, \Phi) = 0 \qquad \text{for all } \Phi \in \mathbf{V}, \tag{9.1}$$

where $B(\Psi, \Phi) := B_1(\Psi, \Phi) + B_3(\Psi, \Psi, \Psi, \Phi)$. For all $\Xi = (\xi_1, \xi_2), \boldsymbol{\eta} = (\eta_1, \eta_2), \Theta = (\theta_1, \theta_2), \Phi = (\varphi_1, \varphi_2) \in \mathbf{X}$, the bilinear and quadrilinear forms given by

$$A(\Theta, \Phi) := \sum_{i=1}^{2} \int_\Omega \nabla\theta_i \cdot \nabla\varphi_i \, \mathrm{d}x, \ \ B_1(\Theta, \Phi) := -2\epsilon^{-2} \sum_{i=1}^{2} \int_\Omega \theta_i \varphi_i \, \mathrm{d}x,$$
$$B_3(\Xi, \boldsymbol{\eta}, \Theta, \Phi) := \frac{2\epsilon^{-2}}{3} \int_\Omega ((\Xi \cdot \boldsymbol{\eta})(\Theta \cdot \Phi) + (\Xi \cdot \Theta)(\boldsymbol{\eta} \cdot \Phi) + (\boldsymbol{\eta} \cdot \Theta)(\Xi \cdot \Phi)) \, \mathrm{d}x.$$

The boundedness of $A : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$, its coercivity in $\mathbf{V} \times \mathbf{V}$, the boundedness of the bilinear form $B_1 : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ and the quadrilinear form $B_3 : \mathbf{X} \times \mathbf{X} \times \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ are discussed in [23]. Set $\mathbf{H}^{1+\alpha}(\Omega) := (H^{1+\alpha}(\Omega))^2$. We approximate the regular solutions $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega)$, $0 < \alpha \le 1$, of (9.1). The discrete formulation that corresponds to (9.1) seeks $\Psi_h \in X_h$ (with $\Psi_h = \mathbf{g}_{\mathrm{C}}$ on $\partial\Omega$ for conforming FEM) such that for all $\Phi_h \in V_h$,

$$N_h(\Psi_h; \Phi_h) := A_h(\Psi_h, \Phi_h) + B(\Psi_h, \Psi_h, \Phi_h) - F_h(\Phi_h) = 0, \tag{9.2}$$

where the discrete forms $A_h(\cdot, \cdot)$, and $F_h(\cdot)$ are defined in Table 3 for each scheme. Here $\mathbf{g}_{\mathrm{C}}$ is the Lagrange P_1 interpolation of $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}+\alpha}(\partial\Omega)$ with $0 < \alpha \le 1$. Set $\mathbf{C}^0(\partial\Omega) := (C^0(\partial\Omega))^2$ and $\mathbf{H}^1(E) := (H^1(E))^2$.

Theorem 9.1 (Error estimates)**.**

(i) *(A priori). There exist $\delta > 0$ (and a sufficiently large $\sigma_{\mathrm{dG}} > 0$ and $\sigma > 0$ for dGFEM and Nitsche scheme, resp.) such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the regular solution $\Psi \in \mathcal{X} \cap \mathbf{H}^{1+\alpha}(\Omega), 0 < \alpha \le 1$ to (9.1) and the unique discrete solution $\Psi_h \in X_h$ to (9.2) satisfy $\|\Psi - \Psi_h\|_{X_h} \lesssim h^\alpha$, where*

$$\Psi_h := \begin{cases} \Psi_{\mathrm{C}} & \text{for the CFEM and } \Psi_{\mathrm{C}} = \mathbf{g}_{\mathrm{C}} \ \text{ on } \partial\Omega; \\ \Psi_{\mathrm{dG}} & \text{for the dGFEM;} \\ \Psi_{\mathrm{N}} & \text{for the Nitsche's method;} \\ \Psi_{\mathrm{P}} & \text{for the WOPSIP method.} \end{cases}$$

Table 3. Overview of notation of discrete spaces and discrete forms for LDG model for NLCs.

Schemes	Spaces	Notation: For $\Theta := (\theta_1,\theta_2), \Phi := (\varphi_1,\varphi_2) \in (P_1(\mathcal{T}))^2$		
		Energy norm ($\|\cdot\|_{X_h}$)	$A_h(\Theta,\Phi)$	$F_h(\Phi)$
CFEM	$X_C := P_1(\mathcal{T}) \cap H^1(\Omega)$, $V_C := P_1(\mathcal{T}) \cap H^1_0(\Omega)$, $X_h := (X_C)^2$, $V_h := (V_C)^2$	$\sum_{i=1}^2 \|\varphi_i\|_1^2$	$A(\Theta,\Phi)$	0
dGFEM	$X_h := V_h := (P_1(\mathcal{T}))^2$	$\sum_{i=1}^2 \|\varphi_i\|_{\mathrm{dG}}^2$	$\sum_{i=1}^2 (a_{\mathrm{pw}}(\theta_i,\varphi_i) + \mathcal{J}(\theta_i,\varphi_i) + \mathcal{J}_{\sigma_{\mathrm{dG}}}(\theta_i,\varphi_i))$	$\sum_{i=1}^2 \left(F_{\mathrm{dG}}^{i,\mathcal{J}}(\varphi_i) + F_{\mathrm{dG}}^{i,\mathcal{J}_{\sigma_{\mathrm{dG}}}}(\varphi_i)\right)$
Nitsche	$X_h := V_h := (X_C)^2$	$\sum_{i=1}^2 \|\varphi_i\|_{\mathrm{N}}^2$	$\sum_{i=1}^2 (a(\theta_i,\varphi_i) + \mathcal{J}(\theta_i,\varphi_i) + \mathcal{J}_{\sigma}(\theta_i,\varphi_i))$	$\sum_{i=1}^2 (F_{\mathrm{N}}^{i,\mathcal{J}}(\varphi_i) + F_h^{i,\mathcal{J}_\sigma}(\varphi_i))$
WOPSIP	$X_h := V_h := (P_1(\mathcal{T}))^2$	$\sum_{i=1}^2 \|\varphi_i\|_{\mathrm{P}}^2$	$\sum_{i=1}^2 (a_{\mathrm{pw}}(\theta_i,\varphi_i) + \mathcal{J}_{\sigma_{\mathrm{P}}}(\theta_i,\varphi_i))$	$\sum_{i=1}^2 F_{\mathrm{P}}^{i,\mathcal{J}_{\sigma_{\mathrm{P}}}}(\varphi_i)$

(ii) *(A posteriori). Let* Ψ *be a regular solution to* (9.1) *with* $\mathbf{g} \in \mathbf{C}^0(\partial\Omega)$ *and* $\mathbf{g}|_E \in \mathbf{H}^1(E)$ *for all* $E \in \mathcal{E}(\partial\Omega)$. *There exist* $\delta, R, C_{\mathrm{rel}}$, *and* $C_{\mathrm{eff}} > 0$ *such that any* $\mathcal{T} \in \mathbb{T}(\delta)$, *and the unique solution* Ψ_h *to* (9.2) *with* $\|\Psi - \Psi_h\|_{X_h} < R$ *satisfy*

$$C_{\mathrm{eff}}^{-2}\vartheta^2 \le \|\Psi - \Psi_h\|_{X_h}^2 \le C_{\mathrm{rel}}^2\Big(\vartheta^2 + \mathrm{data\ app}(\mathbf{g}, \mathcal{E}(\partial\Omega))^2\Big),$$

where $\vartheta^2 := \vartheta_T^2 + \vartheta_E^2$ *with* $\vartheta_T^2 := \sum_{T\in\mathcal{T}} h_T^2 \|\!|2\epsilon^{-2}(|\Psi_h|^2 - 1)\Psi_h\|\!|_{0,T}^2$ *and*

$$\vartheta_E^2 := \begin{cases} \sum_{E\in\mathcal{E}(\Omega)} h_E \|\!|[\nabla\Psi_{\mathrm{C}}\nu_E]_E\|\!|_{0,E}^2 & \text{for CFEM;} \\ \sum_{E\in\mathcal{E}(\Omega)} \Big(h_E\|\!|[\nabla\Psi_{\mathrm{dG}}\nu_E]_E\|\!|_{0,E}^2 + h_E^{-1}\|\!|[\Psi_{\mathrm{dG}}]_E\|\!|_{0,E}^2\Big) & \\ \quad + \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-1}\|\!|\Psi_{\mathrm{dG}} - \mathbf{g}\|\!|_{0,E}^2 & \text{for dGFEM;} \\ \sum_{E\in\mathcal{E}(\Omega)} h_E\|\!|[\nabla\Psi_{\mathrm{N}}\nu_E]_E\|\!|_{0,E}^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-1}\|\!|\Psi_{\mathrm{N}} - \mathbf{g}\|\!|_{0,E}^2 & \text{for Nitsche's method;} \\ \sum_{E\in\mathcal{E}(\Omega)} \Big(h_E\|\!|[\nabla\Psi_{\mathrm{P}}\nu_E]_E\|\!|_{0,E}^2 + h_E^{-1}\|\!|[\Psi_{\mathrm{P}}]_E\|\!|_{0,E}^2 + h_E^{-2}\big|\Pi_E^0[\Psi_{\mathrm{P}}]_E\big|^2\Big) & \\ \quad + \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-2}\big|\Pi_E^0(\Psi_{\mathrm{P}} - \mathbf{g})\big|^2 + h_E^{-1}\|\!|\Psi_{\mathrm{P}} - \mathbf{g}\|\!|_{0,E}^2 & \text{for WOPSIP method.} \end{cases}$$

Remark 9.1. The constants C_{rel} and C_{eff} in Theorem 9.1 will vary for different methods owing to the different penalty parameters, norm definitions, and the associated local efficiency results (see [24], Lem. 3.8).

The proof of Theorem 9.1 follows analogous to the ferronematic case, and are skipped for brevity. Numerical results for both *a priori* and *a posteriori* error estimates for the LDG model for NLCs are discussed in [23, 24].

10. Conclusions

This paper develops a framework of *a priori* and *a posteriori* error analysis for a systems of nonlinear PDEs with polynomial nonlinearities and inhomogeneous Dirichlet boundary conditions for different FE schemes – conforming, dG, Nitsche, and WOPSIP schemes. The abstract theory deals with the discrete approximation of solutions of semilinear elliptic PDEs with nonlinearity in lower-order terms and non-homogeneous Dirichlet boundary conditions, and works for milder regularity of solutions in $\mathbf{H}^{1+\alpha}(\Omega)$, $0 < \alpha \le 1$. This work includes the system of semilinear elliptic PDEs associated with the nematic and ferronematic model problems in applications and we expect to extend the results to broader class of PDEs, for instance, the Robin boundary value problem that appears in weak anchoring LDG minimization problem for NLCs [22], the Q-tensor models in three dimensions [5], and the coupled nonlinear system of PDEs of nematic Q-tensor and smectic density in smectic-A liquid crystals. The application of the general framework studied in this article to LDG theory with more generic elastic energy density beyond one constant approximation [14] (opposed to the isotropic energy we studied in Sect. 9) would be an interesting and challenging exercise. The error estimates in weaker Sobolev norm estimates and adaptive convergence are interesting topics to investigate further.

Appendix A. Proof of discrete inf-sup condition

Proof of Theorem 2.1. Given any $\theta_h \in V_h$ with $\|\theta_h\|_{X_h} = 1$, set $\theta := Q\theta_h$. Define

$$\zeta := \mathscr{A}^{-1}(B_L(\theta_h, \cdot)|_V) \in V, \text{ and } \eta := \mathscr{A}^{-1}(B_L(\theta, \cdot)|_V) \in V.$$

The inf-sup condition in (2.3), $\mathscr{A}\eta = \mathscr{B}_L\theta$, $\|\cdot\|_X \le C_1\|\cdot\|_{\widehat{X}}$, and a triangle inequality lead to

$$\begin{aligned}\beta\|\theta\|_X \le \|\mathscr{A}\theta + \mathscr{B}_L\theta\|_{V^*} = \|\mathscr{A}(\theta+\eta)\|_{V^*} \le \|\mathscr{A}\|\,\|\theta+\eta\|_X \le C_1\|\mathscr{A}\|\,\|\theta+\eta\|_{\widehat{X}} \\ \le C_1\|\mathscr{A}\|\big(\|\theta-\theta_h\|_{\widehat{X}} + \|\theta_h+\zeta\|_{\widehat{X}} + \|\eta-\zeta\|_X\big).\end{aligned} \tag{A.1}$$

The definitions of ζ and η, and the boundedness of the operator $\mathscr{A}^{-1}$ and the bilinear form $B_L|_{\widehat{X}\times V}$ establish

$$\|\eta-\zeta\|_X = \big\|\mathscr{A}^{-1}(B_L(\theta-\theta_h,\cdot)|_V)\big\|_X \le C_2\big\|\mathscr{A}^{-1}\big\|\,\|\theta-\theta_h\|_{\widehat{X}}. \tag{A.2}$$

For $\zeta \in V$, **(A3)** implies $\|\theta-\theta_h\|_{\widehat{X}} \le \Lambda_1\|\theta_h+\zeta\|_{\widehat{X}}$. This, a triangle inequality, and a combination of (A.1), (A.2) reveal

$$1 = \|\theta_h\|_{X_h} \le \|\theta-\theta_h\|_{\widehat{X}} + \|\theta\|_X \le \big(\Lambda_1 + C_1\|\mathscr{A}\|\beta^{-1}\big(1+\Lambda_1\big(1+C_2\big\|\mathscr{A}^{-1}\big\|\big)\big)\big)\|\theta_h+\zeta\|_{\widehat{X}}.$$

The last displayed inequality, equation (2.6) and a triangle inequality imply

$$\widehat{\beta} \le \|\theta_h+\zeta\|_{\widehat{X}} \le \|\theta_h+I_h\zeta\|_{\widehat{X}} + \|I_h\zeta-\zeta\|_{\widehat{X}}. \tag{A.3}$$

For given $\theta_h + I_h\zeta \in V_h$ and for any $0<\tau<\alpha_h$, the inf-sup condition (2.4) implies the existence of some $\varphi_h \in V_h$ with $\|\varphi_h\|_{X_h} = 1$ and

$$(\alpha_h-\tau)\|\theta_h+I_h\zeta\|_{X_h} \le A_h(\theta_h+I_h\zeta,\varphi_h) = DN_h(u;\theta_h,\varphi_h) - B_L(\theta_h,\varphi_h) + A_h(I_h\zeta,\varphi_h), \tag{A.4}$$

where the definition of $DN_h(u;\cdot,\cdot)$ from (2.5) is used in the second step above. Take $\tau\to 0$ and use $A(\zeta,Q\varphi_h) = B_L(\theta_h,Q\varphi_h)$ in (A.4) to obtain

$$\begin{aligned}\alpha_h\|\theta_h+I_h\zeta\|_{X_h} &\le DN_h(u;\theta_h,\varphi_h) + B_L(\theta_h,Q\varphi_h-\varphi_h) - A(\zeta,Q\varphi_h) + A_h(I_h\zeta,\varphi_h) \\ &= DN_h(u;\theta_h,\varphi_h) + B_L(\theta_h,Q\varphi_h-\varphi_h) + A(Q(I_h\zeta)-\zeta,Q\varphi_h) + (A_h(I_h\zeta,\varphi_h) - A(Q(I_h\zeta),Q\varphi_h)),\end{aligned} \tag{A.5}$$

where $A(Q(I_h\zeta),Q\varphi_h)$ is added and subtracted in the last step above. The second term on the right hand side of (A.5) is estimated using **(A2)**. The boundedness of $A(\cdot,\cdot)$, a triangle inequality and **(A3)** leads to

$$\begin{aligned}A(Q(I_h\zeta)-\zeta,Q\varphi_h) &\le \|\mathscr{A}\|\|Q(I_h\zeta)-\zeta\|_X\|Q\varphi_h\|_X \le C_1^2\|\mathscr{A}\|\big(\|Q(I_h\zeta)-I_h\zeta\|_{\widehat{X}} + \|I_h\zeta-\zeta\|_{\widehat{X}}\big)\|Q\varphi_h\|_{\widehat{X}} \\ &\le C_1^2(\Lambda_1+1)^2\|\mathscr{A}\|\|I_h\zeta-\zeta\|_{\widehat{X}}\|\varphi_h\|_{X_h}.\end{aligned}$$

The hypothesis **(A4)** for $\xi_h := I_h\zeta$, and then **(A3)** establishes

$$A_h(I_h\zeta,\varphi_h) - A(Q(I_h\zeta),Q\varphi_h) \le C_A\|I_h\zeta - Q(I_h\zeta)\|_{\widehat{X}}\|\varphi_h\|_{X_h} \le C_A\Lambda_1\|I_h\zeta-\zeta\|_{\widehat{X}}\|\varphi_h\|_{X_h}.$$

A combination of the last two displayed estimates with $\|\varphi_h\|_{X_h} = 1$ and **(A1)** in (A.5) yields

$$\alpha_h\|\theta_h+I_h\zeta\|_{X_h} \le DN_h(u;\theta_h,\varphi_h) + \Big(C_A\Lambda_1 + C_1^2(\Lambda_1+1)^2\|\mathscr{A}\|\Big)\delta_1 + \delta_2.$$

The above expression and **(A1)** applied to (A.3) leads to

$$\alpha_h\widehat{\beta} \leq DN_h(u;\theta_h,\varphi_h) + \Big(\alpha_h + C_A\Lambda_1 + C_1^2(\Lambda_1+1)^2\|\mathscr{A}\|\Big)\delta_1 + \delta_2.$$

The above displayed estimate holds for an arbitrary $\theta_h \in V_h$ with $\|\theta_h\|_{X_h} = 1$ and so proves the discrete inf-sup condition for $\beta_0 := (\alpha_h\widehat{\beta} - ((\alpha_h + C_A\Lambda_1 + C_1^2(\Lambda_1+1)^2\|\mathscr{A}\|)\delta_1 + \delta_2))$. □

Appendix B. Dependency of constants

An overview of the dependencies of constants is presented in Table B.1. The constants from trace inequalities, Sobolev embedding results, enrichment operator estimates, and interpolation estimates are denoted by C_T, C_S, C_{en}, C_I, respectively. The constant C_w is associated with the inequalities in Lemma 5.9(iii).

Table B.1. Summary of various constant dependence in ferronematics application.

Methods	Constants					
	C_1 depend on	C_A, $\widetilde{C}_A$ depends on	C_B, $\widetilde{C}_B$ depends on	L depends on	Λ_1 depends on	Λ_2 depends on
CFEM	$C_1 = 1$	0	$c, \ell, C_I, \lvert\lvert\lvert\Psi\rvert\rvert\rvert_{1+\alpha}$ and C_S	$b, c, \ell, C_I, \lvert\lvert\lvert\Psi\rvert\rvert\rvert_{1+\alpha}$, and C_S	0	0
dGFEM	$\sigma_{\mathrm{dG}}, C_{\mathrm{P}}$	C_T, C_{en}, and σ_{dG}	$c, \ell, C_I, \lvert\lvert\lvert\Psi\rvert\rvert\rvert_{1+\alpha}$ and C_{P}	$b, c, \ell, C_I, \lvert\lvert\lvert\Psi\rvert\rvert\rvert_{1+\alpha}$, and C_{P}	C_{en} and σ_{dG}	$\sigma_{\mathrm{dG}}, C_{en}$
Nitsche	σ, C_{P}	C_T, C_{en}, and σ	$c, \ell, C_I, \lvert\lvert\lvert\Psi\rvert\rvert\rvert_{1+\alpha}$ and C_{P}	$b, c, \ell, C_I, \lvert\lvert\lvert\Psi\rvert\rvert\rvert_{1+\alpha}$, and C_{P}	C_{en} and σ	0
WOPSIP	$\sigma_{\mathrm{P}}, C_{\mathrm{P}}, C_w$	C_T, C_{en}, and σ_{P}	$c, \ell, C_I, \lvert\lvert\lvert\Psi\rvert\rvert\rvert_{1+\alpha}$ and C_w	$b, c, \ell, C_I, \lvert\lvert\lvert\Psi\rvert\rvert\rvert_{1+\alpha}$, and C_w	C_{en} and C_w	–

Appendix C. Proof of Lemma 5.4

Proof of Lemma 5.4(c). For $\eta \in X$, $[\eta]_E = 0$ for all $E \in \mathcal{E}(\Omega)$. This plus Lemma 5.4(a) imply

$$\sum_{m=0}^{1}\left|h_{\mathcal{T}}^{m-1}(J_1\varphi_h - \varphi_h)\right|_{H^m(\mathcal{T})} \lesssim \min_{\eta\in X}\|\varphi_h - \eta\|_h. \tag{C.1}$$

For an interior edge $E \subset \partial T^+ \cap \partial T^-$ with the adjacent triangles T^+ and T^-, Lemma 5.7(i) and Young's inequality lead to

$$\|[J_1\varphi_h - \varphi_h]_E\|_{0,E}^2 = \sum_{T\in\{T^+,T^-\}} \|(J_1\varphi_h - \varphi_h)|_T\|_{0,E}^2 \lesssim \sum_{T\in\{T^+,T^-\}}\Big(h_T^{-1}\|J_1\varphi_h - \varphi_h\|_{0,T}^2 + h_T|J_1\varphi_h - \varphi_h|_{H^1(T)}^2\Big).$$

The definition of $\|\cdot\|_h$ and (C.1) completes the proof of (c). □

Proof of Lemma 5.4(d). The proof of (d) follows analogous to (c) with $[\varphi]_E = 0$ for all $E \in \mathcal{E}$ for $\varphi \in V$. □

Appendix D. Proof of Theorem 6.7

Proof of (a). The proofs of **(A1)–(A3)** follow analogous to step 1 of Theorem 6.4. Set $\Theta_{\mathrm{N}}, \Phi_{\mathrm{N}} \in \mathbf{X}_{\mathrm{C}}$ with $\lvert\lvert\lvert\Theta_{\mathrm{N}}\rvert\rvert\rvert_{\mathrm{N}} = \lvert\lvert\lvert\Phi_{\mathrm{N}}\rvert\rvert\rvert_{\mathrm{N}} = 1$. The definition of $A_{\mathrm{N}}(\cdot,\cdot)$, an algebraic manipulation, and a re-arrangement of terms lead to

$$A_{\mathrm{N}}(\Theta_{\mathrm{N}},\Phi_{\mathrm{N}}) - A(J_2\Theta_{\mathrm{N}}, J_2\Phi_{\mathrm{N}}) = (A(\Theta_{\mathrm{N}},\Phi_{\mathrm{N}} - J_2\Phi_{\mathrm{N}}) + \boldsymbol{\mathcal{J}}(\Theta_{\mathrm{N}},\Phi_{\mathrm{N}})) + A(\Theta_{\mathrm{N}} - J_2\Theta_{\mathrm{N}}, J_2\Phi_{\mathrm{N}}) + \boldsymbol{\mathcal{J}}_\sigma(\Theta_{\mathrm{N}},\Phi_{\mathrm{N}}).$$

We proceed with the techniques as applied to estimate **(A4)** in Theorem 6.4. Let $\Theta_{\rm N} := (\theta_1, \theta_2, \theta_3, \theta_4), \Phi_{\rm N} := (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, and $a(\theta_i, \varphi_i) := \int_\Omega \nabla\theta_i \cdot \nabla\varphi_i \,{\rm d}x, 1 \le i \le 4$. In this case, for the components of the first term of the above identity, we obtain $a(\theta_i, \varphi_i - J_2\varphi_i) + \mathcal{J}(\theta_i, \varphi_i) = -\langle \nabla\varphi_i \cdot \nu_E, \theta_i - J_2\theta_i\rangle_{\partial\Omega}$. A combination of all the components, the Cauchy–Schwarz inequality, and Lemma 5.7(ii) lead to

$$A(\Theta_{\rm N}, \Phi_{\rm N} - J_2\Phi_{\rm N}) + \boldsymbol{\mathcal{J}}(\Theta_{\rm N}, \Phi_{\rm N}) \lesssim \|\!|\Theta_{\rm N} - J_2\Theta_{\rm N}|\!\|_{\rm N} |\!\|\Phi_{\rm N}|\!\|_{\rm N}.$$

Lemmas 5.4(d) and 5.9(ii) yield $A(\Theta_{\rm N} - J_2\Theta_{\rm N}, J_2\Phi_{\rm N}) \lesssim |\!\|\Theta_{\rm N} - J_2\Theta_{\rm N}|\!\|_{\rm N} |\!\|\Phi_{\rm N}|\!\|_{\rm N}$. The definition of $\mathcal{J}_\sigma(\cdot,\cdot)$ in (6.15), and $J_2\theta_i = 0$ for all $E \in \mathcal{E}(\partial\Omega)$, $1 \le i \le 4$, yield $\mathcal{J}_\sigma(\theta_i, \varphi_i) := \sum_{E\in\mathcal{E}(\partial\Omega)} \frac{\sigma}{h_E}\langle \theta_i - J_2\theta_i, \varphi_i\rangle_E$. This and the Cauchy–Schwarz inequality lead to $\boldsymbol{\mathcal{J}}_\sigma(\Theta_{\rm N}, \Phi_{\rm N}) \le |\!\|\Theta_{\rm N} - J_2\Theta_{\rm N}|\!\|_{\rm N} |\!\|\Phi_{\rm N}|\!\|_{\rm N}$. Therefore, a combination of the above estimates completes the proof of **(A4)**. Now, for a sufficiently small choice of the maximal mesh-size h and a sufficiently large choice of the penalty parameter σ, Theorem 2.1 verifies to the discrete inf-sup condition in Theorem 6.7 with $\beta_0 > 0$. This concludes the proof. ☐

Proof of (b). The existence and uniqueness of the discrete solution $\Psi_{\rm N}$ applies Theorem 3.1 and so we verify the required hypotheses. For the proof of the error estimate $|\!\|\Psi - \Psi_{\rm N}|\!\|_{\rm N} \lesssim h^\alpha$, see Theorem 6.5.

Step 1: Verification of **(A5)**–**(A8).** For **(A5)**, choose $z_h := I_{\rm C}\Psi$. Lemmas 5.1 and 5.7(i) imply $\delta_3 \lesssim h^\alpha$. Let $\varphi_h := \Phi_{\rm N} \in \mathbf{X}_{\rm C}$ with $|\!\|\Phi_{\rm N}|\!\|_{\rm N} = 1$. Lemmas 5.8(i) and 5.9(ii) establish **(A6)** with $\delta_4 \lesssim h$. Since $P := id$, **(A7)** holds for $\Lambda_2 = 0$. **(A8)** involves $x_h := \boldsymbol{\eta}_{\rm N}, \varphi_h := \Phi_{\rm N} \in \mathbf{X}_{\rm C}$. The definition of $A_{\rm N}(\cdot,\cdot)$, an addition and subtraction of $A(\boldsymbol{\eta}_{\rm N}, J_2\Phi_{\rm N})$, and re-arrangement of terms lead to

$$\begin{aligned} A_{\rm N}(\boldsymbol{\eta}_{\rm N}, \Phi_{\rm N}) - A(\boldsymbol{\eta}_{\rm N}, J_2\Phi_{\rm N}) - F_{\rm N}(\Phi_{\rm N}) = \big(&A(\boldsymbol{\eta}_{\rm N}, \Phi_{\rm N} - J_2\Phi_{\rm N}) + \boldsymbol{\mathcal{J}}(\boldsymbol{\eta}_{\rm N}, \Phi_{\rm N}) - F_{\rm N}^{\mathcal{J}}(\Phi_{\rm N})\big) \\ &+ \Big(\boldsymbol{\mathcal{J}}_\sigma(\boldsymbol{\eta}_{\rm N}, \Phi_{\rm N}) - F_{\rm N}^{\mathcal{J}_\sigma}(\Phi_{\rm N})\Big). \end{aligned}$$

For $\boldsymbol{\eta}_{\rm N} := (\eta_1, \eta_2, \eta_3, \eta_4)$, $\Theta_{\rm N} := (\theta_1, \theta_2, \theta_3, \theta_4), \Phi_{\rm N} := (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbf{X}_{\rm C}$, and $\mathbf{g} := (g_1, g_2, g_3, g_4)$, the steps as in T_4 of Theorem 6.5 lead to

$$a(\eta_i, \varphi_i - J_2\varphi_i) + \mathcal{J}(\eta_i, \varphi_i) - F_{\rm N}^{i,\mathcal{J}}(\varphi_i) = -\langle \nabla\varphi_i \cdot \nu_E, \eta_i\rangle_{\partial\Omega} + \langle \nabla\varphi_i \cdot \nu_E, g_i\rangle_{\partial\Omega}.$$

Therefore, for all $\boldsymbol{\chi} \in \mathbf{X}$ and $\boldsymbol{\chi} = \mathbf{g}$, a combination of all the components, the Cauchy–Schwarz inequality and Lemma 5.7 imply $A(\boldsymbol{\eta}_{\rm N}, \Phi_{\rm N} - J_2\Phi_{\rm N}) + \boldsymbol{\mathcal{J}}(\boldsymbol{\eta}_{\rm N}, \Phi_{\rm N}) - F_{\rm N}^{\mathcal{J}}(\Phi_{\rm N}) \le |\!\|\boldsymbol{\chi} - \boldsymbol{\eta}_{\rm N}|\!\|_{\rm N} |\!\|\Phi_{\rm N}|\!\|_{\rm N}$. For the bound $\boldsymbol{\mathcal{J}}_\sigma(\boldsymbol{\eta}_{\rm N}, \Phi_{\rm N}) - F_{\rm N}^{\mathcal{J}_\sigma}(\Phi_{\rm N}) \le |\!\|\boldsymbol{\eta}_{\rm N} - \boldsymbol{\chi}|\!\|_{\rm N} |\!\|\Phi_{\rm N}|\!\|_{\rm N}$, see the steps as in T_6 of Theorem 6.5. This concludes the proof of **(A8)**.

Step 2: Verification of **(B1)**, **(B2)** and Lipschitz continuity of $D\mathscr{B}$ on $\overline{B}_{\mathbf{X}_{\rm C}}(I_{\rm C}\Psi, 2b)$**.** We apply Lemma 5.8 (ii) with $z_h := I_{\rm C}\Psi$ and Lemmas 5.9(ii), 5.1 to establish **(B1)**, **(B2)**. For the proof of the Lipschitz continuity, we apply Lemmas 5.8(iv) and 5.9(ii) at the first step, and then the triangle inequalities similar to (6.2) in the energy norm and conforming interpolation estimate in Lemma 5.1 to obtain

$$\langle DB(\boldsymbol{\eta}_{\rm N})\Theta_{\rm N}, \Phi_{\rm N}\rangle - \langle DB(\boldsymbol{\chi}_{\rm N})\Theta_{\rm N}, \Phi_{\rm N}\rangle \le L|\!\|\boldsymbol{\eta}_{\rm N} - \boldsymbol{\chi}_{\rm N}|\!\|_{\rm N} |\!\|\Theta_{\rm N}|\!\|_{\rm N} |\!\|\Phi_{\rm N}|\!\|_{\rm N}.$$

The rest of the proof adopts the techniques utilized in Theorem 6.5, and is skipped here.

☐

Proof of (c). Since $X_h := \mathbf{X}_{\rm C}$ for Nitsche's method, define $\widehat{N} := N$. For $\boldsymbol{\eta} \in B_{\mathbf{X}}(\Psi, R_0)$ with $R_0 > 0$, $\Theta \in \mathbf{X}$ and $\Phi \in \mathbf{V}$, Lemmas 5.8(iii) and 5.9(ii) imply $|\!\|D\mathscr{B}(\boldsymbol{\eta}) - D\mathscr{B}(\Psi)|\!\|_{L(\mathbf{X},\mathbf{V}^*)} \lesssim (1 + R_0 + |\!\|\Psi|\!\|_1)|\!\|\Theta|\!\|_{\rm N} |\!\|\Phi|\!\|_1$, and this proves the locally Lipschitz continuity of the Frechét derivative of $\mathscr{B} : \mathbf{X} \to \mathbf{V}^*$ at $\Psi \in \mathbf{X}$ for the choice of $\gamma := \gamma_{\rm N} \lesssim 1 + R_0 + |\!\|\Psi|\!\|_1$ in (4.2). Here the constant in "$\lesssim$" depends on the parameters ℓ, c, C_S and $C_{\rm P}$. To establish **(AP)**, apply Lemma 5.3 for the discrete solution $\boldsymbol{\eta}_h := \Psi_{\rm N} \in \mathbf{X}_{\rm C}$ to (6.14) and denote $\Psi_{\mathbf{g}} := \mathscr{G}\Psi_{\rm N}$. The definition of $\|\cdot\|_{\rm N}$ and $[\Psi_{\rm N}]_E = 0$ for all $E \in \mathcal{E}(\Omega)$ imply

$$\|\mathscr{G}\Psi_{\rm N} - \Psi_{\rm N}\|_{\rm N}^2 \lesssim \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-1}\|\mathbf{g} - \Psi_{\rm N}\|_{0,E}^2 + (\text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega)))^2.$$

Suppose that δ satisfy Theorem 6.7(b), and if necessary, are chosen smaller such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the unique discrete solution $\Psi_{\rm N} \in \mathbf{X}_{\rm C}$ to (6.14) satisfies $\|\Psi - \Psi_{\rm N}\|_{\rm N} < R < \min(R_0, \frac{\beta}{2\gamma_{\rm N}})$. The above displayed estimate plus the abstract residual in Theorem 4.1(i) with $\eta_h := \Psi_{\rm N}$ yields

$$\|\Psi - \Psi_{\rm N}\|_{\rm N} \lesssim \|N(\Psi)\|_{\mathbf{V}^*} + \Big(1 + \|DN(\Psi)\|_{L(\mathbf{X},\mathbf{V}^*)}\Big)\left(\left(\sum_{E\in\mathcal{E}(\partial\Omega)} (\vartheta_E^{\partial})^2\right)^{\frac{1}{2}} + \text{data app}(\mathbf{g}, \mathcal{E}(\partial\Omega))\right).$$

The estimation of $\|N(\Psi_{\rm N})\|_{\mathbf{V}^*}$ and the proof of reliability follows Theorem 2.3 from [24] and details are skipped. Standard local efficiency estimates and $(\vartheta_E^{\partial})^2 = \sum_{E\in\mathcal{E}(\partial\Omega)} h_E^{-1}\|\mathbf{g} - \Psi_{\rm N}\|_{0,E}^2 \leq \|\Psi - \Psi_{\rm N}\|_{\rm N}^2$ concludes the proof. □

Appendix E. Numerical results

E.1. Uniform refinement for ferronematic example

Table E.1. Numerical errors ($\mathbf{e}_{\rm N}(n) = \|\Psi_{\rm N}^n - \Psi_{\rm N}^{n-1}\|_{\rm N}$ and $\mathbf{e}_{\mathbf{L}^2}(n) = \|\Psi_{\rm N}^n - \Psi_{\rm N}^{n-1}\|_{\mathbf{L}^2}$) and convergence rates for the discrete solutions $\Psi_{\rm N} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$, $\Psi_{\rm N} = (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ for $c = 0.25$ and $\Psi_{\rm N} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1}^1)$, $\Psi_{\rm N} = (\mathbf{Q}_{R4}, \mathbf{M}_{R4}^1)$ for $c = -0.25$ in energy and $\mathbf{L}^2$ norms for the Nitsche's method with $\ell = 0.001$.

Solution	N	h	$c = 0.25$				$c = -0.25$			
			$\mathbf{e}_{\rm N}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order	$\mathbf{e}_{\rm N}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order
$D1$	289	0.0440	2.2765	–	0.5444E−1	–	2.3170	–	0.3828E−1	–
	1089	0.0220	1.2103	0.9114	0.1754E−1	1.6336	1.0879	1.0906	0.1161E−1	1.7213
	4225	0.0110	0.5926	1.0302	0.5014E−2	1.8070	0.5335	1.0279	0.3175E−2	1.8703
	16 641	0.0055	0.2916	1.0228	0.1317E−2	1.9276	0.2659	1.0043	0.8182E−3	1.9564
$R4$	289	0.0440	2.2720	–	1.6339E−1	–	2.4431	–	0.4081E−1	–
	1089	0.0220	1.1403	0.9944	0.1092E−1	3.9028	1.1600	1.0745	0.1248E−1	1.7088
	4225	0.0110	0.5456	1.0635	0.2984E−2	1.8720	0.5704	1.0240	0.3403E−2	1.8748
	16 641	0.0055	0.2661	1.0353	0.7870E−3	1.9227	0.2845	1.0032	0.8753E−3	1.9592

TABLE E.2. Numerical errors ($\mathbf{e}_{\text{dG}}(n) = \|\!|\Psi_{\text{dG}}^n - \Psi_{\text{dG}}^{n-1}|\!\|_{\text{dG}}$ and $\mathbf{e}_{\mathbf{L}^2}(n) = \|\!|\Psi_{\text{dG}}^n - \Psi_{\text{dG}}^{n-1}|\!\|_{\mathbf{L}^2}$) and convergence rates for the discrete solutions $\Psi_{\text{dG}} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$, $\Psi_{\text{dG}} = (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ for $c = 0.25$ and $\Psi_{\text{dG}} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1}^1)$, $\Psi_{\text{dG}} = (\mathbf{Q}_{R4}, \mathbf{M}_{R4}^1)$ for $c = -0.25$ in energy and $\mathbf{L}^2$ norms for the dG scheme with $\ell = 0.001$.

Solution	N	h	$c = 0.25$				$c = -0.25$			
			$\mathbf{e}_{\text{dG}}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order	$\mathbf{e}_{\text{dG}}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order
	289	0.0440	5.1244	–	0.3848E−1	–	4.3300	–	0.9041E−1	–
$D1$	1089	0.0220	2.8183	0.8625	0.1278E−1	1.5893	2.2819	0.9241	0.9041E−2	1.7449
	4225	0.0110	1.4640	0.9448	0.3781E−2	1.7576	1.1877	0.9420	0.2493E−2	1.8583
	16 641	0.0055	0.7464	0.9718	0.1009E−2	1.9051	0.6094	0.9626	0.6458E−3	1.9489
	289	0.0440	4.8139	–	0.1381E−1	–	4.4967	–	0.3218E−1	–
$R4$	1089	0.0220	2.1603	1.1559	0.8666E−2	3.9945	2.3648	0.9271	0.9729E−2	1.7259
	4225	0.0110	1.1124	0.9574	0.2375E−2	1.8670	1.1877	0.9420	0.2493E−2	1.8583
	16 641	0.0055	0.5690	0.9671	0.6324E−3	1.9094	0.6094	0.9626	0.6458E−3	1.9489

E.2. Adaptive refinements for ferronematic example

TABLE E.3. Numerical estimators and convergence rates for adaptive mesh refinements for the discrete solutions $\Psi_{\text{C}} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$ (*left*) for $c = 0.25$ and $\Psi_{\text{C}} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1}^1)$ (*right*) for $c = -0.25$ for conforming FEM with $\ell = 0.001$.

N	ϑ_{C}	Order_ϑ
289	15.71517	–
412	12.3774	0.67329
507	10.3545	0.86004
741	8.52550	0.51218
1049	6.40026	0.82488
1608	5.43721	0.38176
2543	4.31722	0.50322
3630	3.54920	0.55042
5450	2.95075	0.45440
8455	2.40222	0.46833
12 665	1.93712	0.53253
18 643	1.59857	0.49684
27 747	1.32635	0.46943
41 546	1.07577	0.51871

N	ϑ_{C}	Order_ϑ
289	14.2137	–
330	12.1805	1.16357
436	10.4048	0.56567
556	8.99144	0.60050
748	7.48446	0.61841
1134	6.24071	0.43675
1609	5.19778	0.52266
2283	4.25981	0.56879
3584	3.49812	0.43681
5327	2.86334	0.50525
7846	2.33756	0.52393
11 630	1.94224	0.47071
16 988	1.62775	0.46617
25 062	1.34114	0.49807
37 164	1.09601	0.51231

TABLE E.4. Numerical estimators and convergence rates for adaptive mesh refinements for the discrete solutions $\Psi_\mathrm{C} = (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ (*left*) for $c = 0.25$ and $\Psi_\mathrm{C} = (\mathbf{Q}_{R4}, \mathbf{M}^1_{R4})$ (*right*) for $c = -0.25$ for conforming FEM with $\ell = 0.001$.

N	ϑ_C	Order_ϑ	N	ϑ_C	Order_ϑ
289	12.6221	–	289	14.1851	–
357	10.4202	0.90719	334	12.2192	1.03088
432	8.96614	0.78817	449	10.3949	0.54646
608	7.41089	0.55744	565	8.89133	0.67991
959	6.13670	0.41399	753	7.68706	0.50667
1459	4.99236	0.49182	1206	6.31753	0.41658
2146	4.12910	0.49202	1733	5.20412	0.53477
3112	3.46151	0.47450	2432	4.28093	0.57628
4602	2.85567	0.49178	3770	3.53862	0.43442
7069	2.33269	0.47127	5601	2.88776	0.51344
10 076	1.94305	0.51564	8324	2.34790	0.52236
14 943	1.61033	0.47658	12 222	1.95661	0.47464
22 706	1.31867	0.47759	18 115	1.63035	0.46356
33 585	1.08722	0.49302	27 061	1.33472	0.49849
48 266	0.90404	0.50877	39 752	1.09779	0.50817

TABLE E.5. Numerical estimators and convergence rates for uniform and adaptive mesh refinements for the discrete solutions $\Psi_\mathrm{C} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$, $\Psi_\mathrm{C} = (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ for $c = 0.25$ and $\Psi_\mathrm{C} = (\mathbf{Q}_{D1}, \mathbf{M}^1_{D1})$, $\Psi_\mathrm{C} = (\mathbf{Q}_{R4}, \mathbf{M}^1_{R4})$ for $c = -0.25$ for the conforming FEM with $\ell = 0.001$.

	$c = 0.25$						$c = -0.25$					
	Uniform refinement			Adaptive refinement			Uniform refinement			Adaptive refinement		
	N	ϑ_C	Order_ϑ	N	ϑ_C	Order_ϑ	N	ϑ_C	Order_ϑ	N	ϑ_C	Order_ϑ
	289	15.715	–	289	15.715	–	289	14.213	–	289	14.213	–
	1089	10.082	0.3345	1049	6.4002	0.6967	1089	8.4595	0.3911	1134	6.2407	0.6020
$D1$	4225	5.6249	0.4304	3630	3.5492	0.4749	4225	4.5282	0.4609	3584	3.4981	0.5030
	16 641	2.9401	0.4732	12 665	1.9371	0.4845	16 641	2.3473	0.4793	11 630	1.9422	0.4998
	66 049	1.4990	0.4886	27 747	1.3263	0.4829	66 049	1.1996	0.4869	25 062	1.3411	0.4823
	263 169	0.7595	0.4917	41 546	1.0757	0.5187	263 169	0.6111	0.4878	37 164	1.0960	0.5123
	289	12.622	–	289	12.622	–	289	14.185	–	289	14.185	–
	1089	7.6582	0.3766	959	6.1367	0.6012	1089	8.4288	0.3923	1206	6.3175	0.5661
$R4$	4225	4.2016	0.4427	4602	2.8556	0.4877	4225	4.5293	0.4580	3770	3.5386	0.5085
	16 641	2.2004	0.4718	14 943	1.6103	0.4864	16 641	2.3514	0.4782	12 222	1.9566	0.5037
	66 049	1.1290	0.4840	33 585	1.0872	0.4850	66 049	1.2023	0.4865	27 061	1.3347	0.4812
	263 169	0.5765	0.4861	48 266	0.9040	0.5087	263 169	0.6126	0.4877	39 752	1.0977	0.5081

TABLE E.6. Numerical estimators and convergence rates for uniform and adaptive mesh refinements for the discrete solutions $\Psi_{\mathrm{N}} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$, $\Psi_{\mathrm{N}} = (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ for $c = 0.25$ and $\Psi_{\mathrm{N}} = (\mathbf{Q}_{D1}, \mathbf{M}^1_{D1})$, $\Psi_{\mathrm{N}} = (\mathbf{Q}_{R4}, \mathbf{M}^1_{R4})$ for $c = -0.25$ for the Nitsche's method with $\ell = 0.001$.

	$c = 0.25$						$c = -0.25$					
	Uniform refinement			Adaptive refinement			Uniform refinement			Adaptive refinement		
	N	ϑ_{N}	Order$_\vartheta$	N	ϑ_{N}	Order$_\vartheta$	N	ϑ_{N}	Order$_\vartheta$	N	ϑ_{N}	Order$_\vartheta$
	289	10.915	–	289	10.915	–	289	9.6184	–	289	9.6184	–
	1089	7.0153	0.3332	729	5.9246	0.6604	1089	5.8208	0.3785	1219	4.1574	0.5827
$D1$	4225	3.9318	0.4270	2516	3.0096	0.5467	4225	3.1469	0.4536	5723	1.9362	0.4941
	16 641	2.0629	0.4705	12 477	1.3625	0.4949	16 641	1.6401	0.4753	18 132	1.1001	0.4902
	66 049	1.0547	0.4866	41 309	0.7564	0.4916	66 049	0.8417	0.4839	57 563	0.6274	0.4861
	263 169	0.5351	0.4907	60 949	0.6204	0.5092	263 169	0.4297	0.4862	83 313	0.5217	0.4987
	289	8.7141	–	289	8.7141	–	289	10.033	–	289	10.033	–
	1089	5.2452	0.3826	910	4.1858	0.6392	1089	6.0375	0.3828	1260	4.4146	0.5575
$R4$	4225	2.9074	0.4352	4709	1.9513	0.4642	4225	3.2586	0.4548	5640	2.0608	0.5082
	16 641	1.5340	0.4664	15 269	1.1048	0.4835	16 641	1.6954	0.4766	18 053	1.1574	0.4958
	66 049	0.7913	0.4802	49 341	0.6219	0.4899	66 049	0.8687	0.4850	57 024	0.6550	0.4949
	263 169	0.4051	0.4842	72 577	0.5153	0.4869	263 169	0.4430	0.4872	83 978	0.5433	0.4831

TABLE E.7. Numerical estimators and convergence rates for uniform and adaptive mesh refinements for the discrete solutions $\Psi_{\mathrm{dG}} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$, $\Psi_{\mathrm{dG}} = (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ for $c = 0.25$ and $\Psi_{\mathrm{dG}} = (\mathbf{Q}_{D1}, \mathbf{M}^1_{D1})$, $\Psi_{\mathrm{dG}} = (\mathbf{Q}_{R4}, \mathbf{M}^1_{R4})$ for $c = -0.25$ for the dG scheme with $\ell = 0.001$

	$c = 0.25$						$c = -0.25$					
	Uniform refinement			Adaptive refinement			Uniform refinement			Adaptive refinement		
	N	ϑ_{dG}	Order$_\vartheta$	N	ϑ_{dG}	Order$_\vartheta$	N	ϑ_{dG}	Order$_\vartheta$	N	ϑ_{dG}	Order$_\vartheta$
	289	12.420	–	289	12.420	–	289	11.886	–	289	11.886	–
	1089	8.0460	0.3131	1601	4.5036	0.5661	1098	7.1391	0.3677	1194	5.1362	0.5647
$D1$	4225	4.5310	0.4142	12 709	1.6336	0.4834	4225	3.8484	0.4457	12 080	1.6341	0.4869
	16 641	2.3837	0.4633	28 449	1.1047	0.4831	16 641	2.0027	0.4711	38 671	0.9221	0.4887
	66 049	1.2204	0.4829	42 776	0.8987	0.5026	66 049	1.0267	0.4819	56 505	0.7697	0.4743
	289	10.428	–	289	10.428	–	289	12.310	–	289	12.957	–
	1089	6.4081	0.3512	1276	4.4371	0.5485	1089	7.3704	0.3700	1300	5.1664	0.5846
$R4$	4225	3.5564	0.4247	9186	1.7360	0.4683	4225	3.9687	0.4465	13 192	1.6282	0.4903
	16 641	1.8752	0.4616	20 709	1.1705	0.4816	16 641	2.0624	0.4721	28 999	1.1045	0.4904
	66 049	0.9663	0.4782	44 204	0.8022	0.4952	66 049	1.0560	0.4828	42 658	0.9135	0.4888

TABLE E.8. Numerical estimators and convergence rates for uniform and adaptive mesh refinements for the discrete solutions $\Psi_{\rm P} = (\mathbf{Q}_{D1}, \mathbf{M}_{D1})$, $\Psi_{\rm P} = (\mathbf{Q}_{R4}, \mathbf{M}_{R4})$ for $c = 0.25$ and $\Psi_{\rm P} = (\mathbf{Q}_{D1}, \mathbf{M}^1_{D1})$, $\Psi_{\rm P} = (\mathbf{Q}_{R4}, \mathbf{M}^1_{R4})$ for $c = -0.25$ for the WOPSIP scheme with $\ell = 0.001$

	$c = 0.25$						$c = -0.25$					
	Uniform refinement			Adaptive refinement			Uniform refinement			Adaptive refinement		
	N	$\vartheta_{\rm P}$	Order$_\vartheta$	N	$\vartheta_{\rm P}$	Order$_\vartheta$	N	$\vartheta_{\rm P}$	Order$_\vartheta$	N	$\vartheta_{\rm P}$	Order$_\vartheta$
	289	11.571	–	289	11.571	–	289	13.478	–	289	14.398	–
	1089	6.5141	0.4144	1062	5.2149	0.5819	1089	7.2071	0.4515	1251	4.8612	0.7086
$D1$	4225	3.4599	0.4564	3582	2.8733	0.4836	4225	3.8086	0.4601	13 530	1.5144	0.4819
	16 641	1.7824	0.4784	17 589	1.2963	0.4945	16 641	1.9603	0.4790	28 070	1.0553	0.4932
	66 049	0.9081	0.4864	37 329	0.8884	0.4965	66 049	0.9973	0.4874	41 084	0.8805	0.4714
	289	11.105	–	289	11.801	–	289	14.378	–	289	13.874	–
	1089	6.6437	0.3705	1492	4.1669	0.6109	1089	7.2762	0.4912	1182	5.0599	0.6869
$R4$	4225	3.5337	0.4554	7123	1.9445	0.4780	4225	3.8456	0.4599	11959	1.6542	0.4740
	16 641	1.8215	0.4780	21 980	1.1182	0.4866	16 641	1.9791	0.4792	35 831	0.9522	0.5002
	66 049	0.9277	0.4866	45 086	0.7814	0.4953	66 049	1.0066	0.4876	51 645	0.7889	0.5117

E.3. Uniform refinements for L-shape domain with zero load

TABLE E.9. Numerical errors ($\mathbf{e}_{\rm C}(n) = |||\Psi^n_{\rm C} - \Psi^{n-1}_{\rm C}|||_1$ and $\mathbf{e}_{\mathbf{L}^2}(n) = |||\Psi^n_{\rm C} - \Psi^{n-1}_{\rm C}|||_{\mathbf{L}^2}$) and convergence rates for the discrete solutions for the conforming FEM for $c = 0.25$ and $c = -0.25$ with $\ell = 1$.

h	$c = 0.25$				$c = -0.25$			
	$\mathbf{e}_{\rm C}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order	$\mathbf{e}_{\rm C}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order
0.0440	0.2192	–	0.6763E−2	–	0.2165	–	0.6618E−2	–
0.0220	0.1517	0.5306	0.2957E−2	1.1933	0.1499	0.5300	0.2901E−2	1.1897
0.0110	0.1054	0.5257	0.1300E−2	1.1856	0.1042	0.5243	0.1278E−2	1.1822
0.0055	0.0734	0.5215	0.5733E−3	1.1812	0.0727	0.5199	0.5649E−3	1.1783

TABLE E.10. Numerical errors ($\mathbf{e}_{\rm N}(n) = |||\Psi^n_{\rm N} - \Psi^{n-1}_{\rm N}|||_{\rm N}$ and $\mathbf{e}_{\mathbf{L}^2}(n) = |||\Psi^n_{\rm N} - \Psi^{n-1}_{\rm N}|||_{\mathbf{L}^2}$) and convergence rates for the discrete solutions for the Nitsche's method for $c = 0.25$ and $c = -0.25$ with $\ell = 1$.

h	$c = 0.25$				$c = -0.25$			
	$\mathbf{e}_{\rm N}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order	$\mathbf{e}_{\rm N}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order
0.0440	0.1648	–	0.4375E−2	–	0.1629	–	0.4287E−2	–
0.0220	0.1138	0.5343	0.1918E−2	1.1895	0.1124	0.5356	0.1884E−2	1.1861
0.0110	0.0789	0.5279	0.8461E−3	1.1808	0.0780	0.5274	0.8331E−3	1.1775
0.0055	0.0549	0.5231	0.3743E−3	1.1765	0.0543	0.5219	0.3692E−3	1.1738

TABLE E.11. Numerical errors ($\mathbf{e}_{\mathrm{dG}}(n) = |||\Psi_{\mathrm{dG}}^n - \Psi_{\mathrm{dG}}^{n-1}|||_{\mathrm{dG}}$ and $\mathbf{e}_{\mathbf{L}^2}(n) = |||\Psi_{\mathrm{dG}}^n - \Psi_{\mathrm{dG}}^{n-1}|||_{\mathbf{L}^2}$) and convergence rates for the discrete solutions for the dG scheme for $c = 0.25$ and $c = -0.25$ with $\ell = 1$.

h	$c = 0.25$				$c = -0.25$			
	$\mathbf{e}_{\mathrm{dG}}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order	$\mathbf{e}_{\mathrm{dG}}(n)$	Order	$\mathbf{e}_{\mathbf{L}^2}(n)$	Order
0.0440	0.4272	–	0.6483E−2	–	0.4208	–	0.6416E−2	–
0.0220	0.2944	0.5371	0.2726E−2	1.2499	0.2899	0.5375	0.2695E−2	1.2509
0.0110	0.2031	0.5353	0.1172E−2	1.2167	0.2001	0.5344	0.1160E−2	1.2161
0.0055	0.1405	0.5316	0.5123E−3	1.1949	0.1386	0.5299	0.5073E−3	1.1935

E.4. Adaptive refinements for L-shape domain with zero load

TABLE E.12. Numerical estimators and experimental convergence rates for uniform and adaptive mesh refinements for the conforming FEM.

Parameter	Uniform refinement			Adaptive refinement		
	N	ϑ_{C}	$\mathrm{Order}_{\vartheta}$	N	ϑ_{C}	$\mathrm{Order}_{\vartheta}$
$\ell = 1$, $c = 0.25$	225	1.7958	–	225	1.7959	–
	833	1.2392	0.2834	747	0.5081	1.0522
	3201	0.8566	0.2742	3592	0.2260	0.5159
	12 545	0.5938	0.2682	11 594	0.1247	0.5074
	49 665	0.4128	0.2641	24 829	0.0851	0.5011
	197 633	0.2877	0.2612	52 259	0.0586	0.5014
$\ell = 1$, $c = -0.25$	225	1.7703	–	225	1.7703	–
	833	1.2220	0.2831	758	0.4967	1.0464
	3201	0.8453	0.2737	3722	0.2187	0.5155
	12 545	0.5865	0.2675	11 954	0.1220	0.5001
	49 665	0.4083	0.2633	25 811	0.0831	0.4989
	197 633	0.2849	0.2603	54 069	0.0570	0.5099

TABLE E.13. Numerical estimators and experimental convergence rates for uniform and adaptive mesh refinements for the Nitsche's method.

Parameter	Uniform refinement			Adaptive refinement		
	N	ϑ_{N}	$\mathrm{Order}_{\vartheta}$	N	ϑ_{N}	$\mathrm{Order}_{\vartheta}$
$\ell = 1$, $c = 0.25$	225	1.4681	–	225	1.4682	–
	833	1.0150	0.2662	664	0.5316	0.9388
	3201	0.7009	0.2671	3111	0.2423	0.5089
	12 545	0.4849	0.2656	14 720	0.1103	0.5063
	49 665	0.3364	0.2636	45 643	0.0626	0.5010
	197 633	0.2341	0.2616	65 634	0.0520	0.5063
$\ell = 1$, $c = -0.25$	225	1.4471	–	225	1.4472	–
	833	1.0008	0.2660	687	0.5140	0.9273
	3201	0.6913	0.2668	3197	0.2363	0.5053
	12 545	0.4787	0.2651	15 154	0.1084	0.5010
	49 665	0.3325	0.2628	46 770	0.0616	0.5008
	197 633	0.2316	0.2607	67 405	0.0512	0.5080

TABLE E.14. Numerical estimators and experimental convergence rates for uniform and adaptive mesh refinements for the dG scheme.

Parameter	Uniform refinement			Adaptive refinement		
	N	ϑ_{dG}	$\mathrm{Order}_{\vartheta}$	N	ϑ_{dG}	$\mathrm{Order}_{\vartheta}$
$\ell = 1$, $c = 0.25$	65	1.5776	–	65	1.5777	–
	225	1.1122	0.2521	251	0.6684	0.5543
	833	0.7707	0.2645	799	0.3835	0.4614
	3201	0.5323	0.2668	4093	0.1722	0.4799
	12 545	0.3682	0.2658	13 434	0.0954	0.4926
	49 665	0.2554	0.2639	19 860	0.0789	0.4836
$\ell = 1$, $c = -0.25$	65	1.5579	–	65	1.5580	–
	225	1.0981	0.2523	244	0.6777	0.5468
	833	0.7607	0.2647	819	0.3764	0.4670
	3201	0.5254	0.2669	4250	0.1679	0.4817
	12 545	0.3636	0.2655	13 872	0.0933	0.4914
	49 665	0.2524	0.2632	20 430	0.0771	0.4889

TABLE E.15. Numerical estimators and experimental convergence rates for uniform and adaptive mesh refinements for the WOPSIP scheme.

Parameter	Uniform refinement			Adaptive refinement		
	N	ϑ_{P}	Order$_\vartheta$	N	ϑ_{P}	Order$_\vartheta$
$\ell = 1$, $c = 0.25$	65	1.44260	–	65	1.44260	–
	225	1.01044	0.2568	224	0.63839	0.5712
	833	0.69572	0.2692	707	0.36381	0.4674
	3201	0.47830	0.2702	3693	0.16256	0.4780
	12 545	0.32967	0.2684	11 886	0.09062	0.4949
	49 665	0.22799	0.2660	25 249	0.06169	0.5072
$\ell = 1$, $c = -0.25$	65	1.43107	–	65	1.43107	–
	225	0.99988	0.2586	200	0.67548	0.5778
	833	0.68683	0.2709	626	0.38909	0.4585
	3201	0.47156	0.2712	4638	0.14507	0.4833
	12 545	0.32490	0.2687	15 057	0.08091	0.4907
	49 665	0.22476	0.2658	31 644	0.05558	0.5021

E.5. Uniform and adaptive refinements for L-shape domain

TABLE E.16. Numerical errors, estimators and experimental convergence rates for uniform and adaptive mesh refinements for the conforming FEM.

Parameter	Uniform refinement					Adaptive refinement				
	N	Error	Order$_e$	ϑ_{C}	Order$_\vartheta$	N	Error	Order$_e$	ϑ_{C}	Order$_\vartheta$
$\ell = 1$, $c = 0.25$	225	0.3701	–	1.7078	–	225	0.3701	–	1.7078	–
	833	0.2571	0.2782	1.1873	0.2777	1052	0.0744	1.0396	0.3863	0.9635
	3201	0.1789	0.2692	0.8261	0.2694	3428	0.0401	0.5229	0.2101	0.5154
	12 545	0.1247	0.2639	0.5756	0.2644	10 969	0.0220	0.5146	0.1166	0.5064
	49 665	0.0871	0.2605	0.4019	0.2611	23 534	0.0149	0.5083	0.0793	0.5050
	197 633	0.0610	0.2581	0.2811	0.2587	49 550	0.0102	0.5047	0.0546	0.4992
$\ell = 1$, $c = -0.25$	225	0.3701	–	1.7087	–	225	0.3701	–	1.7087	–
	833	0.2571	0.2782	1.1875	0.2779	373	0.1346	2.0008	0.6870	1.8026
	3201	0.1789	0.2692	0.8262	0.2695	2249	0.0495	0.5558	0.2588	0.5433
	12 545	0.1247	0.2639	0.5757	0.2644	7425	0.0268	0.5127	0.1416	0.5049
	49 665	0.0871	0.2605	0.4019	0.2611	23 560	0.0149	0.5075	0.0792	0.5021
	197 633	0.0610	0.2581	0.2811	0.2587	49 628	0.0102	0.5051	0.0546	0.4999

TABLE E.17. Numerical errors, estimators and experimental convergence rates for uniform and adaptive mesh refinements for the Nitsche's method.

Parameter	Uniform refinement					Adaptive refinement				
	N	Error	Order$_e$	ϑ_{N}	Order$_\vartheta$	N	Error	Order$_e$	ϑ_{N}	Order$_\vartheta$
$\ell = 1$ $c = 0.25$	225	0.5064	–	1.3861	–	225	0.5064	–	1.3861	–
	833	0.3514	0.2791	0.9670	0.2750	960	0.0875	1.2100	0.3968	0.8620
	3201	0.2444	0.2695	0.6731	0.2690	9911	0.0238	0.5568	0.1219	0.5053
	12 545	0.1705	0.2638	0.4686	0.2650	31 417	0.0130	0.5212	0.0682	0.5030
	49 665	0.1191	0.2602	0.3267	0.2621	45 551	0.0108	0.5130	0.0566	0.4999
	197 633	0.0834	0.2578	0.2282	0.2597	65 703	0.0089	0.5213	0.0471	0.5055
$\ell = 1$ $c = -0.25$	225	0.5065	–	1.3866	–	225	0.5065	–	1.3866	–
	833	0.3514	0.2792	0.9671	0.2752	637	0.1111	1.4576	0.4886	1.0021
	3201	0.2445	0.2695	0.6731	0.2691	3130	0.0447	0.5712	0.2176	0.5079
	12 545	0.1705	0.2638	0.4686	0.2650	14 728	0.0193	0.5408	0.1002	0.5008
	49 665	0.1191	0.2602	0.3267	0.2621	45 625	0.0107	0.5171	0.0566	0.5046
	197 633	0.0834	0.2578	0.2282	0.2597	65 879	0.0089	0.5180	0.0470	0.5025

TABLE E.18. Numerical errors, estimators and experimental convergence rates for uniform and adaptive mesh refinements for the dG scheme.

Parameter	Uniform refinement					Adaptive refinement				
	N	Error	Order$_e$	ϑ_{dG}	Order$_\vartheta$	N	Error	Order$_e$	ϑ_{dG}	Order$_\vartheta$
$\ell = 1$ $c = 0.25$	65	1.04883	–	1.47777	–	65	1.04883	–	1.47777	–
	225	0.72875	0.2626	1.04639	0.2489	343	0.26840	0.7238	0.53138	0.5431
	833	0.50594	0.2632	0.73238	0.2573	2587	0.09266	0.5122	0.19608	0.4801
	3201	0.35162	0.2624	0.51045	0.2604	5748	0.06128	0.5127	0.13167	0.4937
	12 545	0.24485	0.2610	0.35554	0.2608	12 750	0.04104	0.4987	0.08905	0.4867
	49 665	0.17087	0.2594	0.24790	0.2601	18 577	0.03379	0.5137	0.07361	0.5030
$\ell = 1$ $c = -0.25$	65	1.04942	–	1.47813	–	65	1.04942	–	1.47813	–
	225	0.72888	0.2629	1.04656	0.2490	345	0.26756	0.7233	0.52939	0.5434
	833	0.50597	0.2633	0.73243	0.2574	2752	0.08959	0.5129	0.18974	0.4810
	3201	0.35163	0.2624	0.51046	0.2604	6098	0.05929	0.5135	0.12740	0.4955
	12 545	0.24485	0.2610	0.35555	0.2608	13 439	0.03981	0.5005	0.08639	0.4881
	49 665	0.17087	0.2595	0.24791	0.2601	19 576	0.03277	0.5133	0.07142	0.5025

TABLE E.19. Numerical errors, estimators and experimental convergence rates for uniform and adaptive mesh refinements for the WOPSIP scheme.

Parameter	Uniform refinement					Adaptive refinement				
	N	Error	Order$_e$	ϑ_{P}	Order$_\vartheta$	N	Error	Order$_e$	ϑ_{P}	Order$_\vartheta$
$\ell = 1$, $c = 0.25$	65	0.61779	–	1.31475	–	65	0.61779	–	1.31475	–
	225	0.38684	0.33769	0.94188	0.24059	279	0.27775	0.47867	0.50107	0.57762
	833	0.25108	0.31179	0.65818	0.25852	2128	0.09176	0.52961	0.18357	0.48018
	3201	0.16762	0.29147	0.45701	0.26314	4806	0.05996	0.51699	0.12334	0.48323
	12 545	0.11405	0.27779	0.31717	0.26348	15 546	0.03339	0.49461	0.06905	0.49015
	49 665	0.07854	0.26905	0.22047	0.26234	22 729	0.02713	0.54317	0.05710	0.49682
$\ell = 1$, $c = -0.25$	65	0.61736	–	0.93591	–	65	0.61736	–	1.31258	–
	225	0.38671	0.33744	0.66851	0.24271	279	0.27787	0.47801	0.50088	0.57685
	833	0.25105	0.31165	0.46636	0.25975	2141	0.09146	0.52980	0.18294	0.48021
	3201	0.16761	0.29141	0.32351	0.26382	4827	0.05963	0.52077	0.12284	0.48497
	12 545	0.11404	0.27777	0.22440	0.26385	15 557	0.03317	0.49739	0.06891	0.49007
	49 665	0.07854	0.26904	0.15594	0.26254	22 735	0.02712	0.52626	0.05703	0.49501

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