Verified Bounds on the Imprecise Failure Probability with the SIVIA Algorithm

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Preamble Verified numerical methods empower scientific computing to solve challenging mathematical problems whilst providing error guarantees. Differently from other methods, in verified methods, the error is a constituent component, it is contextually quantified and it can be reduced with more computational effort. These methods have been used to find rigorous—thus proven to be correct—solutions to notoriously difficult problems. An example of a challenging mathematical problem in engineering is the computation of the failure probability. Computing the failure probability of a complex system involves the solution of high dimensional integrals over a domain characterised by the preimage of a nonlinear function. Because of the complexity of this task, analytical rigour is sacrificed to find suitable numerical approximations. With verified methods the failure probability can be bounded numerically, while controlling the magnitude of the approximation.

Motivations Bounding the failure probability of a complex system with the desired precision means being able to provide an interval where the reference solution resides with a given level of confidence. So far, only the Monte Carlo method has been used to provide reference values to failure probability problems whose analytical solution is not possible. Monte Carlo however, has a huge limitation: the desired precision cannot be reached on problems with very small target failure probability. Moreover, the efficiency of Monte Carlo deprecates when the problem is formulated to allow imprecise probability. Several adaptations of Monte Carlo have been proposed to allow for imprecise probability but none with error guarantees [1, 3]. Monte Carlo variations that allow for imprecision, e.g. second-order Monte Carlo, tend to underestimate the incertitude.

The SIVIA algorithm The SIVIA (Set Inversion Via Interval Analysis) algorithm uses interval arithmetic to build a rigorous sub-tiling of the domain whereon the joint density distribution must be integrated to yield the failure probability. This sub-tiling is often also called a sub-paving because its tiles are boxes. Even though this rigorous sub-paving is only possible on problems involving a handful of dimensions (up to seven), in a recent paper [2], we showed how this characterisation can be used to bound the failure probability integral with mathematical certainty on problems involving precise probability.

Imprecise probability In this work, we explore the algorithms making use of SIVIA on failure probability problems formulated with imprecision. Because of the imprecision the integration over the rigorous sub-tiling can no longer be done using the antiderivative of the joint or copula density, a.k.a. h-volume. With random-set independence and on small problems (< three random variables), the imprecise failure probability can be obtained with the space product of marginal focal elements, by evaluating the focal elements' intersections with the failure sub-paving. The joint focal elements that intersect with the failure sub-paving correspond to the Plausibility—failure probability upper bound. The Belief—failure probability lower bound—can be obtained using the conjugate relationship on the remaining joint focal elements. This procedure is also valid when dependence is expressed in the form of a precise copula. On larger problems (four to seven variables), the space product is no longer possible and could be replaced by random slicing. The approximation introduced by the random slicing is controlled by a given level of confidence, which typically decreases the more slices are evaluated.

References

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