Verified bounds on the imprecise failure probability with the SIVIA algorithm

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ISIPTA 2023, July, Oviedo

Abstract

In this work, we explore the use of SIVIA (Set Inversion Via Interval Analysis) on failure probability problems formulated with imprecision. Because of the imprecision the integration over the rigorous sub-paving can no longer be done only using the antiderivative of the joint or copula density, a.k.a. h-volume, because of sub-additivity. Under random-set independence, or precise copulas and on small problems (≤ 3 variables), the imprecise failure probability can be obtained counting the intersections with the sub-pavings of all focal elements in the space product. The joint focal elements that are fully contained in the failure sub-paving correspond to the belief—failure probability lower bound. On larger problems, the space product is no longer possible so it can be replaced by random slicing. The approximation introduced by the random slicing is controlled by a given level of confidence, which typically decreases the more slices are evaluated.

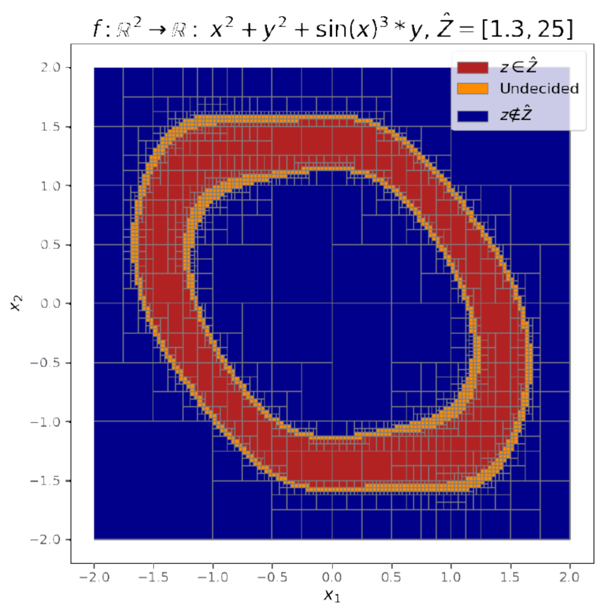
Set Inversion Via Interval Analysis (SIVIA)

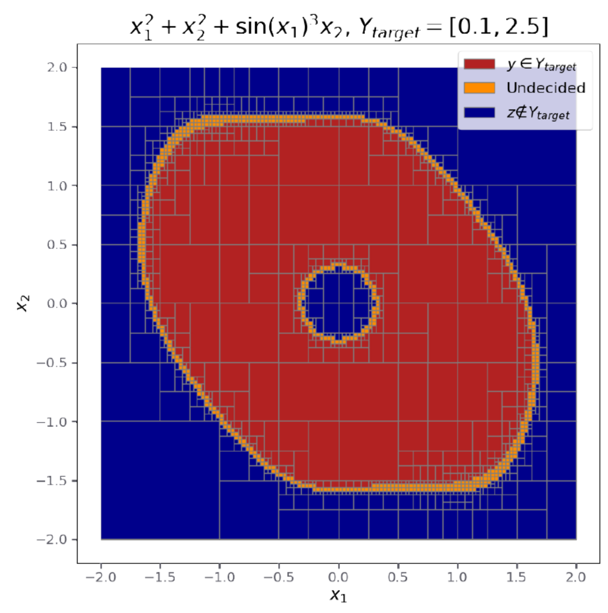
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function, and let $y := f(x)$. Let $X \subset \mathbb{R}^d$ be a n-box and $\mathcal{Y}$ the image of X under f. Now, let us define $Y_{\text{target}} \subset \mathcal{Y}$ a given sub interval of $\mathcal{Y}$. The SIVIA algorithm, by means of an iterative bisection, bounds the preimage $\mathcal{A} \subset X$ such that $f(\mathcal{A}) = Y_{\text{target}}$:

$$\mathcal{A} = \bigcup_{i}^{\infty} \{X_i : f(X_i) \subset Y_{\text{target}}\}. \tag{1}$$

SIVIA then creates a finite tiling of the original domain X known to contain $\mathcal{A}$. On each sub-box X_i of the tiling, f is evaluated with the rules of interval arithmetic. The finite tiling is a rigorous inner approximation of preimage $\mathcal{A}$:

$$\bigcup_{i}^{n} \{X_i : f(X_i) \subset Y_{\text{target}}\} \subset \mathcal{A}. \tag{2}$$





Precise failure probability with SIVIA

The *failure probability* is the probability that the output random variable Y, element of a Borel space, is in the interval $[y_t, \infty)$, that is $p_{\mathrm{F}} = \mathbb{P}_{\mathrm{Y}}(\mathrm{Y} \in [y_t, \infty))$. In reliability, the analytical closed-form distribution of Y is unknown, because f is too complex. Often, because the distribution of the random variable vector X is known, the failure probability is computed as: $p_{\mathrm{F}} = \mathbb{P}_X(\mathrm{X} \in \Omega_{\mathrm{F}})$, where Ω_{F} is the preimage of Y_{target} over f, a.k.a. *failure region*. A lower tiling of the failure domain Ω_F is:

$$\underline{\Omega_{\mathrm{F}}} = \bigcup_{i=1}^{n} \{X_i : f(X_i) \subset Y_{\text{target}}\}, \tag{3}$$

where X_i are the algorithm's sub-boxes. (3) is called an inner tiling because a subset of the actual failure domain, $\underline{\Omega_{\mathrm{F}}} \subset \Omega_{\mathrm{F}}$. The tiling:

$$\Omega_{\mathrm{E}} = \bigcup_{i}^{n} \{X_i : f(X_i) \cap Y_{\text{target}} \neq \varnothing, X_i \notin \underline{\Omega_{\mathrm{F}}}\}, \tag{4}$$

is an outer approximation of the boundary between the failure domain and its complement. The lower bound failure probability $\underline{p_{\mathrm{F}}} = \mathbb{P}_X(\underline{\Omega_{\mathrm{F}}})$, is

$$\mathbb{P}_X(\underline{\Omega_{\mathrm{F}}}) = \sum_{i}^{n} \mathbb{P}_X\left(\{X_i : f(X_i) \subset Y_{\text{target}}\}\right).$$

$$\overline{p_{\mathrm{F}}} = \mathbb{P}_X(\underline{\Omega_{\mathrm{F}}}) + \mathbb{P}_X(\Omega_{\mathrm{E}}) \tag{5}$$

$$\mathbb{P}_X(\Omega_{\mathrm{E}}) = \sum_{i}^{n} \mathbb{P}_X(\{X_i : f(X_i) \cap Y_{\text{target}} \neq \varnothing, X_i \notin \underline{\Omega_{\mathrm{F}}}\}).$$

Imprecise failure probability: Joint mass

$$V_{C_X}([\underline{u}, \overline{u}], [\underline{v}, \overline{v}]) = C_X(\overline{u}, \overline{v}) - C_X(\overline{u}, \underline{v}) - C_X(\underline{u}, \overline{v}) + C_X(\underline{u}, \underline{v})$$

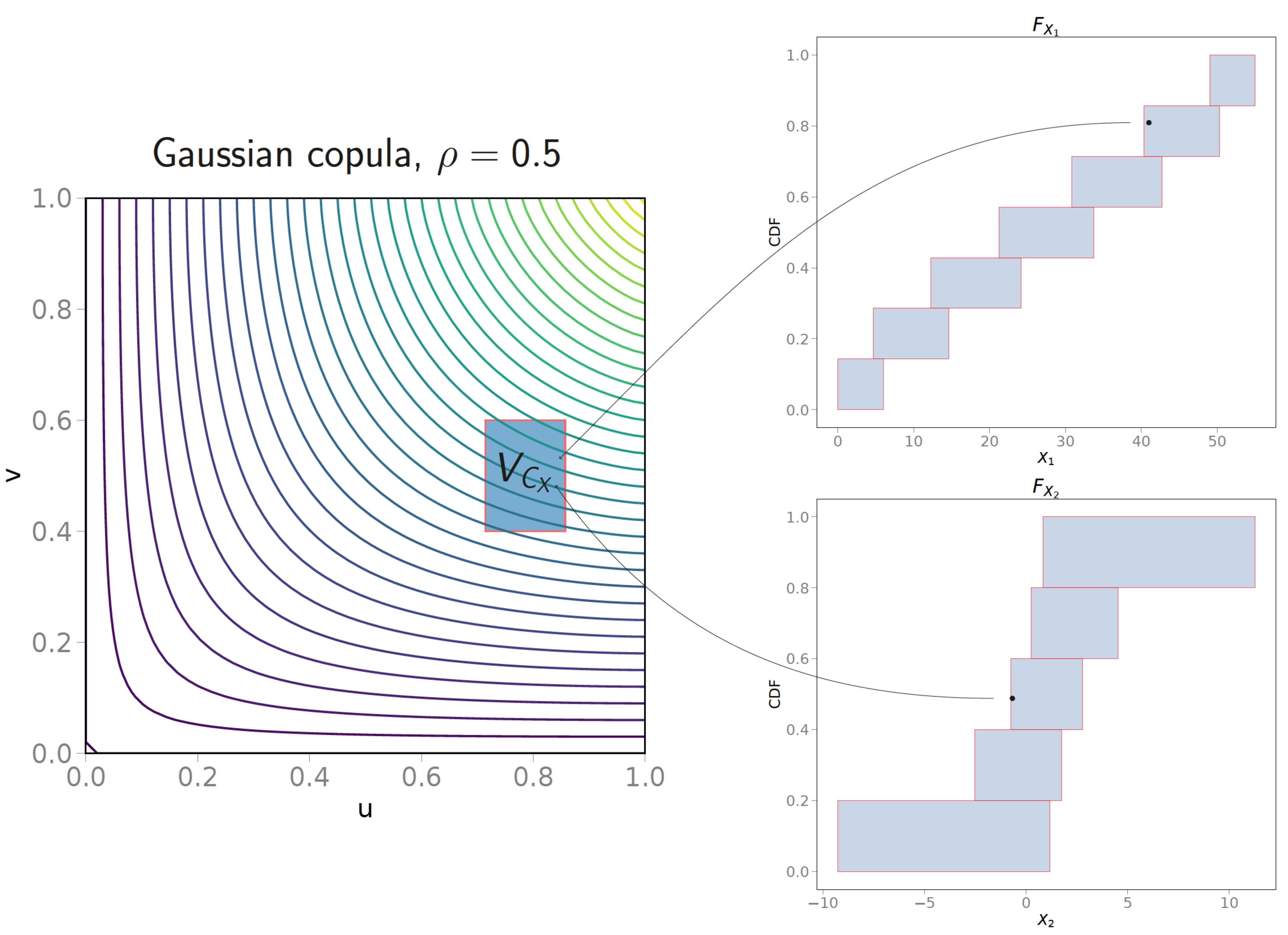


Figure 1: H-volume evaluation in the copula space from the marginal p-boxes.

Example. $p_F = [0.03907,\ 0.05237]$ with 301 marginal focal elements

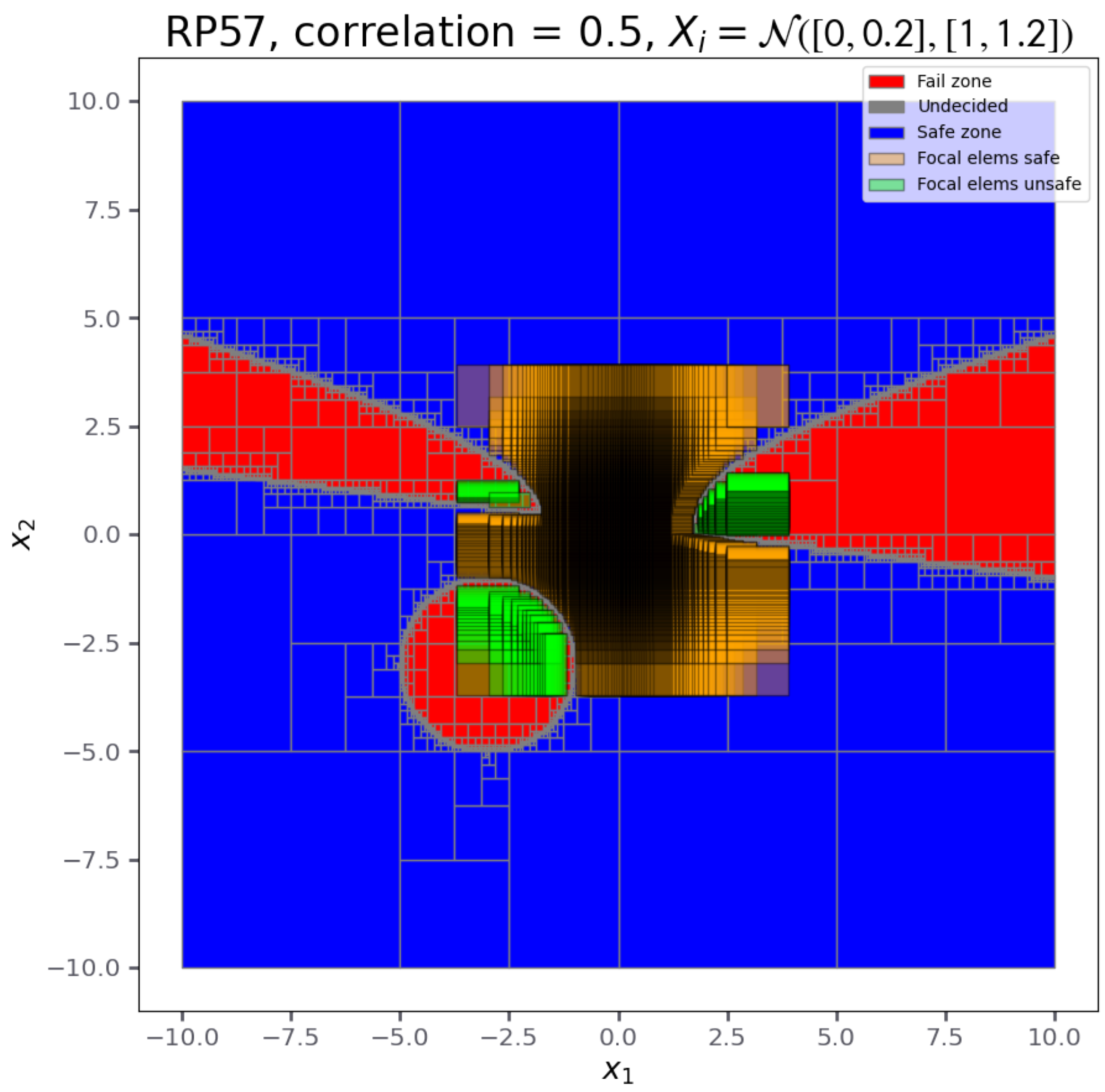


Figure 2: Composite performance function:
$g_1(x) = -x_1^2 + x_2^3 + 3, \quad g_2(x) = 2 - x_1 - 8x_2, \quad g_3(x) = (x_1+3)^2 + (x_2+3)^2 - 4,$
$g(x) = \min(\max(g_1, g_2), g_3),$

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